CLUB

BY:HANAN ALAWAWDA

CALCULUS (2) CHAPTER (8.4&8.7)

8.4 Integration of Rational Functions by Partial Fractions

- **Partial fraction method: Is a method for writing $\frac{f(x)}{g(x)}$ "rational functions"**
- **as a sum of simpler fraction.**
- **"cover up method " can be used when g(x) can be written as a product of distinct linear factors**
- **(the degree of f must be less than the degree of g , if not⇒use long division)**

التكامل باستخدام الكسور الجزئية

- بيكون عندي اقتران نسبي لو حاولت اكامله بالطريقة العادية ما بيزبط و لو عن طريق التعويض برضو بيزبطش لانه مش. موجود عندي اقتران و مشتقته
- انظر الى المقام هل يمكن تحليله الى عوامل خطية (يعني ما يكون في عندي x^2)؟ اذا نعم بنكامل باستخدام الكسور الجزئية بشرط ان تكون اكبر قوة في المقام اكبر من اكبر قوة البسط و اذا لما يتحقق هذا الشرط نقوم بعمل القيمة الطويلة

Example:

- $\int \frac{x^2+4x+1}{(x^2-1)(x+3)} = \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}$ حللت المقام
- $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}$ العوامل يلي طلعت عندي من التحليل وزعتها ع عدة مقامات و بينهم اشارة جمع مع فرض مجاهيل في البسوط
- $\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A(x+1)(x+3)}{(x-1)(x+1)(x+3)} + \frac{B(x-1)(x+3)}{(x-1)(x+1)(x+3)} + \frac{C(x-1)(x+1)}{(x-1)(x+1)(x+3)}$ وحدت المقامات
- $x^2 + 4x + 1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$ ساويت البسط الموجود عندي في البسوط يلي فرضتها
- When x=-1 ⇒-2=-4B ⇒B=$\frac{1}{2}$
- When x=1 ⇒6=8A ⇒A=$\frac{3}{4}$
- When x=-3 ⇒-2=8C ⇒C=$\frac{-1}{4}$

بنحل المعادلة باي طريقة بنعرفها
الطريقة يلي بفضلها هي التعويض مكان x و بحل المعادلات يلي بتطلع معي
عادة اختر رقم لما تعوضه يحذف مجهول او مجهولين

- $\int \frac{x^2+4x+1}{(x^2-1)(x+3)} = \int \frac{3}{4(x-1)} + \int \frac{1}{2(x+1)} + \int \frac{-1}{4(x+3)} = \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3|$ بعوض قيم المجاهيل يلي طلعت معي و بكامل

- OUTLINE SOLUTION

express the integrand as a sum of partial fractions and evaluate the integrals.

12. $\int \frac{2x+1}{x^2-7x+12}\,dx$

$\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3) + B(x-4);\ x = 3 \Rightarrow B = \frac{7}{-1} = -7;\ x = 4 \Rightarrow A = \frac{9}{1} = 9;$

$\int \frac{2x+1}{x^2-7x+12}\,dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3} = 9\ln|x-4| - 7\ln|x-3| + C = \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$

14. $\int_{1/2}^{1} \frac{y+4}{y^2+y}\,dy$

$\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By;\ y = 0 \Rightarrow A = 4;\ y = -1 \Rightarrow B = \frac{3}{-1} = -3;$

$\int_{1/2}^{1} \frac{y+4}{y^2+y}\,dy = 4\int_{1/2}^{1} \frac{dy}{y} - 3\int_{1/2}^{1} \frac{dy}{y+1} = [4\ln|y| - 3\ln|y+1|]_{1/2}^{1} = (4\ln 1 - 3\ln 2) - \left(4\ln\frac{1}{2} - 3\ln\frac{3}{2}\right)$

$= \ln\frac{1}{8} - \ln\frac{1}{16} + \ln\frac{27}{8} = \ln\left(\frac{27}{8}\cdot\frac{1}{8}\cdot 16\right) = \ln\frac{27}{4}$

18. $\int_{-1}^{0} \frac{x^3\,dx}{x^2 - 2x + 1}$

درجة البسط اعلى من درجة المقام
الحل (قسمة مطولة)

$$\frac{x^3}{x^2 - 2x + 1} = (x + 2) + \frac{3x - 2}{(x - 1)^2} \text{ (after long division); } \frac{3x - 2}{(x - 1)^2} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} \Rightarrow 3x - 2 = A(x - 1) + B$$

$$= Ax + (-A + B) \Rightarrow A = 3, -A + B = -2 \Rightarrow A = 3, B = 1; \int_{-1}^{0} \frac{x^3\,dx}{x^2 - 2x + 1}$$

$$= \int_{-1}^{0} (x + 2)\,dx + 3\int_{-1}^{0} \frac{dx}{x - 1} + \int_{-1}^{0} \frac{dx}{(x - 1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x - 1| - \frac{1}{x - 1}\right]_{-1}^{0}$$

$$= \left(0 + 0 + 3\ln 1 - \frac{1}{(-1)}\right) - \left(\frac{1}{2} - 2 + 3\ln 2 - \frac{1}{(-2)}\right) = 2 - 3\ln 2$$

$$\begin{array}{r|l} & X + 2 \\ \hline x^2 - 2x + 1 & x^3 \\ & -x^3 + 2x^2 - x \\ \hline & 2x^2 - x \\ & -2x^2 + 4x - 2 \\ \hline & 3x-2 \end{array}$$

