Chapter -7-

design of digital filter

$x(n) \longrightarrow \boxed{LTI} \longrightarrow y(n)$

complex exp. is an eigenfunction of LTI systems.

$$e^{j\omega_0 n} \longrightarrow H(e^{j\omega_0})\, e^{j\omega_0 n}$$

Changed

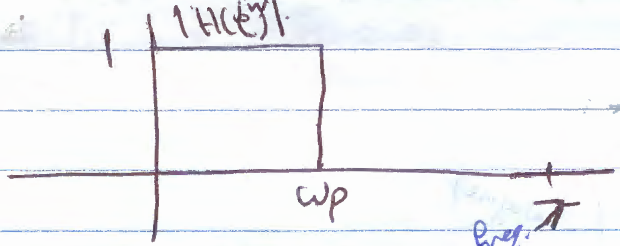
$$\cos \omega_0 n \longrightarrow |H| \cos(\omega_0 n + \theta)$$

input cos → out is also cos but with change in Magnitude and change in Phase

Ex

* assume we have a signal which ~~consists~~ sum of two signals of different frequencies. suppose we want to get at output one signal and suppress the other.

This means, we need $|H| = 1$ for signal of wanted freq. (low) and $|H| = 0$ for the other (high freq.)



This is one class of filters called freq. selective filters (low-pass, high-pass, bandpass, ...etc).

Filtering Problem desire

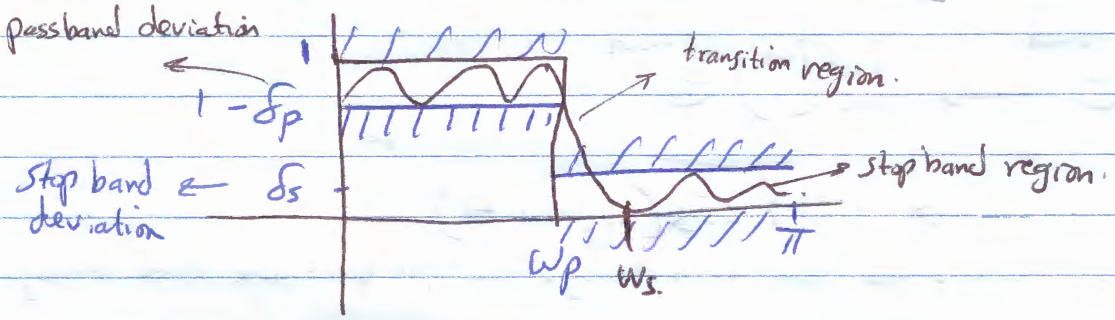
In general, freq. response is linear with freq. (e.g. differentiator)

so, in this chapter, ~~we will design~~ implementation of LTI system a filter (LTI system) which ~~achieve~~ acheive the desire freq.-response

* digital filter design is not ~~for~~ only for selective filters.

* rational sys. function or lin. difference constant eq. ⇒ cannot acheive the ideal selective filter

* So we have to allow some deviation of ideal filter



digital filter design Problem is:

to design a rational transfer function which approximates ~~these specifications~~ this ideal filter maintaining these specifications of passband and stopband deviations.

* Design Techniques

① analytical. (analogue)

② Continuous time → a lot of work (procedures) are done for analogue filters, to take advantage of these procedures.

↳ discrete time

objective is not to approximate analogue by digital filter

③ algorithmic (computer-aided)

* techniques for IIR and others for FIR

called recursive (IIR), non recursive (FIR)

IIR (Mapping from analogue to digital)

* Continuous $\xrightarrow{map}$ discrete

This good filter in analogue domain → so, this will be good filter in digital domain

Laplace $H_a(s) \xrightarrow{Map} H(z)$

Impulse resp. $h_a(t) \longrightarrow h(n)$

Conditions

① $j\Omega$-axis (s-plane) (freq. response) $\Rightarrow$ unit circle (z-plane).

③ $H_a(s)$ stable $\Rightarrow$ $H(z)$ stable.

First method: (Mapping)

Differentials (analogue) $\longrightarrow$ differences (discrete)

analogue $H_a(s)$

$$\sum_{k=0}^{N} c_k \frac{d^k y_a(t)}{dt^k} = \sum_{k=0}^{M} d_k \frac{d^k x_a(t)}{dt^k}$$

linear comb. of derivatives of output — lin. comb. of derivatives of input

$$y_a(t) \longrightarrow y(n)$$

$$\left.\frac{d y_a(t)}{dt}\right|_{t=nT} \longrightarrow \Delta^{(1)}[y(n)] = \frac{y(n+1) - y(n)}{T}$$

(forward difference)

$$\Delta^{(1)}[y(n)] = \frac{y(n+1) - y(n)}{T}$$

we want to replace derivatives by forward differences

1st derivative $\longrightarrow$ first difference

more generally, $\frac{d^k y_a(t)}{dt^k} \longrightarrow \Delta^{(k)}[y(n)]$

kth derivative — kth diff.

$$\Delta^{(k)}[y(n)] = \Delta^{(1)}[\Delta^{(k-1)} y(n)]$$

$$\sum_{k=0}^{N} c_k \Delta^{(k)}[y(n)] = \sum_{k=0}^{M} d_k \Delta^{(k)}[x(n)]$$

lin. k^{th} diff. output   lin. c k^{th} diff input

$$\mathcal{L}\left[\frac{d y_a(t)}{dt}\right] = s Y_a(s)$$

differentiation in time domain → multiplication by s in s-domain

$$Z\left[\frac{y(n+1) - y(n)}{T}\right] = \frac{z-1}{T} Y(z)$$

first diff.

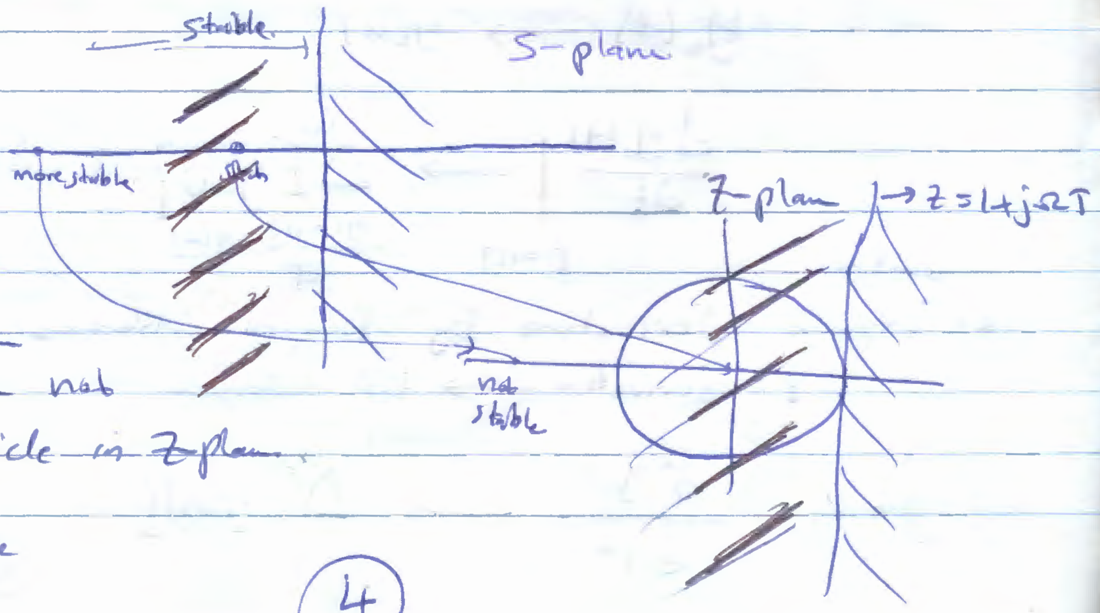
$$H(z) = H_a(s)\Big|_{s=\frac{z-1}{T}} \quad \text{(mapping)}.$$

$$s = \frac{z-1}{T}, \quad z = 1 + sT$$

map from s-plane to z-plane.

at $s = j\Omega$

$z = 1 + j\Omega T$



* This is not good because

① $j\Omega$-axis in s-plane not mapped into unit circle in Z-plane.

