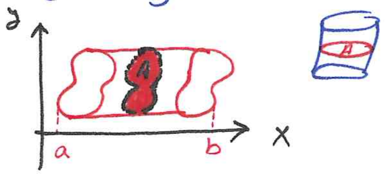
6.1 Volumes Using Cross-Sections

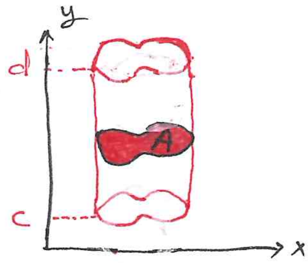
* The volume of cylindrical solid is V = base area × height = Ah



* To Find Volume of a Solid:
  1- Graph the solid
  2- Determine a cross-section of the solid

(a) • If the cross-sectional is perpendicular to the x-axis, then the volume is

$$V = \int_a^b A(x)\,dx$$

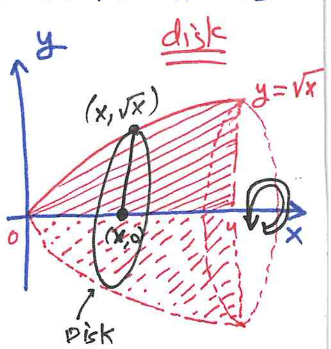


(b) • If the cross-sectional is perpendicular to the y-axis, then the volume is

$$V = \int_c^d A(y)\,dy$$

Disk Method (c) • If the cross-sectional is perpendicular to the x-axis results by rotation about x-axis, then

$$V = \int_a^b \underbrace{A(x)}_{\text{Disk}}\,dx = \int_a^b \pi\,[\underbrace{R(x)}_{\text{radius}}]^2\,dx$$



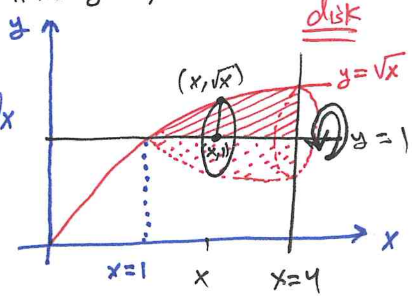
Example: Find the volume of the region between the curve $y=\sqrt{x}$, $0 \le x \le 4$ and the x-axis that is revolved about the x-axis.

$$V = \int_0^4 \pi(\sqrt{x})^2\,dx = \pi\int_0^4 x\,dx = \pi\frac{x^2}{2}\Big|_0^4 = 8\pi$$

Example: Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1$, $x=4$ about the line $y=1$.

$$V = \int_1^4 \pi(\sqrt{x}-1)^2\,dx = \pi\int_1^4 (x - 2x^{\frac{1}{2}} + 1)\,dx$$

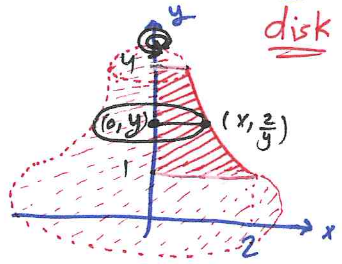
$$= \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{\frac{3}{2}} + x\right]\Big|_1^4 = \frac{7\pi}{6}$$



Disk Method (d) • If the cross-sectional is perpendicular to the y-axis results by rotation about y-axis, then

disk

$$V = \int_c^d \underset{\Downarrow \text{ Disk}}{A(y)}\, dy = \int_c^d \pi [R(y)]^2 dy$$



Example: Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$ about the y-axis.

when $y = 1 \Rightarrow x = 2$

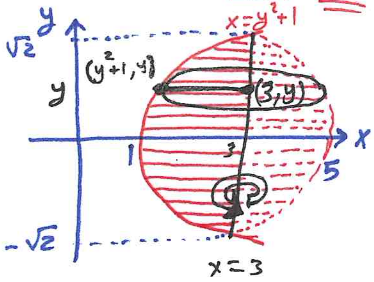
$$V = \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy = \pi \int_1^4 \frac{4}{y^2} dy = 4\pi \left.\frac{y^{-1}}{-1}\right|_1^4 = 3\pi$$

Example: Find the volume of the solid generated by revolving the region between $x = y^2+1$ and the line $x=3$ about the line $x=3$.

disk

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi [3 - y^2 - 1]^2 dy = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (2-y^2)^2 dy$$

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 4y^2 + y^4)\, dy = \frac{64\pi\sqrt{2}}{15}$$



$y^2+1 = 3 \Leftrightarrow y^2 = 2 \Leftrightarrow y = \pm\sqrt{2}$

Washer Method (e) • If the cross-sectional is perpendicular to x-axis results by rotation about x-axis with outer radius $R(x)$ and inner radius $r(x)$, then

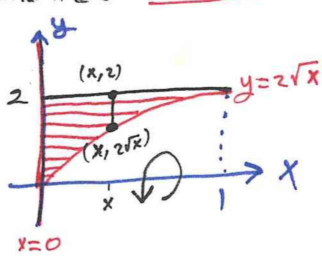
$$V = \int_a^b A(x)\, dx = \int_a^b \pi [R^2(x) - r^2(x)]\, dx$$

Example: Find the volume of the solid generated by revolving the region bounded by $y = 2\sqrt{x}$, $y = 2$ and $x = 0$ about the x-axis.

$$V = \pi \int_0^1 \left[(2)^2 - (2\sqrt{x})^2\right] dx = \pi \int_0^1 [4 - 4x]\, dx$$

$$= 4\pi \int_0^1 (1-x)\, dx = 4\pi \left[x - \frac{x^2}{2}\right]_0^1$$

$$= 2\pi$$

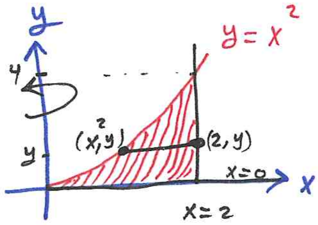


no disk

Washer Method (f): If the cross-sectional is perpendicular to y-axis results by rotation about y-axis with outer radius $R(y)$ and inner radius $r(y)$, then 111

$$V = \int_c^d A(y)\,dy = \int_c^d \pi\left[R^2(y) - r^2(y)\right] dy$$

__Example__: Find the volume of the solid generated by revolving the region, bounded by $y = x^2$, x-axis, and $x = 2$ about the y-axis.
in the first quadrant



$R(y) = 2$, $r(y) = \sqrt{y}$

$$V = \pi \int_0^4 \left[2^2 - (\sqrt{y})^2\right] dy$$

$$= \pi \int_0^4 (4 - y)\,dy = \pi\left[4y - \frac{y^2}{2}\right]\Big|_0^4 = 8\pi$$