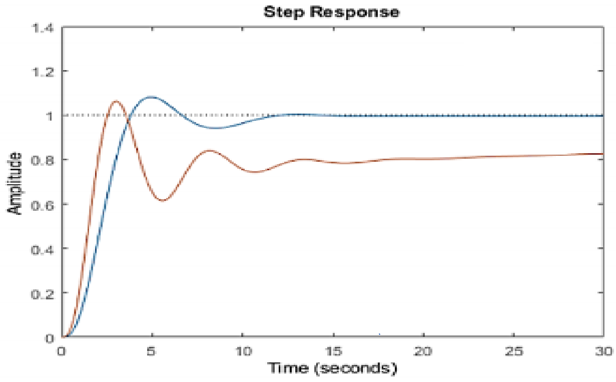
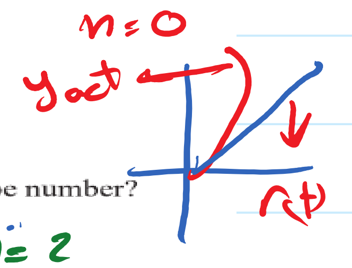
Design of a Servo System (Tracking System):

The tracking system is a control action aims to force the output response y (t) to follow the desired input r(t) with a required performance.





There are two cases for design the tracking system:

- Find the eigenvalues for the open loop system $|sI - A| = 0$
- Check if there is any eigenvalues at the origin or not i.e. find the Type number?
- Remember this:

$n = 2$

TABLE 7.2 Relationships between input, system type, static error constants, and steady-state errors

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

$K_v = \lim_{s \to 0} s G(s)$ $\to \infty$

$e_{ss} = \frac{1}{K_v}$

$K_p = \infty$

$e_{ss} = \frac{1}{K_p + 1}$

If the steady state error is equal zero based on the Type number use the first case. Otherwise use the second case.

$K_p = \lim_{s \to 0} G(s)$

$K_v = \lim_{s \to 0} s\, G(s)$

$K_v = 3$

$e_{ss} = \frac{1}{3}$

Type number = # of Poles at origin
(n)

$$G(s) = \frac{(s+3)}{s^2 (s+1)} \quad n = 2$$

$(n) = 2$

step

$\hookrightarrow K_p = \lim_{s \to 0} G(s)$

$$G(s) = \frac{1}{(s+2)(s+3)} \quad n = 0$$

$$= \lim_{s \to 0} \frac{1}{(2)(3)} = 1/6$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1 + 1/6}$$

Constant

$$k_p = \lim_{s \to 0} G(s), \quad e_{ss}(\infty) = \frac{1}{1+k_p}$$

$$k_v = \lim_{s \to 0} s\, G(s), \quad e_{ss}(\infty) = \frac{1}{k_v}$$

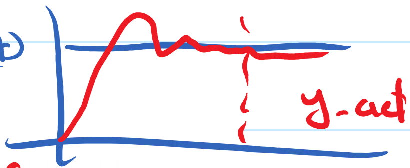
$$k_a = \lim_{s \to 0} s^2 G(s), \quad e_{ss}(\infty) = \frac{1}{k_a}$$

$\dot{x} = Ax + Bu$ —— (1)

$y = Cx$ —— (2)