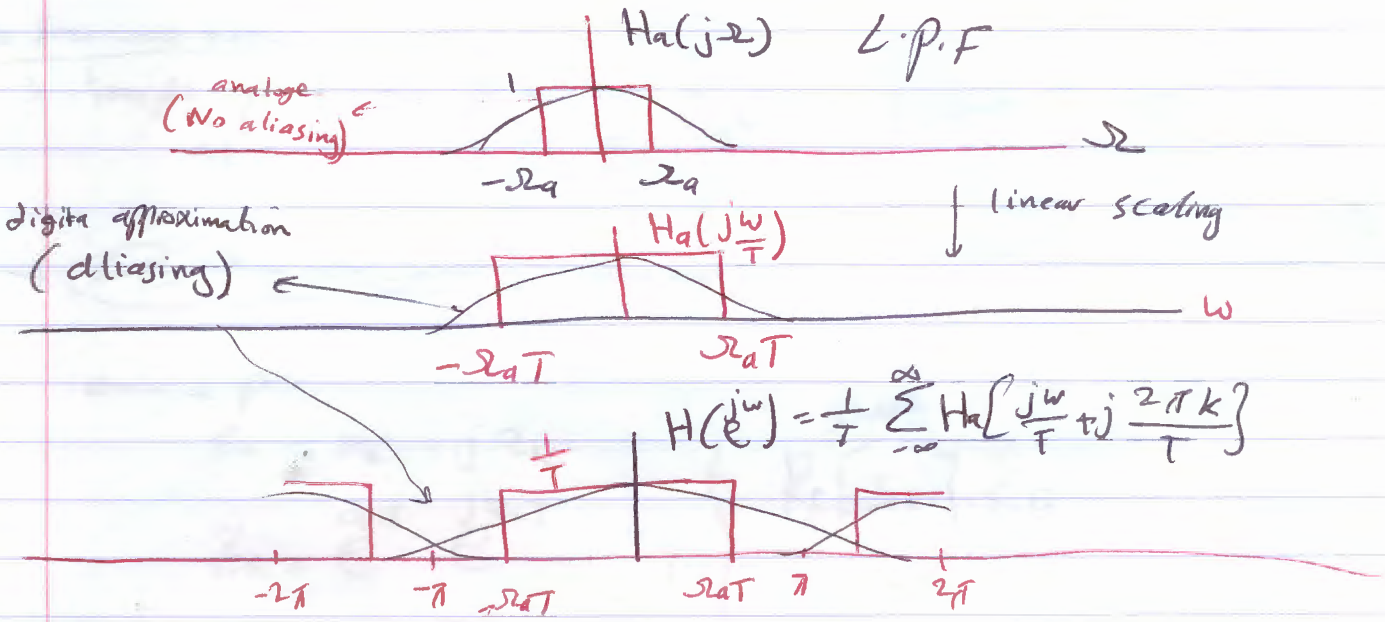
② stable (s) → not stable (Z)

Impulse Invariance

Convert analogue filter to digital filter.
s.t.

$$h(n) = h_a(nT)$$ samples $h_a(t)$, T space between.

$$H(e^{jw}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left[\frac{jw}{T} + j\frac{2\pi k}{T}\right]$$



$$H(e^{jw}) = \frac{1}{T}\sum_{-\infty}^{\infty} H_a\left[\frac{jw}{T} + j\frac{2\pi k}{T}\right]$$

+ problem of this technique:
We start with the desired analogue filter characteristic, and we endup with some distortion on digital filter due to the aliasing. Introduced.

* this technique. objective. → reduce effect of aliasing.

advantages:
→ linear scaling from analogue freq. to digital freq.
→ good characteristics in the anologue ~~charactor~~ filter will be reserved.

look at S-plan ⟶ Z-plane. of this technique

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{S - S_k}$$ (consider. simple poles)

$$h_a(t) = \sum_{k=1}^{N} A_k e^{S_k t} u(t)$$

$$h(n) = h_a(nT) = \sum_{k=1}^{N} A_k e^{S_k nT} u(n)$$

$$= \sum_{k=1}^{N} A_k \left(e^{S_k T}\right)^n u(n)$$

→ by inspection & write sys. func.

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{S_k T} z^{-1}}$$, same coeff. , same poles as for $H_a(s)$.

A_k → residues of poles.

Impulse Invariance method ⇒

→ maps pole at $s = S_k$ to pole in z-plane at $z = e^{S_k T}$

doesn't map $H_a(s) \longrightarrow H(z)$ using this mapping but it maps poles → poles

→ ~~Same thing for zeros~~ not for zeros.

analogue pole

$$S_k = \sigma_k + j\Omega_k$$

$$z_k = e^{\sigma_k T} e^{j\Omega_k T}$$

$$|z_k| = |e^{\sigma_k T}| \, |\cancel{e^{j\Omega_k T}}|$$ → 1

stable

$$Re[S_k] < 0$$

$$\Downarrow$$

$$|z_k| < 1$$

adv. and disadv. (advantages and disadvantages by this Example)

Ex

$$H_a(s) = \frac{s+a}{(s+a)^2 + b^2} = \frac{1/2}{s+a+jb} + \frac{1/2}{s+a-jb}$$

maintain residues, pole $-a-jb$, pole $-a+jb$

map

$$H(z) = \frac{1/2}{1 - e^{-aT} e^{-jbT} z^{-1}} + \frac{1/2}{1 - e^{-aT} e^{jbT} z^{-1}}$$

re-combine

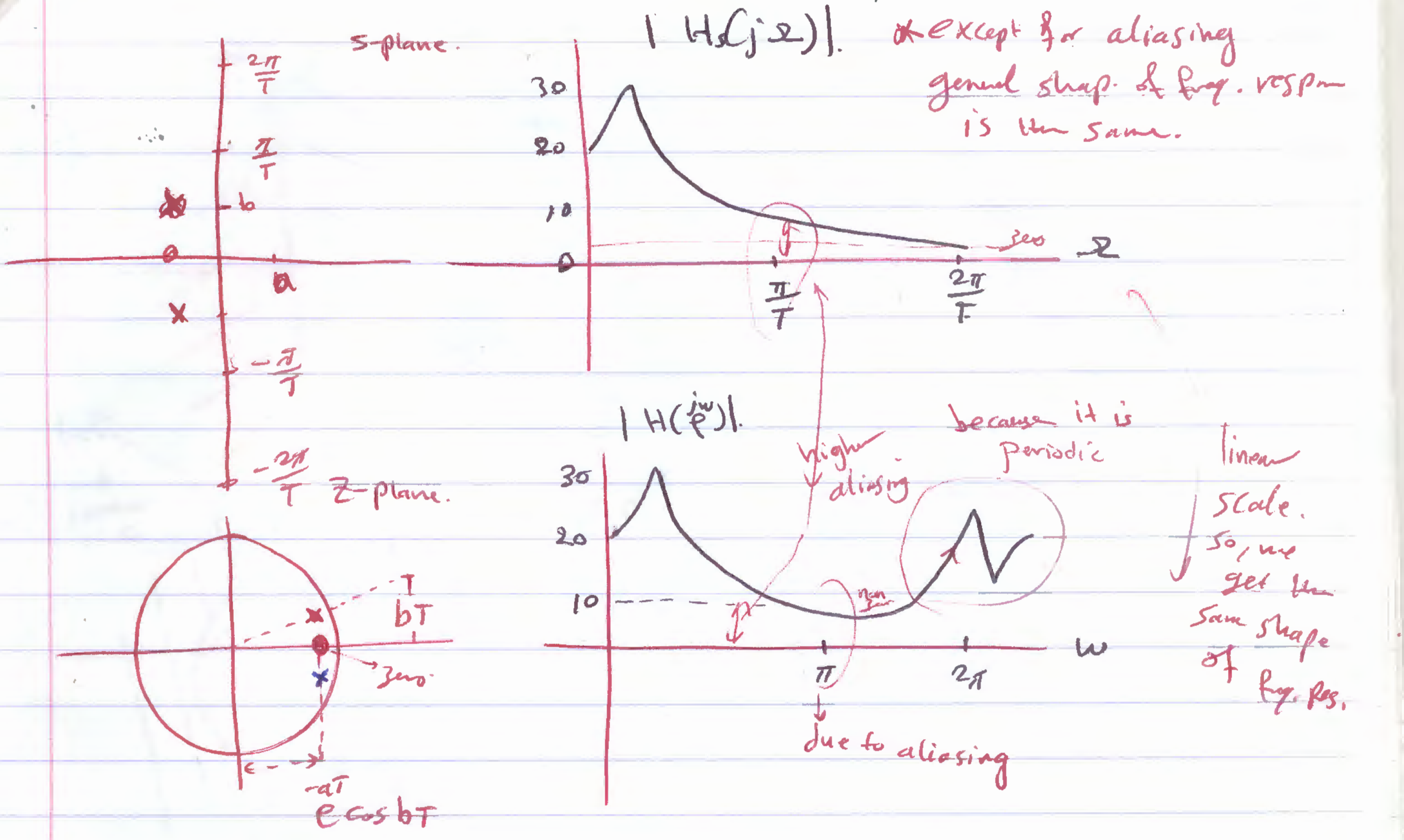
$$= \frac{1 - \left(e^{-aT} \cos bT\right) z^{-1}}{(1 - e^{-aT} e^{-jbT} z^{-1})(1 - e^{-aT} e^{jbT} z^{-1})}$$

