4.2: Transformations of Variables of the discrete type.

Thm: change of variable method (1 variable, discrete type).

let X be a r.v with space $\mathcal{A}$ with p.d.f $f(x)$

let $u: \mathcal{A} \to \mathcal{B}$ such that u is a one to one function

let $Y = u(X)$ then the p.d.f of Y is given by:

$$g(y) = \begin{cases} f(w(y)) & , y \in \mathcal{B} \\ 0 & , \text{elsewhere} \end{cases}$$

where $w(y) = x = u^{-1}(y)$.

expl: $X \sim b(n, p)$ $n = 3$, $p = \frac{2}{3}$

$X \sim b(3, \frac{2}{3})$

X r.v with p.d.f $f(x) = \begin{cases} \binom{3}{x}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{3-x} & , x = 0, 1, 2, 3 \\ 0 & , \text{elsewhere}. \end{cases}$

Find the distribution of $Y = X^2$.

$u(x) = Y = X^2$

$u(x) = y = x^2 \iff u^{-1}(y) = w(y) = x = \sqrt{y}$

$u: \mathcal{A} \to \mathcal{B}$

$\mathcal{A} = \{0, 1, 2, 3\}$.

$\mathcal{B} = \{0, 1, 4, 9\}$

$$g(y) = \begin{cases} f(w(y)) & , y \in \mathcal{B} \\ 0 & , \text{elsewhere} \end{cases} = \begin{cases} f(\sqrt{y}) & , y \in \mathcal{B} \\ 0 & , \text{elsewhere} \end{cases}$$

$$= \begin{cases} \binom{3}{\sqrt{y}}\left(\frac{2}{3}\right)^{\sqrt{y}}\left(\frac{1}{3}\right)^{3-\sqrt{y}} & , y = 0, 1, 4, 9 \\ 0 & , \text{elsewhere} \end{cases}$$

Thm: change of variable method (2 variables / discrete type)

let X_1, X_2 be r.v's with space $\mathcal{A}$ and joint p.d.f $f(x_1, x_2)$

let $U: \mathcal{A} \to \mathcal{B}$ such that U is a one to one transformation

let $U(x_1, x_2) = (Y_1, Y_2)$, then the joint p.d.f of Y_1, Y_2 is given by

$$g(y_1, y_2) = \begin{cases} f(w_1(y_1, y_2), w_2(y_1, y_2)) & , (y_1, y_2) \in \mathcal{B} \\ 0 & , \text{elsewhere} \end{cases}$$

where $U(x_1, x_2) = (u_1(x_1, x_2); u_2(x_1, x_2)) = (Y_1, Y_2)$ and

$w(Y_1, Y_2) = (w_1(Y_1, Y_2), w_2(Y_1, Y_2)) = (u_1^{-1}(Y_1, Y_2), u_2^{-1}(Y_1, Y_2)) = (x_1, x_2)$.

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exp 2: X_1, X_2 indep. r.v have Poisson distribution with mean's μ_1, μ_2 resp.

Find the dist. of $X_1 + X_2$.

$$X_1 \sim \text{Poisson}(\mu_1) \Rightarrow f_1(x_1) = \begin{cases} \dfrac{\mu_1^{x_1} e^{-\mu_1}}{x_1!} & , x_1 = 0, 1, 2, \ldots \\ 0 & , \text{elsewhere} \end{cases}$$

$$X_2 \sim \text{Poisson}(\mu_2) \Rightarrow f_2(x_2) = \begin{cases} \dfrac{\mu_2^{x_2} e^{-\mu_2}}{x_2!} & , x_2 = 0, 1, 2, \ldots \\ 0 & , \text{elsewhere} \end{cases}$$

$$X_1, X_2 \text{ indep.} \Rightarrow f(x_1, x_2) = \begin{cases} f_1(x_1) \cdot f_2(x_2) & , (x_1, x_2) \in \mathcal{A} \\ 0 & , \text{elsewhere} \end{cases}$$

$$= \begin{cases} \dfrac{\mu_1^{x_1} e^{-\mu_1}}{x_1!} \cdot \dfrac{\mu_2^{x_2} e^{-\mu_2}}{x_2!} & , (x_1, x_2) \in \mathcal{A} \\ 0 & , \text{elsewhere} \end{cases}$$