20. $\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

Repeated

$$\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1);\ x=-1$$

$$\Rightarrow C = -\frac{1}{2};\ x = 1 \Rightarrow A = \frac{1}{4};\ \text{coefficient of } x^2 = A + B \Rightarrow A + B = 1 \Rightarrow B = \frac{3}{4};\ \int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$$

$$= \frac{1}{4}\int \frac{dx}{x-1} + \frac{3}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{(x+1)^2} = \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C = \frac{\ln|(x-1)(x+1)^3|}{4} + \frac{1}{2(x+1)} + C$$

23. $\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy$

بنحط في البسط المتغير أقل بدرجة واحدة

Quadrant. And repeated من المتغير يلي في المقام

$$\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2} \Rightarrow y^2+2y+1 = (Ay+B)(y^2+1) + Cy + D$$

$$= Ay^3 + By^2 + (A+C)y + (B+D) \Rightarrow A = 0, B = 1;\ A + C = 2 \Rightarrow C = 2;\ B + D = 1 \Rightarrow D = 0;$$

$$\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy = \int \frac{1}{y^2+1}\,dy + 2\int \frac{y}{(y^2+1)^2}\,dy = \tan^{-1} y - \frac{1}{y^2+1} + C$$

29. $\int \frac{x^2}{x^4-1}\,dx$

$\frac{x^2}{x^4-1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \Rightarrow x^2 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx + D)(x-1)(x+1)$
$= (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x - A + B - D \Rightarrow A+B+C = 0,\ -A+B+D = 1,$
$A+B-C = 0,\ -A+B-D = 0 \Rightarrow$ adding eq(1) to eq (3) gives $2A+2B = 0$, adding eq(2) to eq(4) gives $-2A+2B = 1$, adding these two equations gives $4B = 1 \Rightarrow B = \frac{1}{4}$, using $2A+2B = 0 \Rightarrow A = -\frac{1}{4}$, using $-A+B-D = 0 \Rightarrow D = \frac{1}{2}$, and using $A+B-C = 0 \Rightarrow C = 0$; $\int \frac{x^2}{x^4-1}dx = \int \left(\frac{-1/4}{x+1} + \frac{1/4}{x-1} + \frac{1/2}{x^2+1}\right)dx$
$= -\frac{1}{4}\int \frac{1}{x+1}dx + \frac{1}{4}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx = -\frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + \frac{1}{2}\tan^{-1}x + C = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\tan^{-1}x + C$

30. $\int \frac{x^2+x}{x^4-3x^2-4}\,dx$

$\frac{x^2+x}{x^4-3x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1} \Rightarrow x^2+x = A(x+2)(x^2+1) + B(x-2)(x^2+1) + (Cx + D)(x-2)(x+2)$
$= (A+B+C)x^3 + (2A-2B+D)x^2 + (A+B-4C)x + 2A - 2B - 4D \Rightarrow A+B+C = 0,\ 2A-2B+D = 1,$
$A+B-4C = 1,\ 2A-2B-4D = 0 \Rightarrow$ subtractin eq(1) from eq (3) gives $-5C = 1 \Rightarrow C = -\frac{1}{5}$, subtacting eq(2) from eq(4) gives $-5D = -1 \Rightarrow D = \frac{1}{5}$, substituting for C in eq(1) gives $A+B = \frac{1}{5}$, and substituting for D in eq(4) gives $2A-2B = \frac{4}{5} \Rightarrow A-B = \frac{2}{5}$, adding this equation to the previous equatin gives $2A = \frac{3}{5} \Rightarrow A = \frac{3}{10} \Rightarrow B = -\frac{1}{10}$;
$\int \frac{x^2+x}{x^4-3x^2-4}dx = \int \left(\frac{3/10}{x-2} - \frac{1/10}{x+2} + \frac{(-1/5)x+1/5}{x^2+1}\right)dx = \frac{3}{10}\int \frac{1}{x-2}dx - \frac{1}{10}\int \frac{1}{x+2}dx - \frac{1}{5}\int \frac{x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx$
$\frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln|x^2+1| + \frac{1}{5}\tan^{-1}x + C$

PERFORM LONG DIVISION ON THE INTEGRAND, WRITE THE PROPER FRACTION AS A SUM. OF PARTIAL FRACTIONS, AND THEN EVALUATE THE INTEGRAL.

34. $\int \frac{x^4}{x^2 - 1}\, dx$

$$\frac{x^4}{x^2-1} = (x^2+1) + \frac{1}{x^2-1} = (x^2+1) + \frac{1}{(x+1)(x-1)}\,;\ \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x+1);$$

$$x = -1 \Rightarrow A = -\frac{1}{2}\,;\ x = 1 \Rightarrow B = \frac{1}{2}\,;\ \int \frac{x^4}{x^2-1}\, dx = \int (x^2+1)\, dx - \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$$

$$= \frac{1}{3}x^3 + x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C = \frac{x^3}{3}$$ 37. $\int \frac{y^4+y^2-1}{y^3+y}\, dy$ $\ln\left|\frac{x-1}{x+1}\right| + C$

