Two Non-Commutative Binomial Theorems

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Abstract

We derive two formulae for $(A+B)^n$, where A and B are elements in a non-commutative, associative algebra with identity.

1 Introduction

Let $\mathfrak{A}$ be an associative algebra, not necessarily commutative, with identity. For two elements A and B in $\mathfrak{A}$, that commute, i.e.

$$AB = BA \tag{1}$$

the well-known Binomial Theorem reads

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \tag{2}$$

If A and B do not commute, we find the first formula for $(A+B)^n$ that retains the binomial coefficient. It also gives a representation of $e^{(A+B)}$ that is different from the Campell-Baker-Hausdorff representation [3]. The first formula is then applied to a problem in non-commutative geometry. The second formula for $(A+B)^n$ complements the first one. We apply it to a problem in quantum mechanics.

2 The First Non-Commutative Binomial Theorem

Let $\mathfrak{A}$ be an associative algebra, not necessarily commutative, with identity 1. $L(\mathfrak{A})$ denotes the algebra of linear transformations from $\mathfrak{A}$ to $\mathfrak{A}$.

Definition 1

Let A and X be elements of $\mathfrak{A}$.

1. A can be looked upon as an element in $L(\mathfrak{A})$ by

$$A(X) = AX \tag{3}$$

i.e. leftmultiplication

2. The element d_A in $L(\mathfrak{A})$ is defined by

$$d_A(X) = [A, X] = AX - XA \tag{4}$$

We now have the following trivial relations:

Statements

1. As elements in $L(\mathfrak{A})$, A and d_A commute, i.e.

$$Ad_A(X) = d_A A(X) \tag{5}$$

2. d_A is a derivation on $\mathfrak{A}$, i.e.

$$d_A(XY) = (d_A X)Y + X(d_A Y) \tag{6}$$

3.
$$(A - d_A)X = XA \tag{7}$$

4. Jacobi identity

$$d_A d_B(C) + d_B d_C(A) + d_C d_A(B) = 0 \tag{8}$$

These simple statements are sufficient to prove the following non-commutative Binomial Theorem [1], [2].

Theorem 1

For A and B elements in $\mathfrak{A}$, and 1 being the identity in $\mathfrak{A}$

$$(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} \{(A + d_B)^k 1\} B^{n-k} \tag{9}$$

Proof. The formula holds true for n=1. We now proceed by induction.

$$\begin{aligned}(A + B)^{n+1} &= (A + B)(A + B)^n = (A + d_B + B - d_B)(A + B)^n \\ &= (A + d_B + B - d_B) \sum_{k=0}^{n} \binom{n}{k} \{(A + d_B)^k 1\} B^{n-k}\end{aligned}$$

Using the previous Statements, we get

$$
\begin{aligned}
(A+B)^{n+1} &= \sum_{k=0}^{n} \binom{n}{k} [A\{(A+d_B)^k 1\}B^{n-k} + \{d_B(A+d_B)^k 1\}B^{n-k} + \{(A+d_B)^k 1\}B^{n-k+1}] \\
&= \sum_{k=0}^{n} \binom{n}{k} [\{(A+d_B)^{k+1} 1\}B^{n-k} + \{(A+d_B)^k 1\}B^{n-k+1}] \\
&= \sum_{k=1}^{n} \binom{n}{k} \{(A+d_B)^k 1\}B^{n-k+1} + B^{n+1} \\
&\quad + \sum_{k=1}^{n} \binom{n}{k-1} \{(A+d_B)^k 1\}B^{n-k+1} + \{(A+d_B)^{n+1} 1\}
\end{aligned}
$$

From the identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

we then get

$$(A+B)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \{(A+d_B)^k 1\}B^{n+1-k}$$

□

3 The Essential Non-Commutative Part

We write

$$(A+d_B)^n 1 = A^n + D_n(B,A) \tag{10}$$

For a commutative algebra, $D_n(B,A)$ is identically zero. We thus call $D_n(B,A)$ the essential non-commutative part.
$D_n(B,A)$ satisfies the following recurrence relation

$$D_{n+1}(B,A) = d_B A^n + (A+d_B)D_n(B,A) \tag{11}$$

with

$$D_0(B,A) = 0$$

Definition 2

1.
$$M_n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \tag{12}$$

2.
$$D_k(B,A) = D_k \tag{13}$$

We now have the following obvious corollary.

