

Convergence of Taylor series

Theory:- Taylor formula: There exist $c \in \text{Interval of definition}$ where c is between a and x such that:-

$$f(x) = T_n(x) + R_n(x) \quad \text{Remainder (error)} = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$$

$$f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

And if $\lim_{n \rightarrow \infty} R_n(x) = 0$ Then T.S converges to $f(x)$

$\rightarrow$ usually you prove that using sandwich Theorem

Theory:- Constant M and Remainder Estimation. Th
if there exists a Constant M such that

$$|f^{(n)}(x)| \leq M \quad \text{for } x \in I$$

$$\text{then } |R_n(x)| \leq \frac{M(x-a)^{n+1}}{(n+1)!}$$

In this case:- $\lim_{n \rightarrow \infty} R_n(x) = 0$ and T.S converges to $f(x)$

Estimating Errors

• You can estimate errors using:-

$\rightarrow$ Alternating Series Estimation Theorem A.S.E.T

or
 $\rightarrow$ Remainder Estimation Theorem R.E.T

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How to solve Problems Concerning

T.S, M.S, convergence and Errors

A.T.S/M.S/Conv

- for example: if you want to find Maclaurin series of e^x :-

↳ step 1:- find derivatives of e^x

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$\vdots$$

$$f^n(x) = e^x$$

Note:-

you can use $R_n(x)$ Theory to prove that T.S converges to $f(x)$
knowing that $\lim_{n \rightarrow \infty} R_n(x) = 0$

↳ step 2:- Plug $x=a$ in $f^n(x)$
center

$$f^n(c) = f^n(0) = e^0 = 1$$

↳ step 3:- find $\sum$:-

$$\sum_{n=0}^{\infty} \frac{f^n(0) x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

↳ step 4:- find interval of convergence

We can use Ratio test :-

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n+1} |x| = 0 < 1$$

• So it converges abs for all x ($x \in (-\infty, \infty)$)

↳ step 5: represent your series

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

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M. series of $\cos x$ & $\sin x$:-

$\cos x$:-

1- $f(x) = \cos x$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$f^{(4)}(x) = \cos x$$

2) f^n here has two cases :- order $\begin{cases} \text{even} \\ \text{odd} \end{cases}$

$\rightarrow$ If order is even ($\Rightarrow 2n$) Then $f^{2n}(x) = (-1)^n \cos x$ $\xleftarrow{x=0}$ so it's always 1

$\rightarrow$ If order is odd ($\Rightarrow 2n+1$) Then $f^{2n+1}(x) = (-1)^{n+1} \sin x$ $\xleftarrow{\text{since } x=0}$ it's always zero

3) $\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n \overset{=1}{\cos x}}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

4) To find interval of convergence :- let's use Ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{2n!}{x^{2n}} \right| = \lim_{n \rightarrow \infty} \frac{|x|^2}{(2n+2)(2n+1)} = 0 < 1 \quad \text{so conv for all } x$$

5) Our series :-

$$f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

Note $\rightarrow$ I know that the 0's of the odd order does exist :-

$$\cos x = 1 + 0 - \frac{x^2}{2} + 0 + \frac{x^4}{4!} - \dots$$

$$\text{so here } P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

• for $\sin x$ it's the same way

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Estimating Errors:-

* If the Question Gives you an order $P_n(x)$ / the function and value of x :- & wants Error

Ex: Exer 35/36/ in the book page 595

let's solve 36:-

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$\text{at } x = \frac{1}{2}$$

$$R_4(x) = \frac{f^{(5)}(c)}{5!} x^5$$

$$R_4(x) = \frac{e^c}{5!} x^5$$

1- First c is between x and $a=0$ center

$$0 < c < \frac{1}{2}$$

Take e for all

$$e^0 < e^c < e^{\frac{1}{2}}$$

$$1 < e^c < e^{\frac{1}{2}}$$

$$\frac{x^5}{5!} \approx (e^c < e^{\frac{1}{2}})$$

$$|e^c \frac{x^5}{5!}| < \frac{e^{\frac{1}{2}} x^5}{5!} \quad x = \frac{1}{2} \text{ (Given)}$$

$$R_4(x) < \frac{e^{\frac{1}{2}} (\frac{1}{2})^5}{5!}$$

$$\text{error} < \frac{e^{\frac{1}{2}} (\frac{1}{2})^5}{5!}$$

Note:
if the T.S is Alternating use A.S.E.T

* If the Question gives you the maximum value of error and wants x (you use A.S.E.T) the series is Alternating

Ex: Exer 37

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120}$$

up to 3 terms

$$|\text{Error}| < \frac{5 \times 10^{-4}}{6} \text{ Given}$$

$$|\frac{x^5}{120}| < 5 \times 10^{-4} \Rightarrow \sqrt[5]{-6 \times 10^{-3}} < x < \sqrt[5]{6 \times 10^{-3}}$$

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* If the Question gives you the function and it's replacement and x as a range :- and wants Error.

Ex: Exer 38/39/40,

let's solve Exer 38:-

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

A.S. E.T :-

$$|\text{Error}| < \frac{x^4}{4!}$$

$$|E| < \frac{(0.5)^4}{4!}$$

$$-0.5 < x < 0.5$$

$$0 < x^2 < (0.5)^2$$

$$\frac{x^4}{4!} < \frac{(0.5)^4}{4!}$$

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