37. $\int \frac{y^4 + y^2 - 1}{y^3 + y}\, dy$

$$\frac{y^4+y^2-1}{y^3+y} = y - \frac{1}{y(y^2+1)}\,;\ \frac{1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1} \Rightarrow 1 = A(y^2+1) + (By+C)y = (A+B)y^2 + Cy + A$$

$$7 \Rightarrow A = 1;\ A + B = 0 \Rightarrow B = -1;\ C = 0;\ \int \frac{y^4+y^2-1}{y^3+y}\, dy = \int y\, dy - \int \frac{dy}{y} + \int \frac{y\, dy}{y^2+1}$$

$$= \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln(1+y^2) + C$$

EVALUATE THE INTEGRALS

42. $\int \frac{\sin\theta\, d\theta}{\cos^2\theta + \cos\theta - 2}$

$$\int \frac{\sin\theta\, d\theta}{\cos^2\theta + \cos\theta - 2}\ ;\ [\cos\theta = y]\ \rightarrow\ -\int \frac{dy}{y^2 + y - 2} = \frac{1}{3}\int \frac{dy}{y+2} - \frac{1}{3}\int \frac{dy}{y-1} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C = \frac{1}{3}\ln\left|\frac{\cos\theta + 2}{\cos\theta - 1}\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{2+\cos\theta}{1-\cos\theta}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta - 1}{\cos\theta + 2}\right| + C$$

47. $\int \frac{\sqrt{x+1}}{x}\, dx$

(*Hint*: Let $x + 1 = u^2$.)

$$\int \frac{\sqrt{x+1}}{x}\, dx\ \left[\text{Let } x + 1 = u^2 \Rightarrow dx = 2u\, du\right] \rightarrow \int \frac{u}{u^2-1}\, 2u\, du = \int \frac{2u^2}{u^2-1}\, du = \int \left(2 + \frac{2}{u^2-1}\right) du$$

$$= 2\int du + \int \frac{2}{u^2-1}\, du;\ \frac{2}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A + B = 0,$$

$$-A + B = 2 \Rightarrow B = 1 \Rightarrow A = -1;\ 2\int du + \int \frac{2}{u^2-1}\, du = 2u + \int \left(\frac{-1}{u+1} + \frac{1}{u-1}\right) du = 2u - \int \frac{1}{u+1}\, du + \int \frac{1}{u-1}\, du$$

$$= 2u - \ln|u+1| + \ln|u-1| + C = 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$$

49. $\int \frac{1}{x(x^4+1)}\,dx$

$\left(\textit{Hint}: \text{Multiply by } \frac{x^3}{x^3}.\right)$

$$\int \frac{1}{x(x^4+1)}dx = \int \frac{x^3}{x^4(x^4+1)}dx \left[\text{Let } u = x^4 \Rightarrow du = 4x^3\,dx\right] \to \frac{1}{4}\int \frac{1}{u(u+1)}du;\ \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$\Rightarrow 1 = A(u+1) + Bu = (A+B)u + A \Rightarrow A = 1 \Rightarrow B = -1;\ \frac{1}{4}\int \frac{1}{u(u+1)}du = \frac{1}{4}\int \left(\frac{1}{u} - \frac{1}{u+1}\right)du$$

$$= \frac{1}{4}\int \frac{1}{u}du - \frac{1}{4}\int \frac{1}{u+1}du = \frac{1}{4}\ln|u| - \frac{1}{4}\ln|u+1| + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C$$

SOLVE THE INITIAL VALUE PROBLEM FOR X AS A FUNCTION OF T

54. $(t+1)\frac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = 0$

$$(t+1)\,\frac{dx}{dt} = x^2 + 1 \ \Rightarrow\ \int \frac{dx}{x^2+1} = \int \frac{dt}{t+1} \ \Rightarrow\ \tan^{-1} x = \ln|t+1| + C;\ t = 0 \text{ and } x = 0 \ \Rightarrow\ \tan^{-1} 0 = \ln|1| + C$$

$$\Rightarrow\ C = \tan^{-1} 0 = 0 \ \Rightarrow\ \tan^{-1} x = \ln|t+1| \Rightarrow\ x = \tan(\ln(t+1)),\ t > -1$$

8.7 Improper Integrals

Type 1

احدد حدوده او كلاهما تكون
$\infty, -\infty$

$$\int_{-\infty}^{\infty} \quad \int_{-\infty}^{a} \quad \int_{a}^{\infty}$$

Type 2

$$\int_{a}^{b} \quad dx$$

الاقتران مش متصل عند عدد محدود من النقاط (اصفار الاقتران ،
اعداد اذا عوضتها داخل الجذر بيصير عدد تخيلي)
بشرط تكون هاي النقاط داخل الفترة يلي بكامل عليها

$$\int_{2}^{\infty} \frac{dx}{x-1}$$

Type 1
∞موجود عندي احد حدود التكامل يساوي
صفر الاقتران هو ١ و مش داخل في الفترة يلي
بكامل عليها عشان هيك ما بعتبره
Type 2

$$\int_{0}^{2} \frac{dx}{x-1}$$

Type 2 لان صفر
الاقتران يساوي١ و
داخل في الفترة يلي
بكامل عليها

$$\int_{1}^{\infty} \frac{dx}{x-1}$$

Type 1 and 2
Type 1 لانه عندي احد حدود التكامل ∞
Type 2 لانه صفر الاقتران يساوي ١ و داخل في الفترة يلي
بنكامل عليها

How to find improper integrals

- Type 1
- F is continuous on [a,∞) ⇒ $\int_a^{\infty} f(x)\,\mathrm{d}x$
- = $\lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x$
- في حدود التكامل بشيل ∞ و بحط بدالهاb و بكامل طبيعي بعدها بوجد النهاية عندما b تقترب من ∞

