1.9: Some special Expectations:

Def: let X be a Random with p.d.f. $f(x)$:

1. The mean or the mean value μ is defined as $\mu = E(X)$.

2. The variance of X is defined as: $\sigma^2 = var(X) = E\left[(X-\mu)^2\right]$.

3. The standard deviation of X is defined as: $\sigma = \sqrt{\sigma^2}$.

Proposition: X Random variable with Mean μ and variance σ^2 then

$$\sigma^2 = E(X^2) - \mu^2.$$

proof: $\sigma^2 = E\left[(X-\mu)^2\right] = E\left[X^2 - 2X\mu + \mu^2\right]$

$$= E(X^2) - 2\mu E(X) + \mu^2 \qquad \text{But } E(X) = \mu$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$

Note: $var(X) = E(X^2) - (E(X))^2$.

Note: Ingeneral $(E(X))^2 \neq E(X^2)$.

Note: $E(X)$ = First moment.

$E(X^2)$ = second Moment.

$E(X^k)$ = k^{th} Moment.

Def: let X be a Random variable with p.d.f. $f(x)$, suppose that there is a positive number h such that for $-h<t<h$, the mathematical expectation

$$E(e^{tX}) = \begin{cases} \sum_x e^{tx} f(x) & , x \text{ discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & , x \text{ continuous.} \end{cases}$$

exists then this expectation is called the moment generating function (m.g.f). of X and is denoted $M(t)$.

Remark: $M(t) = E(e^{tX})$, $-h<t<h$.

Remarks: 1. $M(0) = 1$

2. if X discrete then $M(t) = E(e^{tX}) = \sum_x e^{tx} f(x)$.

$$\rightarrow \bar{M}(t) = \frac{d\,M(t)}{dt} = \frac{d}{dt} \sum_{x\in\mathbb{R}} e^{tx} f(x)$$

$$= \sum_{x\in\mathbb{R}} \frac{d}{dt} e^{tx} f(x)$$

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$$\boxed{\bar{M}(x) = \sum_{x\in\mathbb{R}} x e^{tx} f(x)}$$

$$\text{So } \bar{M}(0) = \frac{d\,M(t)}{dt}\bigg|_{t=0} = \sum_{x\in\mathbb{R}} x f(x) = E(x)$$

$$\Rightarrow \bar{M}(0) = E(x)$$

③ if X discrete then $\bar{\bar{M}}(t) = \frac{d^2 M(t)}{dt^2} = \frac{d}{dt}\bar{M}(t) = \frac{d}{dt}\sum_{x\in R} x e^{tx} f(x)$

$\bar{M}(t) = \sum \frac{d}{dt} x e^{tx} f(x)$

$\Rightarrow \bar{\bar{M}}(t) = \sum_{x\in R} x^2 e^{tx} f(x)$.

$\Rightarrow \bar{\bar{M}}(0) = \frac{d M(t)}{dt^2}\Big|_{t=0} = \sum_{x\in R} x^2 f(x) = E(x^2)$.

$\Rightarrow \bar{\bar{M}}(0) = E(x^2)$

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④ $M^{K}(0) = \frac{d^K M(t)}{dt^K}\Big|_{t=0} = E(x^K)$ where $K = 1, 2, 3, \ldots$.

Question : let $M(t) = \frac{1}{1-t}$, $t < 1$

$M(t) = E(e^{tx})$ can we find $f(x)$?!

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example : let X be a Random variable with p.d.f , Find m.g.f :

X :	1	2	3	4	sum	discrete .
$f(x)$:	0.1 ($\frac{1}{10}$)	0.2	0.3	0.4	1	

$E(e^{tx}) = \sum_{x\in R} e^{tx} f(x) = \frac{e^{t}}{10} + \frac{e^{2t}}{10} + \frac{e^{3t}}{10} + \frac{4e^{4t}}{10}$, $t \in R$.

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example : $f(x) = \begin{cases} e^{-x} & , x>0 \\ 0 & , \text{elsewhere} \end{cases}$ is a p.d.f for a random variable x. Find m.g.f .

$$E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{0}^{\infty} e^{tx} e^{-x}\, dx = \int_{0}^{\infty} e^{tx-x}\, dx = \int_{0}^{\infty} e^{x(t-1)}\, dx$$

$$= \frac{e^{(t-1)x}}{(t-1)} \Big|_0^{\infty}$$

$\underline{\text{if}}$ $(t<1) \rightarrow$

$$= \lim_{a\to\infty} \frac{e^{(t-1)a}}{t-1} - \frac{e^{0}}{t-1}$$

$$= 0 - \frac{1}{t-1}$$

$$= \frac{1}{1-t}$$

$$\boxed{\text{so } M(t) = \frac{1}{1-t} \quad , t<1}$$

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example : $f(x) = \begin{cases} e^{-x} & , x>0 \\ 0 & , \text{elsewhere} \end{cases}$ Find μ, σ^2, σ .

$\rightarrow M(t) = (1-t)^{-1} \quad , t<1$

$\bar{M}(t) = +(1-t)^{-2} \quad , t<1$

$\bar{M}(0) = 1$

so $\mu = E(x) = 1$

$\rightarrow \bar{\bar{M}}(t) = 2(1-t)^{-3} \quad , t<1$

$\bar{\bar{M}}(0) = 2$

so $E(x^2) = 2$

$var(x) = \sigma^2 = E(x^2) - (E(x))^2 = 2-1 = 1$

$\therefore \mu = 1$

$\sigma^2 = 1$

$\sigma = 1$

example 1 : $f(x) = \begin{cases} \frac{1}{2}(x+1) & , -1<x<1 \\ 0 & , \text{elsewhere} \end{cases}$ Find μ, σ^2.

1. $E(x) = \mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-1}^{1} x\left(\frac{1}{2}(x+1)\right) dx = \frac{1}{2}\int_{-1}^{1} x^2 + x \; dx$

$$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\Big|_{-1}^{1}\right]$$

$$= \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}.$$

2. $E(x^2) = \frac{1}{2}\int_{-1}^{1} x^2(x+1)\, dx = \frac{1}{2}\int_{-1}^{1} x^3 + x^2 \; dx$

$$= \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\Big|_{-1}^{1}\right]$$

$\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3}$

$$= \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}.$$

But $\sigma^2 = E(x^2) - (E(x))^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}.$

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example 2 : $f(x) = \begin{cases} \frac{1}{x^2} & , x>1 \\ 0 & , \text{elsewhere} \end{cases}$ Find $E(x) = \mu$.

$E(x) = \int_{1}^{\infty} x\left(\frac{1}{x^2}\right) dx = \int_{1}^{\infty} \frac{1}{x}\, dx \rightarrow$ diverge.

check : $E(|x|) = \int_{1}^{\infty} |x| \frac{1}{x^2}\, dx = \int_{1}^{\infty} \frac{1}{x}\, dx \rightarrow$ diverge.

So $E(x)$ is undefined #

