STATICS

Chapter 3: Rigid Bodies' Equivalent systems of Forces

Contents

Introduction

- Treatment of a body as a single particle is not always possible. In general, the size of the body and the specific points of application of the forces must be considered.

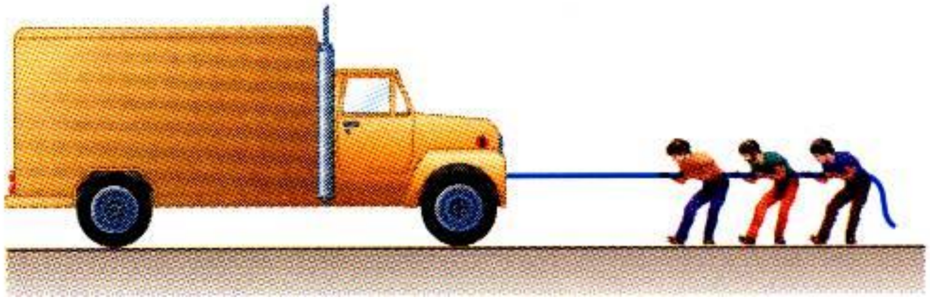
- Most bodies in elementary mechanics are assumed to be rigid, i.e., the actual deformations are small and do not affect the conditions of equilibrium or motion of the body.

- Current chapter describes the effect of forces exerted on a rigid body and how to replace a given system of forces with a simpler equivalent system.
    - moment of a force about a point
    - moment of a force about an axis
    - moment due to a couple

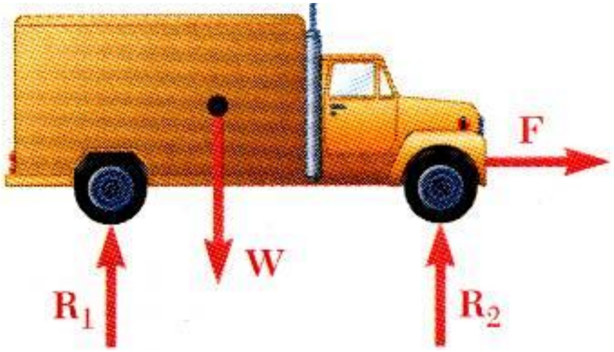
- Any system of forces acting on a rigid body can be replaced by an equivalent system consisting of one force acting at a given point and one couple.

External and Internal Forces

- Forces acting on rigid bodies are divided into two groups:
  - External forces
  - Internal forces



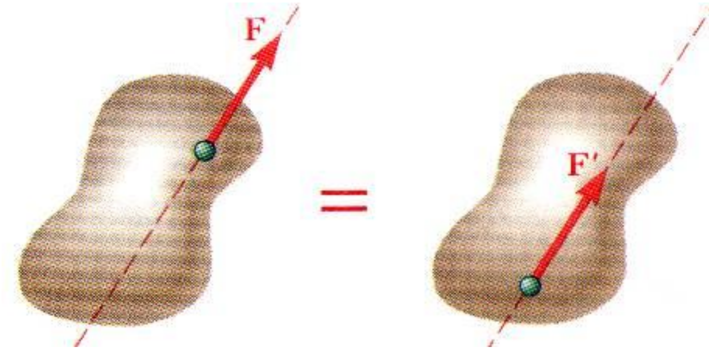
- External forces are shown in a free-body diagram.



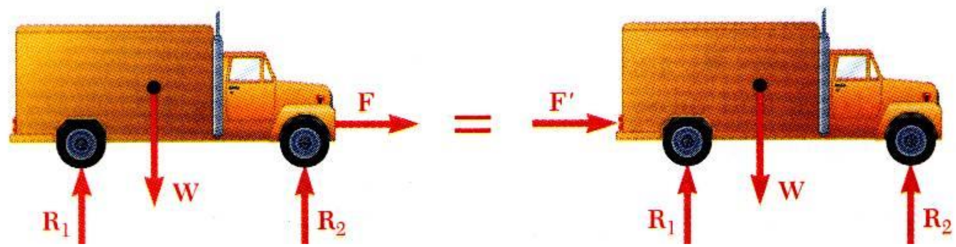
- If unopposed, each external force can impart a motion of translation or rotation, or both.

Principle of Transmissibility: Equivalent Forces

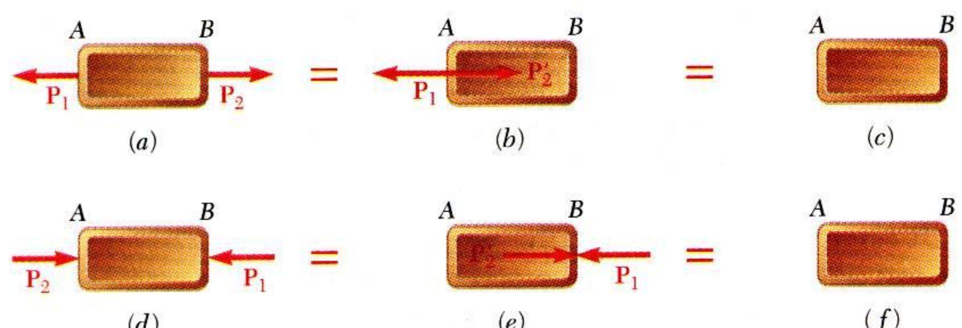
- *Principle of Transmissibility*
  Conditions of equilibrium or motion are not affected by *transmitting* a force along its line of action.
  NOTE: **F** and **F'** are equivalent forces.



- Moving the point of application of the force **F** to the rear bumper does not affect the motion or the other forces acting on the truck.



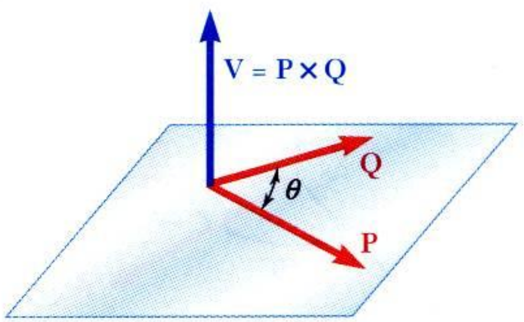
- Principle of transmissibility may not always apply in determining internal forces and deformations.



Vector Product of Two Vectors

- Concept of the moment of a force about a point is more easily understood through applications of the *vector product* or *cross product*.

- Vector product of two vectors ***P*** and ***Q*** is defined as the vector ***V*** which satisfies the following conditions:
  1. Line of action of ***V*** is perpendicular to plane containing ***P*** and ***Q***.
  2. Direction of ***V*** is obtained from the right-hand rule.
  3. Magnitude of V= P Q sin θ



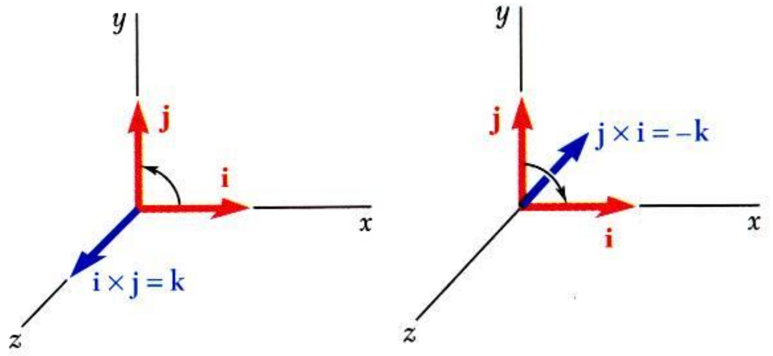
(a)

(b)

- Vector products:
  - are not commutative, $\boldsymbol{Q} \times \boldsymbol{P} = -(\boldsymbol{P} \times \boldsymbol{Q})$
  - are distributive, $\boldsymbol{P} \times (\boldsymbol{Q}_1 + \boldsymbol{Q}_2) = \boldsymbol{P} \times \boldsymbol{Q}_1 + \boldsymbol{P} \times \boldsymbol{Q}_2$
  - are not associative, $(\boldsymbol{P} \times \boldsymbol{Q}) \times \boldsymbol{S} \neq \boldsymbol{P} \times (\boldsymbol{Q} \times \boldsymbol{S})$

Vector Products: Rectangular Components

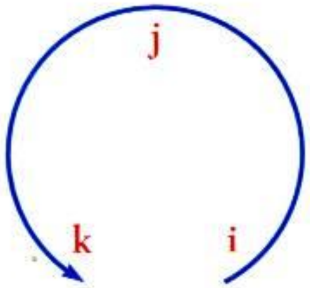
Vector products of Cartesian unit vectors,



$$\vec{i}\times\vec{i}=0 \qquad \vec{j}\times\vec{i}=-\vec{k} \qquad \vec{k}\times\vec{i}=\vec{j}$$

$$\vec{i}\times\vec{j}=\vec{k} \qquad \vec{j}\times\vec{j}=0 \qquad \vec{k}\times\vec{j}=-\vec{i}$$

$$\vec{i}\times\vec{k}=-\vec{j} \qquad \vec{j}\times\vec{k}=\vec{i} \qquad \vec{k}\times\vec{k}=0$$

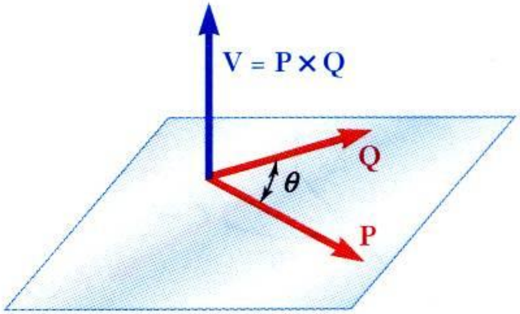


Vector products in terms of rectangular coordinates

$$\vec{V} = \left(P_x\vec{i} + P_y\vec{j} + P_z\vec{k}\right)\times\left(Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k}\right)$$

$$= \left(P_yQ_z - P_zQ_y\right)\vec{i} + \left(P_zQ_x - P_xQ_z\right)\vec{j} + \left(P_xQ_y - P_yQ_x\right)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$



(a)

(b)

Cross Product Revision

Each component can be determined using 2 × 2 determinants.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

For element **i**: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}(A_yB_z - A_zB_y)$

For element **j**: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = -\mathbf{j}(A_xB_z - A_zB_x)$

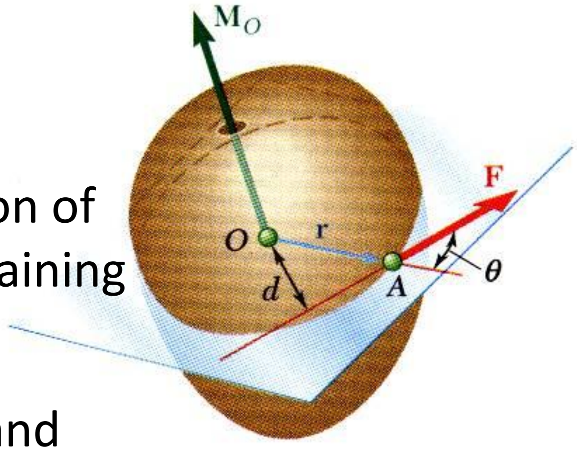
Remember the negative sign

For element **k**: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{k}(A_xB_y - A_yB_x)$

Moment of a Force About a Point

- The *moment* of ***F*** about *O* is defined as

$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$



(a)

- $\boldsymbol{M_O}$ measures the tendency of the force to cause rotation of the body about an axis perpendicular to the plane containing *O* and the force ***F*.**
- The sense of the moment is determined by the right-hand rule.
- The magnitude of the moment is given by the product of the force and the perpendicular distance from point O on the line of action of the force.

(b)

$$M_o = F\, r \sin\theta = F\, d$$

NOTE: Any force ***F'*** that has the same magnitude, direction and line of action as ***F,*** produces the same moment about the point O.

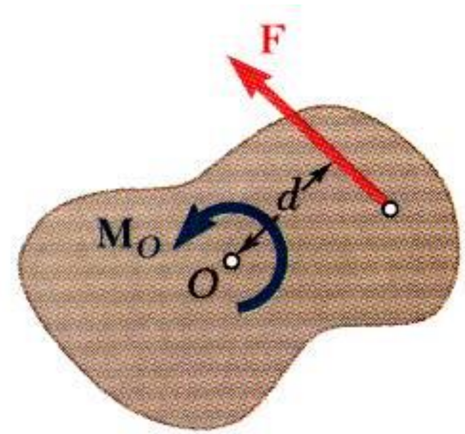
Moment of a Force About a Point: 2d structures

- *Two-dimensional structures* have length and breadth but negligible depth and are subjected to forces contained in the plane of the structure (e.g. x and y plane)

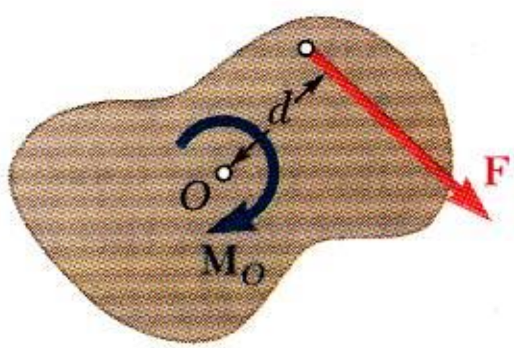
- The plane of the structure contains the point *O* and the force ***F***. ***M$_O$***, the moment of the force about *O* is perpendicular to the plane xy (in the z direction)

- If the force tends to rotate the structure clockwise, the sense of the moment vector is out of the plane of the structure and the magnitude of the moment is positive.

- If the force tends to rotate the structure counterclockwise, the sense of the moment vector is into the plane of the structure and the magnitude of the moment is negative.



(a) $\mathbf{M}_O = +Fd$

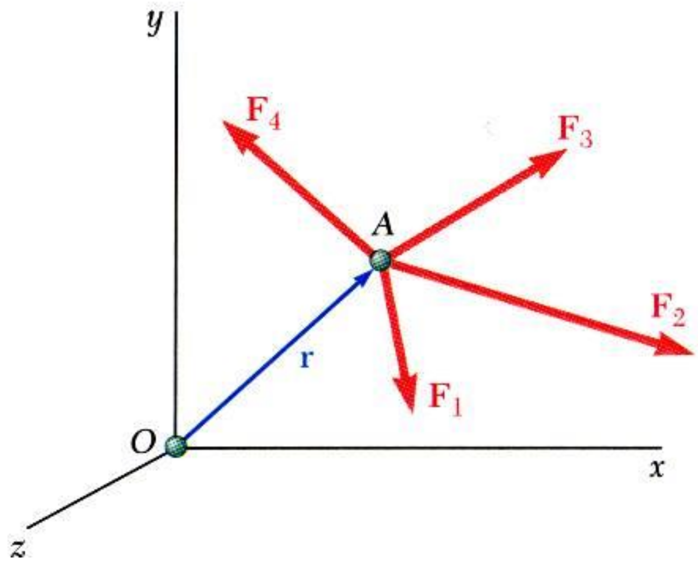


(b) $\mathbf{M}_O = -Fd$

Varignon's Theorem

- The moment about a give point *O* of the resultant of several concurrent forces is equal to the sum of the moments of the various moments about the same point *O*.

$$\vec{r} \times \left(\vec{F}_1 + \vec{F}_2 + \cdots\right) = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \cdots$$



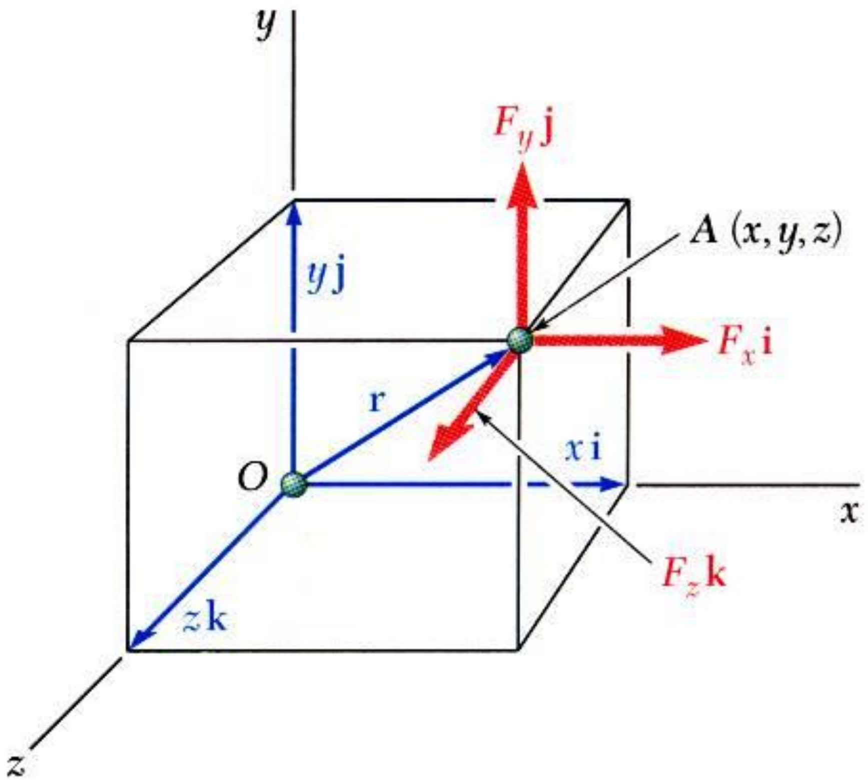
- Varigon's Theorem makes it possible to replace the direct determination of the moment of a force ***F*** by the moments of two or more component forces of ***F***.