To Design the tracking Sys. there are two cases :-

x Case (1) Design tracking Sys. without integral action

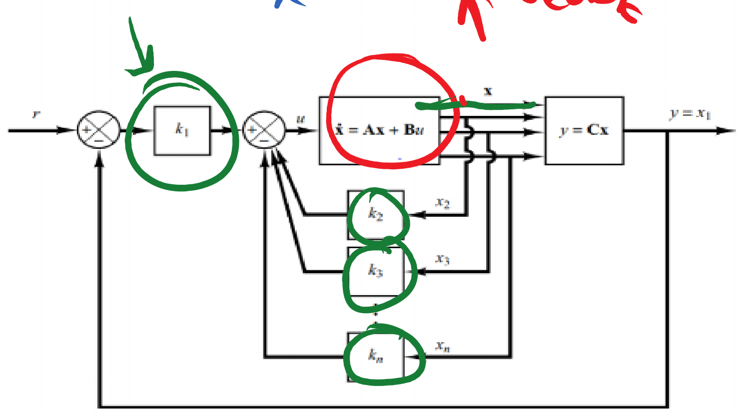


Here we assumed that $y = x_1$

x

states vector

$e_{ss} = 0$



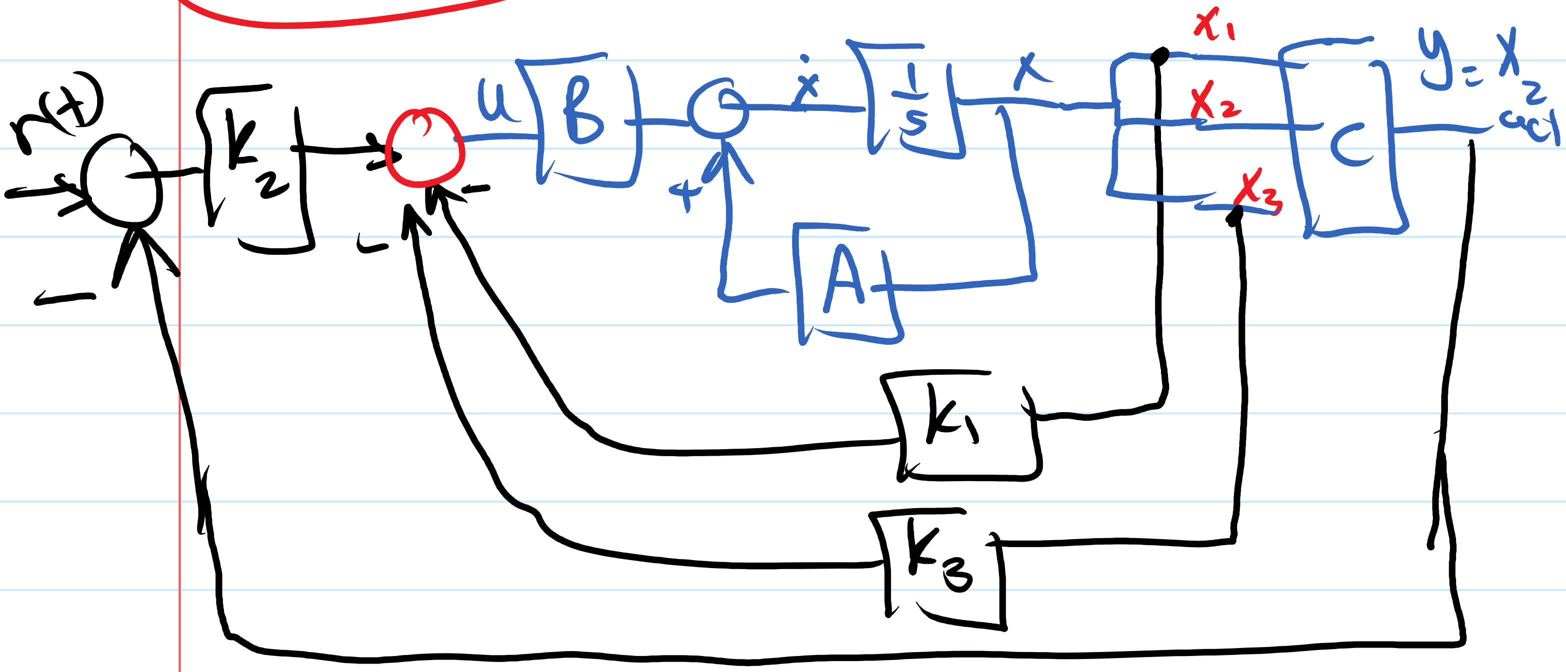
$y = x_1$

$u = -k_1 x_1 - k_2(r - x_2) - k_3 x_3$

$= -[k_1 \cdots k_3] x + k_2 r$

number of state 3

$y = x_2 = r(t)$



* To determine which case is suitable you must check the following

* $|\lambda I - A| = 0$ — (*) eigenvalue for open loop sys.

you can determine the type number (n) from Eq (*)

r(t) :- step input

$\lambda_1 = 3.56$

$\lambda_2 = -0.56$

$\boxed{n = 0}$

$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$

$y = [0 \quad 1] x$

char. Eq

$|[\lambda I - A]| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ -2 & \lambda - 3 \end{vmatrix} = \lambda^2 - 3\lambda - 2 = 0$

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r(t) = Step input , n = 0 , ess (∞) = Constant

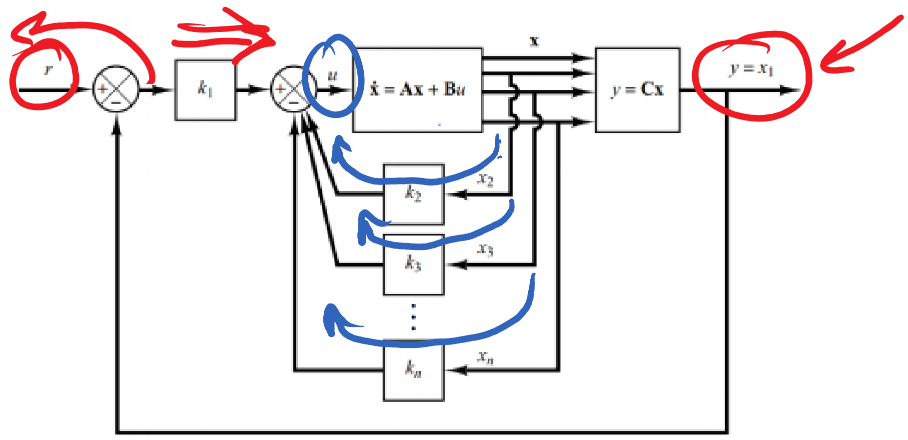
we must use Case (2)