Important notes

poles mapped as indicated

zero at $(-a)$ mapped to $e^{-aT} \cos bT$

This example in freq. domain (7)



* Bilinear Transformation [very important and very common]

$$H_a(s) \overset{map}{\Longrightarrow} H(z)$$

$$S = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \rightarrow \text{bilinear Transform.}$$

comes from applying integration ($\int$)

(book) derivation not req.
differentials eq. ⇒ Integration → approximate $\int$ to by sums ⇓ digital filter ⇐ difference eq.

$$Z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$$

preserves freq. characteristics

Let's ① check if this transformation maps $j\Omega$ axess to unit circle in Z-plane and ② maps stable analogue sys. to stable digital filter.

~~let's~~ look at unit-circle.

for $z = e^{jw}$ (unit circle)

* if z is on unit circle then s is on $j\Omega$ axis

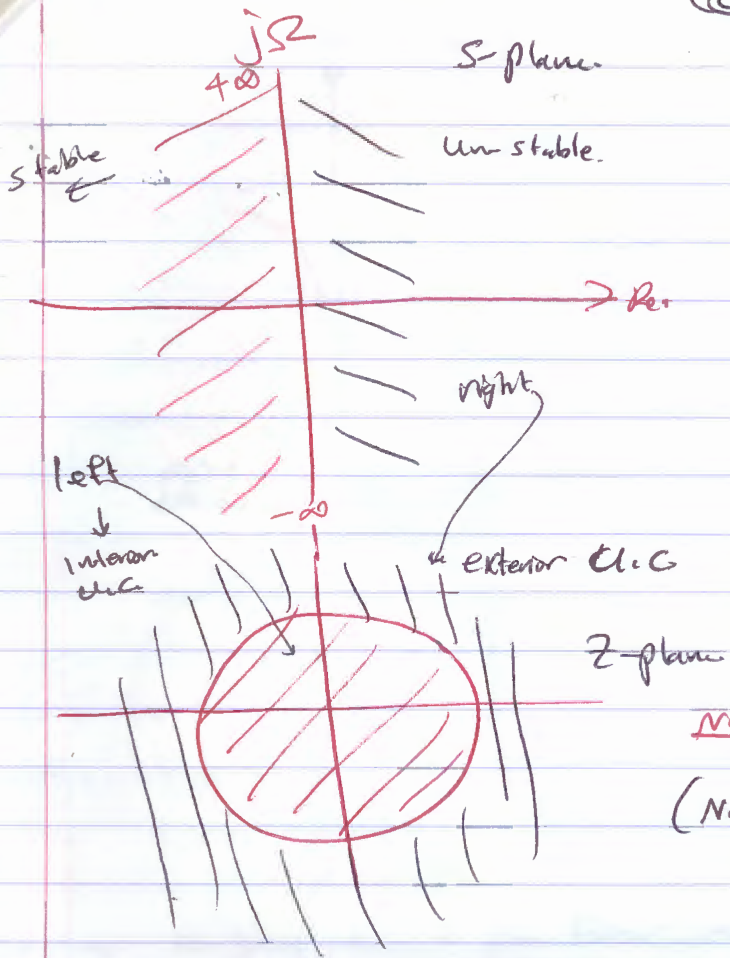
$$S = \frac{2}{T}\left[\frac{1-e^{-jw}}{1+e^{-jw}}\right] \qquad e^{-j\frac{w}{2}}\left[e^{+j\frac{w}{2}} - e^{-j\frac{w}{2}}\right]$$

same thing.

$$= \frac{2}{T} j \tan\frac{w}{2} = j\Omega \text{ real no.}$$

since $w \to 0 \to 2\pi$ always real no.

so, $j\Omega$ axis ⟺ unit circle.

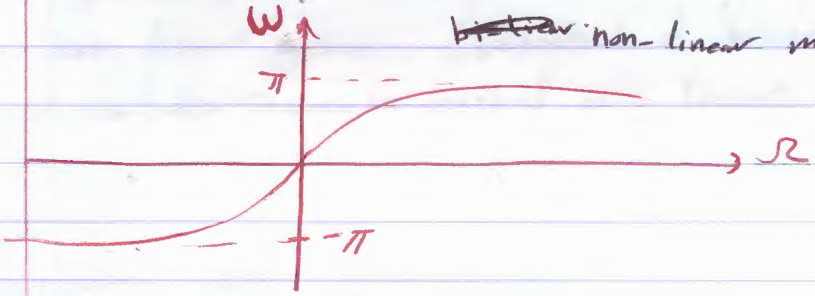


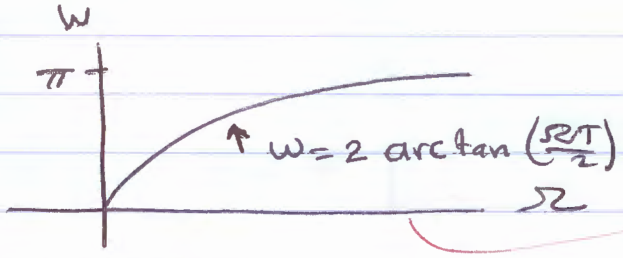
Notes:

(No aliasing).