F is continuous on (-∞, a] ⇒ $\int_{-\infty}^{a} f(x)\,\mathrm{d}x$ = $\lim_{b\to-\infty} \int_b^a f(x)\,\mathrm{d}x$

اي رقم في الكوكب

F is continuous on $(-\infty,\infty)$ ⇒ $\int_{-\infty}^{c} f(x)\,\mathrm{d}x$+ $\int_c^{\infty} f(x)\,\mathrm{d}x$

نفس الرقم

$\Rightarrow \lim_{b\to-\infty} \int_b^c f(x)dx + \lim_{b\to\infty} \int_c^b f(x)\,\mathrm{d}x$

ما عملت اشي جديد غير اني قسمت الفترة الى فترتين

Example 1:

- $\int_0^\infty \frac{dx}{x^2+1} \Rightarrow \int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \tan^{-1} x\big|_0^b = \lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 0$
- $\lim_{b\to\infty} \tan^{-1} b$, $\tan^{-1} \infty = \frac{\pi}{2}$ (convergs to $\frac{\pi}{2}$)
- $\int_{-\infty}^0 \frac{dx}{x^2+1} \Rightarrow \int_b^0 \frac{dx}{x^2+1} = \lim_{b\to-\infty} \int_b^0 \frac{dx}{x^2+1} = \lim_{b\to-\infty} \tan^{-1} x\big|_b^0 =$ $\lim_{b\to-\infty} \tan^{-1} 0 - \tan^{-1} b$
- $\lim_{b\to-\infty} - \tan^{-1} b = \frac{\pi}{2}$ (convergs to $\frac{\pi}{2}$)
- $\int_{-\infty}^\infty \frac{dx}{x^2+1} = \int_{-\infty}^0 \frac{dx}{x^2+1} + \int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ (convergs to π)

Example 2

- $\int_{-\infty}^{-2} \frac{2\ dx}{x^2-1} \Rightarrow \lim_{b\to-\infty} \int_{b}^{-2} \frac{2}{x^2-1}\,\mathrm{dx}$
- $\int_{b}^{-2} \frac{2}{x^2-1}\,\mathrm{dx} \Rightarrow \frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$(تكامل بالاجزاء)
- 2=(x+1)A +(x-1)B
- When x =-1 ⇒ $\mathrm{B} = -1$
- When x=1 ⇒ $\mathrm{A} = 1$
- $\int_{b}^{-2} \frac{2}{x^2-1}\,\mathrm{dx} = \int_{b}^{-2} \frac{1}{x-1} + \int_{b}^{-2} \frac{-1}{x+1}$ = ln|x-1| -ln|x+1|
- $\lim_{b\to-\infty}$[ln|x-1| -ln|x+1|]$_b^{-2}$ ⇒ $\lim_{b\to-\infty}$ ln$\left|\frac{x-1}{x+1}\right|]_b^{-2}$ ⇒ $\lim_{b\to-\infty}$ ln$\left|\frac{-3}{-1}\right|$- ln$\left|\frac{b-1}{b+1}\right|$
- = ln3- $\lim_{b\to-\infty}$ ln$\left|\frac{b-1}{b+1}\right|$=ln3 - ln $\lim_{b\to-\infty}$ $\left|\frac{b-1}{b+1}\right|$(استخدم لوبيتال)
- = ln3 – ln1 = ln3 (converges to ln3)

Remark

- If the limit is finite (exists) ⇒ converge to الجواب النهائي يلي طلع معي
- Converges.المثال الاول كانو كل النهايات موجودة يعني
- If the limit is infinite (does not exists) ⇒ diverge
- Example : $\int_{1}^{\infty} \frac{\mathrm{d}x}{x} \Rightarrow \lim_{b\to\infty} \int_{1}^{b} \frac{\mathrm{d}x}{x} \Rightarrow \lim_{b\to\infty} \ln|x|\Big]_{1}^{b}$ =ln∞ = ∞ ⇒diverge

- Exp * $\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & \text{if } p>1 \quad \text{Converge} \\ \infty, & \text{if } p\leq 1 \quad \text{Diverge} \end{cases}$

هي الاجوبة النهائية للنهاية $\frac{1}{p-1}/\infty$

Exp ** $\int_0^{1} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & \text{if } p<1 \quad \text{Converge} \\ \infty, & \text{if } p\geq 1 \quad \text{Diverge} \end{cases}$

بديش احكي. احفظوهم بس (احفظوهم)

Exp 3 $\int_1^{\infty} \frac{dx}{x^3} = \frac{1}{3-1} = \frac{1}{2}$ type 1 converge to $\frac{1}{2}$

$Exp\ 4 \int_1^{\infty} \frac{dx}{x^{\frac{2}{3}}} = \infty$ diverge

Exp 5 $\int_0^{1} \frac{dx}{\sqrt{x}} \Rightarrow \frac{1}{1-\frac{1}{2}} = 2$ converge. To. 2

How to find improper integrals

- Type 2
- If f is discontinuous at a , $\int_a^b f(x)\,dx \Rightarrow \lim_{c \to a^+} \int_c^b f(x)\,dx$
- بشيل a و بحط. c حيث ان c تقترب من. a من. اليمين
- If f is discontinuous at b , $\int_a^b f(x)\,dx \Rightarrow \lim_{c \to b^-} \int_a^c f(x)\,dx$
- بشيل. b و بحط. c حيث ان. c تقترب من b من. اليسار
- If f is discontinuous at c , a<c<b (نقطة داخلية)
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ → نقطة عدم الاتصال
- $\Rightarrow \lim_{d \to c^-} \int_a^d f(x)\,dx + \lim_{d \to c^+} \int_d^b f(x)\,dx$