Design Tracking sys if $y_{act} = x_1 = r(t)$

Here we assumed that $y = x_1$

r-y

$r - x_1$



$$u = -k_2 x_2 - k_3 x_3 - \cdots - k_n x_n + k_1 (r - x_1)$$

$$= -k_1 x_1 - k_2 x_2 - \cdots - k_n x_n + k_1 r$$

$$= -[k_1 \cdots k_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + k_1 r$$

$$= -Kx + k_1 r$$

$K = [k_1 \cdots k_n]$, x: state vector

$r(t)$ r
step input

$r(t) = r$, $r(\infty) = r$

$u = -Kx + K_1 r$ — (1) (Control input)

$\dot{x} = Ax + Bu$ — (2)

$y = Cx$

Sub Eq (1) into Eq (2)

$\dot{x} = Ax + B(-Kx + K_1 r)$

closed loop dynamic at time t

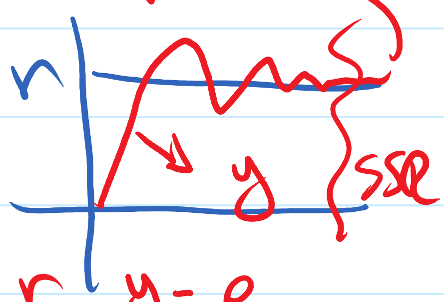
$\rightarrow \dot{x}(t) = (A - BK)x(t) + BK_1 r(t)$ — (3)

Note: To design sys. will be Asy stable
$y(\infty)$ will approach constant value r,
and $u(\infty)$ will approach zero, when
r is step input

at $t = \infty$

$\rightarrow \dot{x}(\infty) = (A - BK)x(\infty) + BK_1 r(\infty)$ — (4)

SSR.



$r - y = e$
$r - y_{act} = 0$
$r = y_{act}$

(3) − (4)

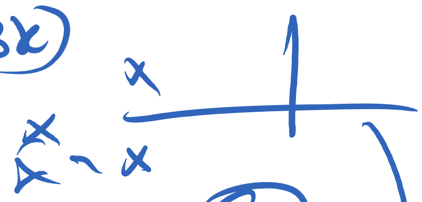
$\dot{x}(t) - \dot{x}(\infty) = (A - BK)(x(t) - x(\infty)) + \cancel{BK_1 r(t)} - \cancel{BK_1 r(\infty)}$

$\dot{e}(t) = (A - BK)e(t)$

$e(t) := x(t) - x(\infty)$

$\dot{e}(t) := \dot{x}(t) - \dot{x}(\infty)$

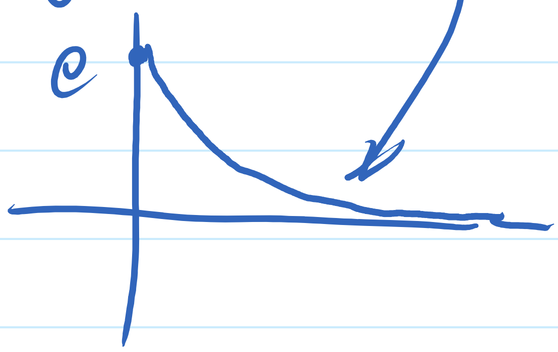
$(A-BK)$



$$\dot{e}(t) = (A-BK)\,e(t) \quad (5)$$

This is simillar to regulator equation

if the eigenvalues of $(A-BK)$ in the Left side plane (Asystable)



There fore $e(t)$ will goes to Zero

$e(t) = r(t) - y(t)$

if we have three states

$y_{out} = r(t) = x_1$

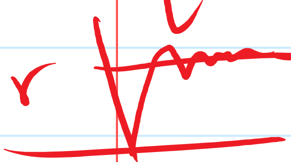
$$e(t) = [x(t) - x(\infty)]$$

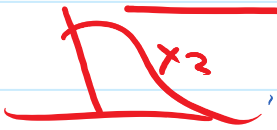
$$\begin{bmatrix} e_1 \\ \\ \end{bmatrix} = \begin{bmatrix} x_1(t) - x_1(\infty) \\ x_2(t) - x_2(\infty) \\ x_3(t) - x_3(\infty) \end{bmatrix}$$

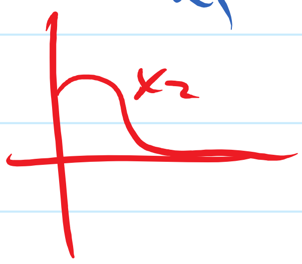
$x_1(\infty) \rightarrow r(t)$

$x_2(\infty) \rightarrow$ Zero

$x_3(\infty) \rightarrow$ Zero







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Case 2:- Tracking Sys with an integral action

* This Case is used when the $e_{ss}(\infty)$ is Constant and we would like to improve the $e_{ss}(\infty)$ to Zero.