Entire $j\Omega$-axis → Unit circle once
no overlapping

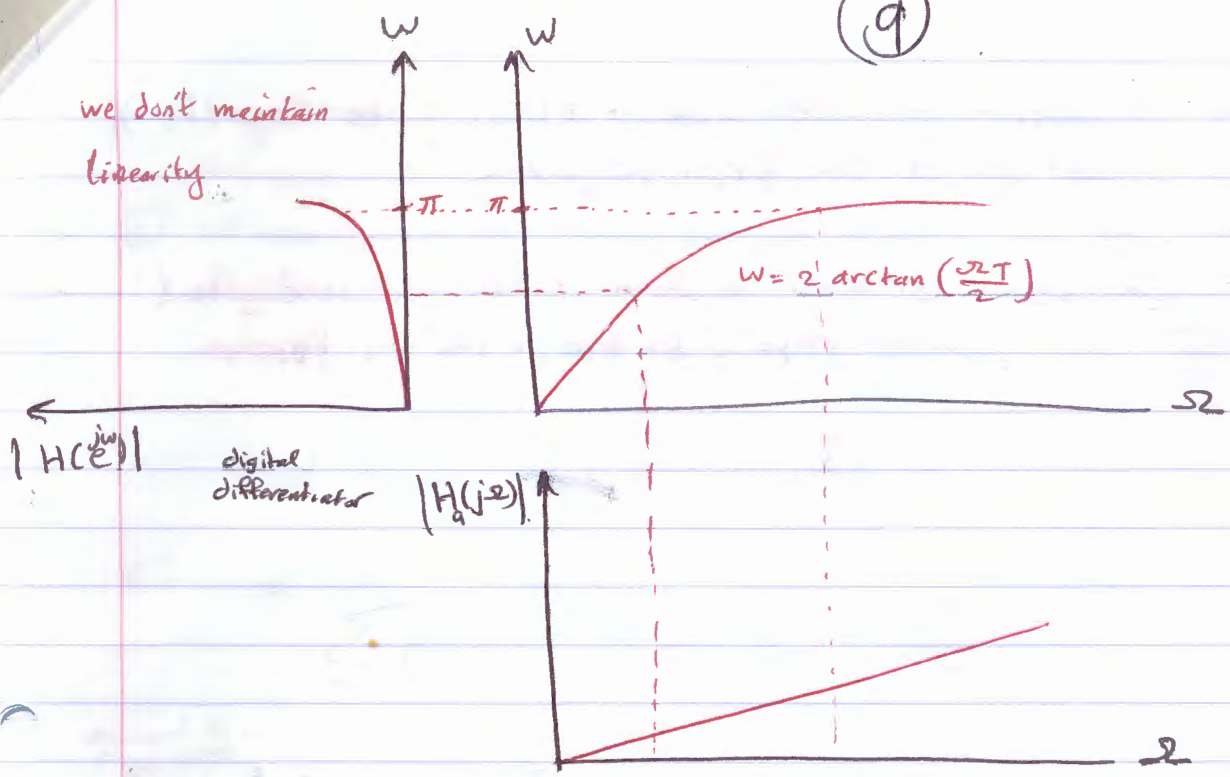
~~bilinear~~ non-linear mapping





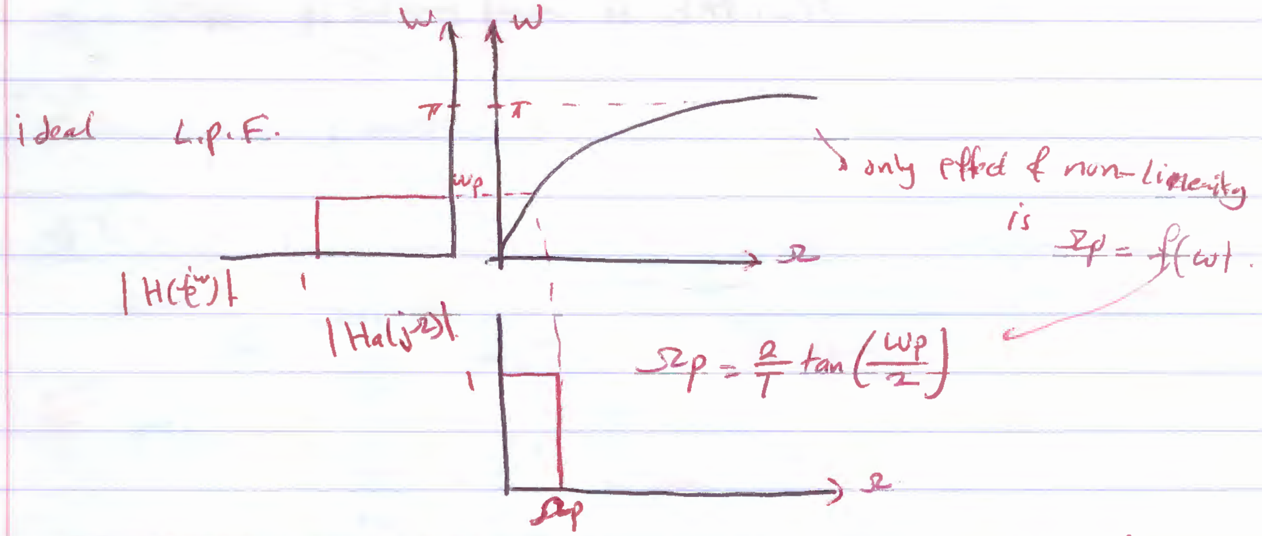
mapped to upper half or unit circle

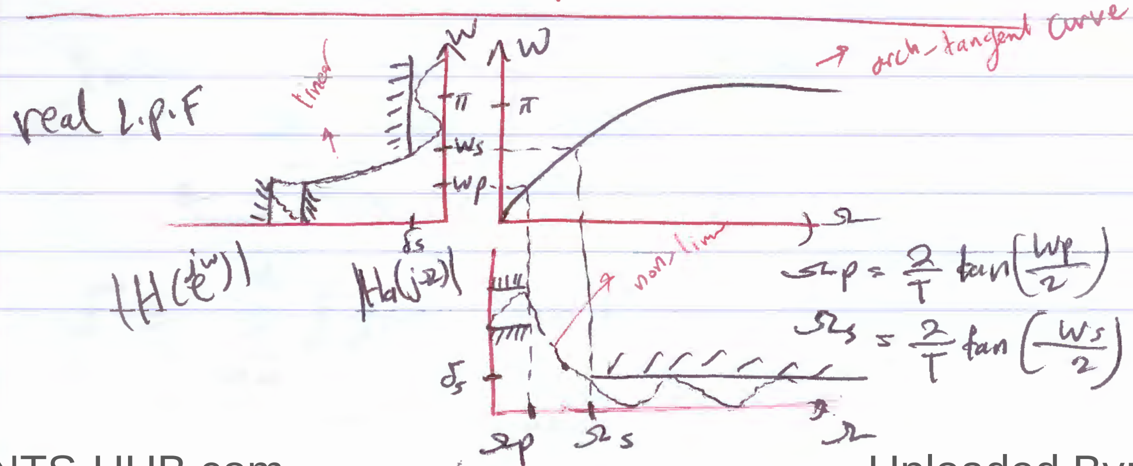
(there is some distortion in this transformation).



analogue linear differentiator

because bi-liear is a non-linear mapping, linear analogue is mapped to a non-linear system. (Drawback of bilinear transformation).
but impulse Invarianca method is a linear mapping.





* bi-linear transform → we avoid aliasing in the price of distortion on the freq. response

* so, we can use Bi-linear transform to design a digital filter that can tolerate these distortion on the freq. response

* Alegorithmic techniques for desiging IIR. ther are alogrithmic techniques used for designing analogue filters that can be used also for designing digital filters

① Minimization of mean square error. between the desired freq. response and the actual one. So, No need to design it on analogue then map it to digit.

* we apply these technique to design digital filter.

$H_d(e^{jw}) =$ Desired Freq. Response

$H_d(e^{jw_i}) \quad i = 1, 2, \ldots, M$

desire freq

$$\text{Error} = \sum_{i=1}^{M} \left[\left| \underset{\text{actual}}{H(e^{jw_i})} \right| - \left| \underset{\text{desired}}{H_d(e^{jw_i})} \right| \right]^2$$

general form of H(z)

rational function

$$H(z) = A_k \prod_{k=1}^{k} \frac{1 + a_k z^{-1} + b_k z^{-2}}{1 + c_k z^{-1} + d_k z^{-2}} \quad \text{(Cascade form)}$$

① specify order of the filter

Choose parameters of H(z) to minimize Error.