Example 6:

- $\int_0^4 \frac{dx}{\sqrt{4-x}} \Rightarrow \lim_{C\to 4^-} \int_0^C \quad (4-x)^{\frac{1}{2}} \Rightarrow \lim_{C\to 4^-}$ -2$\sqrt{4-x}\Big|_0^c \Rightarrow \lim_{C\to 4^-}$ -2$\sqrt{4-c}$--4
- =-2$\sqrt{4-4} + 4$ = 4 (converges to 4)

- Example 7:

- $\int_0^1 \frac{dx}{\sqrt{x}} \Rightarrow \int_0^1 \frac{dx}{x^{1/2}} \Rightarrow \frac{1}{1-\frac{1}{2}}$=2

Example 8

- $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}} \mathrm{d}\theta$
- $\theta^2 + 2\theta > 0$ if $\theta = 0, -2 \Rightarrow$ اصفار اقتران

عند ال -٢ مش مشكلة لان مش داخل في الفترة يلي بكامل عليها

$\lim_{C\to 0^+} \int_C^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}} \mathrm{d}\theta$ let u = $\theta^2 + 2\theta \Rightarrow$du= 2θ +2=2(θ+1)

u(1)=3 , u(c)=$C^2 + 2C$

$\lim_{C\to 0} \frac{1}{2} \int_C^1 \frac{1}{(u)^{\frac{1}{2}}} \Rightarrow \lim_{C\to +0} \frac{1}{2} \int_C^1 u^{-\frac{1}{2}} \Rightarrow \lim_{C\to +0} \frac{1}{2} 2\sqrt{u}]_{c^2+c}^{3} \Rightarrow \lim_{C\to +0} \sqrt{u}]_{c^2+c}^{3}$

$\Rightarrow \lim_{C\to +0} \sqrt{3}$-$\sqrt{C^2 + 2C}$ = $\sqrt{3} - \sqrt{0-0} = \sqrt{3}$ converges to $\sqrt{3}$

Example 9

- $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$, type 1&2

اي رقم في الكوكب بس ما يكون صفر اقتران

- $\int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}}$ بقسم التكامل لنصين
- $\lim_{c\to1^+} \int_c^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{b\to\infty} \int_2^b \frac{dx}{x\sqrt{x^2-1}}$
- $\lim_{c\to1^+} \sec^{-1}|x|]_c^2 + \sec^{-1}|x|]_2^b$
- $\sec^{-1} 2 - \sec^{-1}$ c+ $\sec^{-1}$ b- $\sec^{-1}$ 2 = - $\sec^{-1}$ 1+ $\sec^{-1} \infty$
- = 0 +$\frac{\pi}{2} = \frac{\pi}{2}$ converges to $\frac{\pi}{2}$

Example 10

- $\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,\mathrm{d}x \Rightarrow \lim_{b\to\infty}\int_0^b \frac{16\tan^{-1}x}{1+x^2}$, let u = $\tan^{-1}x$ ⇒du=$\frac{\mathrm{d}x}{1+x^2}$
- $\lim_{b\to\infty}\int_0^b 16\,u\,du \;\Rightarrow\; \lim_{b\to\infty} 8u^2$ ⇒$8\left(\tan^{-1}x\right)^2\Big|_0^b$ =$8\left(\tan^{-1}b\right)^2 - 0$
- = $8\left(\tan^{-1}\infty\right)^2 = 8\,\frac{\pi^2}{4} = 2\pi^2$

WE HAVE TWO TESTS TO CHECK CONVERGENCE AND DIVERGENCE

Direct comparison Test (DCT)

F , g are continuous on [a,∞) such that $0 \le f(x) \le g(x)$, $x \in [a, \infty)$, than if

$\int_a^\infty g(x)\,dx$ converge $\Rightarrow \int_a^\infty f(x)\,dx$ is converge too

If $\int_a^\infty f(x)\,dx$ diverge than $\int_a^\infty g(x)\,dx$ is diverge too

Ex : $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

$$\frac{\sin^2 x}{x^2} < \frac{1}{x^2}$$

Sin اكبر قيمة لل

$$\int_1^\infty \frac{dx}{x^2} = \frac{1}{1} = 1 \Rightarrow converges$$

Than $\int_1^\infty \frac{\sin^2 x}{x^2}$ is converges too by DCT

Limit comparison Test (LCT)

F(x), g(x) are positive and continuous on [a,∞) and $\lim_{x\to\mp\infty} \frac{f}{g}$ or $\lim_{x\to number} \frac{f}{g}$=L where 0<L<∞ , then $\int_a^\infty f(x)\,dx$ *and* $\int_a^\infty g(x)\,dx$ both are diverging or both are converging

Ex: $\int_1^\infty \frac{dx}{1+x^2}$

بفرض. زي ما بدي بس يكون فرض ذكي بحيث اني اكون بعرف هل هو.