Rectangular Components of the Moment of a Force

The moment of ***F*** about *O (origin),*

$$\vec{M}_O = \vec{r} \times \vec{F}, \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$



$$\vec{M}_O = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

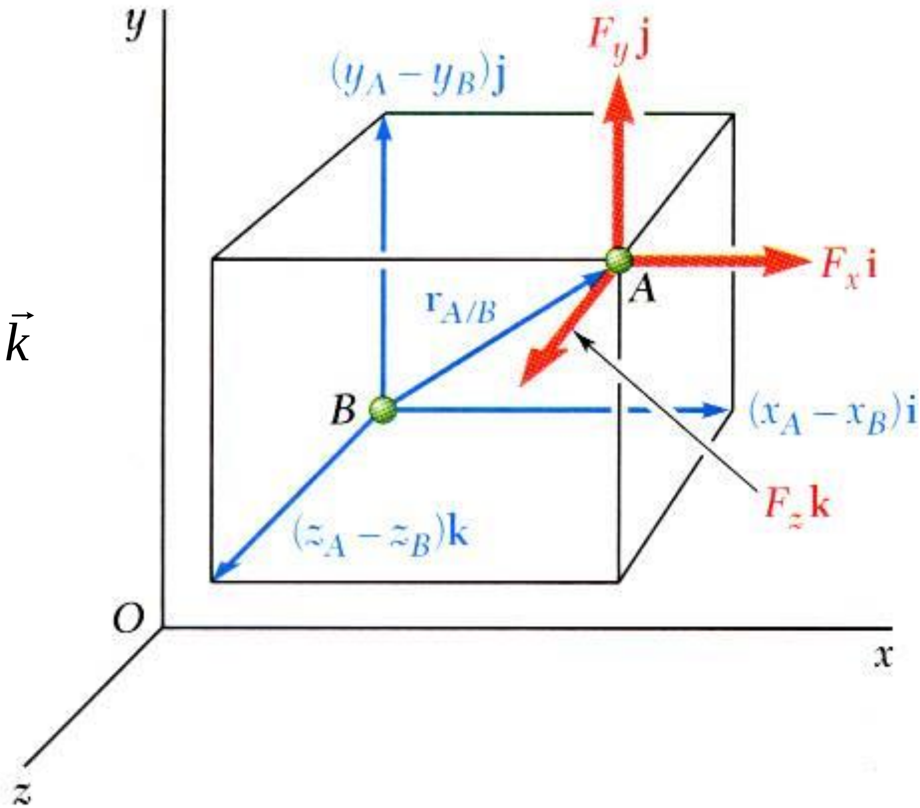
$$= \left(yF_z - zF_y\right)\vec{i} + \left(zF_x - xF_z\right)\vec{j} + \left(xF_y - yF_x\right)\vec{k}$$

The moment of ***F*** about *B (not origin)*

$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F}$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$
$$= (x_A - x_B)\vec{i} + (y_A - y_B)\vec{j} + (z_A - z_B)\vec{k}$$
$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$\vec{M}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{vmatrix}$$



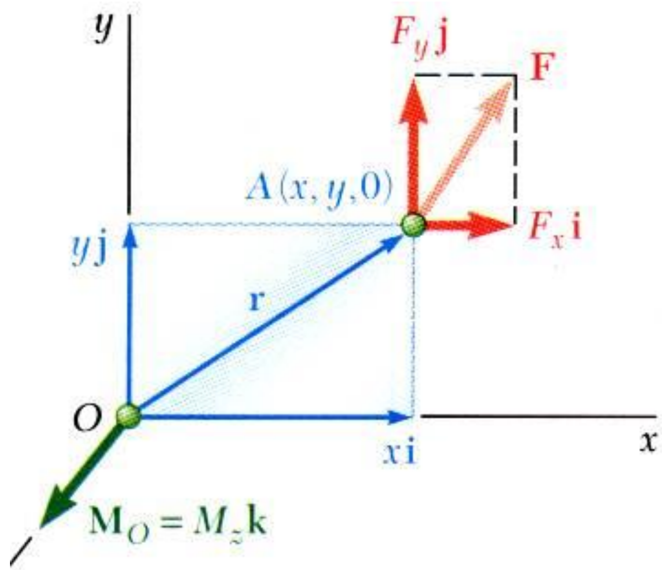
Vector product can be used to find the perpendicular distance (d) from a point (Say point B) to a line (say Force F) as the magnitude of the moment M_B divided by the magnitude of the force $d = M_B/F$

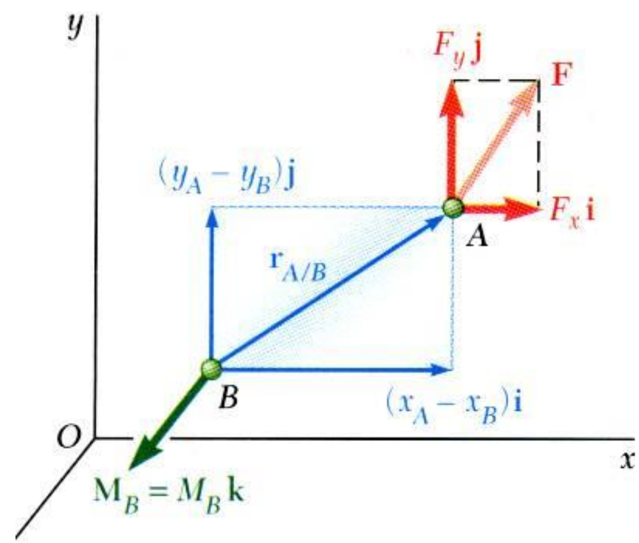
For two-dimensional structures,

$$\vec{M}_O = \left(xF_y - yF_x\right)\vec{k}$$
$$M_O = M_Z$$
$$= xF_y - yF_x$$

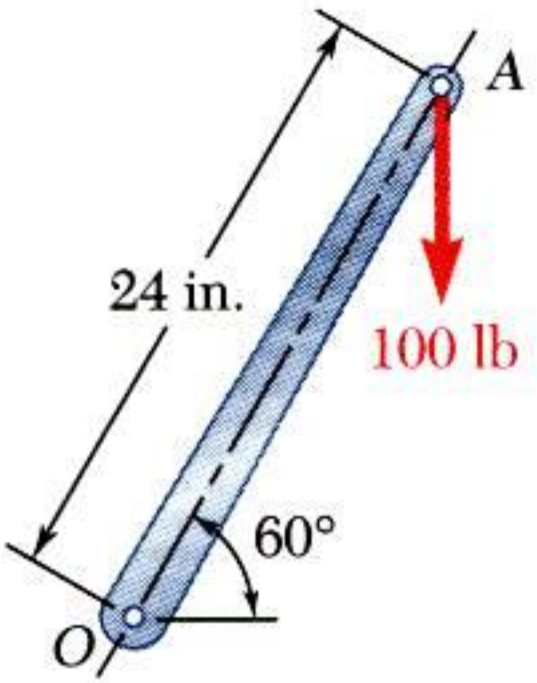
$$\vec{M}_O = \left[(x_A - x_B)F_y - (y_A - y_B)F_x\right]\vec{k}$$
$$M_O = M_Z$$
$$= (x_A - x_B)F_y - (y_A - y_B)F_x$$

The moment is the sum of the product of each force component and its perpendicular distance to it from the point of reference.





Sample Problem 3.1



A 100-lb vertical force is applied to the end of a lever which is attached to a shaft at *O*.

Determine:

a) moment about *O*,

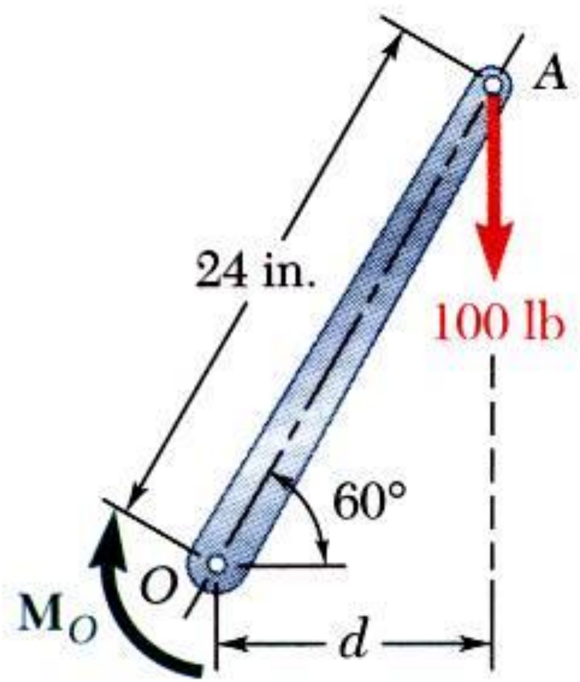
b) horizontal force at *A* which creates the same moment,

c) smallest force at A which produces the same moment,

d) location for a 240-lb vertical force to produce the same moment,

e) whether any of the forces from b, c, and d is equivalent to the original force.

Sample Problem 3.1



a) Moment about *O* is equal to the product of the force and the perpendicular distance between the line of action of the force and *O.* Since the force tends to rotate the lever clockwise, the moment vector is into the plane of the paper.

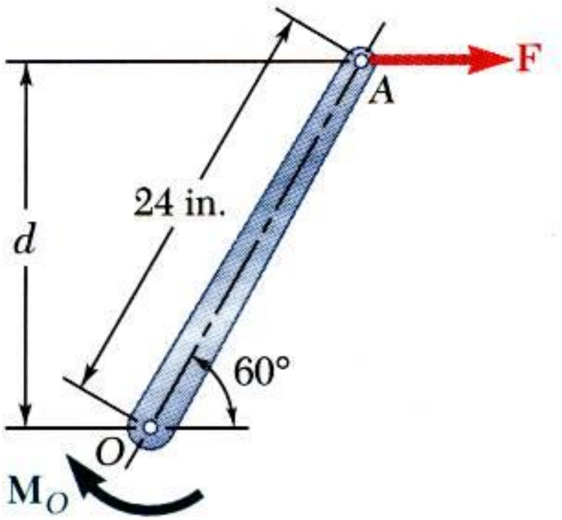
$$M_O = Fd$$

$$d = (24\,\text{in.})\cos 60^\circ = 12\,\text{in.}$$

$$M_O = (100\,\text{lb})(12\,\text{in.})$$

$$\boxed{M_O = 1200\,\text{lb}\cdot\text{in}}$$

Sample Problem 3.1



b) Horizontal force at *A* that produces the same moment,

$$d = (24\,\text{in.})\sin 60° = 20.8\,\text{in.}$$

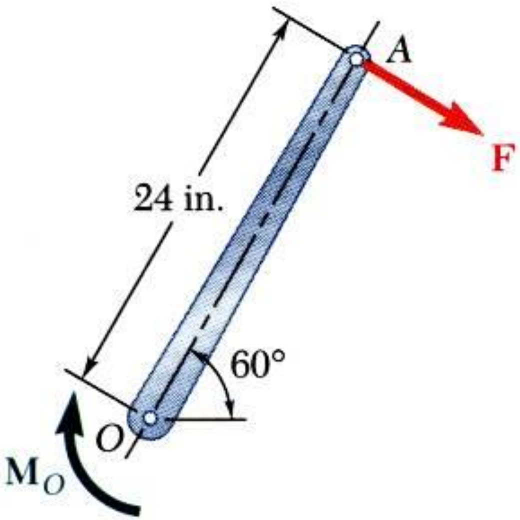
$$M_O = Fd$$

$$1200\,\text{lb}\cdot\text{in.} = F(20.8\,\text{in.})$$

$$F = \frac{1200\,\text{lb}\cdot\text{in.}}{20.8\,\text{in.}}$$

$$\boxed{F = 57.7\,\text{lb}}$$

Sample Problem 3.1



c) The smallest force *A* to produce the same moment occurs when the perpendicular distance is a maximum or when *F* is perpendicular to *OA*.

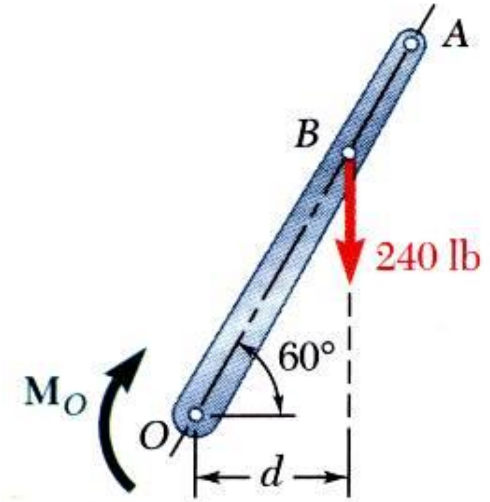
$$M_O = Fd$$

$$1200\,\text{lb}\cdot\text{in.} = F(24\,\text{in.})$$

$$F = \frac{1200\,\text{lb}\cdot\text{in.}}{24\,\text{in.}}$$

$$\boxed{F = 50\,\text{lb}}$$

Sample Problem 3.1



d) To determine the point of application of a 240 lb force to produce the same moment,

$$M_O = Fd$$

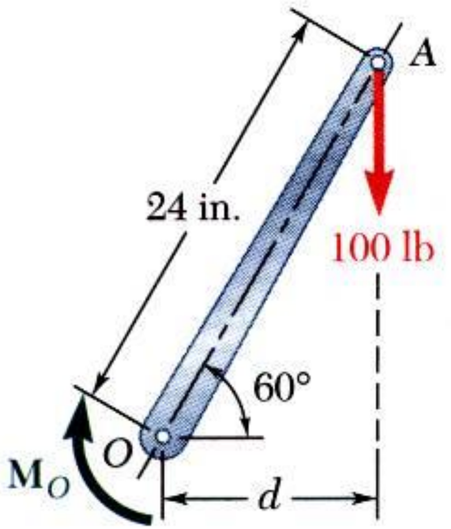
$$1200\,\text{lb}\cdot\text{in.} = (240\,\text{lb})d$$

$$d = \frac{1200\,\text{lb}\cdot\text{in.}}{240\,\text{lb}} = 5\,\text{in.}$$

$$OB\cos 60° = 5\,\text{in.}$$

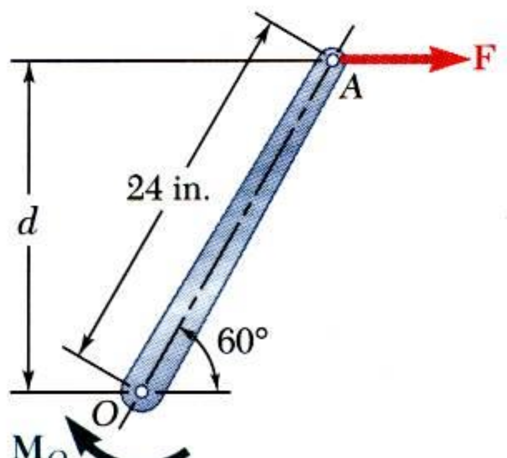
$$\boxed{OB = 10\,\text{in.}}$$

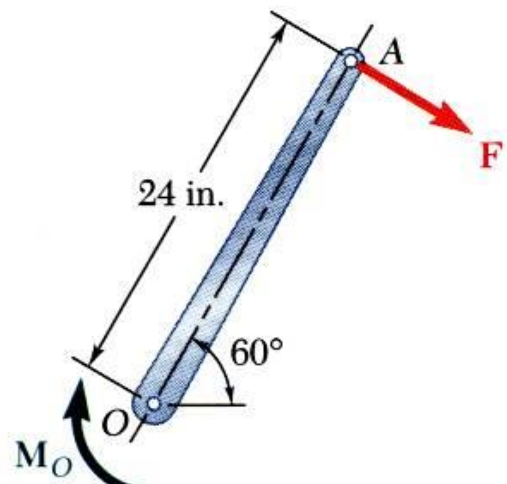
Sample Problem 3.1

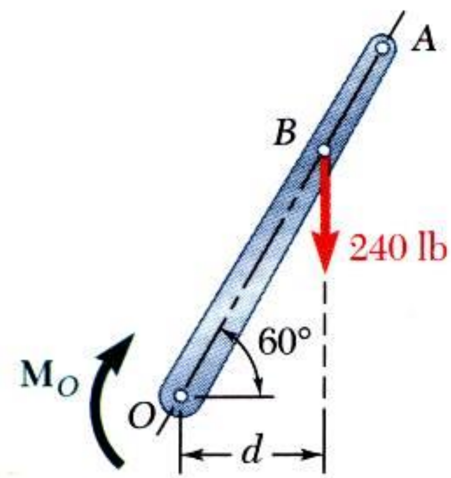


e) Although each of the forces in parts b), c), and d) produces the same moment as the 100 lb force, none are of the same magnitude and sense, or on the same line of action.

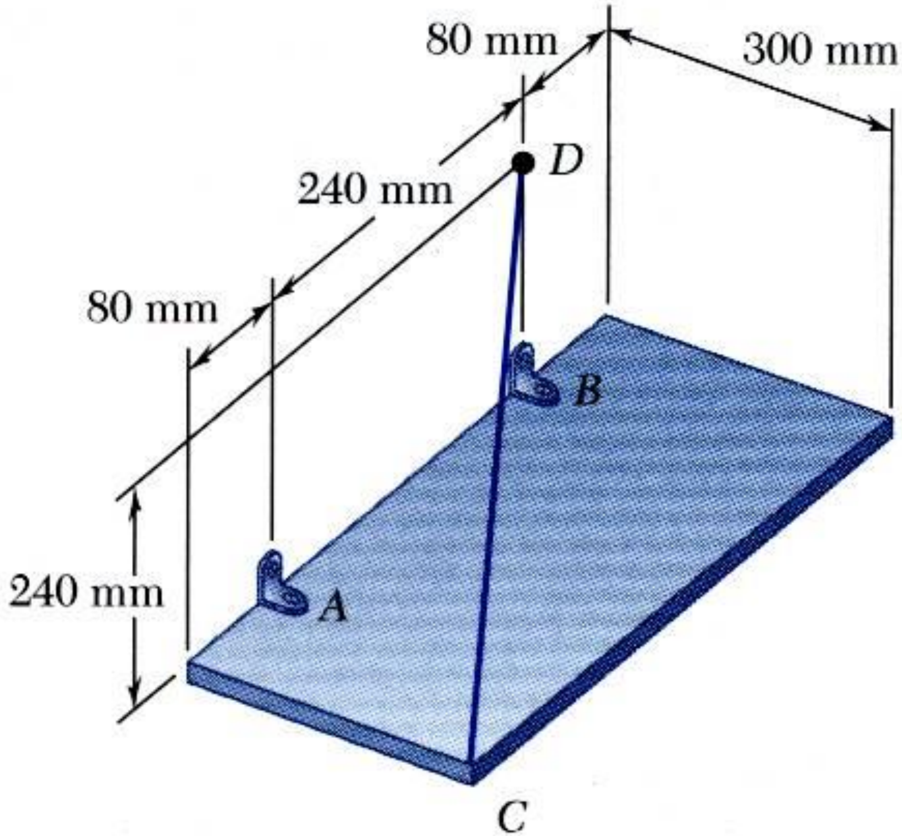
None of the forces is equivalent to the 100 lb force.