Difficulty: rational func. in Error. eq. yeilds non-linear equations. Solving them algorithmically is difficult.

(Commonly used method).

② least ~~Invariance~~ squares Inverse design.

specify hd(n)

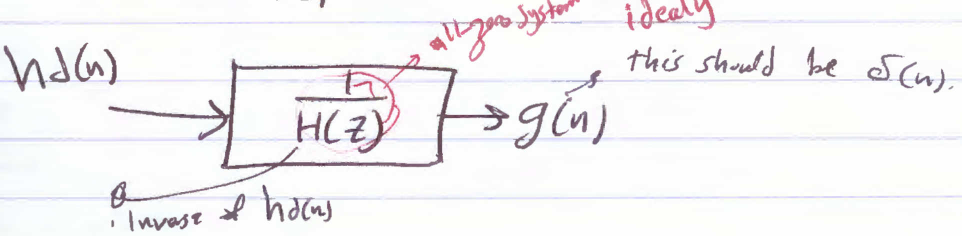
e.g

$$H(z) = \frac{b_0}{1 - \sum_{k=1}^{N} a_k z^{-k}} \Leftrightarrow h(n)$$

No zeros

impulse response

only poles

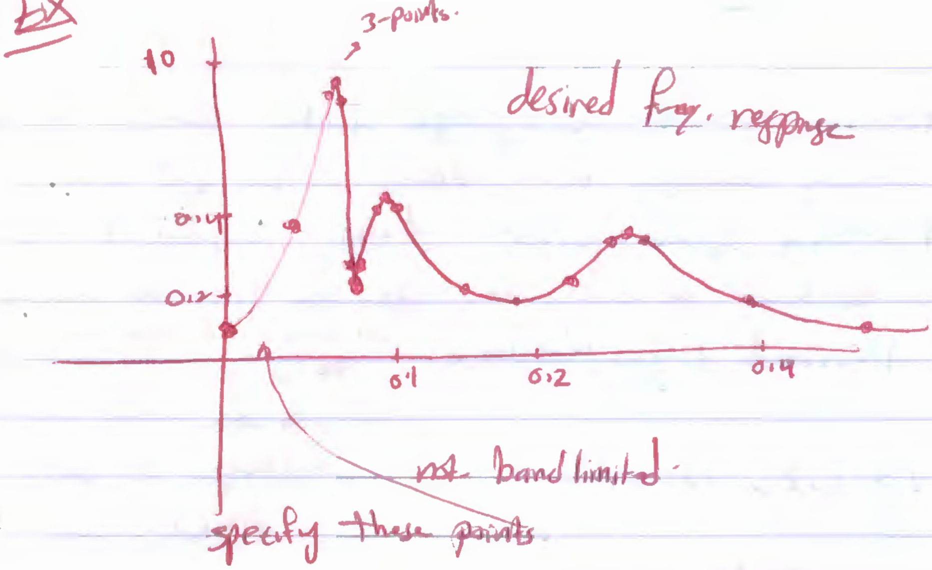


$$E = \sum_{n=0}^{\infty} [g(n) - \delta(n)]^2$$

ideal

Linear equations ⇒ simpler to solve

Ex



3-order second sections
(hard example) → time 2.5 min. computations

* this method allows for generic freq. response

* non-linear eq's. ⇒ difficult to solve.

Design.

Example using Impulse Invariance and bi-Linear Transform

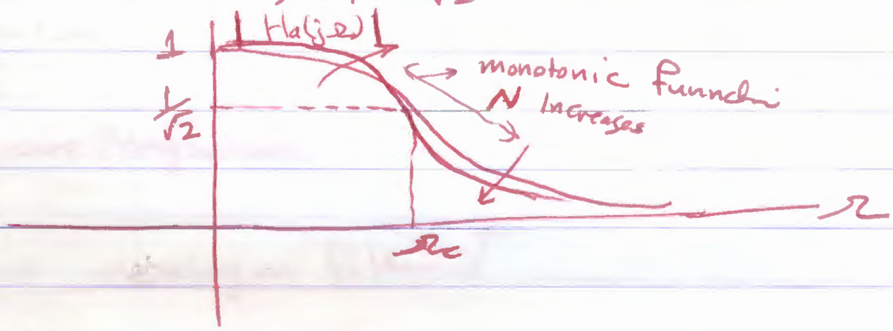
The class of analogue Butterworth Filter (L.P.F.)

$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

two parameters: ① Ω_c → cutoff freq. ② N ? → order of filter

at $\Omega = 0 \Rightarrow |\ |^2 = 1$

at $\Omega = \Omega_c \Rightarrow |\ | = \frac{1}{\sqrt{2}}$



effect of N → affects shape

larger N → more flatly passband, slower transition region.

$$H_a(s)H_a(-s) = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$$

$s = j\Omega$

has poles at

$$S_p = (-1)^{1/2N}(j\Omega_c)$$

$$S_p = \underbrace{e^{j\left[\frac{\pi+2k\pi}{2N}\right]}}_{\text{roots of } (-1)} \; \underbrace{e^{j\frac{\pi}{2}}}_{j} \; \Omega_c$$

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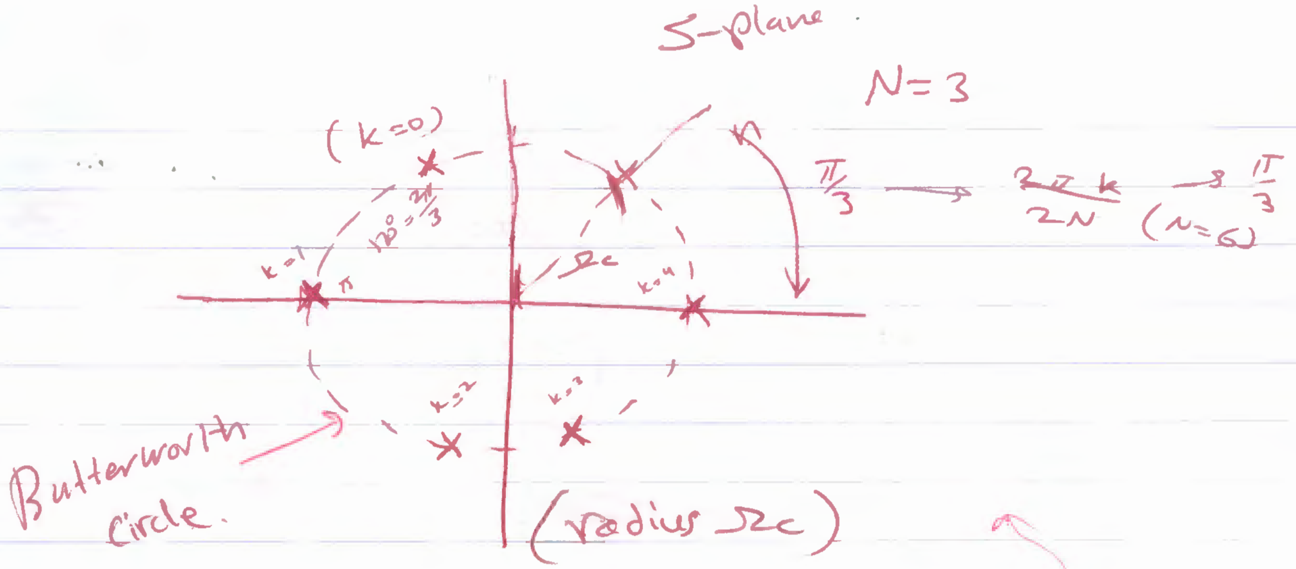


Similar to z-plane, But it is s-plane.