Diverge او Coverage

F = $\frac{dx}{1+x^2}$, g=$\frac{1}{x^2} \Rightarrow \int_1^\infty \frac{1}{x^2} = \frac{1}{2-1} = 1 \Rightarrow$ converge (by exp*)

$$\lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}} \Rightarrow \lim_{x\to\infty} \frac{x^2}{1+x^2} = 1 \Rightarrow L$$

Than f(x) is convergs too by LCT

- OUTLINE SOLUTION

EVALUATE THE INTEGRALS

1. $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1}$

$$\int_0^{\infty} \frac{dx}{x^2+1} = \lim_{b \to \infty} \int_0^{b} \frac{dx}{x^2+1} = \lim_{b \to \infty} \left[\tan^{-1} x\right]_0^b = \lim_{b \to \infty} \left(\tan^{-1} b - \tan^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

5. $\displaystyle\int_{-1}^{1} \frac{dx}{x^{2/3}}$

$$\int_{-1}^{1} \frac{dx}{x^{2/3}} = \int_{-1}^{0} \frac{dx}{x^{2/3}} + \int_{0}^{1} \frac{dx}{x^{2/3}} = \lim_{b \to 0^-} \left[3x^{1/3}\right]_{-1}^{b} + \lim_{c \to 0^+} \left[3x^{1/3}\right]_{c}^{1}$$

$$= \lim_{b \to 0^-} \left[3b^{1/3} - 3(-1)^{1/3}\right] + \lim_{c \to 0^+} \left[3(1)^{1/3} - 3c^{1/3}\right] = (0+3) + (3-0) = 6$$

11. $\displaystyle\int_2^{\infty} \frac{2}{v^2 - v}\, dv$

$$= \lim_{b \to \infty} \left[2 \ln \left|\frac{v-1}{v}\right|\right]_2^b = \lim_{b \to \infty} \left(2 \ln \left|\frac{b-1}{b}\right| - 2 \ln \left|\frac{2-1}{2}\right|\right) = 2 \ln(1) - 2 \ln\left(\frac{1}{2}\right) = 0 + 2 \ln 2 = \ln 4$$

14. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$

$$\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}} = \int_{-\infty}^{0} \frac{x\,dx}{(x^2+4)^{3/2}} + \int_{0}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}\,;\ \begin{bmatrix} u = x^2+4 \\ du = 2x\,dx \end{bmatrix} \rightarrow \int_{\infty}^{4} \frac{du}{2u^{3/2}} + \int_{4}^{\infty} \frac{du}{2u^{3/2}}$$

$$= \lim_{b\to\infty} \left[-\frac{1}{\sqrt{u}}\right]_b^4 + \lim_{c\to\infty} \left[-\frac{1}{\sqrt{u}}\right]_4^c = \lim_{b\to\infty} \left(-\frac{1}{2} + \frac{1}{\sqrt{b}}\right) + \lim_{c\to\infty} \left(-\frac{1}{\sqrt{c}} + \frac{1}{2}\right) = \left(-\frac{1}{2} + 0\right) + \left(0 + \frac{1}{2}\right) = 0$$

17. $\displaystyle\int_{0}^{\infty} \frac{dx}{(1+x)\sqrt{x}}$

$$\int_{0}^{\infty} \frac{dx}{(1+x)\sqrt{x}}\,;\ \begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \rightarrow \int_{0}^{\infty} \frac{2\,du}{u^2+1} = \lim_{b\to\infty} \int_{0}^{b} \frac{2\,du}{u^2+1} = \lim_{b\to\infty} \left[2\tan^{-1} u\right]_0^b$$

$$= \lim_{b\to\infty} \left(2\tan^{-1} b - 2\tan^{-1} 0\right) = 2\left(\frac{\pi}{2}\right) - 2(0) = \pi$$

21. $\displaystyle\int_{-\infty}^{0} \theta e^{\theta}\,d\theta$

$$\int_{-\infty}^{0} \theta e^{\theta}\,d\theta = \lim_{b\to-\infty} \left[\theta e^{\theta} - e^{\theta}\right]_b^0 = \left(0 \cdot e^0 - e^0\right) - \lim_{b\to-\infty} \left[be^b - e^b\right] = -1 - \lim_{b\to-\infty} \left(\frac{b-1}{e^{-b}}\right)$$

$$= -1 - \lim_{b\to-\infty} \left(\frac{1}{-e^{-b}}\right) \quad \text{(l'Hôpital's rule for } \tfrac{\infty}{\infty} \text{ form)}$$

$$= -1 - 0 = -1$$

24. $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

$$\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \int_{-\infty}^{0} 2xe^{-x^2}\,dx + \int_{0}^{\infty} 2xe^{-x^2}\,dx = \lim_{b \to -\infty} \left[-e^{-x^2}\right]_b^0 + \lim_{c \to \infty} \left[-e^{-x^2}\right]_0^c$$

$$= \lim_{b \to -\infty} \left[-1 - \left(-e^{-b^2}\right)\right] + \lim_{c \to \infty} \left[-e^{-c^2} - (-1)\right] = (-1 - 0) + (0 + 1) = 0$$

26. $\int_{0}^{1} (-\ln x)\,dx$

$$\int_0^1 (-\ln x)\,dx = \lim_{b \to 0^+} \left[x - x\ln x\right]_b^1 = [1 - 1\ln 1] - \lim_{b \to 0^+} [b - b\ln b] = 1 - 0 + \lim_{b \to 0^+} \frac{\ln b}{\left(\frac{1}{b}\right)} = 1 + \lim_{b \to 0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$$

$$= 1 - \lim_{b \to 0^+} b = 1 - 0 = 1$$

32. $\int_{0}^{2} \frac{dx}{\sqrt{|x-1|}}$

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{b \to 1^-} \left[-2\sqrt{1-x}\right]_0^b + \lim_{c \to 1^+} \left[2\sqrt{x-1}\right]_c^2$$

$$= \lim_{b \to 1^-} \left(-2\sqrt{1-b}\right) - \left(-2\sqrt{1-0}\right) + 2\sqrt{2-1} - \lim_{c \to 1^+} \left(2\sqrt{c-1}\right) = 0 + 2 + 2 - 0 = 4$$