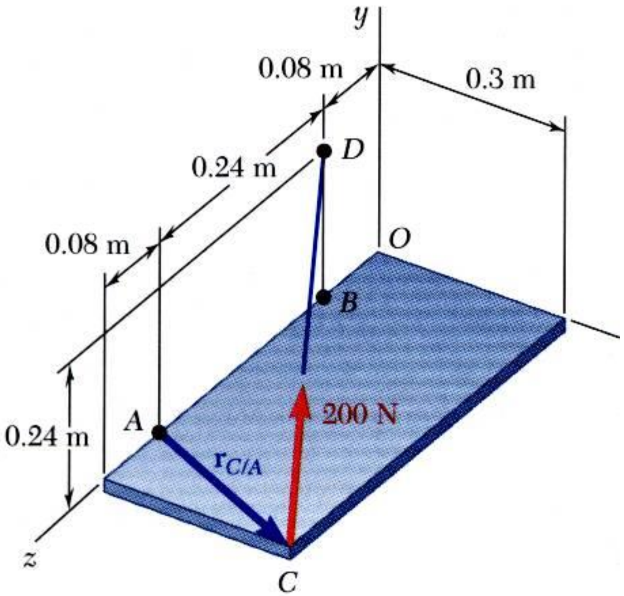
Sample Problem 3.2

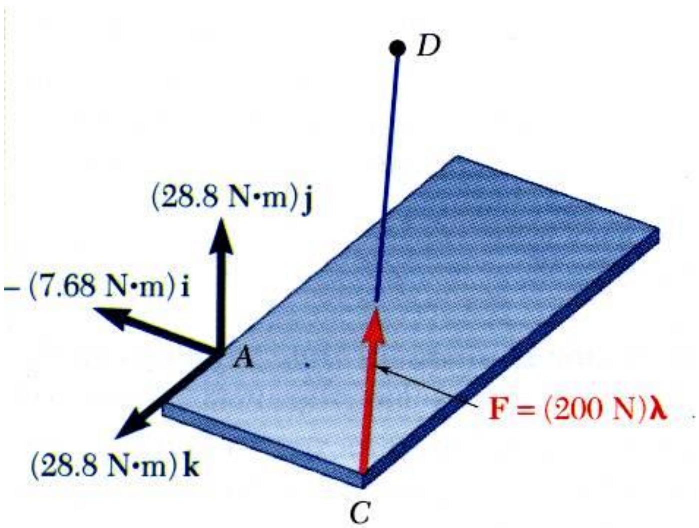


The rectangular plate is supported by the brackets at *A* and *B* and by a wire *CD*.

Knowing that the tension in the wire is 200 N, determine the moment about *A* of the force exerted by the wire at *C*.

Sample Problem 3.2





SOLUTION:

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

$$\vec{r}_{C/A} = \vec{r}_C - \vec{r}_A = (0.3\,\mathrm{m})\vec{i} + (0.08\,\mathrm{m})\vec{j}$$

$$\vec{F} = F\vec{\lambda} = (200\,\mathrm{N})\frac{\vec{r}_{C/D}}{r_{C/D}}$$

Note error– should be $r_{D/C}$ not $r_{C/D}$ (that is r_{CD})

$$= (200\,\mathrm{N})\frac{-(0.3\,\mathrm{m})\vec{i} + (0.24\,\mathrm{m})\vec{j} - (0.32\,\mathrm{m})\vec{k}}{0.5\,\mathrm{m}}$$

$$= -(120\,\mathrm{N})\vec{i} + (96\,\mathrm{N})\vec{j} - (128\,\mathrm{N})\vec{k}$$

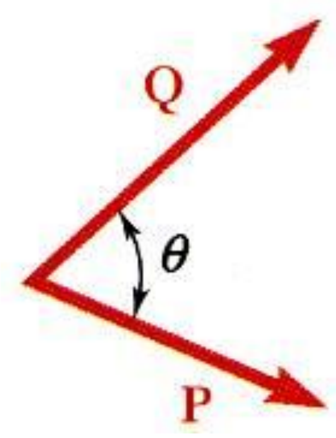
$$\vec{M}_A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

$$\boxed{\vec{M}_A = -(7.68\,\mathrm{N\cdot m})\vec{i} + (28.8\,\mathrm{N\cdot m})\vec{j} + (28.8\,\mathrm{N\cdot m})\vec{k}}$$

Scalar Product of Two Vectors

- The *scalar product* or *dot product* between two vectors ***P*** and ***Q*** is defined as

$$\vec{P} \bullet \vec{Q} = PQ\cos\theta \quad (\text{scalar result})$$



- Scalar products:
  - are commutative, $\vec{P} \bullet \vec{Q} = \vec{Q} \bullet \vec{P}$
  - are distributive, $\vec{P} \bullet \left(\vec{Q}_1 + \vec{Q}_2\right) = \vec{P} \bullet \vec{Q}_1 + \vec{P} \bullet \vec{Q}_2$
  - are not associative, $\left(\vec{P} \bullet \vec{Q}\right) \bullet \vec{S} = \text{undefined}$

- Scalar products with Cartesian unit components,

$$\vec{P} \bullet \vec{Q} = \left(P_x\vec{i} + P_y\vec{j} + P_z\vec{k}\right) \bullet \left(Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k}\right)$$

$$\vec{i} \bullet \vec{i} = 1 \quad \vec{j} \bullet \vec{j} = 1 \quad \vec{k} \bullet \vec{k} = 1 \quad \vec{i} \bullet \vec{j} = 0 \quad \vec{j} \bullet \vec{k} = 0 \quad \vec{k} \bullet \vec{i} = 0$$

$$\vec{P} \bullet \vec{Q} = P_xQ_x + P_yQ_y + P_zQ_z$$

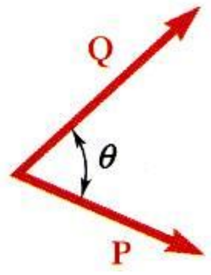
$$\vec{P} \bullet \vec{P} = P_x^2 + P_y^2 + P_z^2 = P^2$$

Scalar Product of Two Vectors: Applications

- Find the angle between two vectors:

$$\vec{P} \bullet \vec{Q} = PQ\cos\theta = P_x Q_x + P_y Q_y + P_z Q_z$$

$$\cos\theta = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{PQ}$$

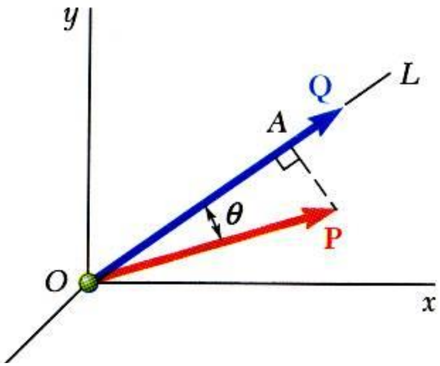


- Find the Projection of a vector on a given axis:

$$P_{OL} = P\cos\theta = \text{projection of } P \text{ along } OL$$

$$\vec{P} \bullet \vec{Q} = PQ\cos\theta$$

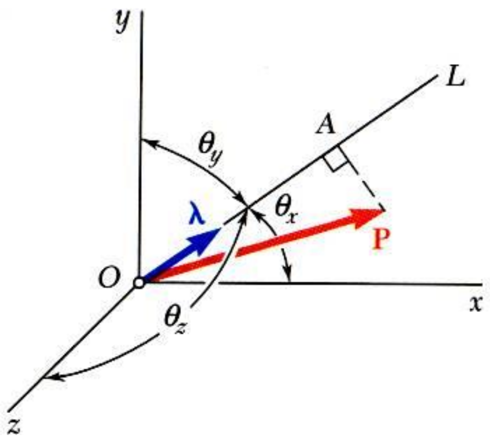
$$\frac{\vec{P} \bullet \vec{Q}}{Q} = P\cos\theta = P_{OL}$$



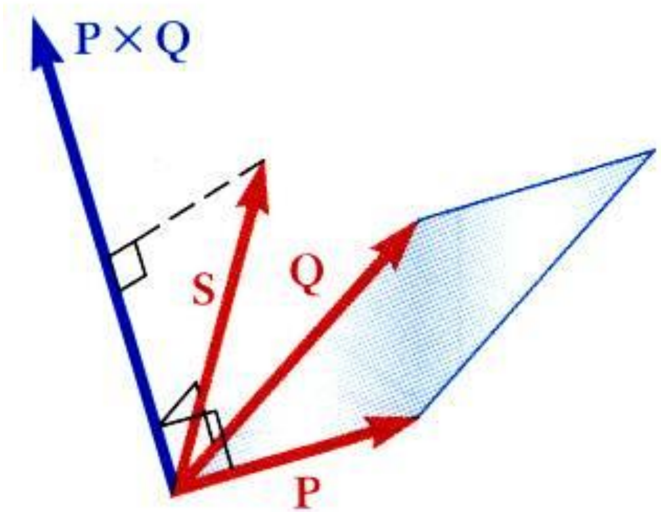
- For an axis defined by a unit vector:

$$P_{OL} = \vec{P} \bullet \vec{\lambda}$$

$$= P_x \cos\theta_x + P_y \cos\theta_y + P_z \cos\theta_z$$



Mixed Triple Product of Three Vectors



- Mixed triple product of three vectors,

$$\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = \text{scalar result}$$

$$\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = S_x\left(P_y Q_z - P_z Q_y\right) + S_y\left(P_z Q_x - P_x Q_z\right) + S_z\left(P_x Q_y - P_y Q_x\right)$$

$$= \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

- The six mixed triple products formed from ***S***, ***P***, and ***Q*** have equal magnitudes but not the same sign,

$$\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = \vec{P} \bullet \left(\vec{Q} \times \vec{S}\right) = \vec{Q} \bullet \left(\vec{S} \times \vec{P}\right)$$
$$= -\vec{S} \bullet \left(\vec{Q} \times P\right) = -\vec{P} \bullet \left(\vec{S} \times \vec{Q}\right) = -\vec{Q} \bullet \left(\vec{P} \times \vec{S}\right)$$

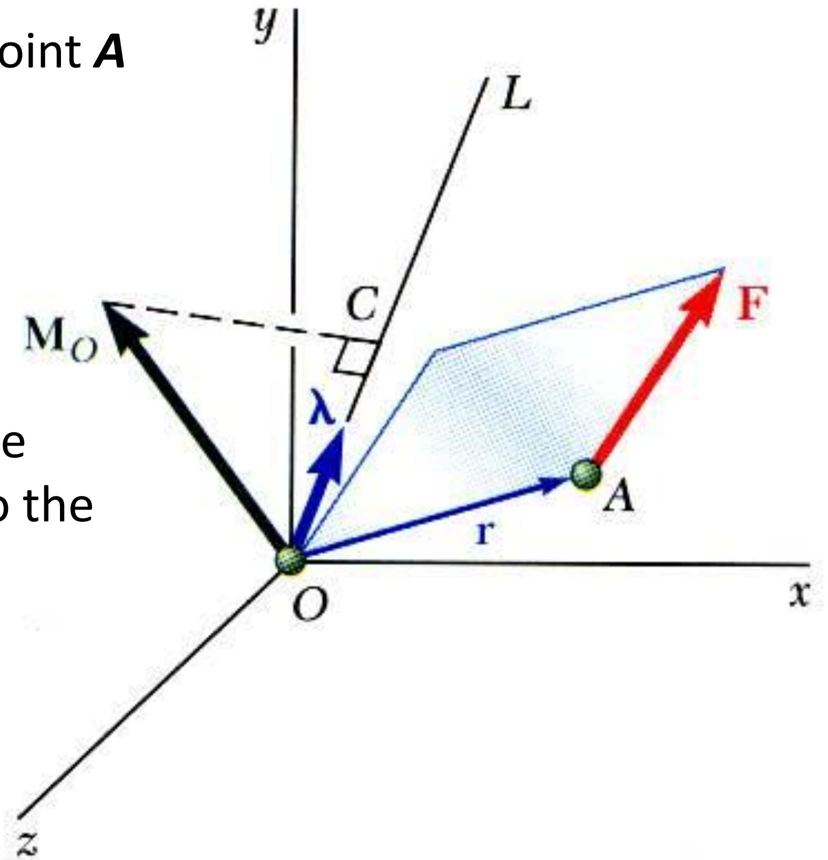
Triple product is equivalent to the moment of a Force about a given axis

- Moment $\boldsymbol{M_O}$ of a force $\boldsymbol{F}$ applied at the point $\boldsymbol{A}$ about a point $\boldsymbol{O}$,

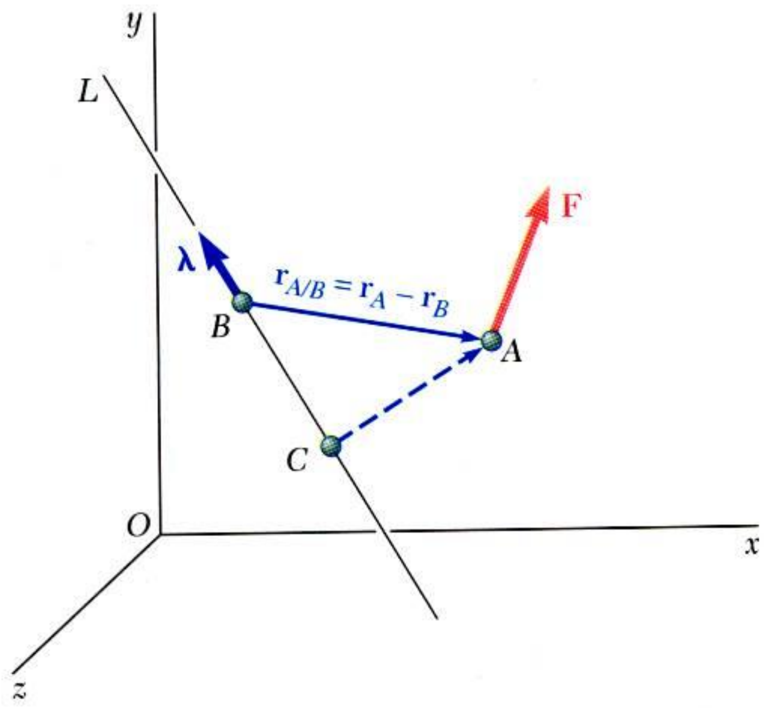
$$\vec{M}_O = \vec{r} \times \vec{F}$$

- Scalar moment M_{OL} about an axis $\boldsymbol{OL}$ is the projection of the moment vector $\boldsymbol{M_O}$ onto the axis,

$$M_{OL} = \vec{\lambda} \bullet \vec{M}_O = \vec{\lambda} \bullet \left(\vec{r} \times \vec{F}\right)$$



Moment of a Force About a Given Axis



- Moment of a force about an arbitrary axis L which has a unit vector λ can be determined by choosing an arbitrary point B along that axis:

$$M_{BL} = \vec{\lambda} \bullet \vec{M}_B$$
$$= \vec{\lambda} \bullet \left(\vec{r}_{A/B} \times \vec{F}\right)$$
$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

- The result is independent of the point *B* along the given axis.

Perpendicular distance between 2 lines

1. Only **(F_{Normal})** the perpendicular (normal) component of the force **(F)** on the axis **L** is responsible for the moment about the axis **M_{BL}** ; such that

$$M_{BL} = F_{Normal}\, d$$

where d is the perpendicular distance between the two lines (L and the force F)

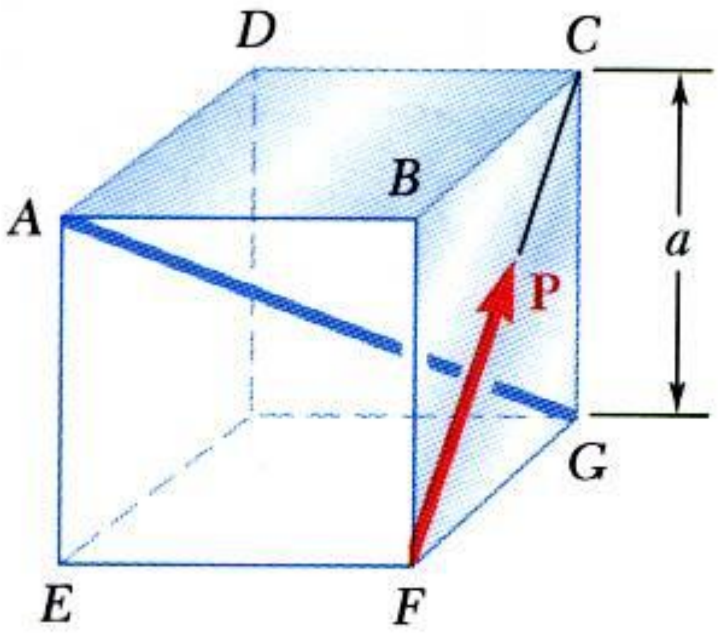
2. The projection of F on L **($F_{Parallel}$)** is given by dot product

$$F_{Parallel} = F \cdot \lambda$$

3. And the normal component of the force can be calculated from

$$F^2 = F^2_{Normal} + F^2_{Parallel}$$

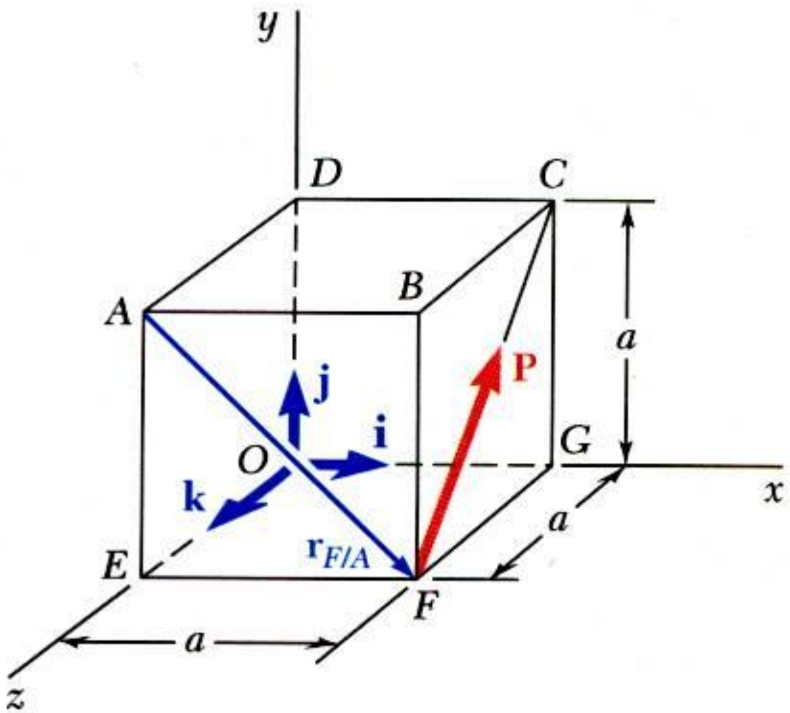
Sample Problem 3.3



A cube is acted on by a force ***P*** as shown. Determine the moment of ***P***

a) about *A*
b) about the edge *AB* and
c) about the diagonal *AG* of the cube.
d) Determine the perpendicular distance between *AG* and *FC*.

