

8.7 Improper Integrals.

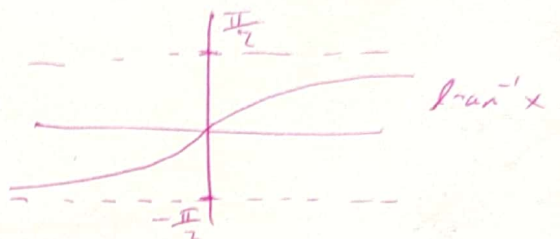
$$\text{1) } \int_0^{\infty} \frac{dx}{x^2+1}$$

$$= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{x^2+1}$$

$$= \lim_{b \rightarrow \infty} \tan^{-1} x \Big|_0^b$$

$$= \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 0)$$

$$= \boxed{\frac{\pi}{2}} \rightarrow \text{converges}$$



$$\text{4) } \int_0^4 \frac{dx}{\sqrt{4-x}}$$

$$4-x=0 \text{ if } x=4.$$

$$= \lim_{b \rightarrow 4^-} \int_0^b \frac{dx}{\sqrt{4-x}} = \lim_{b \rightarrow 4^-} \left(-2\sqrt{4-x} \Big|_0^b \right)$$

$$= \lim_{b \rightarrow 4^-} \left(-2\sqrt{4-b} - (-2\sqrt{4-0}) \right)$$

$$= \lim_{b \rightarrow 4^-} (-2\sqrt{4-b} + 4)$$

$$= 0 + 4$$

$$= \boxed{4} \rightarrow \text{converges}$$

$$\underline{\underline{7)}} \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

$$1-x^2=0 \Rightarrow \underline{\underline{x=1}}$$

$$\lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{b \rightarrow 1^-} \left(\sin^{-1}(b) - \sin^{-1}(0) \right)$$

$$= \frac{\pi}{2} - 0$$

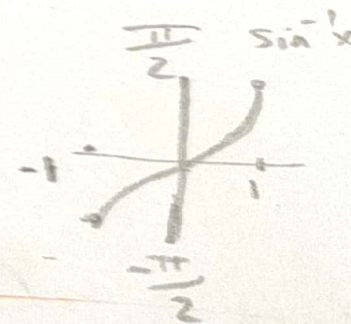
$$= \frac{\pi}{2}$$

Converges.

Remember:

$$\int \frac{1}{\sqrt{a^2-x^2}} dx$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + C$$



$$10) \int_{-\infty}^2 \frac{2 dx}{x^2 + 4}$$

$$= \lim_{a \rightarrow -\infty} \left(\int_a^2 \frac{2 dx}{x^2 + 4} \right)$$

$$= \lim_{a \rightarrow -\infty} \left(2 \cdot \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) \Big|_a^2 \right)$$

$$= \lim_{a \rightarrow -\infty} \left(\tan^{-1} \left(\frac{2}{2} \right) - \tan^{-1} \left(\frac{a}{2} \right) \right)$$

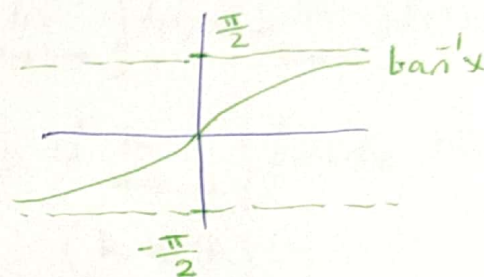
$$= \frac{\pi}{4} - -\frac{\pi}{2}$$

$$= \frac{\pi}{4} + \frac{2\pi}{4}$$

$$= \boxed{\frac{3\pi}{4}}$$

converges.

remember $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right) + C$



$$14) \int_{-\infty}^{\infty} \frac{x dx}{(x^2 + 4)^{3/2}}$$

$$= \int_{-\infty}^0 \frac{x dx}{(x^2 + 4)^{3/2}} + \int_0^{\infty} \frac{x dx}{(x^2 + 4)^{3/2}}$$

$$= \int_0^4 \frac{1}{2(u)^{3/2}} du + \int_4^{\infty} \frac{1}{2(u)^{3/2}} du$$

$$= -\frac{1}{2} \int_4^{\infty} u^{-3/2} du + \frac{1}{2} \int_4^{\infty} u^{-3/2} du = 0$$

let $u = x^2 + 4$

$$du = 2x dx$$

$$\frac{du}{2} = x dx$$

$$x = -\infty \Rightarrow u = \infty$$

$$x = \infty \Rightarrow u = \infty$$

$$x = 0 \Rightarrow u = 4$$

$$\Rightarrow -\frac{1}{2} \int_4^{\infty} u^{-3/2} du + \frac{1}{2} \int_4^{\infty} u^{-3/2} du$$