- choose Ω_c and N, then from poles of $H(s)\,H(-s)$

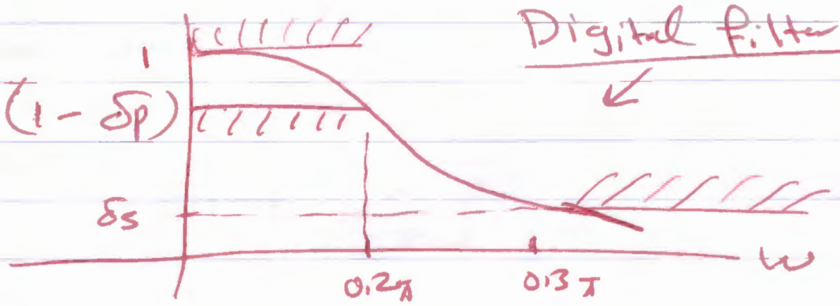
left-hand plane — right — image of

- Square Magnitude function $H_a(s)\,H_a(-s)$

This is analogue filter

now, we want ~~to~~ design a digital filter
- design analogue filter and map it to digital filter.

Digital filter specs



specs
① $1-\delta_p \geq -1$ dB (within 1 dB from 1) (deviation)

② $\delta_s \leq -15$ dB

③ freq. characteristic is monotonic

$$20 \log_{10} |H(e^{j0.2\pi})| \geq -1$$

or

$$|H(e^{j0.2\pi})| \geq 10^{-0.05} \quad (\text{freq. } 0 \rightarrow 0.2\pi)$$

also:

$$20 \log_{10} |H(e^{j0.3\pi})| \leq -15$$

or

$$|H(e^{j0.3\pi})| \leq 10^{-0.75}$$

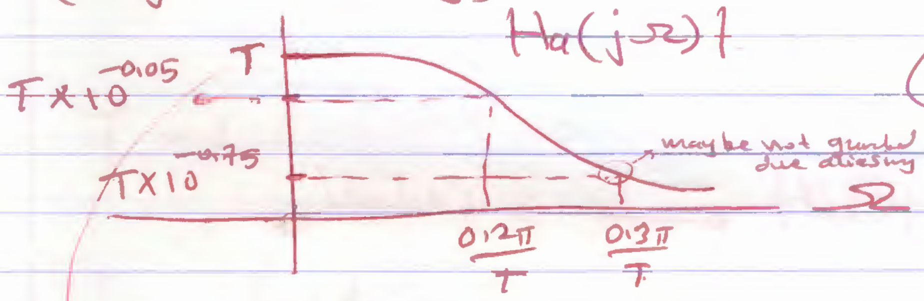
(1) Impulse Invariance Method

We know,

$$H(e^{jw}) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} H_a\left[\frac{jw}{T} + j\frac{2\pi k}{T}\right]$$

$$\Omega = \frac{w}{T}$$

now, Initially, (Neglect Aliasing)



(desiging Analogue filter).

from eq. we have $\frac{1}{T}$, in order to get digit filt unity, we design analogue filter to have T dc.

$$|H_a(j\Omega)|^2 = \frac{T^2}{1 + \left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$$1 + \left(\frac{j\frac{0.2\pi}{T}}{j\Omega_c}\right)^{2N} = 10^{0.1} \quad \text{--- (1)}$$

$$1 + \left(\frac{j\frac{0.3\pi}{T}}{j\Omega_c}\right)^{2N} = 10^{1.5} \quad \text{--- (2)}$$

Solve. (1) and (2)

$N = 5.8858 \longrightarrow$ should be Integer $\Rightarrow N = 6$ (round-up)

$\Omega_c T = 0.70474$

Due to aliasing, we know stopband specs will not be met: ⇒ so, we exceed δ_s specs and meet δ_p.

Choose $N = 6$

$$1 + \left(j \frac{0.2\pi/T}{j\Omega_c}\right)^{2\times 6} = 10$$

$$\Omega_c T = 0.7032$$

$$H_a(s)H_a(-s) = \frac{T^2}{1 + \left(\frac{sT}{j\,0.7032}\right)^{2\times 6}}$$ T disappears

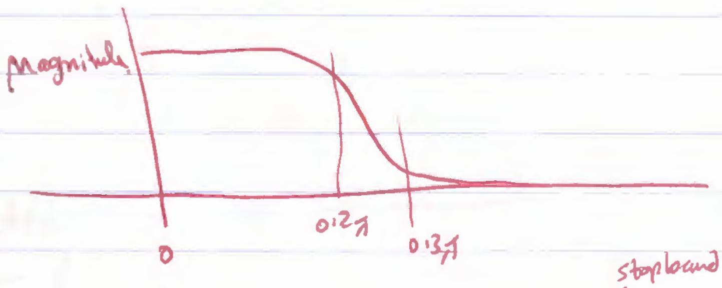
$T = 1$

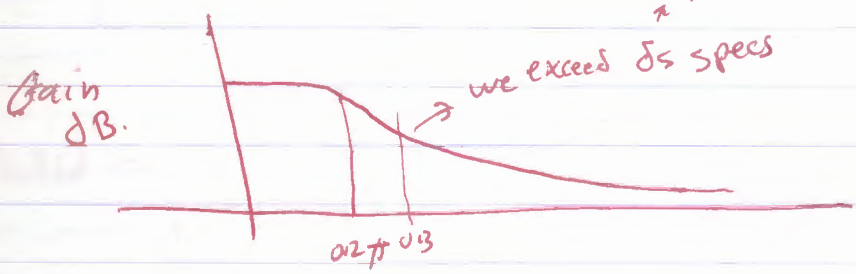
$$h_a(t) \longleftrightarrow H_a(s)$$

$$h_a(t/T) \longleftrightarrow T\,H_a(sT)$$

$$h(n) = h_a(nT) = \underline{h_a(n)}.$$ → T disappears.

* specs of digital filter: all freqs. are ΩT so, T is irrelevant. (T is irrelevant in digital specs).





5)

* For $N=6$ and $\Omega_c = 0.7032 \Rightarrow 12$ Poles (6 pole-pairs) on the Butterworth circle.

$$\left(e\right)^{j\frac{\pi+2\pi k}{12}} e^{j\frac{\pi}{2}}$$

$k=0$ $\quad \Omega_c e^{j\frac{7}{12}\pi}$

$k=1$ $\quad \Omega_c e^{j\frac{3\pi}{4}}$

Three left pole pairs:

pole pair 1: $-0.182 \mp j0.679$

pole pair 2: $-0.497 \mp j0.497$

pole pair 3: $-0.679 \mp j0.182$

$$\cancel{H(s) = \frac{p_1 p_1^* p_2 p_2^* p_3 p_3^*}{[(s-p_1)(s-p_1^*)][(s-p_2)(s-p_2^*)][(s-p_3)(s-p_3^*)]}}$$

Set $T=1$

* We know $\quad |H_a(s) H_a(-s)| = \dfrac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$

$$|H_a(s)| = \frac{\Omega_c^N}{\prod_{k=1}^{N}(s-p_k)}$$

$$p_k = \Omega_c e^{j(\pi(N+2k-1)/2N)}, \quad k=1,2,\dots,N$$

So,

$$H_a(s) = \frac{(0.7032)^6}{\{(s-p_1)(s-p_1^*)\}\{(s-p_2)(s-p_2^*)\}\{(s-p_3)(s-p_3^*)\}}$$

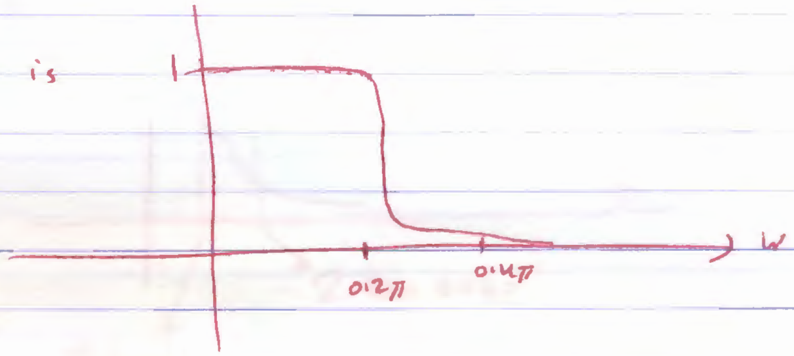
$$H_a(s) = \frac{0.12092}{(s^2+0.364s+0.4945)(s^2+0.9945s+0.4945)(s^2+1.3585s+0.4945)}$$