Sample Problem 3.3



a) Moment of ***P*** about *A*,

$$\vec{M}_A = \vec{r}_{F/A} \times \vec{P}$$

$$\vec{r}_{F/A} = a\vec{i} - a\vec{j} = a(\vec{i} - \vec{j})$$

$$\boldsymbol{P} = \frac{P}{\sqrt{2}}(\boldsymbol{j} - \boldsymbol{k})$$

$$\boldsymbol{M_A} = \frac{Pa}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})\, X (\boldsymbol{j} - \boldsymbol{k})$$

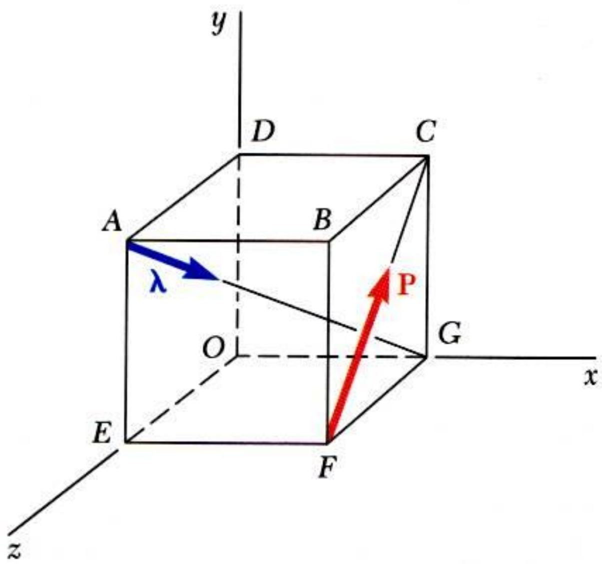
$$\boldsymbol{M_A} = \frac{Pa}{\sqrt{2}}(i + j + k)$$

b) Moment of ***P*** about *AB,*

$$M_{AB} = \vec{i} \bullet \vec{M}_A$$

$$= \boldsymbol{i}.\ \frac{\boldsymbol{Pa}}{\sqrt{\boldsymbol{2}}}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = \frac{\boldsymbol{Pa}}{\sqrt{\boldsymbol{2}}}$$

Sample Problem 3.3



c) Moment of ***P*** about the diagonal *AG*,

$$M_{AG} = \vec{\lambda} \bullet \vec{M}_A$$

$$\vec{\lambda} = \frac{\vec{r}_{A/G}}{r_{A/G}} = \frac{a\vec{i} - a\vec{j} - a\vec{k}}{a\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right)$$

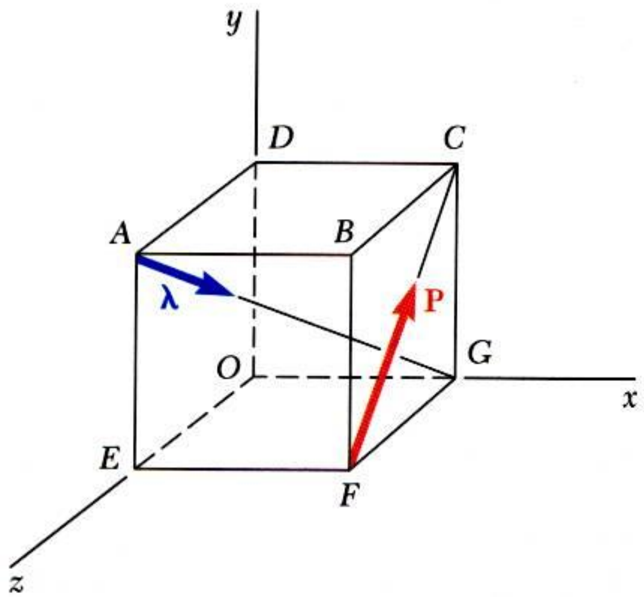
$$\vec{M}_A = \frac{aP}{\sqrt{2}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

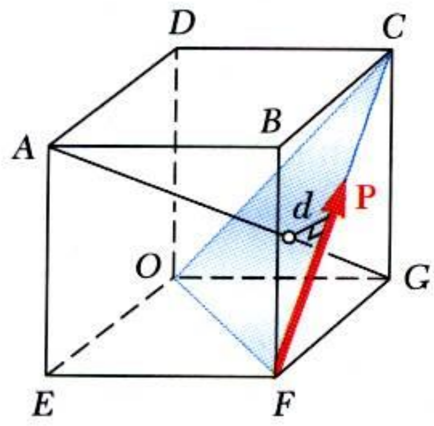
$$M_{AG} = \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right) \bullet \frac{aP}{\sqrt{2}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

$$= \frac{aP}{\sqrt{6}}(1 - 1 - 1)$$

$$\boxed{M_{AG} = -\frac{aP}{\sqrt{6}}}$$

Sample Problem 3.3





d) Perpendicular distance between *AG* and *FC*,

$$\vec{P}\bullet\vec{\lambda} = \frac{P}{\sqrt{2}}\left(\vec{j}-\vec{k}\right)\bullet\frac{1}{\sqrt{3}}\left(\vec{i}-\vec{j}-\vec{k}\right) = \frac{P}{\sqrt{6}}(0-1+1)$$
$$= 0$$

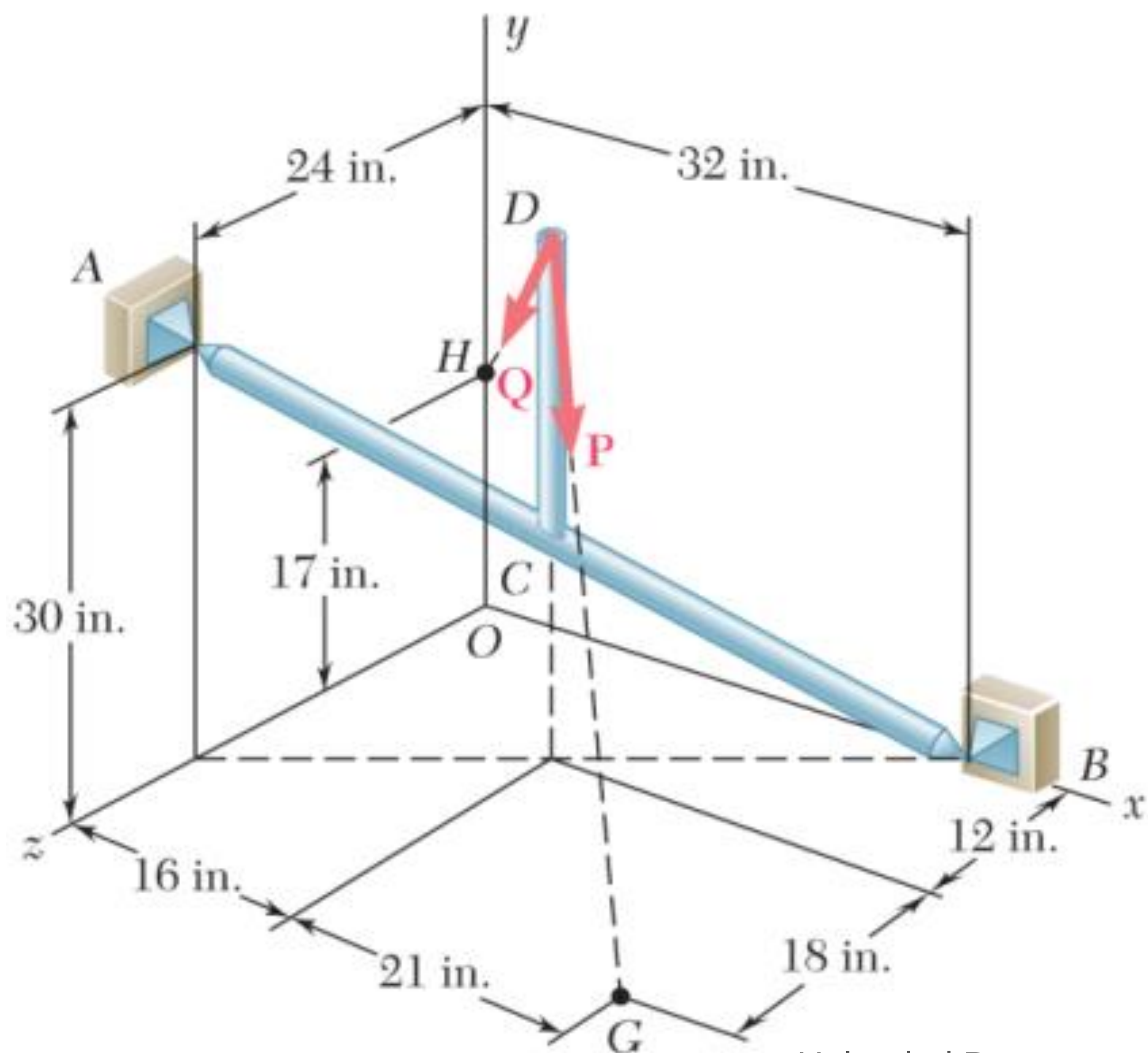
P is perpendicular to *AG*; hence **P = P_{Normal}**

M_{AG} = P_{Normal} d

$$|M_{AG}| = \frac{aP}{\sqrt{6}} = Pd$$

$$\boxed{d = \frac{a}{\sqrt{6}}}$$

PROBLEM 3.56 The 23-in. vertical rod *CD* is welded to the midpoint *C* of the 50-in. rod *AB.* Determine the moment about *AB* of the 174-lb force **Q**.



$$\overline{AB} = (32 \text{ in.})\mathbf{i} - (30 \text{ in.})\mathbf{j} - (24 \text{ in.})\mathbf{k}$$

$$AB = \sqrt{(32)^2 + (-30)^2 + (-24)^2} = 50 \text{ in.}$$

$$\boldsymbol{\lambda}_{AB} = \frac{\overline{AB}}{AB} = 0.64\mathbf{i} - 0.60\mathbf{j} - 0.48\mathbf{k}$$

$$\mathbf{r}_{H/B} = -(32 \text{ in.})\mathbf{i} + (17 \text{ in.})\mathbf{j}$$

$$\overline{DH} = -(16 \text{ in.})\mathbf{i} - (21 \text{ in.})\mathbf{j} - (12 \text{ in.})\mathbf{k}$$

$$DH = \sqrt{(16)^2 + (-21)^2 + (-12)^2} = 29 \text{ in.}$$

$$\mathbf{Q} = \frac{\overline{DH}}{DH} = (174 \text{ lb})\frac{-16\mathbf{i} - 21\mathbf{j} - 12\mathbf{k}}{29}$$

$$Q = -(96 \text{ lb})\mathbf{i} - (126 \text{ lb})\mathbf{j} - (72 \text{ lb})\mathbf{k}$$

$$\mathbf{M}_{AB} = \boldsymbol{\lambda}_{AB} \cdot (\mathbf{r}_{H/B} \times \mathbf{Q}) = \begin{vmatrix} 0.64 & -0.60 & -0.48 \\ -32 \text{ in.} & 17 \text{ in.} & 0 \\ -96 \text{ lb} & -126 \text{ lb} & -72 \text{ lb} \end{vmatrix}$$

$$\begin{aligned} \mathbf{M}_{AB} &= 0.64[(17 \text{ in.})(-72 \text{ lb}) - 0] \\ &\quad - 0.60[(0 - (-32 \text{ in.})(-72 \text{ lb})] \\ &\quad - 0.48[(-32 \text{ in.})(-126 \text{ lb}) - (17 \text{ in.})(-96 \text{ lb})] \\ &= -2119.7 \text{ lb} \cdot \text{in.} \end{aligned}$$

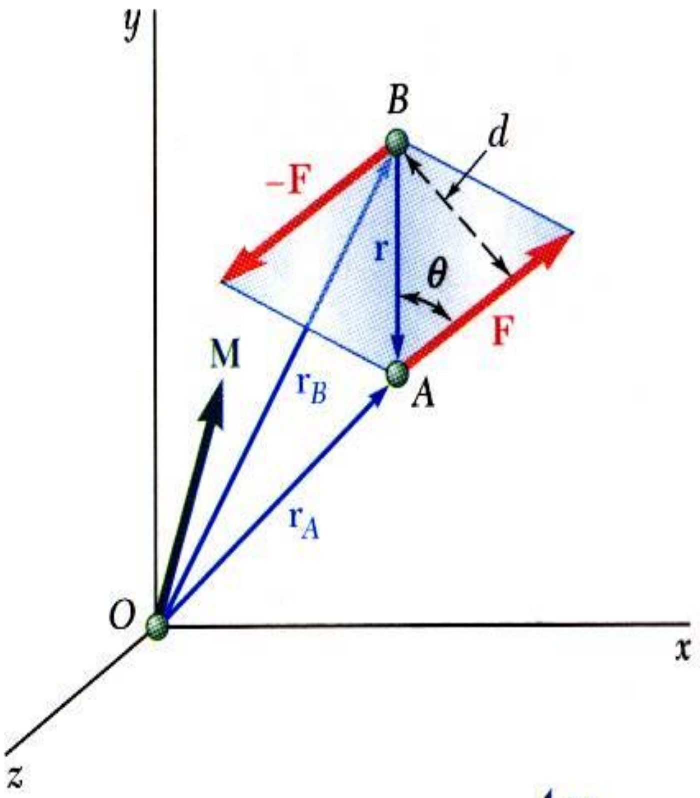
$Q_{\text{parallel}} = Q \,.\, \lambda_{AB} = -96(0.64) + 126(0.6) + 72\,(0.48) = 48.7\text{lb}$

$Q_{\text{perpendicular}} = 167.04$ lb

$d = M_{AB}/Q_{\text{perpendicular}} = 12.69$ in

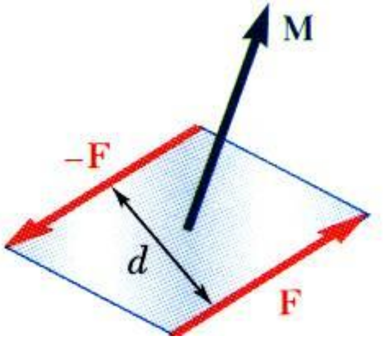
Moment of a Couple

- Two forces ***F*** and -***F*** having the same magnitude, parallel lines of action, and opposite sense are said to form a *couple*.



$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right)$$
$$= \left(\vec{r}_A - \vec{r}_B\right) \times \vec{F}$$
$$= \vec{r} \times \vec{F}$$
$$M = rF \sin\theta = Fd$$

- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a *free vector* that can be applied at any point with the same effect.



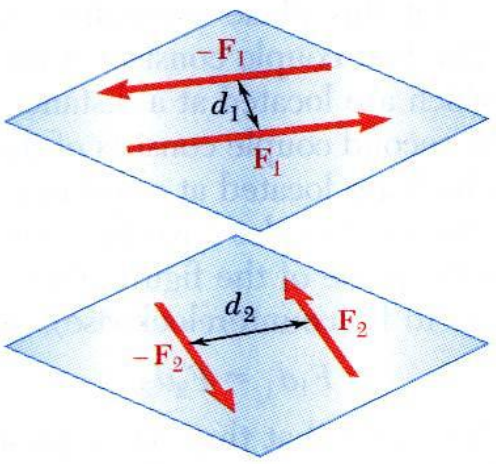
Equivalent Couples

Two couples will have equal moments if

$$F_1 d_1 = F_2 d_2$$

Provided that:

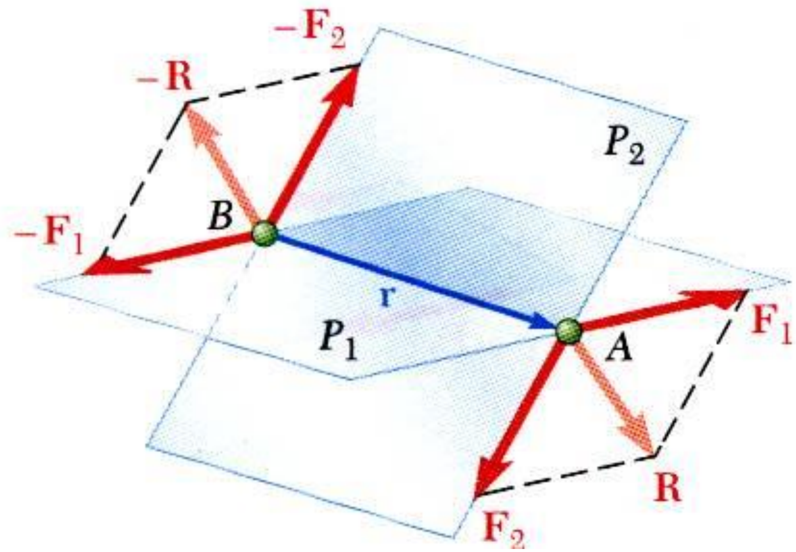
1. the two couples lie in parallel planes;
2. the two couples have the same sense or the tendency to cause rotation in the same direction.