$\mathcal{A} = \{ (x_1, x_2) : x_1 = 0, 1, \ldots, x_2 = 0, 1, \ldots \}$

continue,

y_1 لازم دايماً تكون أكبر من y_2

عشان هيك $y_2 = 0, 1, \dots, y_1$

$$\left\{ \begin{array}{l} Y_1 = X_1 + X_2 \\ Y_2 = X_2 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} X_1 = Y_1 - Y_2 \\ X_2 = Y_2 \, . \end{array} \right\}$$

$$B = \{ (y_1, y_2) : y_1 = 0, 1, \dots, \; y_2 = 0, 1, \dots, y_1 \}$$

$$g(y_1, y_2) = \begin{cases} f(w_1(y_1, y_2), w_2(y_1, y_2)) & , (y_1, y_2) \in B \\ 0 & , \text{elsewhere.} \end{cases}$$

$$= \begin{cases} f(y_1 - y_2, y_2) & , (y_1, y_2) \in B \\ 0 & , \text{elsewhere} \end{cases}$$

$$g(y_1, y_2) = \begin{cases} \dfrac{\mu_1^{y_1 - y_2} e^{-\mu_1}}{(y_1 - y_2)!} \cdot \dfrac{\mu_2^{y_2} e^{-\mu_2}}{y_2!} & , (y_1, y_2) \in B . \\ 0 & , \text{elsewhere} \end{cases}$$

$$\to g_1(y_1) = \sum_{y_2} g(y_1, y_2)$$

$$= \sum_{y_2 = 0}^{y_1} \frac{\mu_1^{y_1 - y_2} e^{-\mu_1}}{(y_1 - y_2)!} \cdot \frac{\mu_2^{y_2} e^{-\mu_2}}{y_2!}$$

يتبع $\binom{n}{x}$

$$= \frac{e^{-\mu_1} e^{-\mu_2}}{y_1!} \sum_{y_2 = 0}^{y_1} \frac{y_1!}{(y_1 - y_2)! \, y_2!} \mu_1^{y_1 - y_2} \mu_2^{y_2}$$

$$= \frac{e^{-(\mu_1 + \mu_2)}}{y_1!} \left[\sum_{y_2 = 0}^{y_1} \binom{y_1}{y_2} \mu_2^{y_2} \mu_1^{y_1 - y_2} \right] \to \text{binomial formula .}$$

$$\to g_1(y_1) = \begin{cases} \dfrac{e^{-(\mu_1 + \mu_2)}}{y_1!} (\mu_2 + \mu_1)^{y_1} & , y_1 = 0, 1, \dots \\ 0 & , \text{elsewhere} \end{cases}$$

عبارة عن Poisson

$Y_1 \sim$ Poisson $(\mu_1 + \mu_2)$

$$\left.\begin{array}{l} X_1 \sim \text{Poisson}(\mu_1) \\ X_2 \sim \text{Poisson}(\mu_2) \\ X_1, X_2 \text{ indep.} \end{array}\right\} \Rightarrow X_1 + X_2 \sim \text{Poisson}(\mu_1 + \mu_2).$$

Note : Binomial formula

$$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m}$$

Exercises.

Q17: let X have a p.d.f $f(x) = \begin{cases} \frac{1}{3}, & x = 1, 2, 3. \\ 0, & \text{elsewhere} \end{cases}$

Find the p.d.f of $Y = 2X+1$.

$Y = 2X+1 \rightsquigarrow X = \frac{y-1}{2}$

$U: A \to B$, $A = 1, 2, 3$

$B = 3, 5, 7$.

$$\Rightarrow g(y) = \begin{cases} f\left(\frac{y-1}{2}\right), & y = 3, 5, 7 \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{1}{3}, & y = 3, 5, 7 \\ 0, & \text{elsewhere} \end{cases}$$