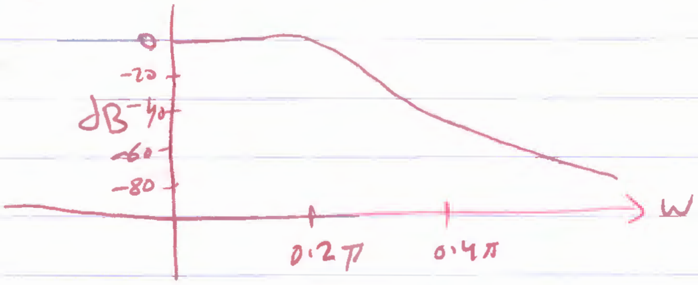
express $H_a(s)$ as partial fraction expansion, then perform transformation from s to z, and then combine complex-conjugate terms

$$H(z) = \frac{0.2871 - 0.4466 z^{-1}}{(1 - 1.2971 z^{-1} + 0.6949 z^{-2})} + \frac{-2.1428 + 1.1456 z^{-1}}{1 - 1.0691 z^{-1} + 0.3699 z^{-2}}$$

$$+ \frac{1.8557 - 0.6303 z^{-1}}{1 - 0.9972 z^{-1} + 0.25702 z^{-2}}$$ (parallel form).

freq. Response is

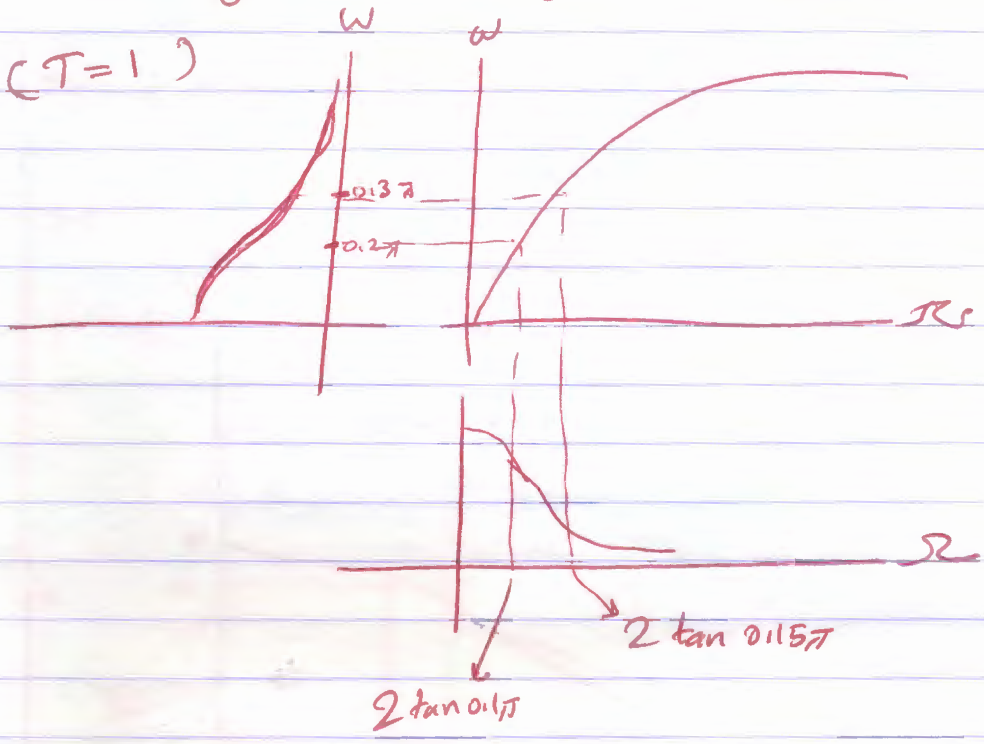




* Same Example using Bi-linear Transformation

We choose digital specs, warp them through bi-linear to analogue filter, design.

$(T=1)$



$$20\log_{10}|H_a(j\,\underbrace{2\tan 0.1\pi}_{\text{freq. dig.}})| \geq -1$$

$$20\log_{10}|H_a(j2\tan 0.15\pi)| \leq -15$$

$$1+\left[\frac{j2\tan(.1\pi)}{j\Omega_c}\right]^{2N} = 10^{0.1} \quad \text{——— (1)}$$

$$1+\left[\frac{j2\tan(0.15\pi)}{j\Omega_c}\right]^{2N} = 10^{1.5} \quad \text{——— (2)}$$

Same

Solve (1) and (2)

$\Rightarrow$ $N = 5.30466$ $\Rightarrow$ should be Integer — rounding down will not meet specs.

round-up $N = 6$.

choose $N=6$ (* exactly meet passband specs)

(*) exceed stopband specs

- for Impulse invariance $\Rightarrow$ because of aliasing we exceed stopband specs.

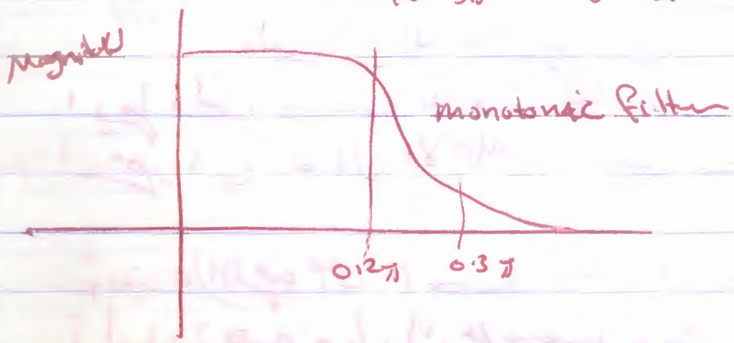
in Bilinear (no aliasing) $\Rightarrow$ we are free to exceed ...

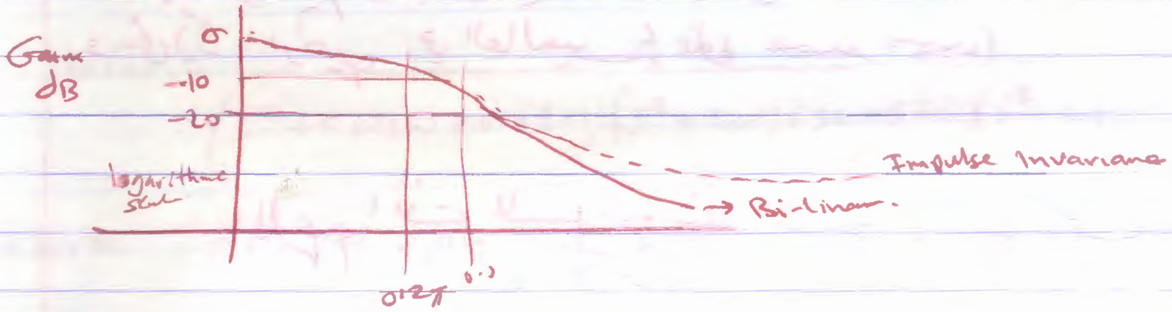
$$1 + \left[\frac{j2\tan(0.15\pi)}{j\Omega_c}\right]^{2\times 6} = 10^{1.5}$$

$$\Rightarrow \Omega_c = 0.76622.$$

Square Magnitude $\underbrace{H_a(s)}_{\text{left side}}\ \underbrace{H_a(-s)}_{\text{right side}} \Longrightarrow$ Map it to $H(z)$

digital filter.