Addition of Couples

- Consider two intersecting planes P_1 and P_2 with each containing a couple

$$\vec{M}_1 = \vec{r} \times \vec{F}_1 \text{ in plane } P_1$$
$$\vec{M}_2 = \vec{r} \times \vec{F}_2 \text{ in plane } P_2$$

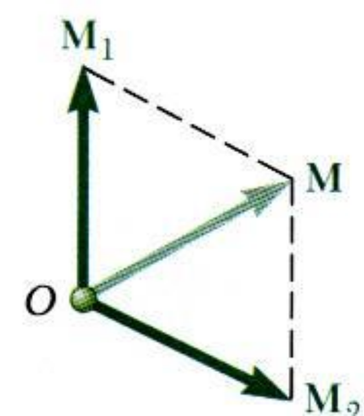


- Resultants of the vectors also form a couple

$$\vec{M} = \vec{r} \times \vec{R} = \vec{r} \times \left(\vec{F}_1 + \vec{F}_2\right)$$

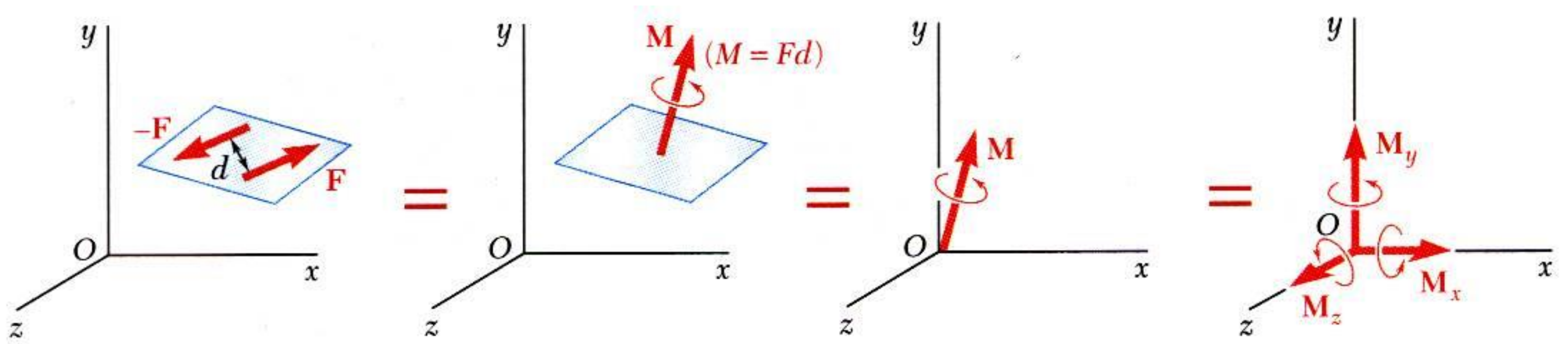
- By Varignon's theorem

$$\vec{M} = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2$$
$$= \vec{M}_1 + \vec{M}_2$$



- Sum of two couples is also a couple that is equal to the vector sum of the two couples

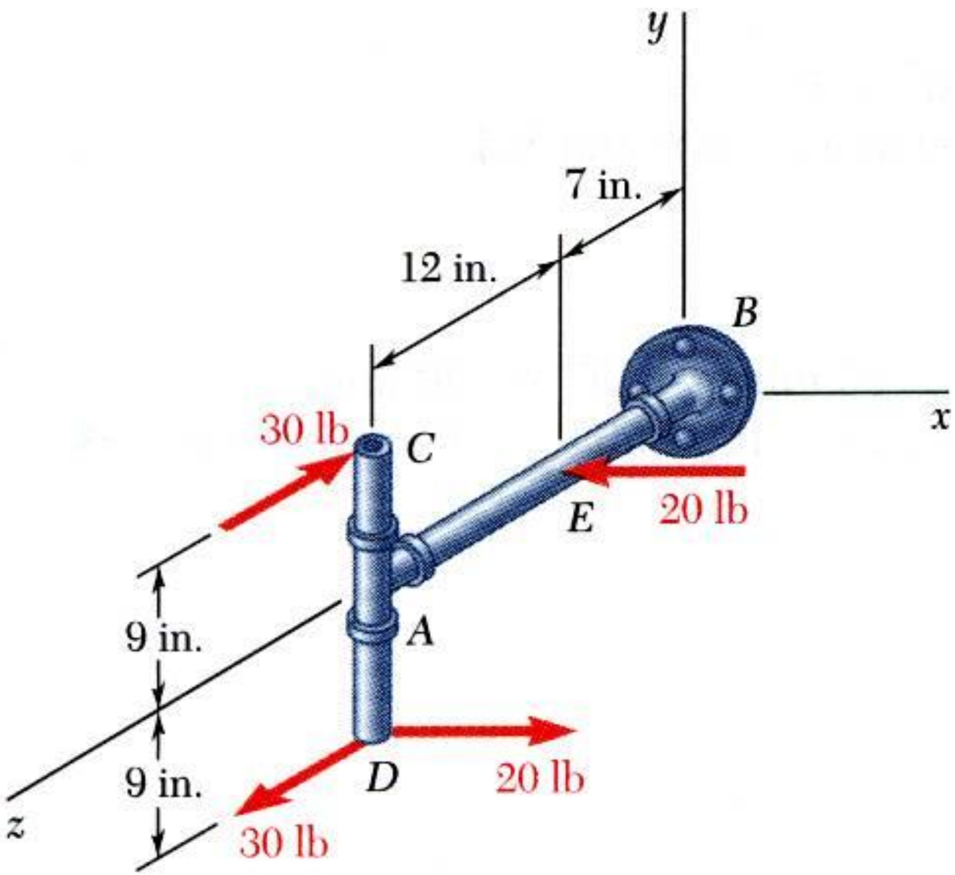
Couples Can Be Represented by Vectors



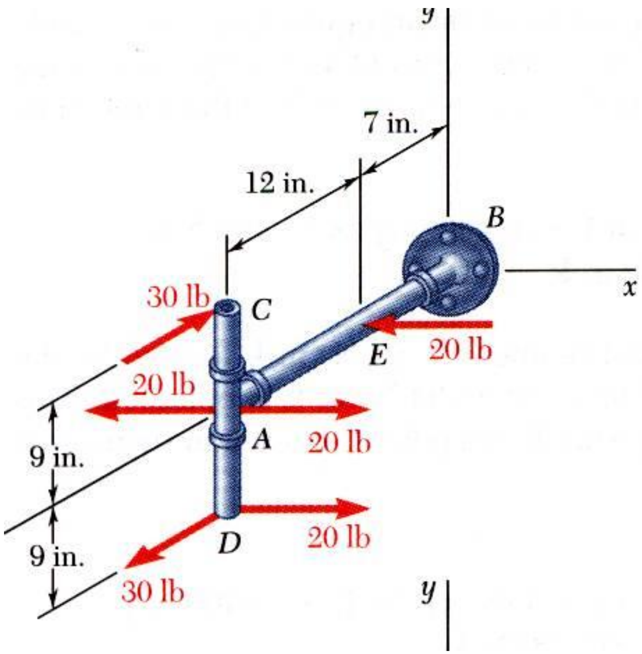
- A couple can be represented by a vector with magnitude and direction equal to the moment of the couple.
- *Couple vectors* obey the law of addition of vectors.
- Couple vectors are free vectors, i.e., the point of application is not significant.
- Couple vectors may be resolved into component vectors.

Sample Problem 3.4

Determine the components of the single couple equivalent to the couples shown.



Sample Problem 3.4



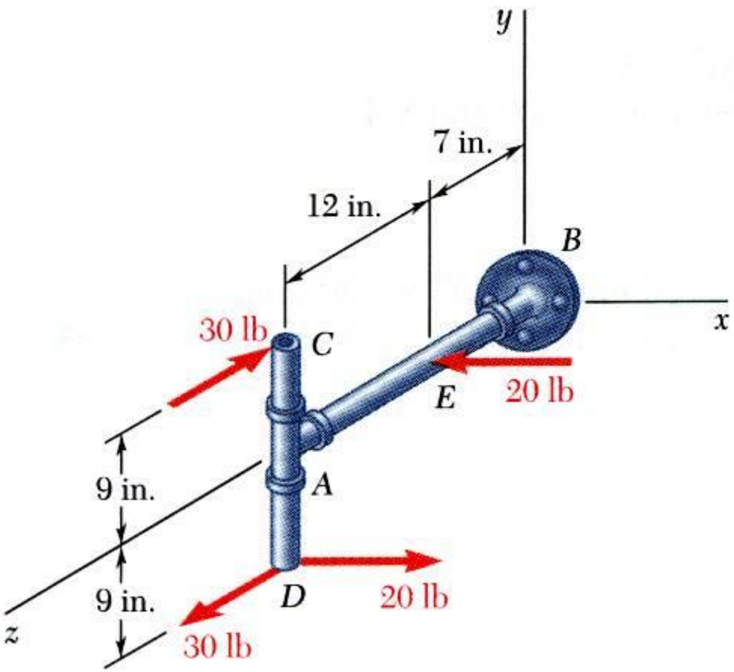
- Attach equal and opposite 20 lb forces in the $\pm x$ direction at *A*

- The three couples may be represented by three couple vectors,

$$M_x = -(30\,\text{lb})(18\,\text{in.}) = -540\,\text{lb}\cdot\text{in.}$$
$$M_y = +(20\,\text{lb})(12\,\text{in.}) = +240\text{lb}\cdot\text{in.}$$
$$M_z = +(20\,\text{lb})(9\,\text{in.}) = +180\,\text{lb}\cdot\text{in.}$$

$$\boxed{\vec{M} = -(540\,\text{lb}\cdot\text{in.})\vec{i} + (240\text{lb}\cdot\text{in.})\vec{j} + (180\,\text{lb}\cdot\text{in.})\vec{k}}$$

Sample Problem 3.4

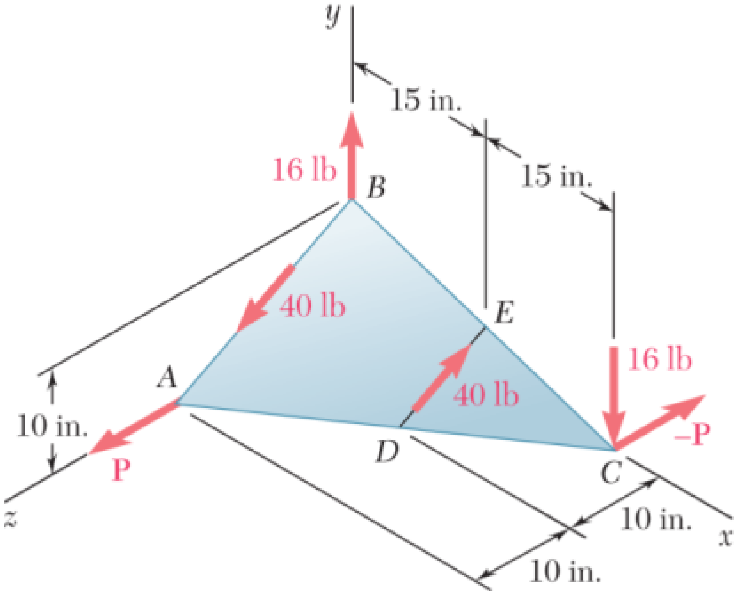


- Alternatively, compute the sum of the moments of the four forces about *D* (or any other point, the answer will be the same!)

- Only the forces at *C* and *E* contribute to the moment about *D*.

$$\vec{M} = \vec{M}_D = (18\,\text{in.})\vec{j} \times (-30\,\text{lb})\vec{k} + \left[(9\,\text{in.})\vec{j} - (12\,\text{in.})\vec{k}\right] \times (-20\,\text{lb})\vec{i}$$

$$\boxed{\vec{M} = -(540\,\text{lb}\cdot\text{in.})\vec{i} + (240\text{lb}\cdot\text{in.})\vec{j} + (180\,\text{lb}\cdot\text{in.})\vec{k}}$$



PROBLEM 3.77

If $P = 20$ lb, replace the three couples with a single equivalent couple, specifying its magnitude and the direction of its axis.

SOLUTION

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2; \quad F_1 = 16 \text{ lb}, \quad F_2 = 40 \text{ lb}$$

$$\mathbf{M_1} = \mathbf{r}_C \times \mathbf{F}_1 = (30 \text{ in.})\mathbf{i} \times [-(16 \text{ lb})\mathbf{j}] = -(480 \text{ lb} \cdot \text{in.})\mathbf{k}$$

$$\mathbf{M_2} = \mathbf{r}_{E/B} \times \mathbf{F}_2; \quad \mathbf{r}_{E/B} = (15 \text{ in.})\mathbf{i} - (5 \text{ in.})\mathbf{j}$$

$$d_{DE} = \sqrt{(0)^2 + (5)^2 + (10)^2} = 5\sqrt{5} \text{ in.}$$

$$F_2 = \frac{40 \text{ lb}}{5\sqrt{5}}(5\mathbf{j} - 10\mathbf{k})$$

$$= 8\sqrt{5}[(1 \text{ lb})\mathbf{j} - (2 \text{ lb})\mathbf{k}]$$

$$\mathbf{M}_2 = 8\sqrt{5}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 15 & -5 & 0 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 8\sqrt{5}[(10 \text{ lb} \cdot \text{in.})\mathbf{i} + (30 \text{ lb} \cdot \text{in.})\mathbf{j} + (15 \text{ lb} \cdot \text{in.})\mathbf{k}]$$

SOLUTION

From the solution to Problem. 3.78:

16-lb force: $M_1 = -(480 \text{ lb} \cdot \text{in.})\mathbf{k}$

40-lb force: $M_2 = 8\sqrt{5}[(10 \text{ lb} \cdot \text{in.})\mathbf{i} + (30 \text{ lb} \cdot \text{in.})\mathbf{j} + (15 \text{ lb} \cdot \text{in.})\mathbf{k}]$

$P = 20$ lb

$$\begin{aligned} M_3 &= \mathbf{r}_C \times P \\ &= (30 \text{ in.})\mathbf{i} \times (20 \text{ lb})\mathbf{k} \\ &= (600 \text{ lb} \cdot \text{in.})\mathbf{j} \end{aligned}$$

$$\begin{aligned} \mathbf{M} &= \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 \\ &= -(480)\mathbf{k} + 8\sqrt{5}(10\mathbf{i} + 30\mathbf{j} + 15\mathbf{k}) + 600\mathbf{j} \\ &= (178.885 \text{ lb} \cdot \text{in.})\mathbf{i} + (1136.66 \text{ lb} \cdot \text{in.})\mathbf{j} - (211.67 \text{ lb} \cdot \text{in.})\mathbf{k} \end{aligned}$$

$$\begin{aligned} M &= \sqrt{(178.885)^2 + (113.66)^2 + (211.67)^2} \\ &= 1169.96 \text{ lb} \cdot \text{in.} \end{aligned}$$

$M = 1170 \text{ lb} \cdot \text{in.}$ ◀

$$\lambda_{\text{axis}} = \frac{\mathbf{M}}{M} = 0.152898\mathbf{i} + 0.97154\mathbf{j} - 0.180921\mathbf{k}$$

$$\cos\theta_x = 0.152898$$

$$\cos\theta_y = 0.97154$$

$$\cos\theta_z = -0.180921$$

$\theta_x = 81.2°$ $\theta_y = 13.70°$ $\theta_z = 100.4°$ ◀

Sample Problem 3.8

If the resultant couple moment acting on the triangular block is to be zero, determine the magnitudes F and P.

U = 10 (-4/5**j** + 3/5***k***)

M_u = (-3**i**) x 10(-4/5**j**+3/5**k**)

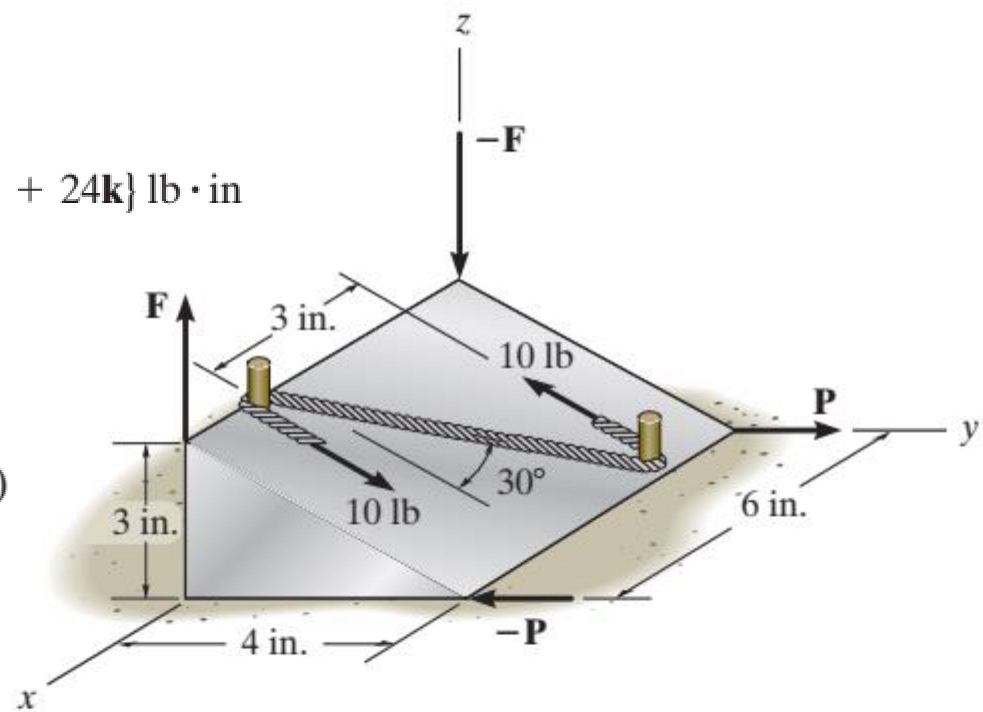
$$\mathbf{M} = M\mathbf{u} = 10(3)\left(\frac{3}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}\right) = \{18\mathbf{j} + 24\mathbf{k}\}\ \text{lb}\cdot\text{in}$$

$$\mathbf{M}_F = -6F\mathbf{j} \qquad \mathbf{M}_P = -6P\mathbf{k}$$

$$\mathbf{0} = \mathbf{M} + \mathbf{M}_F + \mathbf{M}_P$$

$$\mathbf{0} = (18\mathbf{j} + 24\mathbf{k}) + (-6F\mathbf{j}) + (-6P\mathbf{k})$$