Corollary 1

$$(A+B)^n = M_n + \sum_{k=0}^{n} \binom{n}{k} D_k B^{n-k} \tag{14}$$

4 Exponentials

We have as a consequence of the first non-commutative Binomial Theorem

Corollary 2

$$e^{A+B} = [e^{A+d_B} 1] e^B \tag{15}$$

Proof.

$$\begin{aligned} e^{A+B} &= \sum_{n=0}^{\infty} \frac{1}{n!}(A+B)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \{(A+d_B)^k 1\} B^{n-k} \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{k!(n-k)!} \{(A+d_B)^k 1\} B^{n-k} \\ e^{A+B} &= [e^{A+d_B} 1] e^B \end{aligned} \tag{16}$$

□

By splitting of the essential non-commutative part we get

Corollary 3

$$e^{A+B} = e^A e^B + \sum_{n=0}^{\infty} \frac{1}{k!} D_k e^B \tag{17}$$

This is different from the Campell-Baker-Hausdorff formula.

5 Application of Theorem 1 for

$$d_B A = hA^2 \tag{18}$$

Definition 3

For h a scalar and n an integer we introduce

$$\gamma_n(h) = [1+h][1+2h]\cdots[1+(n-1)h], \quad \gamma_0(h) = 1 \tag{19}$$

Lemma 1

The following properties hold

1. $$\gamma_1(h) = 1, \gamma_n(0) = 1, \gamma_n(1) = n!$$

2. $$\gamma_{k+1}(h) = (1 + kh)\gamma_k(h)$$

Proof. Direct verification

□

Now, from Corollary 1 (14)

$$(A+B)^n = M_n + \sum_{k=2}^{n} \binom{n}{k} D_k B^{n-k}$$
$$D_k = d_B A^{k-1} + (A + d_B)D_{k-1}, \quad D_2 = d_B A$$

we find

Lemma 2

1. $$d_B A^k = khA^{k+1}$$

2. $$D_k = \{\gamma_k(h) - 1\}A^k$$

Proof.

1. $d_B A = hA^2$
   Since d_B is a derivation we have by induction
   $d_B A^k = (d_B A^{k-1})A + A^{k-1}(d_B A)$
   $= (k-1)hA^{k+1} + A^{k-1}hA^2 = khA^{k+1}$

2. By induction and $D_2 = hA^2$, we find

$$\begin{aligned}
D_k &= d_B A^{k-1} + (A + d_B)\{\gamma_{k-1}(h) - 1\}A^{k-1} \\
&= d_B A^{k-1} + \{\gamma_{k-1}(h) - 1\}A^k + \gamma_{k-1}(h) d_B A^{k-1} - d_B A^{k-1} \\
&= \{\gamma_{k-1}(h) - 1\}A^k + \gamma_{k-1}(h)(k-1)hA^k \\
&= \{[1 + (k-1)h]\gamma_{k-1}(h) - 1\}A^k \\
D_k &= \{\gamma_k(h) - 1\}A^k
\end{aligned}$$

□

Now

$$
\begin{aligned}
(A+B)^n &= M_n + \sum_{k=2}^{n} \binom{n}{k} D_k B^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} + \sum_{k=2}^{n} \binom{n}{k} \{\gamma_k(h) - 1\} A^k B^{n-k} \\
&= B^n + \binom{n}{1} AB^{n-1} + \sum_{k=2}^{n} \binom{n}{k} \gamma_k(h) A^k B^{n-k}
\end{aligned}
$$

Finally,

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} \gamma_k(h) A^k B^{n-k} \tag{20}$$

The result can also be found in [4]

Note: For $h = 1$, i.e. $d_B A = A^2$, we find

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} k! A^k B^{n-k}$$

$$(A+B)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!} A^k B^{n-k} \tag{21}$$

Also, if on the vector space of infinitely often differentiable function on $\mathbb{R}$ we introduce the operators

$$A = x, \quad B = x^2 \frac{d}{dx} \tag{22}$$

we have $d_B A = A^2$. Thus the representation (21) applies.

