## 1.7: properties of the distribution Function

Recall :

- Def: $X: \mathcal{C} \to \mathcal{D}$

- $P_X(A) = P(X \in A$

Def: $P(A) = \begin{cases} \sum_{x \in A} f(x) , & X \text{ discrete} \\ \int_A f(x)\,dx , & X \text{ continuous} . \end{cases}$

$\rightarrow F(x) = P_r(X \leq x)$ .

Def: $F(x) = P_r(X \leq x) = \begin{cases} \sum_{w \leq x} f(w) , & X \text{ discrete} . \\ \int_{-\infty}^{x} f(w)\,dw , & X \text{ continuous} . \end{cases}$

Proposition :

1. $0 \leq F(x) \leq 1$
2. $F$ is non-decreasing .
3. $F(\infty) = 1$ , $F(-\infty) = 0$ .
4. $F$ is Right continuous .

Proof :

1. $F(x) = Pr(X \leq x)$

$0 \leq Pr(X \leq x) \leq 1$

$\Rightarrow 0 \leq F(x) \leq 1$ for all $x$.

2. let $a<b$

$(-\infty, b] = (-\infty, a] \cup (a, b]$.

$Pr(X \in (-\infty, b]) = Pr(X \in (-\infty, a]) + Pr(X \in (a, b])$

$Pr(X \leq b) = Pr(X \leq a) + Pr(a < X \leq b)$.

$F(b) = F(a) + Pr(a < X \leq b)$

$F(b) - F(a) = Pr(a < X \leq b)$

But $Pr(a < X \leq b) \geq 0$

$\Rightarrow F(b) - F(a) \geq 0$

$\Rightarrow F(a) \leq F(b)$ ∎

Note 1 : $F$ nondecreasing : $a<b \Rightarrow F(a) \leq F(b)$.

Note 2 : $Pr(a<X \leq b) = F(b) - F(a)$ for all $a<b$.

3. $F(\infty) = \lim_{b \to \infty} F(b)$

$= \lim_{b \to \infty} Pr(X \leq b)$

$= \lim_{b \to \infty} Pr(X \in (-\infty, b])$

$= Pr(X \in \mathbb{R})$

$= 1$

$F(-\infty) = \lim_{a \to -\infty} F(a)$

$= \lim_{a \to -\infty} Pr(X \leq a)$

$= \lim_{a \to -\infty} Pr(X \in (-\infty, a])$

$= Pr(X \in \emptyset)$

$= 0$

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4. $F$ right continuous

$F(a^+) - F(a)$

$= \lim_{x \to a^+} F(x) - F(a)$

$= \lim_{h \to 0^+} F(a+h) - F(a)$

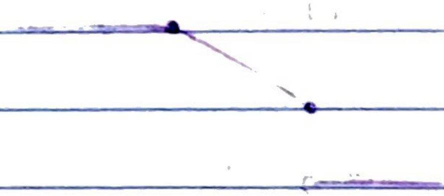
$= \lim_{h \to 0^+} \{ F(a+h) - F(a) \}$

$= \lim_{h \to 0^+} Pr(a < X \leq a+h)$

$= \lim_{h \to 0^+} Pr(X \in (a, a+h])$

$= Pr(X \in \emptyset)$

$= 0$



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Proposition :

$Pr(X=b) = F(b) - F(b^-) = F(b) - \lim_{x \to b^-} F(x)$

$= Pr(X \in \{b\})$

$= Pr(X = b)$

$= \begin{cases} f(b), & X \text{ discrete.} \\ 0, & X \text{ continuous.} \end{cases}$

**Proposition :**

$X$ continuous ,

$\bar{F}(x) = f(x)$ , $\forall x$ where $F$ is differantiable .

**example 1 :** $X$ random variable with C.D.F

$$F(x) = \begin{cases} 0 & , \; x < 0 \\ \frac{x+1}{2} & , \; 0 \le x < 1 \\ 1 & , \; 1 \le x \end{cases}$$

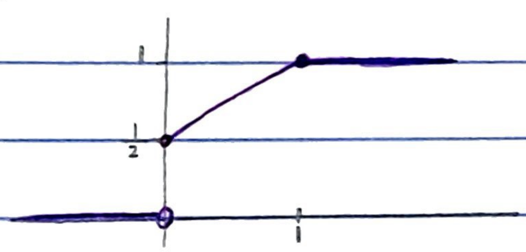
مش step-function لأ disc. ولا cont.

هو خليط من cont. و disc

بنحكي اسمه censored r.v .

لـ تنطبق عليه الخصائص .