34. $\int_0^{\infty} \frac{dx}{(x+1)(x^2+1)}$

$$\int_0^{\infty} \frac{dx}{(x+1)(x^2+1)} = \lim_{b \to \infty} \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x\right]_0^b = \lim_{b \to \infty} \left[\frac{1}{2}\ln\left(\frac{x+1}{\sqrt{x^2+1}}\right) + \frac{1}{2}\tan^{-1}x\right]_0^b$$

$$= \lim_{b \to \infty} \left[\frac{1}{2}\ln\left(\frac{b+1}{\sqrt{b^2+1}}\right) + \frac{1}{2}\tan^{-1}b\right] - \left[\frac{1}{2}\ln\frac{1}{\sqrt{1}} + \frac{1}{2}\tan^{-1}0\right] = \frac{1}{2}\ln 1 + \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\ln 1 - \frac{1}{2}\cdot 0 = \frac{\pi}{4}$$

USE INTEGRATION, THE DIRECT COMPARISON TEST, OR THE LIMIT COMPARISON TEST TO TEST THE INTEGRALS FOR CONVERGENCE. IF MORE THAN ONE METHOD APPLIES, USE WHATEVER METHOD YOU PREFER

37. $\int_0^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{\pi-\theta}}$

$\int_0^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{\pi-\theta}}$; $[\pi-\theta = x] \rightarrow -\int_{\pi}^{0} \frac{\sin x\, dx}{\sqrt{x}} = \int_0^{\pi} \frac{\sin x\, dx}{\sqrt{x}}$. Since $0 \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ for all $0 \le x \le \pi$ and $\int_0^{\pi} \frac{dx}{\sqrt{x}}$ converges, then $\int_0^{\pi} \frac{\sin x}{\sqrt{x}}$ dx converges by the Direct Comparison Test.

39. $\int_0^{\ln 2} x^{-2}e^{-1/x}\, dx$

$$\int_0^{\ln 2} x^{-2}e^{-1/x}\, dx; \left[\frac{1}{x} = y\right] \rightarrow \int_{\infty}^{1/\ln 2} \frac{y^2 e^{-y}\, dy}{-y^2} = \int_{1/\ln 2}^{\infty} e^{-y}\, dy = \lim_{b \to \infty} \left[-e^{-y}\right]_{1/\ln 2}^{b} = \lim_{b \to \infty} \left[-e^{-b}\right] - \left[-e^{-1/\ln 2}\right]$$

$= 0 + e^{-1/\ln 2} = e^{-1/\ln 2}$, so the integral converges.

42. $\int_0^1 \frac{dt}{t - \sin t}$ (*Hint:* $t \geq \sin t$ for $t \geq 0$)

$\int_0^1 \frac{dt}{t-\sin t}$; let $f(t) = \frac{1}{t-\sin t}$ and $g(t) = \frac{1}{t^3}$, then $\lim_{t \to 0} \frac{f(t)}{g(t)} = \lim_{t \to 0} \frac{t^3}{t-\sin t} = \lim_{t \to 0} \frac{3t^2}{1-\cos t} = \lim_{t \to 0} \frac{6t}{\sin t}$ $= \lim_{t \to 0} \frac{6}{\cos t} = 6$. Now, $\int_0^1 \frac{dt}{t^3} = \lim_{b \to 0^+} \left[-\frac{1}{2t^2}\right]_b^1 = -\frac{1}{2} - \lim_{b \to 0^+} \left[-\frac{1}{2b^2}\right] = +\infty$, which diverges $\Rightarrow \int_0^1 \frac{dt}{t-\sin t}$ diverges by the Limit Comparison Test.

46. $\int_{-1}^{1} -x \ln |x| \, dx$

$\int_{-1}^{1} (-x \ln |x|)\, dx = \int_{-1}^{0} [-x \ln(-x)]\, dx + \int_0^1 (-x \ln x)\, dx = \lim_{b \to 0^+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_b^1 - \lim_{c \to 0^+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_c^1$ $= \left[\frac{1}{2} \ln 1 - \frac{1}{4}\right] - \lim_{b \to 0^+} \left[\frac{b^2}{2} \ln b - \frac{b^2}{4}\right] - \left[\frac{1}{2} \ln 1 - \frac{1}{4}\text{-}\right] + \lim_{c \to 0^+} \left[\frac{c^2}{2} \ln c - \frac{c^2}{4}\right] = -\frac{1}{4} - 0 + \frac{1}{4} + 0 = 0 \Rightarrow$ the integral converges (see Exercise 25 for the limit calculations).

48. $\int_4^{\infty} \frac{dx}{\sqrt{x} - 1}$

$\int_4^{\infty} \frac{dx}{\sqrt{x}-1}$; $\lim_{x \to \infty} \frac{\left(\frac{1}{\sqrt{x}-1}\right)}{\left(\frac{1}{\sqrt{x}}\right)} = \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x}-1} = \lim_{x \to \infty} \frac{1}{1-\frac{1}{\sqrt{x}}} = \frac{1}{1-0} = 1$ and $\int_4^{\infty} \frac{dx}{\sqrt{x}} = \lim_{b \to \infty} \left[2\sqrt{x}\right]_4^b = \infty$, which diverges $\Rightarrow \int_4^{\infty} \frac{dx}{\sqrt{x}-1}$ diverges by the Limit Comparison Test.