6 The Second Non-Commutative Binomial Theorem

Let A and B be in $\mathfrak{A}$. With

$$M_n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \in \mathfrak{A} \tag{23}$$

we have

Lemma 3

1.
$$M_0 = 1, M_1 = A + B \tag{24}$$

2.
$$M_1 M_n = M_{n+1} + d_B M_n \tag{25}$$

Proof.

1. Obvious

2.

$$\begin{aligned}
M_1 M_n &= (A+B)\sum_{k=0}^{n}\binom{n}{k}A^k B^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}A^{k+1}B^{n-k} + \sum_{k=0}^{n}\binom{n}{k}BA^k B^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}A^{k+1}B^{n-k} + \sum_{k=0}^{n}\binom{n}{k}\{d_B A^k + A^k B\}B^{n-k} \\
&= \sum_{s=1}^{n+1}\binom{n}{s-1}A^s B^{n+1-s} + \sum_{k=0}^{n}\binom{n}{k}A^k B^{n+1-k} + \sum_{k=0}^{n}\binom{n}{k}\{d_B A^k\}B^{n-k} \\
&= A^{n+1} + B^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k-1} + \binom{n}{k}\right]A^k B^{n+1-k} + d_B\sum_{k=0}^{n}\binom{n}{k}A^k B^{n-k} \\
&= A^{n+1} + B^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}A^k B^{n+1-k} + d_B M_n \\
M_1 M_n &= M_{n+1} + d_B M_n
\end{aligned}$$

□

Lemma 4

$$M_1^n = M_n + \sum_{k=0}^{n-2} M_1^k d_B M_{n-1-k} \tag{26}$$

Proof. This is true for $n = 2$,

$$M_1^2 = M_1 M_1 = M_2 + d_B M_1$$

Now by induction

$$M_1^{n-1} = M_{n-1} + \sum_{k=0}^{n-3} M_1^k d_B M_{n-2-k}$$
$$M_1^n = M_1 M_1^{n-1} = M_1 M_{n-1} + \sum_{k=0}^{n-3} M_1^{k+1} d_B M_{n-2-k}$$
$$= M_n + d_B M_{n-1} + \sum_{s=1}^{n-2} M_1^s d_B M_{n-1-s}$$
$$M_1^n = M_n + \sum_{k=0}^{n-2} M_1^k d_B M_{n-1-k}$$

□

Theorem 2

$$(A+B)^n = M_n + \sum_{k=0}^{n-2} (A+B)^k d_B M_{n-1-k} \tag{27}$$

Proof. This is lemma 4 with $M_1 = A + B$

7 Application of Theorem 2 for the case

$$d_B A = d_B M_1 = C, \text{ and } d_A C = d_B C = 0 \tag{28}$$

Then

$$d_B A^k = kCA^{k-1}, d_B M_n = nCM_{n-1} \tag{29}$$

and

$$(A+B)^n = M_n + \sum_{k=0}^{n-2} (n-1-k) C M_1^k M_{n-2-k}$$

Ansatz

$$(A+B)^n = \sum_{k=0}^{[\frac{n}{2}]} M_{n-2k} A_{n,k} \tag{30}$$

with

$$A_{n,0} = 1 \tag{31}$$

and $A_{n,k}$ commuting with A and B.
$[\frac{n}{2}]$ denotes the greatest integer less than $\frac{n}{2}$.