$$= \lim_{b \rightarrow \infty} -\frac{1}{2} \int_4^b u^{-3/2} du + \lim_{b \rightarrow \infty} \frac{1}{2} \int_4^b u^{-3/2} du$$

$$\Rightarrow \lim_{b \rightarrow \infty} \frac{1}{2} \int_4^b u^{-3/2} du$$

$$= \lim_{b \rightarrow \infty} \left(\frac{1}{2} \cdot u^{-1/2} - 2 \cdot \frac{1}{4} \right)$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{1}{\sqrt{u}} \Big|_4^b \right)$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{1}{\sqrt{b}} + \frac{1}{2} \right)$$

$$= \frac{1}{2}$$

$$\text{So, } -\frac{1}{2} \int_4^{\infty} u^{-3/2} du + \frac{1}{2} \int_4^{\infty} u^{-3/2} du$$

$$= 0$$

$$\therefore \int_{-\infty}^{\infty} \frac{x dx}{(x^2+4)^{3/2}}$$

Converges.

$$16 \int_0^2 \frac{s+1}{\sqrt{4-s^2}} ds.$$

$$= \underbrace{\int_0^2 \frac{s}{\sqrt{4-s^2}} ds}_{I_1} + \underbrace{\int_0^2 \frac{1}{\sqrt{4-s^2}} ds}_{I_2}$$

$$= 2 + \frac{\pi}{2}$$

$$= \frac{4 + \pi}{2}$$

$$I_1 = \int_0^2 \frac{s}{\sqrt{4-s^2}} ds$$

$$\text{let } u = 4 - s^2$$

$$du = -2s ds$$

$$\frac{-du}{2} = s ds$$

$$s=0 \Rightarrow u=4$$

$$s=2 \Rightarrow u=0$$

$$\boxed{I_1} = -\frac{1}{2} \int_4^0 \frac{1}{\sqrt{u}} du.$$

$$= \frac{1}{2} \int_0^4 \frac{1}{\sqrt{u}} du$$

$$= \lim_{a \rightarrow 0^+} \frac{1}{2} \int_a^4 u^{-1/2} du.$$

$$= \lim_{a \rightarrow 0^+} \left(\frac{1}{2} \cdot 2 \sqrt{u} \Big|_a^4 \right)$$

$$= \lim_{a \rightarrow 0^+} (2 - \sqrt{a})$$

$$= 2 - 0$$

$$= \boxed{2}$$

$$\boxed{I_2} = \int_0^2 \frac{1}{\sqrt{4-s^2}} ds$$

$$= \lim_{b \rightarrow 2^-} \int_0^b \frac{1}{\sqrt{4-s^2}} ds.$$

$$= \lim_{b \rightarrow 2^-} \left(\sin^{-1} \left(\frac{s}{2} \right) \Big|_0^b \right)$$

$$= \lim_{b \rightarrow 2^-} \left(\sin^{-1} \left(\frac{b}{2} \right) - \sin^{-1}(0) \right)$$

$$= \frac{\pi}{2} - 0$$

$$= \boxed{\frac{\pi}{2}}$$

$$\textcircled{21} \int_{-\infty}^0 x e^x dx.$$

$$= \lim_{a \rightarrow -\infty} \int_a^0 x e^x dx.$$

by parts

$$\begin{aligned} u &= x & dv &= e^x dx \\ du &= dx & v &= e^x \end{aligned}$$

$$\rightarrow \lim_{a \rightarrow -\infty} \left[x e^x \Big|_a^0 - \int_a^0 e^x dx \right].$$

$$= \lim_{a \rightarrow -\infty} \left[0 - a e^a - e^0 + e^a \right].$$

$$= \lim_{a \rightarrow -\infty} \left[a e^a + e^a - 1 \right]$$

$$= [0 + 0 - 1]$$

$$= \boxed{-1}$$

25) $\int_0^1 x \ln x \, dx$

$$= \lim_{a \rightarrow 0^+} \int_a^1 x \ln x \, dx$$

by parts $\Rightarrow$

$$u = \ln x$$

$$u = \frac{dx}{x}$$

$$dv = x \, dx$$

$$v = \frac{x^2}{2}$$

$$= \lim_{a \rightarrow 0^+} \left[\ln x \cdot \frac{x^2}{2} \Big|_a^1 - \int_a^1 \frac{x^2}{2} \cdot \frac{dx}{x} \right]$$