$n = 0$ r: step input $\Rightarrow e_{ss}(\infty) = \text{constant}$

$\boxed{n=1} \longleftarrow e_{ss}(\infty) = \text{Zero}$

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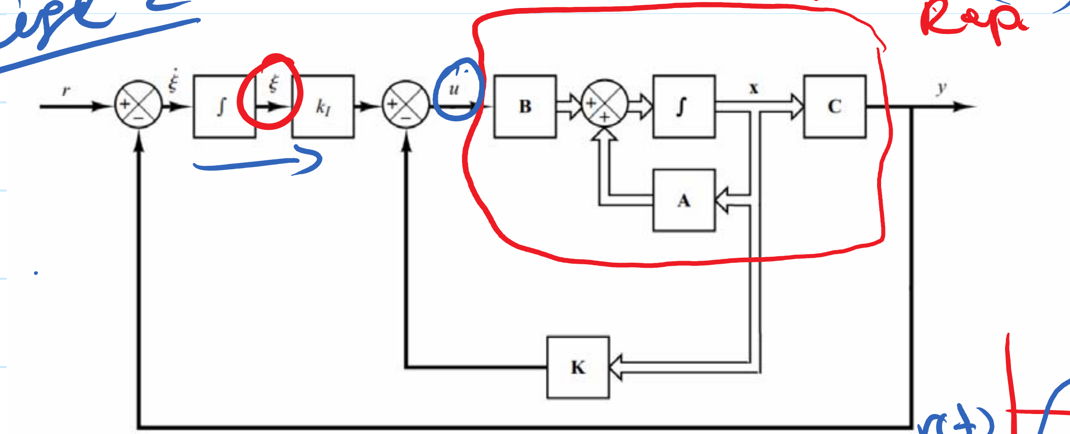
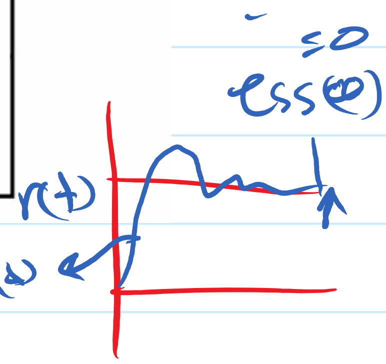
* To determine which Case is required

$|\lambda I - A| = 0 \longrightarrow$ # of (type number) (n)

Case 2

state space Rep





$y = x_1 = r(t)$

$\dot{x} = Ax + Bu$ —— (1)

$y = Cx$ —— (2)

$\rightarrow u = -Kx + k_i \xi$ —— (3) $K = [k_1 - - - k_n]$

k_i: gain integrator

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$$\dot{x} = Ax + Bu \quad \text{—— (1)}$$

from the tracking Control Scheme

$$\dot{\xi} = r - y = r - Cx \quad \text{—— (4)}$$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} \underset{n\times n}{A} & \underset{n\times 1}{\vec{0}} \\ \underset{p\times n}{-C} & \underset{p\times 1}{0} \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} + \underset{n\times r}{\begin{bmatrix} B \\ 0 \end{bmatrix}} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \quad \text{—— (5)}$$

at $t \to (\infty)$

$$\begin{bmatrix} \dot{x}(\infty) \\ \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} A & \vec{0} \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(\infty) \\ \xi(\infty) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(\infty) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(\infty) \quad \text{—— (6)}$$

(5) − (6)

$$\begin{bmatrix} \dot{x}(t) - \dot{x}(\infty) \\ \dot{\xi}(t) - \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} A & \vec{0} \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(t) - x(\infty) \\ \xi(t) - \xi(\infty) \end{bmatrix}$$

$$+ \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - u(\infty) + \cancel{\begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) - r(\infty)}$$

at time $t \to \infty$ $\quad r(t) = r(\infty) = r$

$u(\infty) \longrightarrow$ zero

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$$\begin{bmatrix}\dot{\mathbf{x}}(t)-\dot{\mathbf{x}}(\infty)\\ \dot{\xi}(t)-\dot{\xi}(\infty)\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ -\mathbf{C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)-\mathbf{x}(\infty)\\ \xi(t)-\xi(\infty)\end{bmatrix}+\begin{bmatrix}\mathbf{B}\\ 0\end{bmatrix}[u(t)-u(\infty)] \quad 7$$