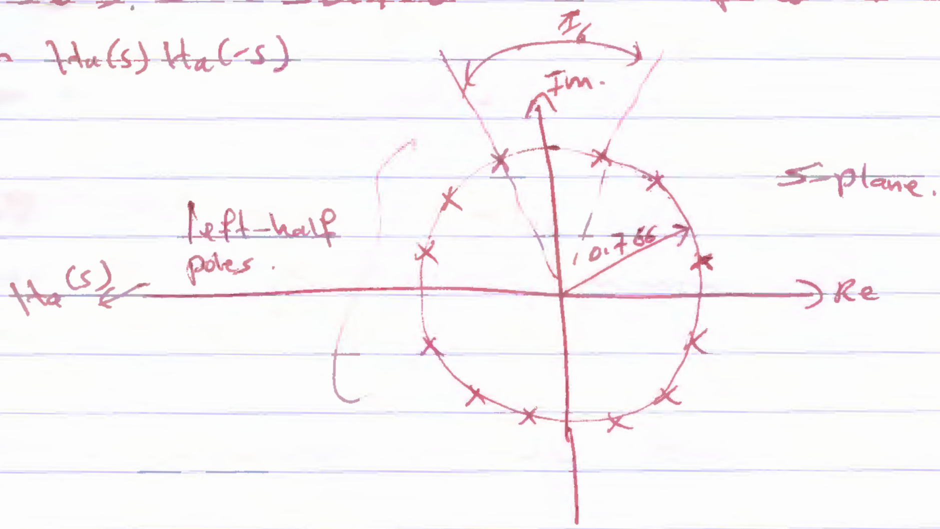


Example Cheppi

* most of our examples were low-pass filters, but the same techniques can be applied to bandpass and stop~~band~~pass filters.

* also, there are techniques which can be used to convert design of L.P.F to any other filter.

$N = 6$, $\Omega_c = 0.766$ $\Rightarrow$ 12 poles of magnitude-squared function $H_a(s)H_a(-s)$



$$H_a(s) = \frac{0.1202}{(s^2+0.3996s+0.5871)(s^2+1.0836s+0.5871)(s^2+1.4802s+0.5871)}$$

* with $T=1$, and applying bi-linear Transform we get

$$H(z) = \frac{0.0007378(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})}$$

$$\times \frac{1}{1-0.9044z^{-1}+0.2155z^{-2})}$$

$$|H(e^{jw})|^2 = \frac{1}{1+\left(\frac{\tan(\frac{w}{2})}{\tan(\frac{w_c}{2})}\right)^{2N}}$$, $\tan\left(\frac{w_c}{2}\right) = \Omega_c T/2$

Butterworth discrete-time filter

Design FIR Digital Filter:

* can be designed to have linear phase

* ~~FIR~~ design digital FIR simpler than design analogue FIR filter. Map from analogue to digital is not benficial.

* Algorithmic procedures for IIR are still applicable for FIR.

* Three types of design techniques for FIR:

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \quad (\text{finit-length})$$

→ non-zero samples

polynomial of z^{-1} has zero's except at $z=0$

* For linear phase $h(n) = h(N-1-n)$ (symmetrical about mid-point)

Basic Design methods:

1. Windows method
2. Frequency Sampling method
3. Equiripple Filters Design (algorithmic)

1 Window Method

Design of FIR filters using windows

Start with Desired unit sample response: $h_d(n)$ (ideal low-pass filter)

→ truncate infinit $h(n)$ to finite by multiplying with $W(n)$

$$h(n) = W(n)\, h_d(n)$$

$\frac{\sin x}{x}$, $-\infty \to \infty$ doesn't converge to FIR

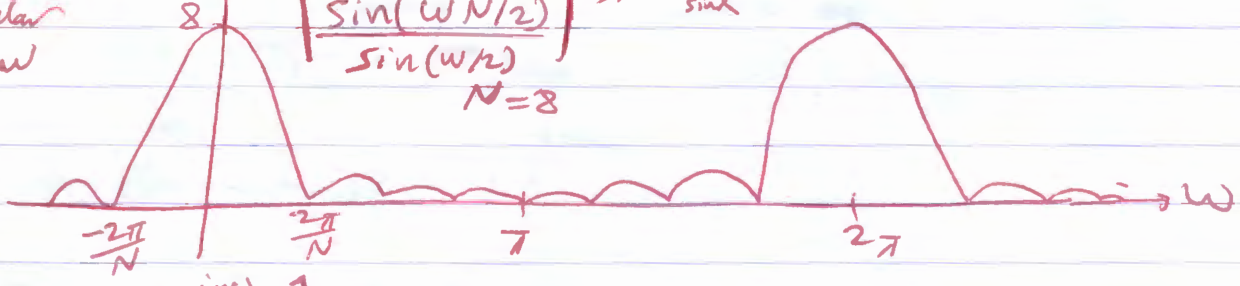
$W(n) = 0 \quad n<0,\ n > (N-1)$ → conv in freq-domain

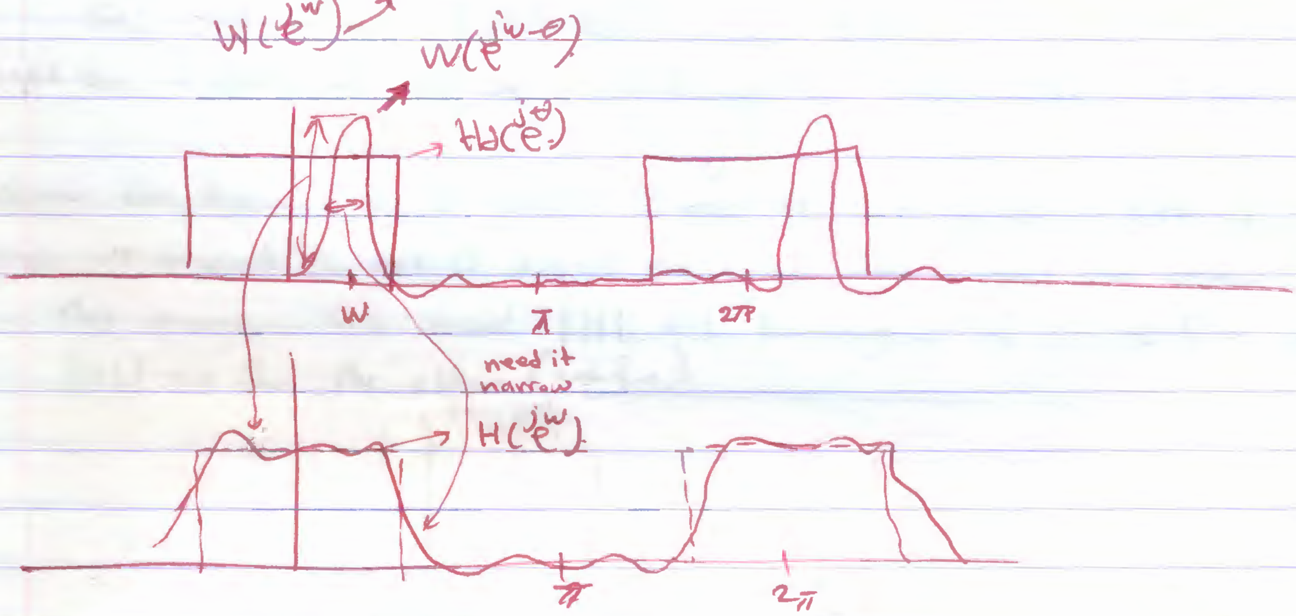
$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, W[e^{j(\omega-\theta)}]\, d\theta$$] periodic convolution

* e.g. rectangular window $w(n)$

$$\left|\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right| \quad N=8$$

$\rightarrow \frac{\sin Nx}{\sin x}$ curve.





— We want $W(e^{j\omega})$ to be an impulse $\Rightarrow$ impulse * Ideal = Ideal.

Ideal L.P.F Multiplied by 51 point rectangular window