$$= \lim_{a \rightarrow 0^+} \left[0 - \ln a \cdot \frac{a^2}{2} - \int_a^1 \frac{x}{2} \, dx \right]$$

$$= \lim_{a \rightarrow 0^+} \left[\frac{a^2}{2} \ln a - \frac{x^2}{4} \Big|_a^1 \right]$$

$$= \lim_{a \rightarrow 0^+} \left[\frac{a^2}{2} \ln a - \frac{1}{4} + \frac{a^2}{4} \right]$$

$$= 0 - \frac{1}{4} + 0$$

$$= \boxed{-\frac{1}{4}}$$

$$\lim_{a \rightarrow 0^+} \frac{a^2}{2} \ln a \quad 0 \cdot -\infty$$

$$= \lim_{a \rightarrow 0^+} \frac{\ln a}{\frac{2}{a^2}} \quad \frac{-\infty}{\infty}$$

$$\frac{1}{a^2}$$

$$= \lim_{a \rightarrow 0^+} \frac{\frac{1}{2a}}{-\frac{2}{a^3}}$$

$$= \lim_{a \rightarrow 0^+} \frac{-\frac{a^2}{4}}{1} = 0$$

$$\textcircled{32} \int_0^2 \frac{dx}{\sqrt{|x-1|}}$$

$$= \int_0^1 \frac{dx}{\sqrt{|x-1|}} + \int_1^2 \frac{dx}{\sqrt{|x-1|}}$$

$$= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{\sqrt{|x-1|}} + \lim_{a \rightarrow 1^+} \int_a^2 \frac{dx}{\sqrt{|x-1|}}$$

$$= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{\sqrt{1-x}} + \lim_{a \rightarrow 1^+} \int_a^2 \frac{dx}{\sqrt{x-1}}$$

$$= \lim_{b \rightarrow 1^-} \left[-2\sqrt{1-x} \right]_0^b + \lim_{a \rightarrow 1^+} \left[2\sqrt{x-1} \right]_a^2$$

$$= \lim_{b \rightarrow 1^-} \left[-2\sqrt{1-b} + 2\sqrt{1} \right] + \lim_{a \rightarrow 1^+} \left[2\sqrt{2-1} - 2\sqrt{a-1} \right]$$

$$= 2 + 2$$

$$= \boxed{4}$$

$$|x-1|$$

$$x-1=0$$

$$x=1$$

$$\begin{array}{c} - \quad + \\ | \end{array}$$

$$|x-1| = \begin{cases} 1-x, & x < 1 \\ x-1, & x > 1 \end{cases}$$

$$\textcircled{40} \int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$$

$$= \int_0^1 2e^{-u} du$$

$$= -2e^{-u} \Big|_0^1$$

$$= -2[e^{-1} - e^0]$$

$$= \boxed{-\frac{2}{e} + 2}$$

$$\text{let } u = \sqrt{x}$$

$$du = \frac{1}{2\sqrt{x}} dx$$

$$2du = \frac{dx}{\sqrt{x}}$$

$$\text{when } x=0 \Rightarrow u=0$$

$$x=1 \Rightarrow u=1$$

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$$\int_0^{\pi} \frac{\sin \theta \, d\theta}{\sqrt{\pi - \theta}}$$

Test the integral for convergence.

$$\frac{\sin \theta}{\sqrt{\pi - \theta}} \leq \frac{1}{\sqrt{\pi - \theta}}$$

⇒ by D.C.T.

$$\int_0^{\pi} \frac{1}{\sqrt{\pi - \theta}} \, d\theta$$

$$= \lim_{b \rightarrow \pi^-} \int_0^b \frac{1}{\sqrt{\pi - \theta}} \, d\theta$$

$$= \lim_{b \rightarrow \pi^-} \left(-2\sqrt{\pi - \theta} \Big|_0^b \right)$$

$$= \lim_{b \rightarrow \pi^-} \left(-2\sqrt{\pi - b} + 2\sqrt{\pi} \right)$$

$$= 0 + 2\sqrt{\pi}$$

$$= 2\sqrt{\pi} \quad \underline{\text{Converges.}}$$

$$\therefore \int_0^{\pi} \frac{\sin \theta}{\sqrt{\pi - \theta}} \, d\theta \quad \underline{\underline{\text{also converges.}}}$$

41) $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$, best D.C.T integral Converges.

remember when $0 \leq t \leq \pi$
 $\Rightarrow 0 \leq \sin t \leq 1$

so, $\Rightarrow \sqrt{t} \leq \sqrt{t} + \sin t \leq 1 + \sqrt{t}$

$\Rightarrow \frac{1}{\sqrt{t}} \geq \frac{1}{\sqrt{t} + \sin t}$

$\Rightarrow$ Now, we can apply Direct Comparison test (D.C.T).