$$\mathbf{0} = (18 - 6F)\mathbf{j} + (24 - 6P)\mathbf{k}$$

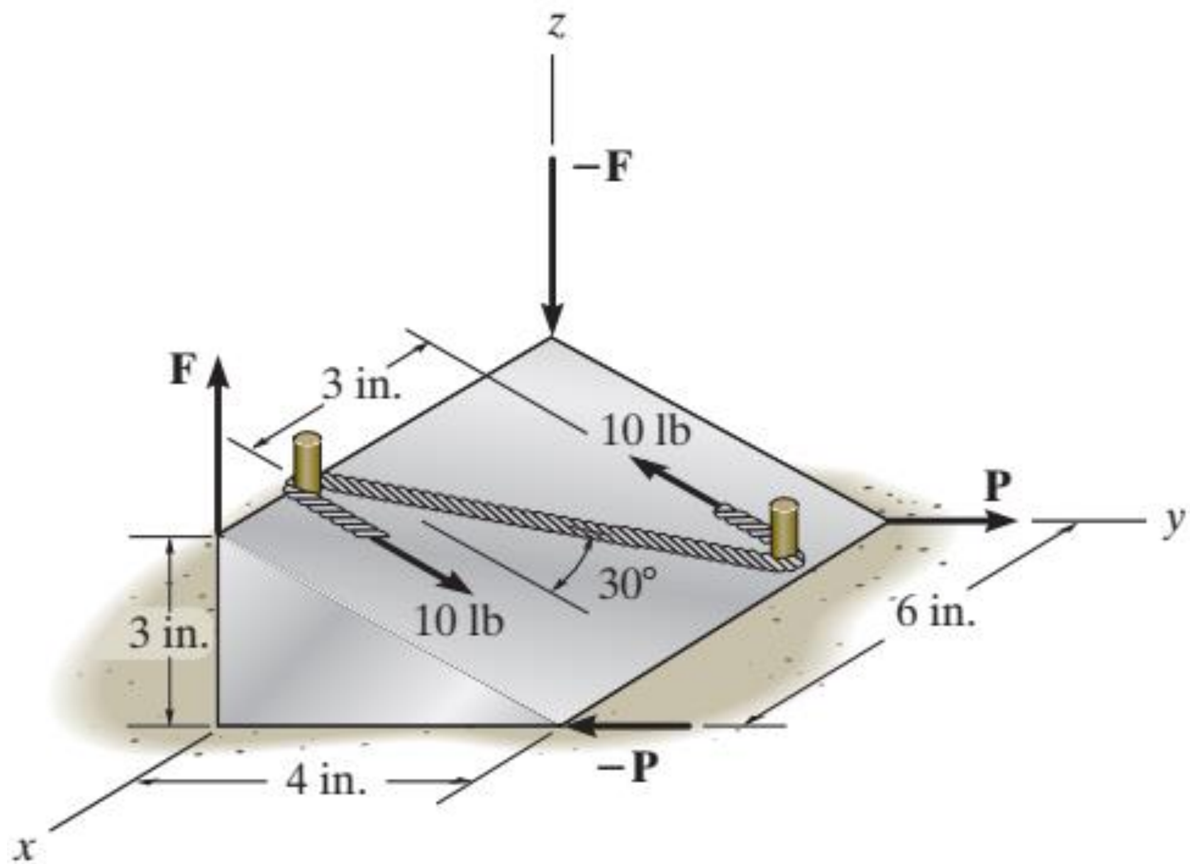


$$(18 - 6F) = 0 \qquad F = 3\ \text{lb}$$

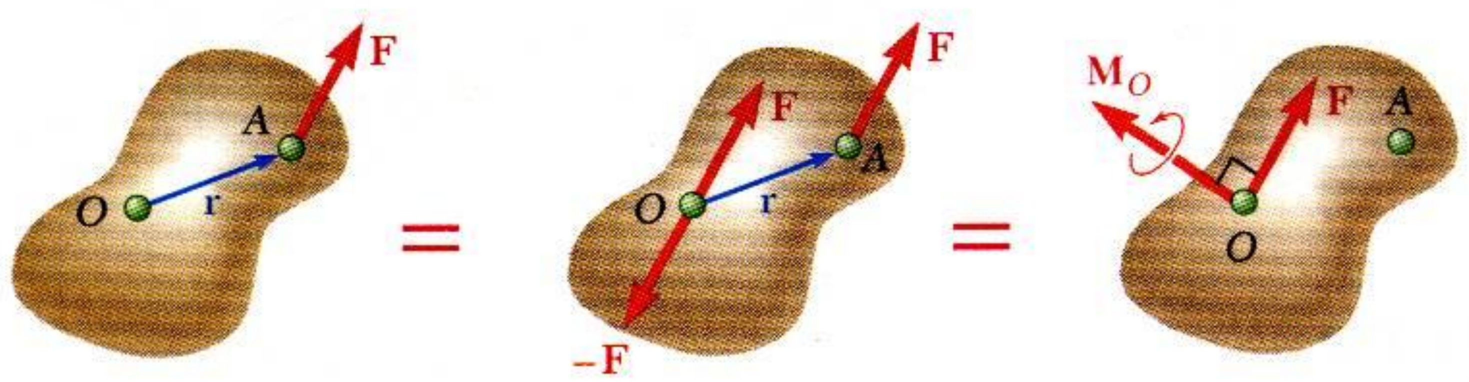
$$(24 - 6P) = 0 \qquad P = 4\ \text{lb}$$

Sample Problem 3.8

If the resultant couple moment acting on the triangular block is to be zero, determine the magnitudes *F* and *P*.

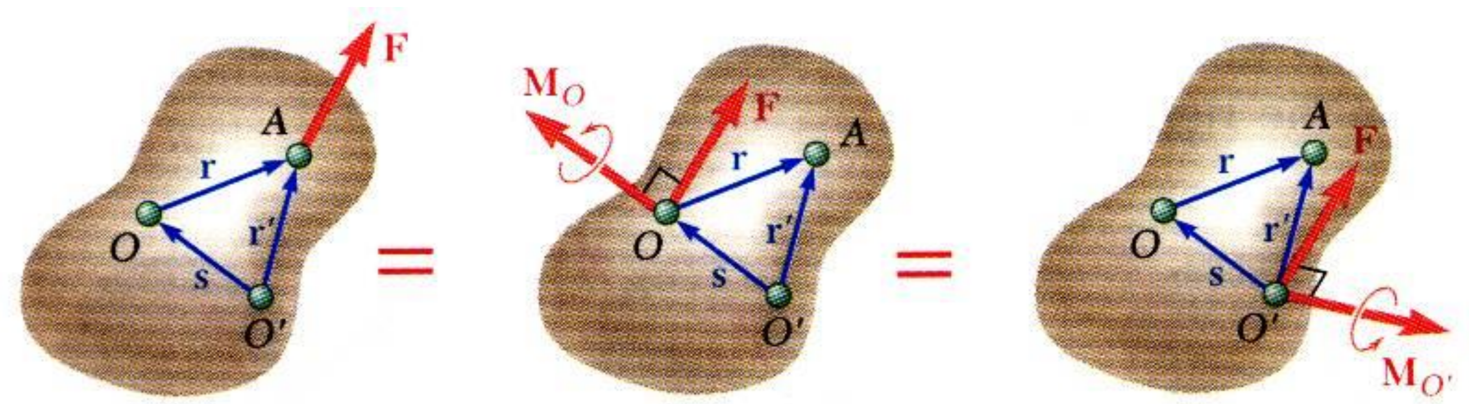


Force-Couple System



- Force vector ***F*** can not be simply moved to *O* without modifying its action on the body.
- Attaching equal and opposite force vectors at *O* produces no net effect on the body.
- The three forces may be replaced by an equivalent force vector and couple vector, i.e, a *force-couple system*.

Resolution of a Force Into a Force at *O* and a Couple



- Moving ***F*** from *A* to a different point *O'* requires the addition of a different couple vector $\boldsymbol{M_{O'}}$

$$\vec{M}_{O'} = \vec{r}\,' \times \vec{F}$$

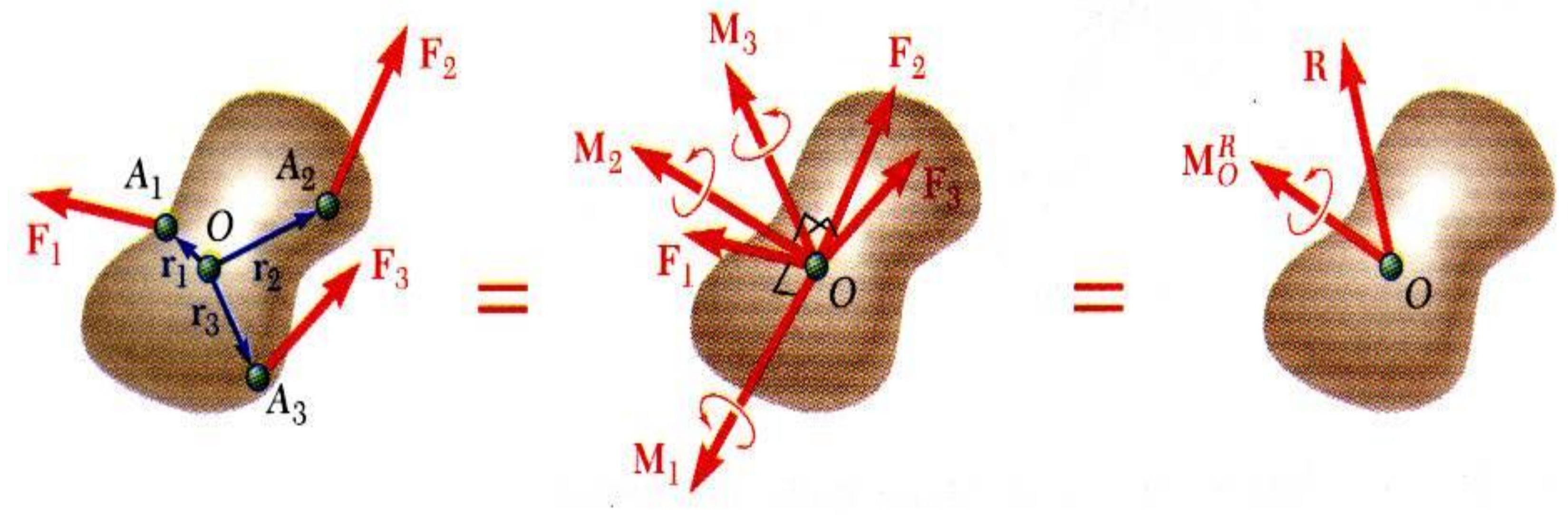
  - The moments of ***F*** about O and *O'* are related,

$$\vec{M}_{O'} = \vec{r}\,'\times\vec{F} = \left(\vec{r} + \vec{s}\right)\times\vec{F} = \vec{r}\times\vec{F} + \vec{s}\times\vec{F}$$
$$= \vec{M}_O + \vec{s}\times\vec{F}$$

- Moving the force-couple system from *O* to *O'* requires the addition of the moment of the force at *O* about *O'*.

Reduction to Force-Couple System

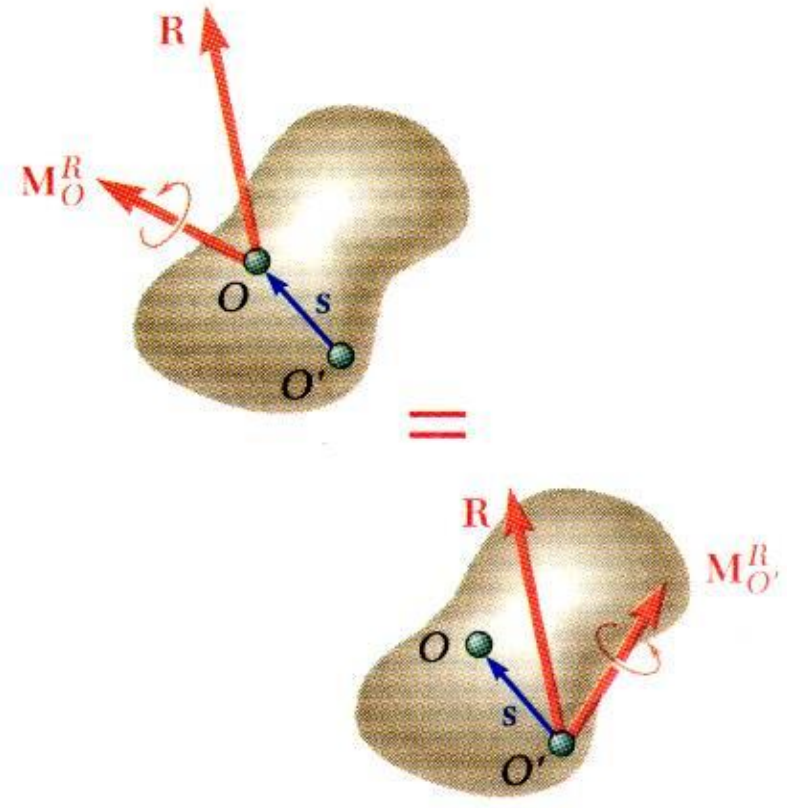
- A system of forces may be replaced by a collection of force-couple systems acting at a given point *O.*



$$\vec{R} = \sum \vec{F} \qquad \vec{M}_O^R = \sum \left(\vec{r} \times \vec{F} \right)$$

Reduction to Force-Couple System

The force-couple system at *O* may be moved to *O'* with the addition of the moment of ***R*** about *O'* ,

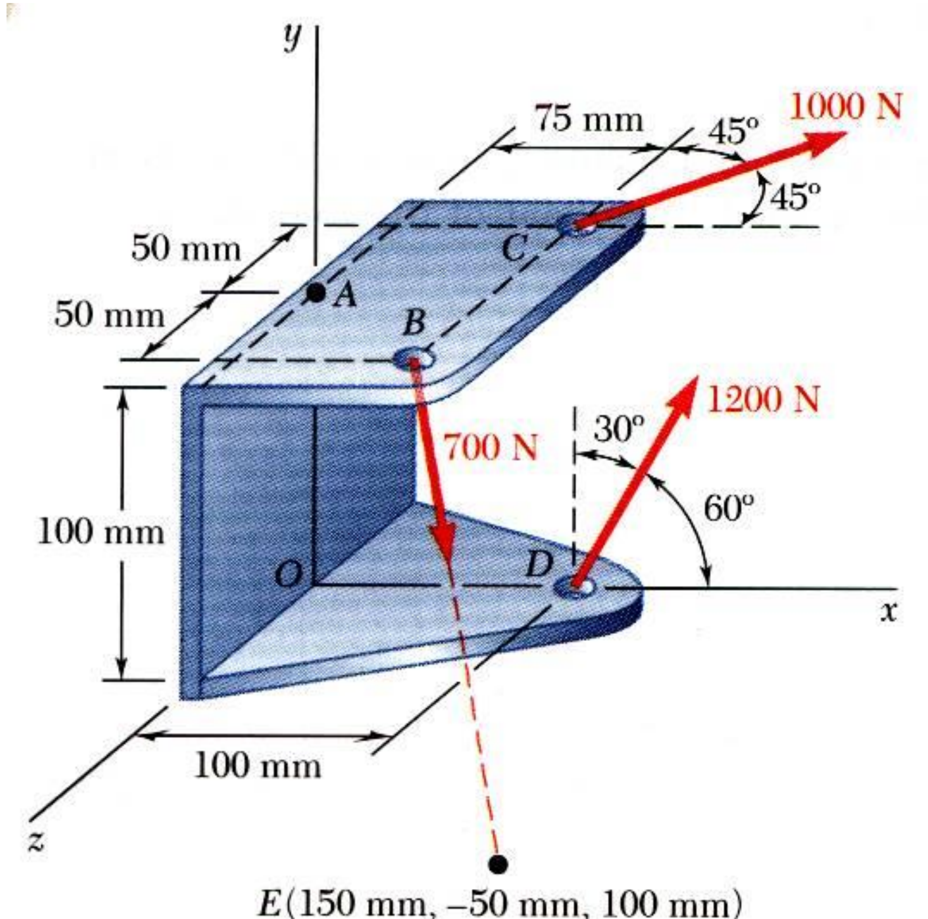


$$\vec{M}_{O'}^{R} = \vec{M}_{O}^{R} + \vec{s} \times \vec{R}$$

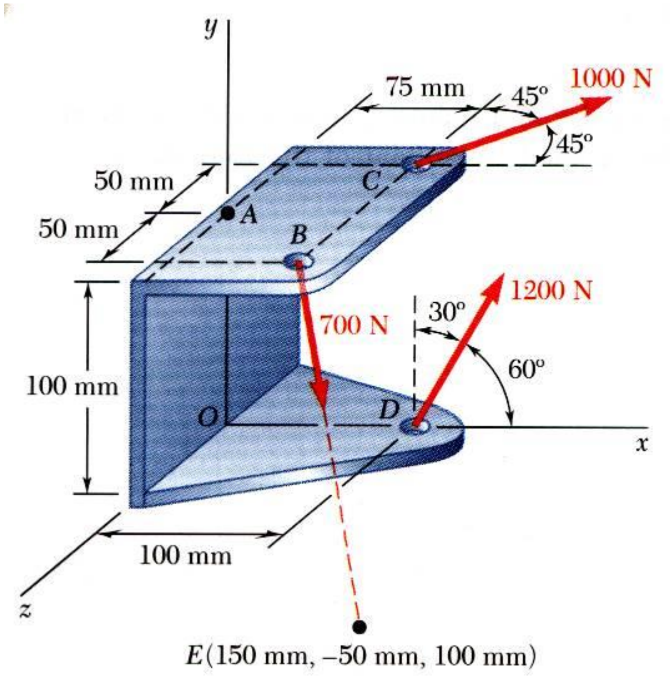
- Two systems of forces are equivalent if they can be reduced to the same force-couple system.

Sample Problem 3.5

Three cables are attached to the bracket as shown. Replace the forces with an equivalent force-couple system at *A*.