From

$$(A+B)^{n+1} = M_1(A+B)^n$$

we have

$$\sum_{k=0}^{[\frac{n+1}{2}]} M_{n+1-2k}A_{n+1,k} = M_1 \sum_{k=0}^{[\frac{n}{2}]} M_{n-2k}A_{n,k}$$

or

$$M_{n+1} + \sum_{k=1}^{[\frac{n+1}{2}]} M_{n+1-2k}A_{n+1,k} = M_1\{M_n + \sum_{k=1}^{[\frac{n}{2}]} M_{n-2k}A_{n,k}\}$$

From (25) and (23) we find

$$M_1M_n = M_{n+1} + nCM_{n-1}$$

resulting in

$$\sum_{k=1}^{[\frac{n+1}{2}]} M_{n+1-2k}A_{n+1,k} = nCM_{n-1} + \sum_{k=1}^{[\frac{n}{2}]} M_{n+1-2k}A_{n,k} + \sum_{k=1}^{[\frac{n}{2}]}(n-2k)CM_{n-1-2k}A_{n,k} \tag{32}$$

For n even, $n = 2N$, (32) reads

$$\sum_{k=1}^{N} M_{2N+1-2k}A_{2N+1,k} = 2NCM_{2N-1} + \sum_{k=1}^{N} M_{2N+1-2k}A_{2N,k} + \sum_{k=1}^{N-1}(2N-2k)CM_{2N-1-2k}A_{2N,k}$$

or

$$M_{2N-1}A_{2N+1,1} + \sum_{k=2}^{N} M_{2N+1-2k}A_{2N+1,k} = 2NCM_{2N-1} + M_{2N-1}A_{2N,1} + \sum_{k=2}^{N} M_{2N+1-2k}A_{2N,k} + \sum_{k=2}^{N} M_{2N+1-2k}(2N+2-2k)A_{2N,k-1}$$

Comparing coefficients gives the recurrence relation

$$A_{2N+1,k} = A_{2N,k} + (2N+2-2k)CA_{2N,k-1}$$

or

$$A_{n+1,k} = A_{n,k} + (n+2-2k)CA_{n,k-1}, k \geq 1 \tag{33}$$

Note, that for n odd, $n = 2N+1$, we get the same relation □

Lemma 5

The recurrence relation (33) with $A_{n,0} = 1$ has the solution

$$A_{n,k} = \frac{n!}{(n-2k)!k!2^k} C^k \tag{34}$$

and (30) becomes

$$(A+B)^n = \sum_{k=0}^{[\frac{n}{2}]} M_{n-2k} \frac{n!}{(n-2k)!k!2^k} C^k \tag{35}$$

Proof. by direct verification
This result can also be found in [5]

Note: On the vector space of infinitely often differentiable function on $\mathbb{R}$ we introduce the operators

$$A = x, B = \lambda \frac{d}{dx}, \text{where } \lambda \text{ is a scalar.} \tag{36}$$

Then $d_B A = \lambda$, or $C = \lambda 1$. Thus the above representation (35) applies.

□

In particular

$$(x + \lambda \frac{d}{dx})^n = \sum_{k=0}^{[\frac{n}{2}]} M_{n-2k} \frac{n!}{(n-2k)!k!2^k} \lambda^k$$

where

$$M_n = \sum_{r=0}^{n} \binom{n}{r} x^r \frac{d^{n-r}}{dx^{n-r}}, \qquad M_n 1 = x^n$$

resulting in

$$(x + \lambda \frac{d}{dx})^n 1 = \sum_{k=0}^{[\frac{n}{2}]} x^{n-2k} \frac{n!}{(n-2k)!k!2^k} \lambda^k \tag{37}$$

For $\lambda = -1$, we get

$$(x - \frac{d}{dx})^n 1 = n! \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{x^{n-2k}}{(n-2k)!k!2^k} \tag{38}$$

The right-hand side are the Hermite polynomials.

Thus

$$He_n(x) = (x - \frac{d}{dx})^n 1 \tag{39}$$

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