$\int_0^{\pi} \frac{1}{\sqrt{t}} dt$

$= \lim_{a \rightarrow 0^+} \int_a^{\pi} \frac{1}{\sqrt{t}} dt = \lim_{a \rightarrow 0^+} \left(2\sqrt{t} \Big|_a^{\pi} \right)$

$= \lim_{a \rightarrow 0^+} (2\sqrt{\pi} - 2\sqrt{a})$

$= 2\sqrt{\pi} - 0$

$= 2\sqrt{\pi}$

converges.

So, by D.C.T $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$ converges.

47 $\int_1^{\infty} \frac{dx}{x^3+1}$

(Test the integral converges??)

$$x^3+1 \geq x^3 \quad (x \geq 1)$$

$$\frac{1}{x^3+1} \leq \frac{1}{x^3}$$

by direct comparison test (D.C.T).

$$0 \leq \frac{1}{x^3+1} \leq \frac{1}{x^3} \quad \text{for } x \geq 1.$$

$$\int_1^{\infty} \frac{1}{x^3} dx.$$

$$= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^3} dx.$$

$$= \lim_{b \rightarrow \infty} \left(\frac{1}{-2x^2} \Big|_1^b \right) = \lim_{b \rightarrow \infty} \left[\frac{1}{-2b^2} + \frac{1}{2} \right]$$

$$= \boxed{\frac{1}{2}}$$

converges

So, $\int_1^{\infty} \frac{dx}{x^3+1}$ also converges.

50 Test the integrals converges.

$$\int_0^{\infty} \frac{dx}{1+e^x}$$

$$1+e^x \geq e^x \quad (x \geq 0)$$

$$\frac{1}{1+e^x} \leq \frac{1}{e^x}$$

$$0 \leq \frac{1}{1+e^x} \leq \frac{1}{e^x}, \quad \text{by D.C.T.} \Rightarrow$$

$$\int_0^{\infty} \frac{1}{e^x} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx.$$

$$= \lim_{b \rightarrow \infty} \left(-e^{-x} \Big|_0^b \right).$$

$$= \lim_{b \rightarrow \infty} \left(-e^{-b} - (-e^0) \right)$$

$$= 0 + 1$$

$$= \boxed{1} \text{ converges.}$$

$\therefore \int_0^{\infty} \frac{dx}{1+e^x}$ converges.

58 $\int_2^{\infty} \frac{1}{\ln x} dx$. Test the integral converges.

we know that- $x > \ln x$
 $\Rightarrow \frac{1}{x} < \frac{1}{\ln x}$.

$$\begin{aligned}\int_2^{\infty} \frac{1}{x} dx &= \lim_{b \rightarrow \infty} \left(\int_2^b \frac{1}{x} dx \right) \\ &= \lim_{b \rightarrow \infty} \left(\ln|x| \Big|_2^b \right) \\ &= \lim_{b \rightarrow \infty} \left(\ln|b| - \ln|2| \right) \\ &= \infty \quad (\text{diverges}).\end{aligned}$$

So, by direct comparison test is

$$\int_2^{\infty} \frac{1}{\ln x} dx \quad \underline{\underline{\text{diverges.}}}$$

56 $\int_{\pi}^{\infty} \frac{1 + \sin x}{x^2} dx$

$$\begin{aligned}\sin x &\leq 1 \\ 1 + \sin x &\leq 2 \\ \frac{1 + \sin x}{x^2} &\leq \frac{2}{x^2}\end{aligned}$$

$$\frac{1 + \sin x}{x^2} \leq \frac{2}{x^2}$$

by D.C.T.

$$\int_{\pi}^{\infty} \frac{2}{x^2} dx = \lim_{b \rightarrow \infty} \int_{\pi}^b \frac{2}{x^2} dx = \lim_{b \rightarrow \infty} \left(\frac{-2}{x} \right) \Big|_{\pi}^b$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{2}{b} - \left(-\frac{2}{\pi} \right) \right)$$

$$= \boxed{\frac{2}{\pi}} \text{ Converges.}$$

So, $\int_{\pi}^{\infty} \frac{1 + \sin x}{x^2} dx$ also converges.