Sample Problem 3.5



SOLUTION:

- Determine the relative position vectors with respect to *A*.

$$\vec{r}_{B/A} = 0.075\vec{i} + 0.050\vec{k}\ (\mathrm{m})$$

$$\vec{r}_{C/A} = 0.075\vec{i} - 0.050\vec{k}\ (\mathrm{m})$$

$$\vec{r}_{D/A} = 0.100\vec{i} - 0.100\vec{j}\ (\mathrm{m})$$

- Resolve the forces into rectangular components.

$$\vec{F}_B = (700\ \mathrm{N})\vec{\lambda}$$

$$\vec{\lambda} = \frac{\vec{r}_{E/B}}{r_{E/B}} = \frac{75\vec{i} - 150\vec{j} + 50\vec{k}}{175}$$

$$= 0.429\vec{i} - 0.857\vec{j} + 0.289\vec{k}$$

$$\vec{F}_B = 300\vec{i} - 600\vec{j} + 200\vec{k}\ (\mathrm{N})$$

$$\vec{F}_C = (1000\ \mathrm{N})(\cos 45\,\vec{i} - \cos 45\,\vec{k})$$

$$= 707\,\vec{i} - 707\,\vec{k}\ (\mathrm{N})$$

$$\vec{F}_D = (1200\ \mathrm{N})(\cos 60\vec{i} + \cos 30\vec{j})$$

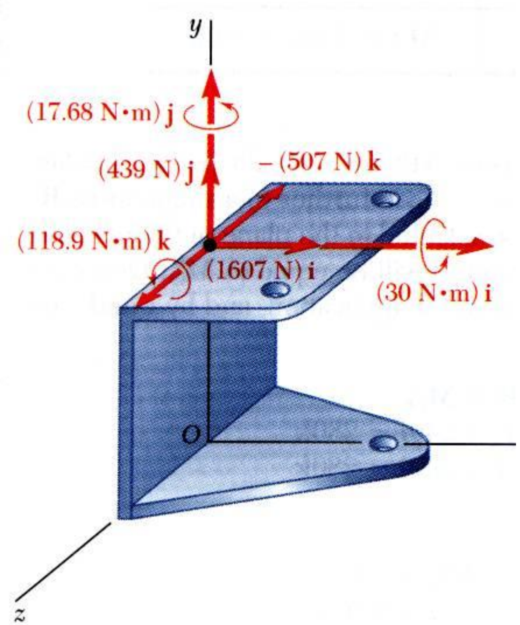
$$= 600\vec{i} + 1039\vec{j}\ (\mathrm{N})$$

Sample Problem 3.5

- Compute the equivalent force,

$$\vec{R} = \sum \vec{F}$$
$$= (300 + 707 + 600)\vec{i} + (-600 + 1039)\vec{j} + (200 - 707)\vec{k}$$

$$\boxed{\vec{R} = 1607\vec{i} + 439\vec{j} - 507\vec{k} \ \ (\mathrm{N})}$$



- Compute the equivalent couple,

$$\vec{M}_A^R = \sum \left(\vec{r} \times \vec{F}\right)$$

$$\vec{r}_{B/A} \times \vec{F}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & 0.050 \\ 300 & -600 & 200 \end{vmatrix} = 30\vec{i} - 45\vec{k}$$

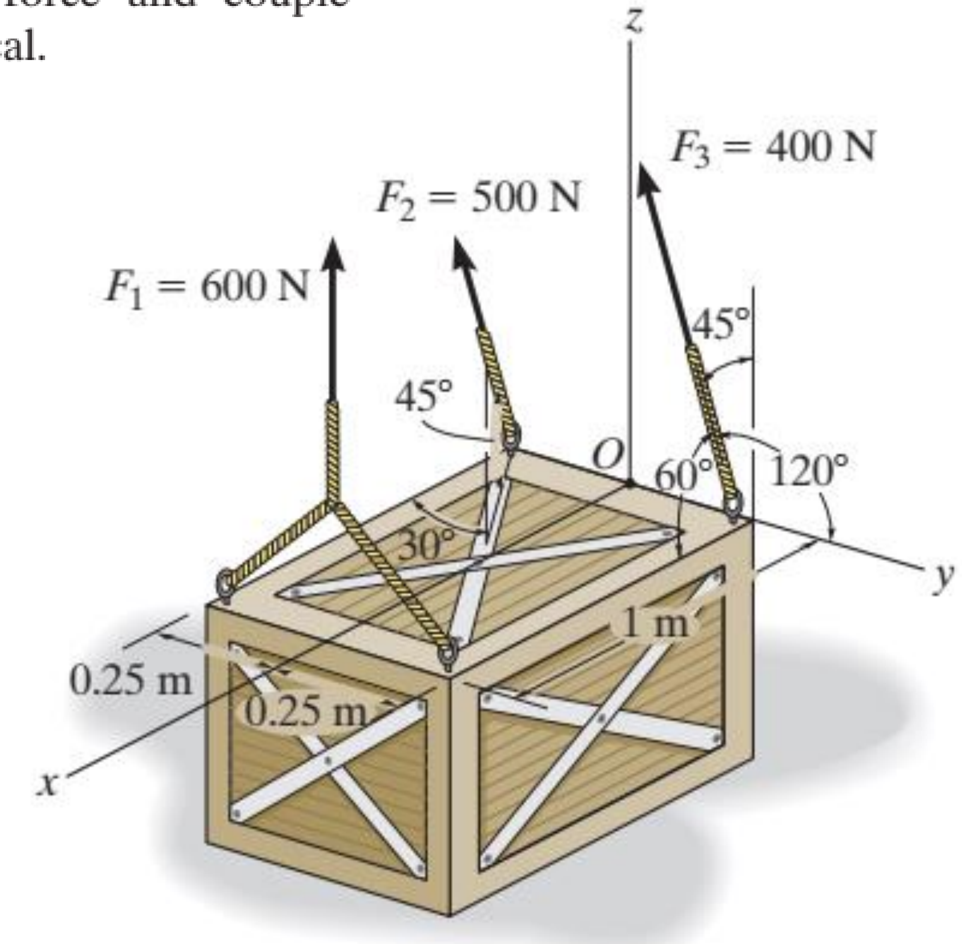
$$\vec{r}_{C/A} \times \vec{F}_c = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & -0.050 \\ 707 & 0 & -707 \end{vmatrix} = 17.68\vec{j}$$

$$\vec{r}_{D/A} \times \vec{F}_D = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.100 & -0.100 & 0 \\ 600 & 1039 & 0 \end{vmatrix} = 163.9\vec{k}$$

$$\boxed{\vec{M}_A^R = 30\vec{i} + 17.68\vec{j} + 118.9\vec{k}}$$

Sample Problem

The crate is on the ground and is to be hoisted using the three slings shown. Replace the system of forces acting on the slings by an equivalent resultant force and couple moment at point O. The force $\boldsymbol{F_1}$ is vertical.



$$\mathbf{F}_1 = \{600\mathbf{k}\}\ \text{N}$$

$$\mathbf{F}_2 = 500(\cos 45^\circ \cos 30^\circ \mathbf{i} + \cos 45^\circ \sin 30^\circ \mathbf{j} + \sin 45^\circ \mathbf{k})$$

$$= \{306.2\mathbf{i} + 176.8\mathbf{j} + 353.6\mathbf{k}\}\ \text{N}$$

$$\mathbf{F}_3 = 400\,(\cos 60^\circ \mathbf{i} + \cos 120^\circ \mathbf{j} + \cos 45^\circ \mathbf{k})$$

$$= \{200\mathbf{i} - 200\mathbf{j} + 282.8\mathbf{k}\}\ \text{N}$$

$$\mathbf{F}_R = \Sigma\mathbf{F}; \quad \mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

$$= (600\mathbf{k}) + (306.2\mathbf{i} + 176.8\mathbf{j} + 353.6\mathbf{k}) + (200\mathbf{i} - 200\mathbf{j} + 282.8\mathbf{k})$$

$$\mathbf{F}_R = \{506\mathbf{i} - 23.2\mathbf{j} + 1236\mathbf{k}\}\ \text{N} \qquad \textbf{Ans.}$$

$$\mathbf{M}_{R_O} = \Sigma\mathbf{M}; \quad \mathbf{M}_{R_O} = (1\mathbf{i}) \times (600\mathbf{k}) + (-0.25\mathbf{j}) \times (306.2\mathbf{i} + 176.8\mathbf{j} + 353.6\mathbf{k})$$

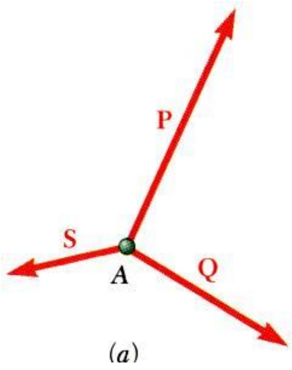
$$+ (0.25\mathbf{j}) \times (200\mathbf{i} - 200\mathbf{j} + 282.8\mathbf{k})$$

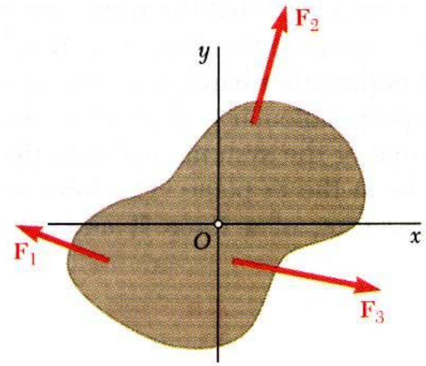
$$\mathbf{M}_{R_O} = \{-17.7\mathbf{i} - 600\mathbf{j} + 26.5\mathbf{k}\}\ \text{N}\cdot\text{m} \qquad \textbf{Ans.}$$

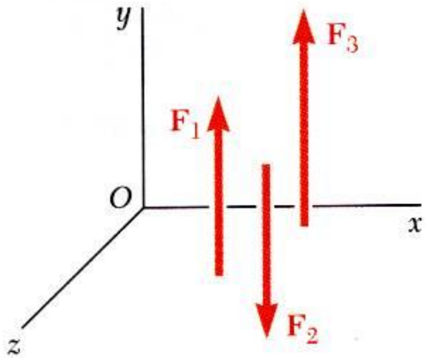
Reduction to a single Force –Special cases only

Condition: If the resultant force and couple at *O* are mutually perpendicular, they can be replaced by a single force acting along a new line of action.

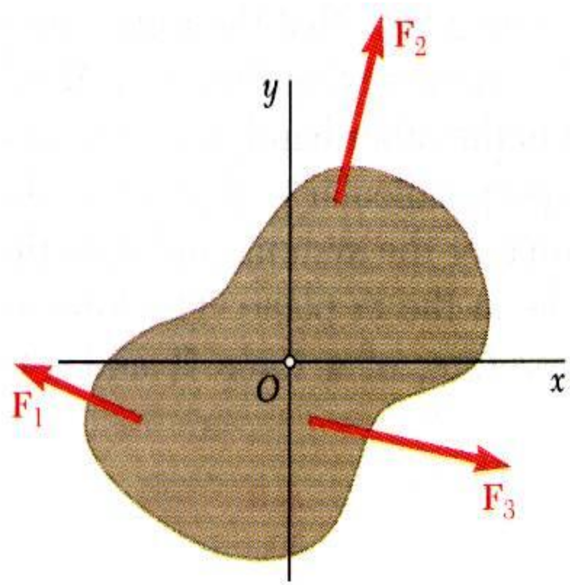
- The resultant force-couple system for a system of forces will be mutually perpendicular if:
  1) the forces are concurrent,
  2) the forces are coplanar, or
  3) the forces are parallel.

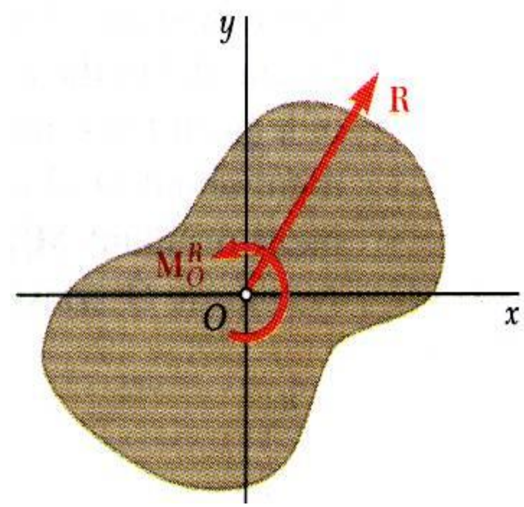




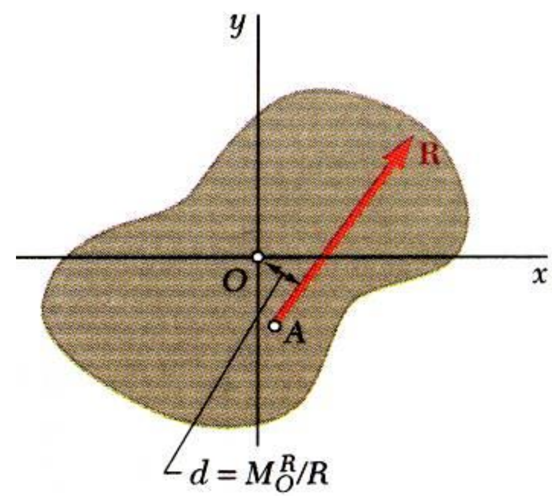


System of coplanar forces is reduced to a force-couple system $\vec{R}$ and $\vec{M}_O^R$ that is mutually perpendicular.



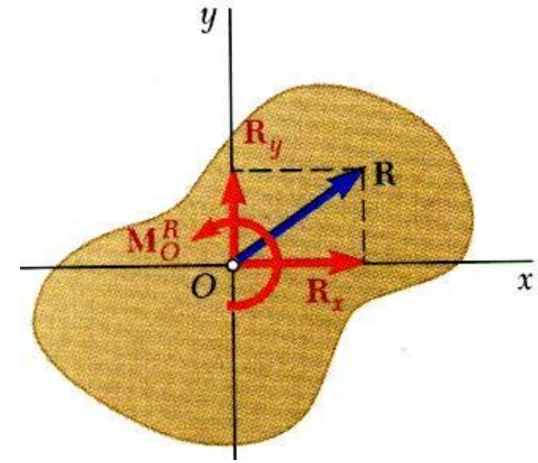


System can be reduced to a single force by moving the line of action of $\vec{R}$ until its moment about *O* becomes zero.

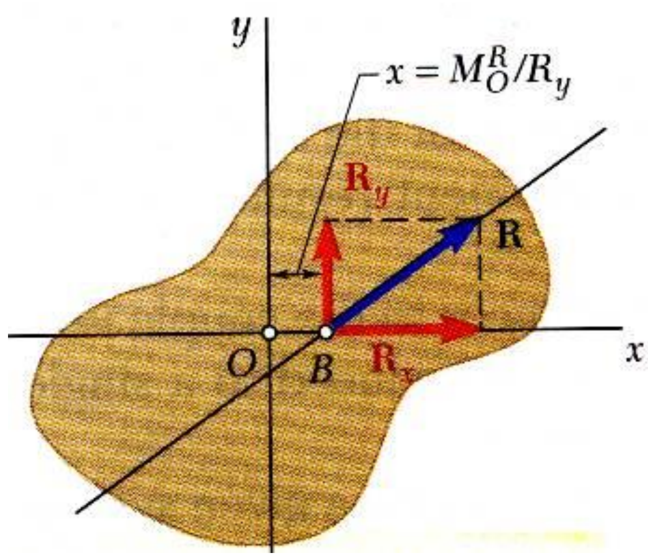


Location of Single Force

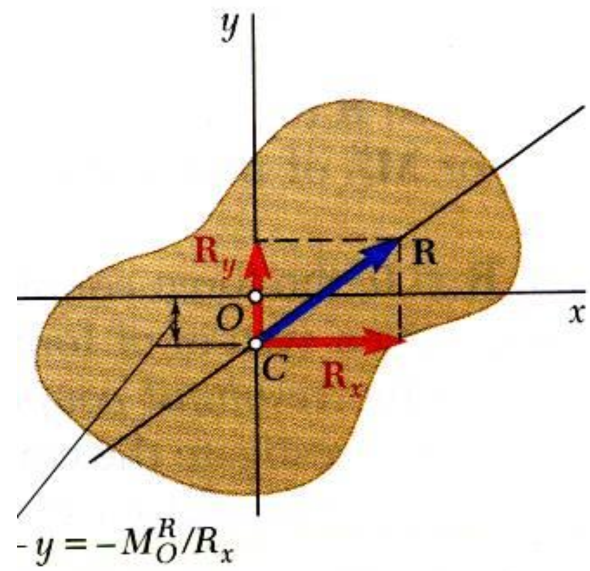
System can be reduced to a single force by moving the line of action of $\vec{R}$ until its moment about *O* becomes zero.



$$M_B = 0 = M_o^R - R_y\, x$$

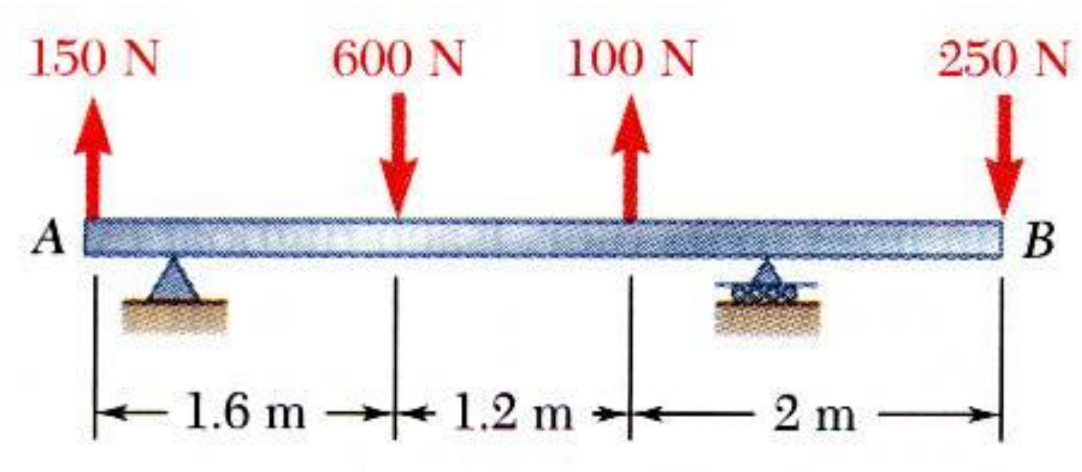


$$M_c = 0 = M_o^R - R_x y$$

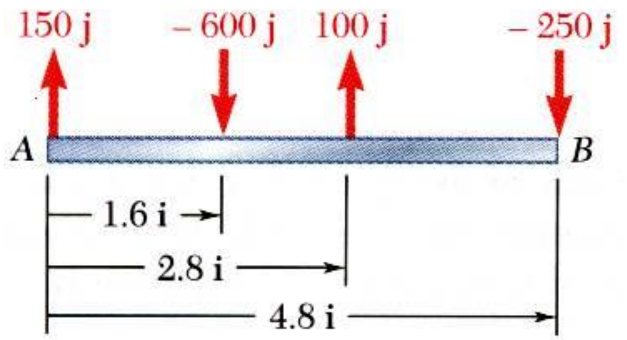


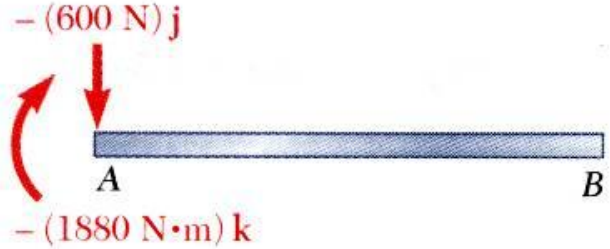
Sample Problem 3.6

For the beam, reduce the system of forces shown to
(a) an equivalent force-couple system at *A*,
(b) an equivalent force couple system at *B*, and
(c) a single force or resultant.
Note: Since the support reactions are not included, the given system will not maintain the beam in equilibrium.



