## 4.6: distribution of order statistic.

**Result 4.6.1:**

let $X_1, \dots, X_n$ be a Random sample with density $f(x)$ on $[a,b]$

let $Y_1 < Y_2 < \dots < Y_n$ be order statistic of $X_1, \dots, X_n$

i.e, $Y_1 = \min(X_1, \dots, X_n) = 1^{st}$ order statistic

$Y_2 = 2^{nd}$ order statistic.

$Y_n = \max(X_1, \dots, X_n) = n^{th}$ order statistic.

the density of $Y_k$, $k = 1, \dots, n$ ($k^{th}$ order statistic) is given by

$$g_k(y_k) = \frac{n!}{(k-1)!\,(n-k)!} \left(F(y_k)\right)^{k-1} \left(1 - F(y_k)\right)^{n-k} f(y_k), \text{ where } a<y_k<b, \text{ Zero else}$$

and $F$ is the c.d.f corresponding to $f$.

exp2: $X_1, X_2, X_3, X_4$ Random sample with p.d.f $f(x) = \begin{cases} 2x & , 0<x<1 \\ 0 & , \text{elsewhere} \end{cases}$

Find the density of $Y_3$.

Find $Pr(\frac{1}{2} < Y_3)$.

$$\rightarrow F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & , x<0 \\ x^2 & , 0 \leq x < 1 \\ 1 & , 1 \leq x \end{cases}$$

$\rightarrow n = 4$, $k = 3$, $a = 0$, $b = 1$

$$\rightarrow g_3(y_3) = \frac{4!}{2!\,1!} \left(y_3^2\right)^2 \left(1 - y_3^2\right)^1 (2y_3) \quad \text{where } 0<y_3<1, \text{ Zero else}$$

$\rightarrow$

$\rightsquigarrow g_3(y_3) = \begin{cases} 24\,(y_3^5 - y_3^7) & , \ 0 < y_3 < 1 \\ 0 & , \ \text{else} \end{cases}$

$\rightsquigarrow Pr\left(\frac{1}{2} < y_3\right) = \int_{\frac{1}{2}}^{\infty} g_3(y_3)\, dy_3 = \int_{\frac{1}{2}}^{1} 24\,(y_3^5 - y_3^5)\, dy_3$

$= \frac{243}{256}.$

Result 4.6.2 :

let $X_1, \dots, X_n$ be a Random sample with density $f(x)$ on $[a,b]$ and c.d.f $F(x)$,

let $y_1 < y_2 < \dots < y_n$ be order statistics of $X_1, \dots, X_n$.

i.e, $Y_1 = \min(X_1, \dots, X_n) = 1^{st}$ order statistic.

$Y_2 = 2^{nd}$ order statistic.

$Y_n = \max(X_1, \dots, X_n) = n^{th}$ order statistic.

$\rightsquigarrow$ The joint density of $Y_i < Y_j$ is given by.

$g_{ij}(y_i, y_j) = \begin{cases} \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} \left(F(y_i)\right)^{i-1} \left(F(y_j) - F(y_i)\right)^{j-i-1} \left(1 - F(y_j)\right)^{n-j} \cdot f(y_i) \cdot f(y_j) & , \ a < y_i < y_j < b. \\ 0 & , \ \text{else}. \end{cases}$

exp 3: $X_1, X_2, X_3$ Random sample with density $f(x) = \begin{cases} 1 & , 0<x<1 \\ 0 & , \text{elsewhere} \end{cases}$

Find the p.d.f of the Range.

$f(x) = \begin{cases} 1 & , 0<x<1 \\ 0 & , \text{else} \end{cases}$ , $F(x) = \begin{cases} 0 & , x<0 \\ x & , 0 \le x < 1 \\ 1 & , 1 \le x \end{cases}$

→ Range: $Y_3 - Y_1$

→ $Y_1 < Y_2 < Y_3$ order statistic

joint p.d.f of $Y_1, Y_3$ → $g_{13}(y_1, y_3) = \begin{cases} \dfrac{3!}{0!\,1!\,0!} (y_1)^0 (y_3 - y_1)^1 (1-y_3)^0 \cdot 1 \cdot 1 & , 0<y_1<y_3<1 \\ 0 & , \text{elsewhere} \end{cases}$

$= \begin{cases} 6(y_3 - y_1) & , 0<y_1<y_3<1 \\ 0 & , \text{elsewhere} \end{cases}$

→ Now $\begin{cases} Z_1 = Y_3 - Y_1 \\ Z_2 = Y_3 \end{cases} \Leftrightarrow \begin{cases} Y_1 = Z_2 - Z_1 \\ Y_3 = Z_2 \end{cases}$

→ $J = \begin{vmatrix} \frac{\partial y_1}{\partial z_1} & \frac{\partial y_1}{\partial z_2} \\ \frac{\partial y_3}{\partial z_1} & \frac{\partial y_3}{\partial z_2} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1$

joint density of $Z_1, Z_2$ → $h(z_1, z_2) = g(z_2 - z_1, z_2) \cdot |J|$

$= \begin{cases} 6z_1 & , 0<z_1<z_2<1 \\ 0 & , \text{else} \end{cases}$

→

p.d.f of $Z_1$

$$\rightarrow h_1(t_1) = \int_{-\infty}^{\infty} h(Z_1, Z_2)\, dZ_2 = \int_{Z_1}^{1} 6Z_1\, dZ_2$$

$$= \begin{cases} 6Z_1(1-Z_1), & 0 < Z_1 < 1 \\ 0, & \text{elsewhere} \end{cases}$$