50. $\int_0^\infty \frac{d\theta}{1 + e^\theta}$

$\int_0^\infty \frac{d\theta}{1+e^\theta}$; $0 \le \frac{1}{1+e^\theta} \le \frac{1}{e^\theta}$ for $0 \le \theta < \infty$ and $\int_0^\infty \frac{d\theta}{e^\theta} = \lim_{b \to \infty} \left[-e^{-\theta}\right]_0^b = \lim_{b \to \infty} \left(-e^{-b} + 1\right) = 1 \Rightarrow \int_0^\infty \frac{d\theta}{e^\theta}$ converges $\Rightarrow \int_0^\infty \frac{d\theta}{1+e^\theta}$ converges by the Direct Comparison Test.

54. $\int_2^\infty \frac{x\,dx}{\sqrt{x^4 - 1}}$

$\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$; $\lim_{x \to \infty} \frac{\left(\frac{x}{\sqrt{x^4-1}}\right)}{\left(\frac{x}{\sqrt{x^4}}\right)} = \lim_{x \to \infty} \frac{\sqrt{x^4}}{\sqrt{x^4-1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1-\frac{1}{x^4}}} = 1$; $\int_2^\infty \frac{x\,dx}{\sqrt{x^4}} = \int_2^\infty \frac{dx}{x} = \lim_{b \to \infty} \left[\ln x\right]_2^b = \infty$, which diverges $\Rightarrow \int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$ diverges by the Limit Comparison Test.

56. $\int_\pi^\infty \frac{1 + \sin x}{x^2}\,dx$

$\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$; $0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$ for $x \ge \pi$ and $\int_\pi^\infty \frac{2}{x^2}\,dx = \lim_{b \to \infty} \left[-\frac{2}{x}\right]_\pi^b = \lim_{b \to \infty} \left(-\frac{2}{b} + \frac{2}{\pi}\right) = \frac{2}{\pi}$ $\Rightarrow \int_\pi^\infty \frac{2\,dx}{x^2}$ converges $\Rightarrow \int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$ converges by the Direct Comparison Test.

58. $\int_2^\infty \frac{1}{\ln x}\,dx$

$\int_2^\infty \frac{dx}{\ln x}$; $0 < \frac{1}{x} < \frac{1}{\ln x}$ for $x > 2$ and $\int_2^\infty \frac{dx}{x}$ diverges $\Rightarrow \int_2^\infty \frac{dx}{\ln x}$ diverges by the Direct Comparison Test.

62. $\int_1^\infty \frac{1}{e^x - 2^x}\,dx$

$\int_1^\infty \frac{dx}{e^x - 2^x}$; $\lim_{x \to \infty} \frac{\left(\frac{1}{e^x - 2^x}\right)}{\left(\frac{1}{e^x}\right)} = \lim_{x \to \infty} \frac{e^x}{e^x - 2^x} = \lim_{x \to \infty} \frac{1}{1 - \left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = 1$ and $\int_1^\infty \frac{dx}{e^x} = \lim_{b \to \infty} \left[-e^{-x}\right]_1^b$

$= \lim_{b \to \infty} \left(-e^{-b} + e^{-1}\right) = \frac{1}{e} \Rightarrow \int_1^\infty \frac{dx}{e^x}$ converges $\Rightarrow \int_1^\infty \frac{dx}{e^x - 2^x}$ converges by the Limit Comparison Test

64. $\int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}}$

$\int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}} = 2\int_0^\infty \frac{dx}{e^x + e^{-x}}$; $0 < \frac{1}{e^x + e^{-x}} < \frac{1}{e^x}$ for $x > 0$; $\int_0^\infty \frac{dx}{e^x}$ converges $\Rightarrow 2\int_0^\infty \frac{dx}{e^x + e^{-x}}$ converges by the Direct Comparison Test.

66. $\int_{-\infty}^{\infty} f(x)\,dx$ **may not equal** $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$ Show that

$$\int_{0}^{\infty} \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty} \int_{-b}^{b} \frac{2x\,dx}{x^2+1} = 0.$$

$\int_0^{\infty} \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \left[\ln(x^2+1)\right]_0^b = \lim_{b\to\infty} \left[\ln(b^2+1)\right] - 0 = \lim_{b\to\infty} \ln(b^2+1) = \infty \Rightarrow$ the integral $\int_{-\infty}^{\infty} \frac{2x}{x^2+1}\,dx$ diverges. But $\lim_{b\to\infty} \int_{-\infty}^{b} \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \left[\ln(x^2+1)\right]_{-b}^{b} = \lim_{b\to\infty} \left[\ln(b^2+1) - \ln(b^2+1)\right] = \lim_{b\to\infty} \ln\left(\frac{b^2+1}{b^2+1}\right)$
$= \lim_{b\to\infty} (\ln 1) = 0$

- "اللهم وكلت أمري إليك، وألجأت ظهري إليك، وفوّضت أمري إليك، لا رادّ لفضلك، ولا مانع لعطائك، لا إله إلا أنت سبحانك إني كنت من الظالمين"