Sample Problem 3.6





SOLUTION:

a) Compute the resultant force and the resultant couple at *A*.

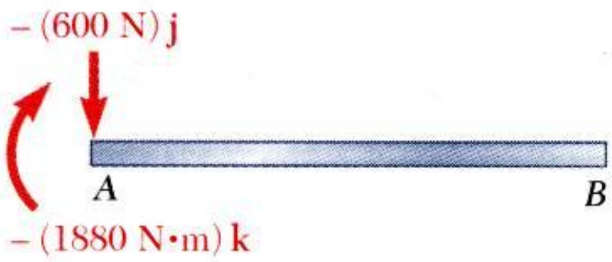
$$\vec{R} = \sum \vec{F}$$
$$= (150\ \mathrm{N})\vec{j} - (600\ \mathrm{N})\vec{j} + (100\ \mathrm{N})\vec{j} - (250\ \mathrm{N})\vec{j}$$

$$\boxed{\vec{R} = -(600\ \mathrm{N})\vec{j}}$$

$$\vec{M}_A^R = \sum\left(\vec{r} \times \vec{F}\right)$$
$$= \left(1.6\vec{i}\right)\times\left(-600\vec{j}\right) + \left(2.8\vec{i}\right)\times\left(100\vec{j}\right)$$
$$+ \left(4.8\vec{i}\right)\times\left(-250\vec{j}\right)$$

$$\boxed{\vec{M}_A^R = -(1880\ \mathrm{N}\cdot\mathrm{m})\vec{k}}$$

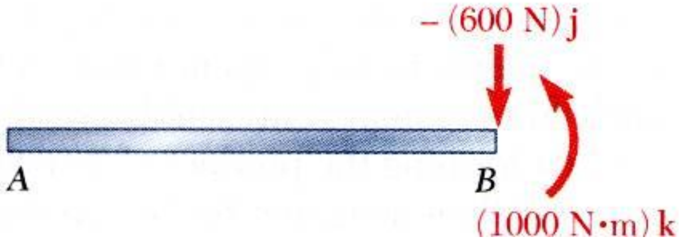
Sample Problem 3.6



b) equivalent force-couple system at *B* based on the force-couple system at *A*:

The force is unchanged by the movement of the force-couple system from *A* to *B*.

$$\boxed{\vec{R} = -(600\,\mathrm{N})\vec{j}}$$



The couple at *B* is equal to the moment about *B* of the force-couple system found at *A*.

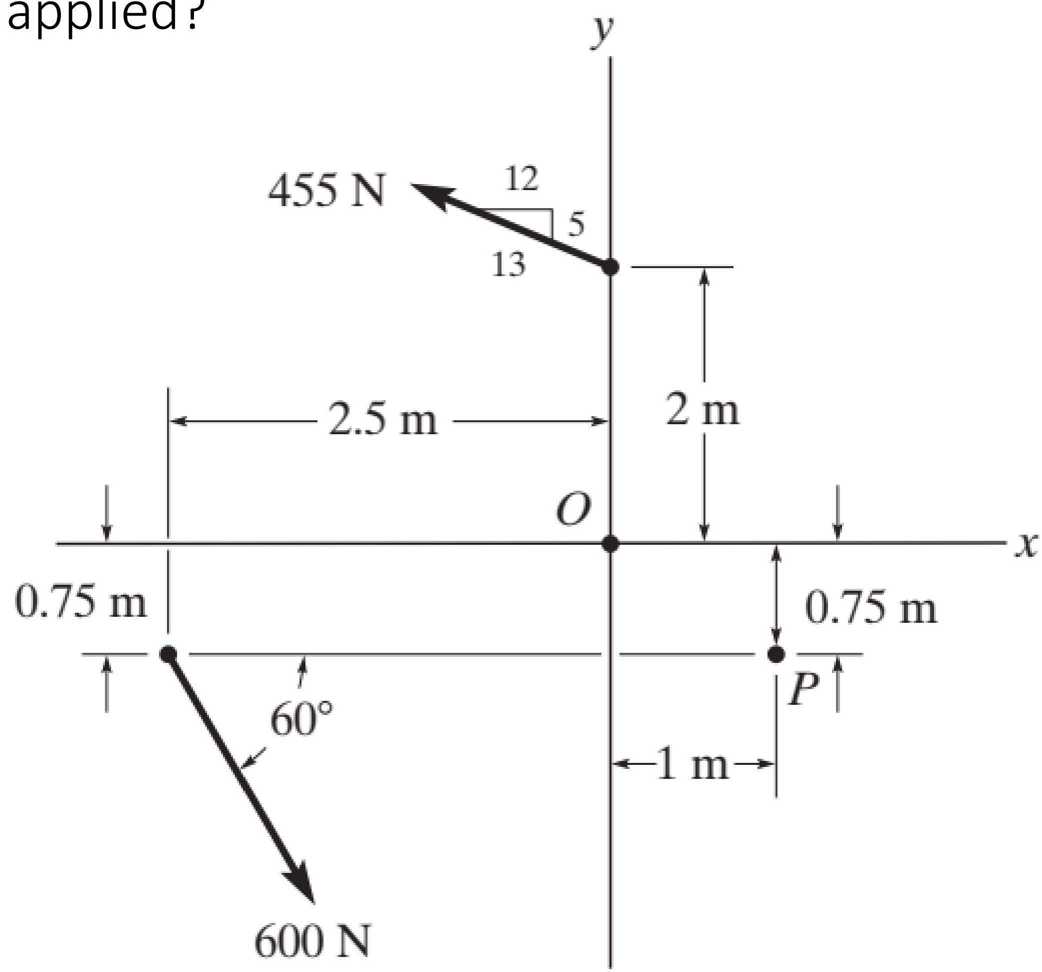
$$\begin{aligned}\vec{M}_B^R &= \vec{M}_A^R + \vec{r}_{B/A} \times \vec{R} \\ &= -(1880\,\mathrm{N}\cdot\mathrm{m})\vec{k} + (-4.8\,\mathrm{m})\vec{i} \times (-600\,\mathrm{N})\vec{j} \\ &= -(1880\,\mathrm{N}\cdot\mathrm{m})\vec{k} + (2880\,\mathrm{N}\cdot\mathrm{m})\vec{k}\end{aligned}$$

$$\boxed{\vec{M}_B^R = +(1000\,\mathrm{N}\cdot\mathrm{m})\vec{k}}$$

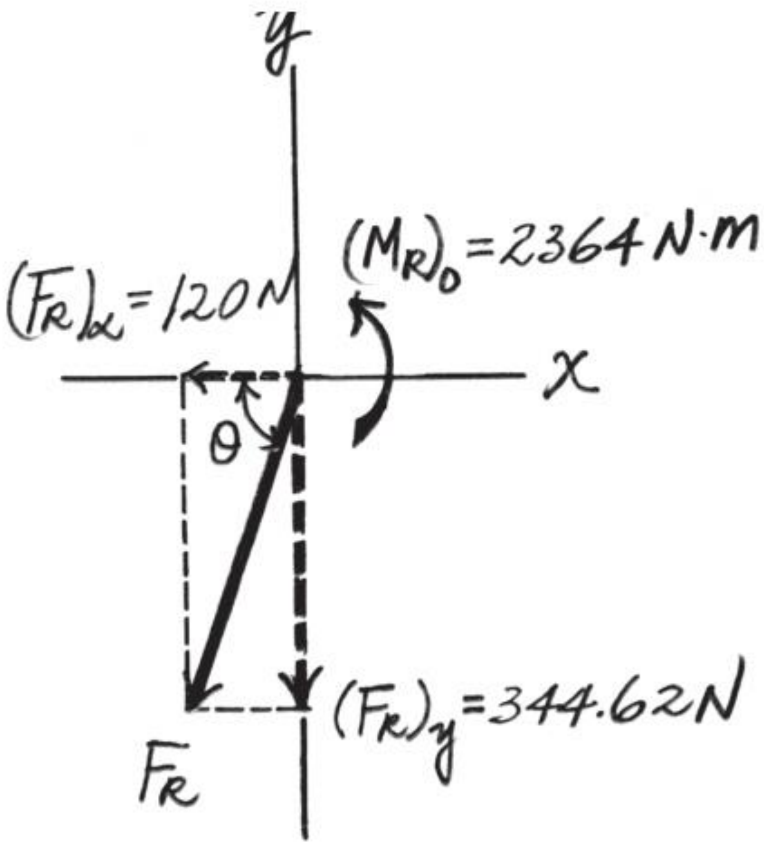
c) Both F and M are mutually perpendicular, x = 1000/600 = 1.67 m to the left of B

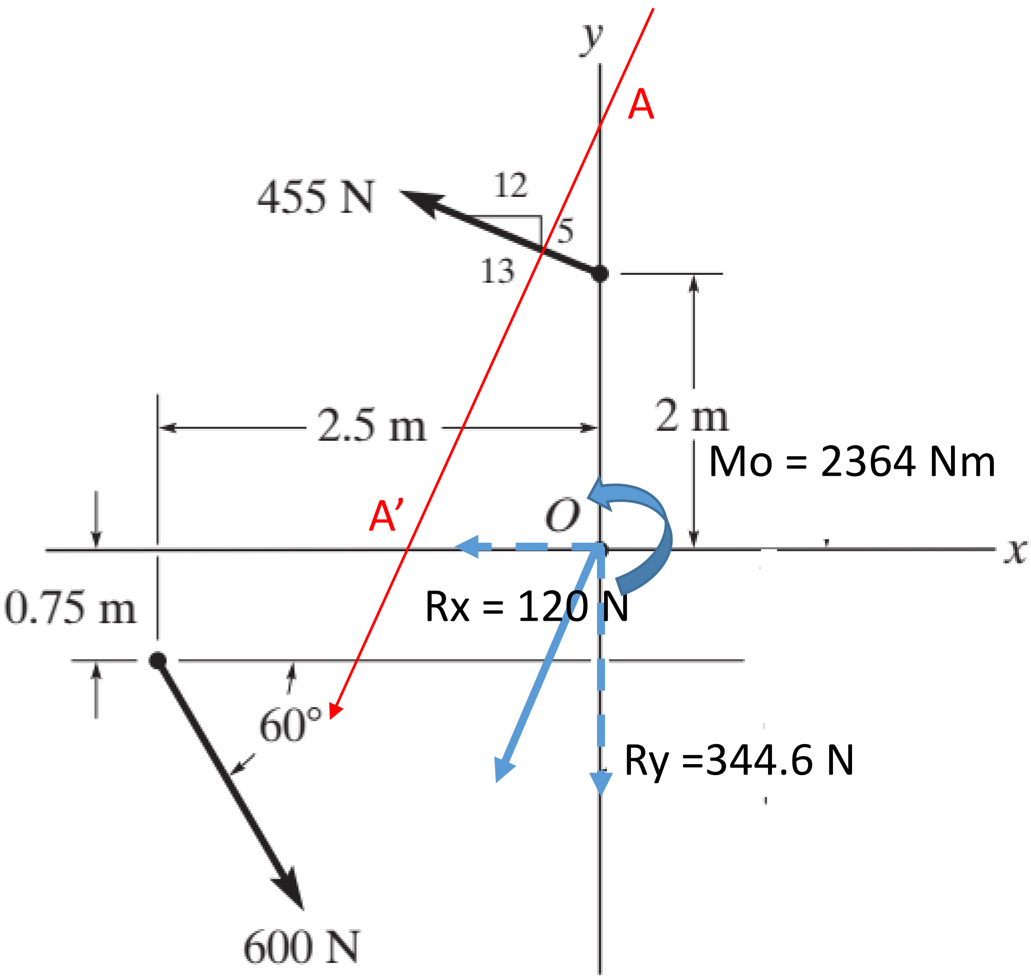
Sample Problem 3.7

1. Replace the force system by an equivalent resultant force and a couple: a) at point O 2
2. . Can the force system be reduced to a single force resultant? Where should it be applied?



At Point O



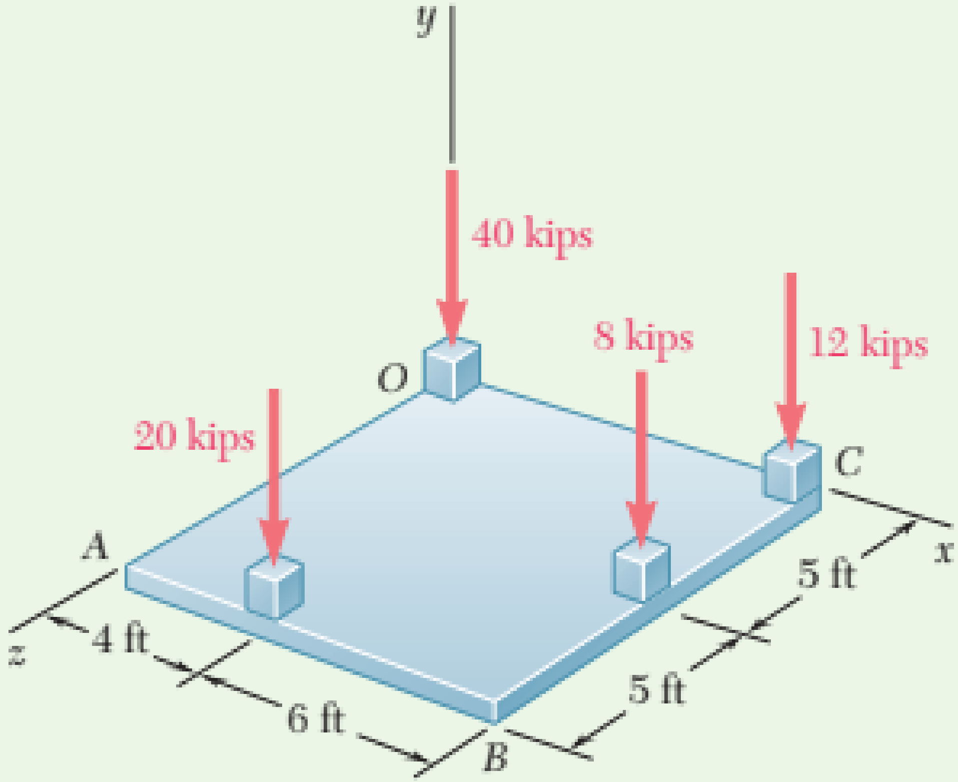


OA’= X = 2364/334.6 = 6.86 m (negative x-direction)

OA = Y = 2364/120 = 19.7 m (positive y-direction)

Sample Problem 3.11

A square foundation mat supports the four columns shown. Determine the magnitude and point of application of the resultant of the four loads.



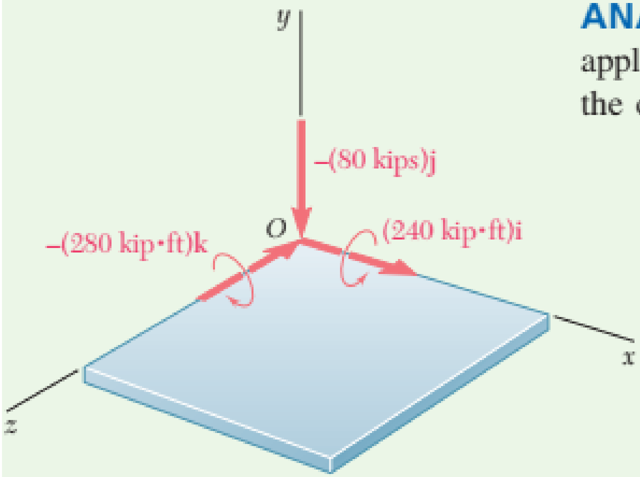


Fig. 1 Force-couple system at O that is equivalent to given force system.

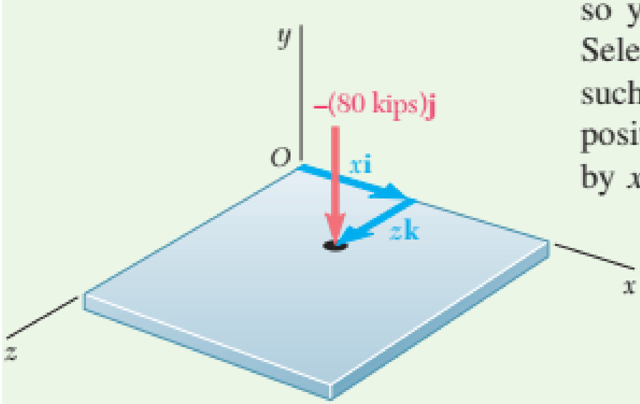


Fig. 2 Single force that is equivalent to given force system.

ANALYSIS: After determining the position vectors of the points of application of the various forces, you may find it convenient to arrange the computations in tabular form. The results are shown in Fig. 1.

r, ft	**F, kips**	**r × F, kip·ft**
0	$-40\mathbf{j}$	0
$10\mathbf{i}$	$-12\mathbf{j}$	$-120\mathbf{k}$
$10\mathbf{i} + 5\mathbf{k}$	$-8\mathbf{j}$	$40\mathbf{i} - 80\mathbf{k}$
$4\mathbf{i} + 10\mathbf{k}$	$-20\mathbf{j}$	$200\mathbf{i} - 80\mathbf{k}$
	$\mathbf{R} = -80\mathbf{j}$	$\mathbf{M}_O^R = 240\mathbf{i} - 280\mathbf{k}$

The force $\mathbf{R}$ and the couple vector $\mathbf{M}_O^R$ are mutually perpendicular, so you can reduce the force-couple system further to a single force $\mathbf{R}$. Select the new point of application of $\mathbf{R}$ in the plane of the mat and in such a way that the moment of $\mathbf{R}$ about O is equal to $\mathbf{M}_O^R$. Denote the position vector of the desired point of application by $\mathbf{r}$ and its coordinates by x and z (Fig. 2). Then

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O^R$$
$$(x\mathbf{i} + z\mathbf{k}) \times (-80\mathbf{j}) = 240\mathbf{i} - 280\mathbf{k}$$
$$-80x\mathbf{k} + 80z\mathbf{i} = 240\mathbf{i} - 280\mathbf{k}$$

It follows that

$$-80x = -280 \qquad 80z = 240$$
$$x = 3.50 \text{ ft} \qquad z = 3.00 \text{ ft}$$

The resultant of the given system of forces is

$$\mathbf{R} = 80 \text{ kips}\downarrow \qquad \text{at } x = 3.50 \text{ ft},\ z = 3.00 \text{ ft}$$