$P=1$

Define

$$\mathbf{x}(t)-\mathbf{x}(\infty)=\mathbf{x}_e(t)$$
$$\xi(t)-\xi(\infty)=\xi_e(t)$$
$$u(t)-u(\infty)=u_e(t)$$

$$\begin{bmatrix}\dot{x}_e(t)\\ \dot{\xi}_e(t)\end{bmatrix}=\begin{bmatrix}A & \vec{0}\\ -C & 0\end{bmatrix}\underbrace{\begin{bmatrix}x_e(t)\\ \xi_e(t)\end{bmatrix}}_{n\times 1}+\begin{bmatrix}B\\ 0\end{bmatrix}u_e(t) \quad (8)$$

$e(t) := (n+1)\times 1$

where $u_e(t) = -K x_e(t) + k_i \xi_e(t)$ — (9)

So will define a new $(n+P)$th order of error vector

$$e(t)=\begin{bmatrix}x_e(t)\\ \xi_e(t)\end{bmatrix}=(n+P)\times 1$$

where: n # of states, P := # of outputs

$$\dot{e}(t)=\underbrace{\begin{bmatrix}A & \vec{0}\\ -C & 0\end{bmatrix}}_{\hat{A}\ (n+1)\times(n+1)}e(t)+\underbrace{\begin{bmatrix}B\\ 0\end{bmatrix}}_{\hat{B}\ (n+1)\times 1}u_e(t) \quad (10)$$

when $P=1$ (output number)

$$u_e(t) = -K\,x_e + \xi_e\, k_i \quad (9)$$

$$u_e(t)=\begin{bmatrix}-K & k_i\end{bmatrix}\begin{Bmatrix}x_e(t)\\ \xi_e(t)\end{Bmatrix} \quad (11)$$

$$e(t)=\begin{Bmatrix}x_e(t)\\ \xi_e(t)\end{Bmatrix}$$

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$e(t) = \begin{bmatrix} X_e(t) \\ \xi_e(t) \end{bmatrix}$

$u_e(t) = -\underbrace{[K \;\vdots\; k_i]}_{\hat{K}} e(t)$ — (12)

$u_e(t) = -\hat{K} e(t)$ — (13)

Sub Eq (13) into Eq. (11)

$\dot{e}(t) = \underbrace{\begin{bmatrix} A & \vec{0} \\ -C & 0 \end{bmatrix}}_{\hat{A}} e(t) + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\hat{B}} u_e(t)$ — (11)

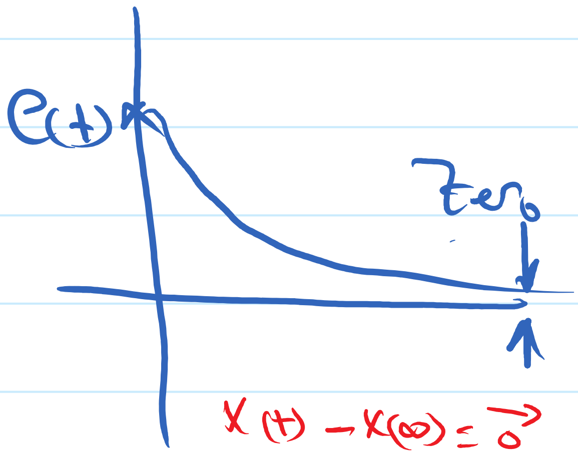
$\dot{e}(t) = \hat{A} e(t) + \hat{B} u_e(t)$

$= \hat{A} e(t) + \hat{B}(-\hat{K}) e(t)$

$\dot{e}(t) = \underbrace{(\hat{A} - \hat{B}\hat{K})}_{A_{CL}} e(t)$ — (14)

closed loop dynamics for case 2 of tracking sys

if the eigenvalues of $A_{CL} = \hat{A} - \hat{B}\hat{K}$ at left side plane. Then the sys is Asy-stable. So the error goes to zero



$X(t) - X(\infty) = \vec{0}$

$X(t) = X(\infty)$

4 states $\quad y_1 = x_1 = r(t)$

$e(t) = \begin{bmatrix} X_e(t) \\ \xi_e(t) \end{bmatrix} = \begin{bmatrix} X(t) - X(\infty) \\ \xi(t) - \xi(\infty) \end{bmatrix}$