62 $\int_1^{\infty} \frac{1}{e^x - 2^x} dx$

(apply Limit Comparison Test)

let $f(x) = \frac{1}{e^x - 2^x}$ & $g(x) = \frac{1}{e^x}$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{e^x - 2^x}}{\frac{1}{e^x}} = \lim_{x \rightarrow \infty} \frac{e^x}{e^x - 2^x}$$

$$\lim_{x \rightarrow \infty} \frac{e^x}{e^x - 2^x} = \lim_{x \rightarrow \infty} \frac{e^x}{\frac{e^x - 2^x}{e^x}} = \lim_{x \rightarrow \infty} \frac{1}{1 - \left(\frac{2}{e}\right)^x} = \frac{1}{1 - 0} = \boxed{1}$$

$$\begin{aligned}\& \int_1^{\infty} \frac{1}{e^x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{e^x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_1^b = \lim_{b \rightarrow \infty} \left[-e^{-b} + \frac{1}{e} \right] \\ &= 0 + \frac{1}{e} = \boxed{\frac{1}{e}}\end{aligned}$$

$$\Rightarrow \int_1^{\infty} \frac{1}{e^x} dx \text{ converges} \Rightarrow \int_1^{\infty} \frac{1}{e^x - 2^x} dx \text{ converges.}$$

(by Limit Comparison Test).

65) Find the values of p for which each integral converges.

a) $\int_1^2 \frac{dx}{x(\ln x)^p}$, b) $\int_2^\infty \frac{dx}{x(\ln x)^p}$

let $u = \ln x$

$du = \frac{dx}{x}$

, when $x = 1 \Rightarrow u = 0$

$x = 2 \Rightarrow u = \ln 2$

$x = \infty \Rightarrow u = \infty$

a) $\int_0^{\ln 2} \frac{du}{u^p}$

b) $\int_{\ln 2}^\infty \frac{du}{u^p}$

a) $\int_0^{\ln 2} \frac{du}{u^p}$

for $p = 1 \Rightarrow \int_0^{\ln 2} \frac{du}{u}$

$= \lim_{a \rightarrow 0^+} \int_a^{\ln 2} \frac{du}{u} = \lim_{a \rightarrow 0^+} \ln|u| \Big|_a^{\ln 2}$
 $= \lim_{a \rightarrow 0^+} (\ln 2 - \ln a)$

for $p \neq 1 \Rightarrow \int_0^{\ln 2} \frac{du}{u^p} = \lim_{a \rightarrow 0^+} \int_a^{\ln 2} \frac{du}{u^p} = \lim_{a \rightarrow 0^+} \frac{u^{-p+1}}{-p+1} \Big|_a^{\ln 2}$
 $= \lim_{a \rightarrow 0^+} \left(\frac{(\ln 2)^{1-p}}{1-p} - \frac{a^{1-p}}{1-p} \right)$

$= \begin{cases} \frac{(\ln 2)^{1-p}}{1-p}, & p < 1 \\ \infty, & p > 1 \end{cases}$

$$\text{So, } \int_1^2 \frac{dx}{x(\ln x)^p} = \begin{cases} \frac{(\ln 2)^{1-p}}{1-p} & , p < 1 \\ \infty & , p \geq 1 \end{cases}$$

Converges for $p < 1$.

diverges for $p \geq 1$.

b) $\int_{\ln 2}^{\infty} \frac{du}{u^p}$

for $p=1 \Rightarrow \int_{\ln 2}^{\infty} \frac{du}{u} = \lim_{b \rightarrow \infty} \int_{\ln 2}^b \frac{du}{u} = \lim_{b \rightarrow \infty} \ln|u| \Big|_{\ln 2}^b$

$$= \lim_{b \rightarrow \infty} \ln b - \ln 2$$

$$= \infty$$

for $p \neq 1 \Rightarrow \int_{\ln 2}^{\infty} \frac{du}{u^p} = \lim_{b \rightarrow \infty} \int_{\ln 2}^b \frac{du}{u^p}$

$$= \lim_{b \rightarrow \infty} \frac{u^{-p+1}}{-p+1} \Big|_{\ln 2}^b$$

$$= \lim_{b \rightarrow \infty} \frac{b^{-p+1}}{-p+1} - \frac{\ln 2^{-p+1}}{-p+1}$$

$$= \begin{cases} \frac{(\ln 2)^{1-p}}{p-1} & , p > 1 \\ \infty & , p < 1 \end{cases}$$

$$\text{So, } \int_2^{\infty} \frac{dx}{x(\ln x)^p} = \begin{cases} \frac{(\ln 2)^{1-p}}{p-1} & , p > 1 \\ \infty & , p \leq 1 \end{cases}$$

Converges for $p > 1$.

diverges for $p \leq 1$.