1. Plot $F(x)$ :




2. What is the type of $X$ ?

is a mixtrue of discrete and cont. and is called censored Random variable .

3. Find $Pr(x=0)$ : $Pr(x=0) = F(0) - F(0^-)$

الاحتمال عند نقطة هيك قانونه

$= \frac{1}{2} - 0 \qquad = \frac{1}{2}$ .

4. Find $Pr(x=1) = F(1) - F(1^-) = 1 - 1 = 0$ .

5. Find $Pr(-3 < x \le \frac{1}{2}) = F(\frac{1}{2}) - F(-3) = \frac{3}{4} - 0 = \frac{3}{4}$ .

check ex 2

Example 2 :

$$F(x) = \begin{cases} 0 & , -\infty < x < 0 \\ 1 - \left(\frac{10}{10+x}\right)^3 & , 0 \leq x < \infty \end{cases}$$

Example 3 : let $f(x) = \begin{cases} \frac{1}{2} & , -1 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$ be the p.d.f of the Random variable $X$

→ continuous .

define the r.v $Y = X^2$ and Find the p.d.f of $Y$ :

| Random variable | p.d.f | c.d.f |
|---|---|---|
| $X$ ✓ | $f$ ✓ | $F$ |
| $Y$ ✓ | $g$ ? | $G$ |

$$\rightarrow G(y) = Pr(Y \leq y) = Pr(X^2 \leq y)$$

$$= Pr(-\sqrt{y} \leq X \leq \sqrt{y}) .$$

$x^2 \leq y \Leftrightarrow |x| \leq \sqrt{y}$

$\Leftrightarrow -\sqrt{y} \leq x \leq \sqrt{y}$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f(x) \, dx$$

→

cont.

$$G(y)=\begin{cases} 0 & , \quad y<0 \\ \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\,dx = \frac{2\sqrt{y}}{2} = \sqrt{y} & , \quad 0\le y<1 \\ \int_{-\sqrt{y}}^{-1} 0\,dx + \int_{-1}^{1} \frac{1}{2}\,dx + \int_{1}^{\sqrt{y}} 0\,dx = 1 & , \quad 1\le y \end{cases}$$

$$G(y)=\begin{cases} 0 & , \quad y<0 \\ \sqrt{y} & , \quad 0\le y<1 \\ 1 & , \quad 1\le y \end{cases}$$

→ Now Find $\bar{G}(x)$ :

$$\bar{G}(y)=\begin{cases} 0 & , \quad y<0 \\ DNE & , \quad y=0 \\ \frac{1}{2\sqrt{y}} & , \quad 0<y<1 \\ DNE & , \quad y=1 \\ 0 & , \quad y>1 \end{cases} = \begin{cases} \frac{1}{2\sqrt{y}} & , \quad 0<y<1 \\ 0 & , \quad y>1 \text{ and } y<0 \\ DNE & , \quad y=0 \text{ and } y=1 \end{cases}$$

→ بشوف مد اليمين واليسار وبطلعو مش متساويات فبتكون DNE

Now: $g(y)=\underline{\bar{G}(y)}$ if direvative exists so

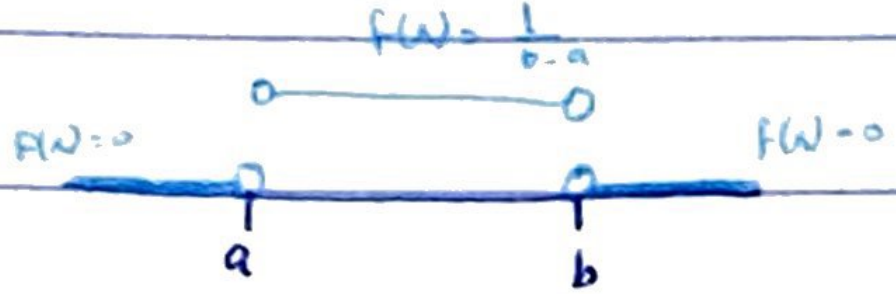
$$g(y)=\begin{cases} \frac{1}{2\sqrt{y}} & , \quad 0<y<1 \\ 0 & , \quad \text{elsewhere.} \end{cases}$$ → This is a p.d.f of $Y$.

→ Distribution Function Technique. (c.d.f method).

## Remark :

uniformly ← $X \sim U(a,b)$

$$f(x) = \begin{cases} \frac{1}{b-a} & , \ a < x < b \\ 0 & , \ \text{elsewhere} \end{cases}$$




$X \sim U[a,b]$

$$f(x) = \begin{cases} \frac{1}{b-a} & , \ a \le x \le b \\ 0 & , \ \text{elsewhere} \end{cases}$$