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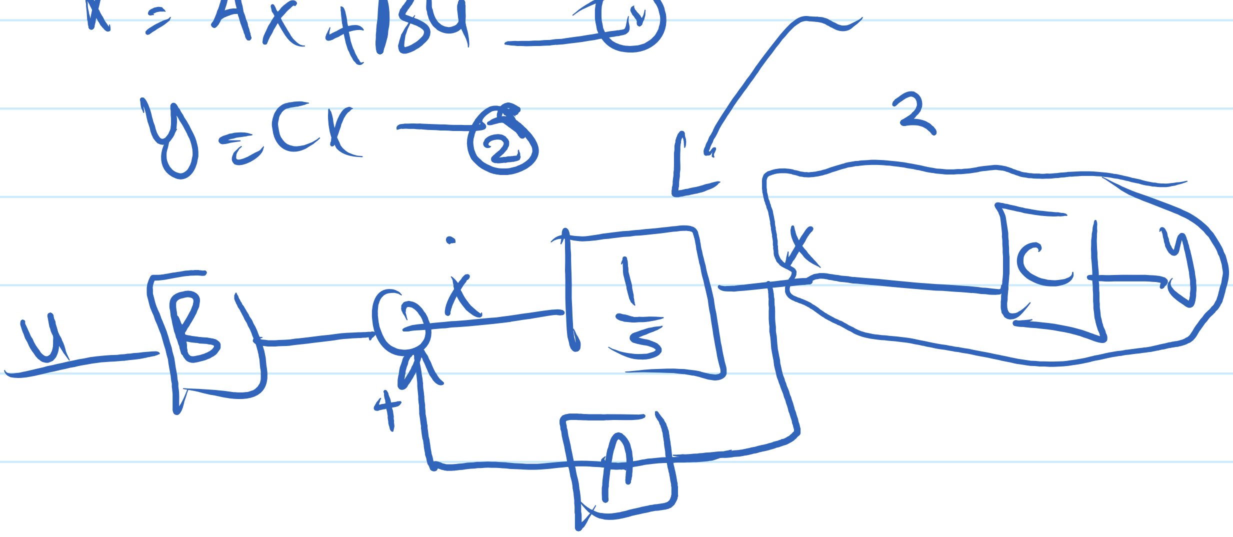
Now to design Asy. stable Sys. the eigenvalues of $\hat{A}-\hat{B}\hat{K}$ must lie on the left side plane. You can do the previous step by pole placement or LQR.

IC [ ]

$\dot{x} = Ax + Bu$ — (1)

$y = Cx$ — (2)



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Design tracking Sys.

The required performance is $T_s = 1$ seconds and $\%OS = \%10$. Assume that the system configuration is the same as that shown in Figure below and the reference input r is a step function. Obtain the unit-step response of the designed system.

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0]$$

r: step input

① $|\lambda I - A| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} \right| =$

① $\begin{vmatrix} (\lambda - 1) & -2 \\ -5 & (\lambda - 10) \end{vmatrix} = (\lambda - 1)(\lambda - 10) - 10 = 0$

$\lambda^2 - 11\lambda + \cancel{10} - \cancel{10} = 0$

$\lambda^2 - 11\lambda = 0 \Rightarrow \lambda(\lambda - 11) = 0$

$a_1 = -11$

$a_2 = 0$

$\lambda_1 = 0 \qquad \lambda_2 = 11 \qquad \boxed{n = 1}$

② $M = [B \quad AB] = \begin{bmatrix} 0 & 2 \\ 1 & 10 \end{bmatrix}$

$\det(M) = -2 \neq 0$ fully state Controllable

③ Will use Case 2 for Bass-Gura Approach because the sys is not in the form of the first Canonical for

$n = 2$

$T = MW \quad , \quad W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -11 & 1 \\ 1 & 0 \end{bmatrix}$

$$T = \begin{bmatrix} 0 & 2 \\ 1 & 10 \end{bmatrix} \begin{bmatrix} -11 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$$

(2) $T_s = 1sec$ $\%os = \%10 =$ $\xi = 0.59$ (under damped)

$$\omega_n = \frac{4}{T_s \xi} = \frac{4}{0.59 \times 1} = 6.78 \text{ rad/s}$$

$$\mu_{1,2} = -\xi \omega_n \pm \omega_n \sqrt{1-\xi^2}\, j$$

$$\mu_{1,2} = -4 \pm 5.47j$$ (desired eigen values)

$$(s-\mu_1)(s-\mu_2) = s^2 + 8s + 45.968$$

$$\alpha_1 = 8, \quad \alpha_2 = 45.968$$

$n = 2$

$$K = \left[(\alpha_2 - a_2) \;\vdots\; (\alpha_1 - a_1) \right] T^{-1}$$

$$\left[(45.968 - 0) \;:\; (8 + 11) \right] \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} 32.48 & 19 \end{bmatrix}$$

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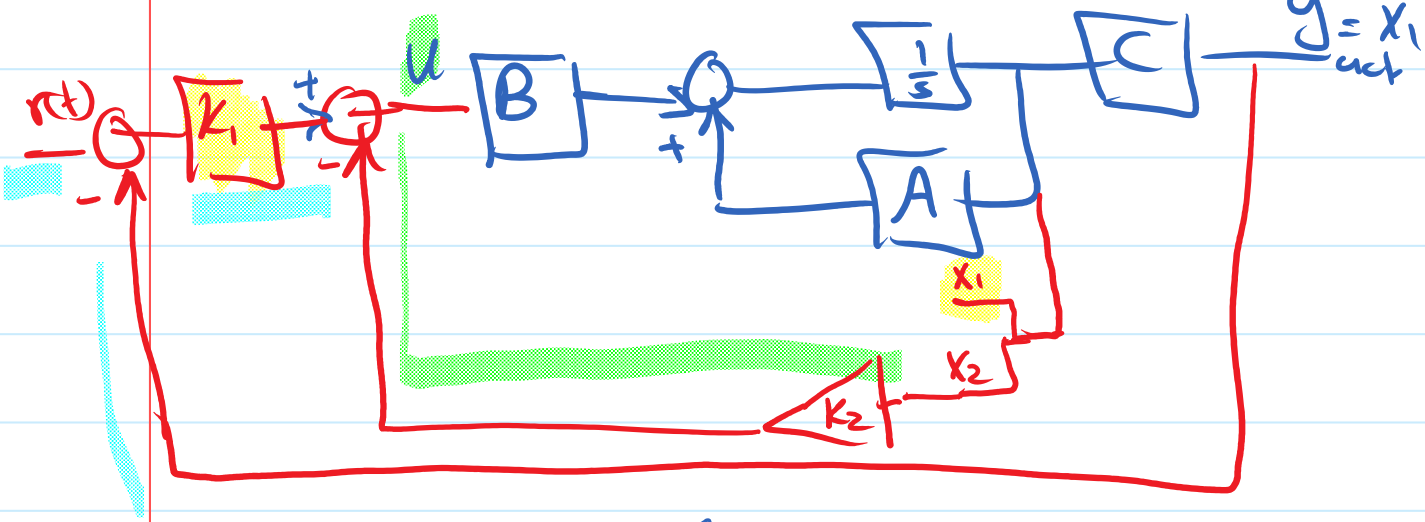
r := step $y = x_1$ $y = Cx + \cancel{Du}$



$$u = -K_2 x_2 + K_1(r - y) = -K_2 x_2 + K_1 r - K_1 x_1$$

$$= -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + K_1 r$$

Example : Consider the system below:

$$\dot{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$
$$y = \boldsymbol{C}\boldsymbol{x} + Du$$

Where:

$A = \begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

a) Study the stability for the system.

b) Design a tracking system for the plant by using the pole-placement (**Bass-Gura Approach -second method**). The desired input for the control scheme is a step input and the controller must achieve the following requirements:
-settling time (T_s) is one second.
-critical damped (ζ).
-the steady state error (ess) must be equal to zero.

Note: ~~in the case you need to approximate the system to second order system~~ use (s_3= -30) if it is necessary.

① $|sI - A| = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$

$= \left| \begin{bmatrix} s & -1 \\ -1 & s-5 \end{bmatrix} \right| = s^2 - 5s - 1 = 0$

internal stability

$s_1 = -0.19$

$s_2 = 5.19$

unstable

$type(F) = 0$

So we will use Case 2 for the tracking sys.

$$\hat{A} = \begin{bmatrix} A_{2\times2} & \vec{0} \\ -C_{1\times2} & 0 \end{bmatrix}_{n+p \times n+p} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 5 & 0 \\ -1 & 0 & 0 \end{bmatrix}_{3\times3}$$

$$\hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(3) $M = \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B} \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & 5 \\ 1 & 5 & 26 \\ 0 & 0 & -1 \end{bmatrix}, \quad |M| = 1$$

fully state Controllable.

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 5 & 0 \\ -1 & 0 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \hat{u}$$

This Sys is not in the form of the first Connincal form (Case 2 - Bass gura - Approach)

$$|\lambda I - \hat{A}| = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 5 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$s^3 - 5s^2 - s$$

$\boxed{a_1 = -5}$ $\boxed{a_2 = -1}$ $\boxed{a_3 = 0}$

$n=3$

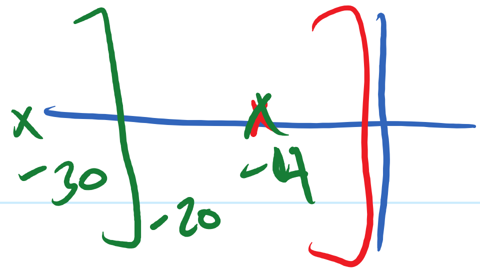
$T = MW$, $W = \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$W = \begin{bmatrix} -1 & -5 & 1 \\ -5 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

T =

$T = MW = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

(4) $T_s = 1\,sec$ $\quad \zeta = 1$



$$\omega_n = \frac{4}{T_s \zeta} = 4 \text{ rad/s}$$

$$\mu_{1,2} = -\omega_n \qquad \mu_{1,2} = -\omega_n\zeta \pm \omega_n\cancel{\sqrt{1-\zeta^2}}\,j = -\omega_n$$

$1-1$

$$\mu_{1,2} = -4$$

$$\mu_3 = -30$$

Desired char. Eq.

$$(s-\mu_1)(s-\mu_2)(s-\mu_3)$$

$$(s+4)(s+4)(s+30) =$$

desired char. Eq $=$ `s^3 + 38.0*s^2 + 256.0*s + 480.0`

$$= s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$$

$$\boxed{\alpha_1 = 38} \quad \boxed{\alpha_2 = 256} \quad \boxed{\alpha_3 = 480}$$

$$K = \left[(\alpha_n - a_n) \vdots \cdots \vdots (\alpha_1 - a_1)\right] T^{-1}$$

$$\left[(\alpha_3 - a_3) \vdots (\alpha_2 - a_2) \vdots (\alpha_1 - a_1)\right] T^{-1}$$

$$\left[(480-0) \vdots (256+1) \vdots (38+5)\right] T^{-1}$$

$$= \left[\underbrace{257}_{k_1} \quad \underbrace{43}_{k_2} \quad \underbrace{-480}_{-k_i}\right]$$

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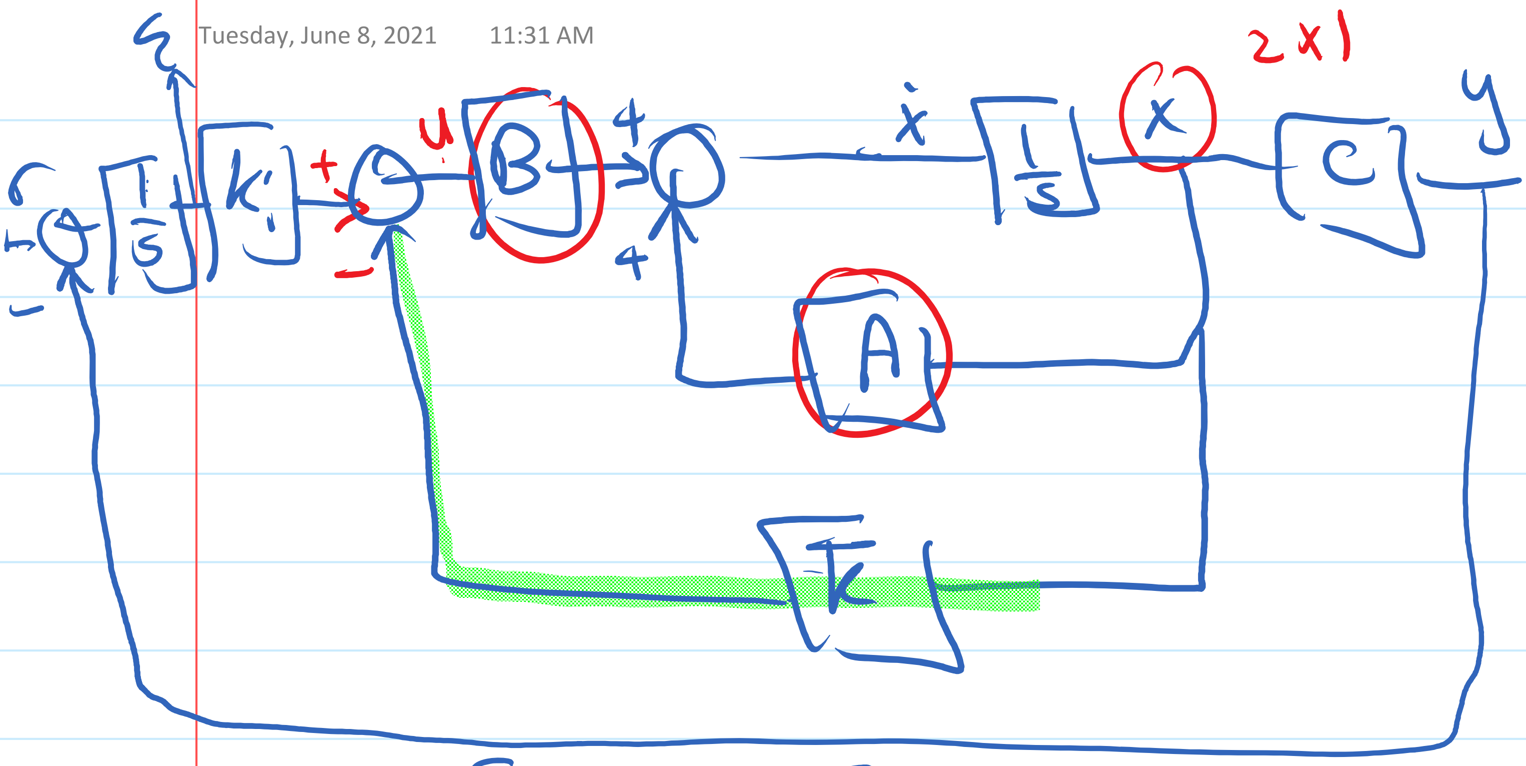

$$\bar{K} = [k_1 \quad k_2]$$

$$u = -\bar{K}x + k_i \xi$$

$$= -k_1 x_1 - k_2 x_2 + k_i \xi$$