ch.4.1 linear transformation

By Rawan Alfares

Definition And Examples

Df. A mapping L from vector space V into a vector space W is said to be linear transformation iff

1. $L(V_1+V_2) = L(V_1) + L(V_2)$, $\forall V_1, V_2 \in V$.
2. $L(\alpha V) = \alpha L(V)$, $\forall V \in V$, $\forall \alpha \in \mathbb{R}$

$$L(\alpha V_1 + \beta V_2) = \alpha L(V_1) + \beta L(V_2).$$

Notation: $L: V \longrightarrow W$

Remark:- if $V = W$ then $L: V \longrightarrow V$ is said to be linear Operator.

Example 1: let L be the operator defined by $L(x) = 3x$, for each $x \in \mathbb{R}^2$

Ans: $L: \mathbb{R}^2 \to \mathbb{R}^2$

let $V_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $V_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, $V_1, V_2 \in V$,

$$L(\alpha V_1 + \beta V_2) = L\left(\begin{bmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \end{bmatrix}\right)$$

$$= 3\begin{bmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \end{bmatrix} = \begin{bmatrix} 3\alpha x_1 \\ 3\alpha y_1 \end{bmatrix} + \begin{bmatrix} 3\beta x_2 \\ 3\beta y_2 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 3x_1 \\ 3y_1 \end{bmatrix} + \beta \begin{bmatrix} 3x_2 \\ 3y_2 \end{bmatrix}$$

$$= \alpha L(V_1) + \beta L(V_2)$$

So L is a linear transformation.

Example 2 let L be the operator defined by $L(x) = (x_1, -x_2)^T$ for each $(x_1, x_2)^T$ in $\mathbb{R}^2$.

let $v_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, v_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in V,$

(1) $L(v_1 + v_2) = L\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -y_1 - y_2 \end{bmatrix}$

$$= \begin{bmatrix} x_1 \\ -y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ -y_2 \end{bmatrix}$$

$$= L(v_1) + L(v_2) \quad \#$$

(2) $L(\alpha v_1) = L\left(\alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha x_1 \\ \alpha y_1 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 \\ -\alpha y_1 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ -y_1 \end{bmatrix}$

$$= \alpha L(v_1)$$

$\#$

So L is linear transformation.

Example 3 The mapping $L: \mathbb{R}^2 \to \mathbb{R}^1$ defined by $L(x) = x_1 + x_2$

$v_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$L(\alpha v_1 + \beta v_2) = L\left(\alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \beta \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = L\begin{pmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \end{pmatrix}$

$$= \alpha x_1 + \alpha y_1 + \beta x_2 + \beta y_2$$

$$= \alpha(x_1 + y_1) + \beta(x_2 + y_2).$$

$$= \alpha L(v_1) + \beta L(v_2)$$

$\#$ So L is a linear transformation

Example 4 the mapping L from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined By $L(x) = (x_2, x_1, x_1+x_2)^T$

(1) let $v_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in V$, $\alpha \in \mathbb{R}$

$$L\left(\alpha\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha a \\ \alpha b \\ \alpha c \end{bmatrix}\right) = \begin{bmatrix} \alpha b \\ \alpha a \\ \alpha a+\alpha b \end{bmatrix} = \alpha\begin{bmatrix} b \\ a \\ a+b \end{bmatrix} = \alpha L(v)$$

\#

(2) $$L(v_1+v_2) = L\left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \end{bmatrix}\right) = \begin{bmatrix} b_1+b_2 \\ a_1+a_2 \\ a_1+b_1+a_2+b_2 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 \\ a_1 \\ a_1+b_1 \end{bmatrix} + \begin{bmatrix} b_2 \\ a_2 \\ a_2+b_2 \end{bmatrix}$$

$$= L(v_1) + L(v_2)$$

\#

So L is linear transformation.

Example 5 let L be the mapping from $C[a,b]$ to $\mathbb{R}^1$ defined by $L(f) = \int_a^b f(x)\,dx$.

$$L(\alpha v_1 + \beta v_2) = \int_a^b (\alpha v_1 + \beta v_2)\, dv = \int_a^b \alpha v_1 dv + \int_a^b \beta v_2 dv$$

$$= \alpha \int_a^b v_1 dv + \beta \int_a^b v_2 dv$$

$$= \alpha L(v_1) + \beta L(v_2)$$

\#

So L. is L.T. \#

Example 6 let D be the linear transformation mapping $C'[a,b]$ defined by $D(f) = f'$ (the derivative of f).

$$D(\alpha f + \beta g) = (\alpha f + \beta g)' = (\alpha f)' + (\beta g)' = \alpha f' + \beta g'$$
$$= \alpha D(f) + \beta D(g)$$

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So D is a linear transformation.

Remark:

if L is a linear transformation mapping a vector space V into a vector space W, then

(i) $L(0_V) = 0_W$, where 0_V and 0_W are the zero vectors in V and W, respectivly.

(ii) if $v_1, \dots, v_n$ are elements of V and $\alpha_1, \dots, \alpha_n$ are scalors, then,
$$L(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) = \alpha_1 L(v_1) + \alpha_2 L(v_2) + \dots + \alpha_n L(v_n)$$

(iii) $L(-v) = -L(v)$

Proof (1):-

$$L(0_v) = L(0 \cdot 0_v) = 0 \cdot L(0_v) = 0 \cdot 0_w = 0_w$$

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Proof (2):-

$$L(\alpha_1 v_1 + (\alpha_2 v_2 + \dots + \alpha_n v_n)) = L(\alpha_1 v_1) + L(\alpha_2 v_2 + (\dots + \alpha_n v_n)).$$
$$= L(\alpha_1 v_1) + L(\alpha_2 v_2) + \dots + L(\alpha_n v_n)$$
$$= \alpha_1 L(v_1) + \alpha_2 L(v_2) + \dots + \alpha_n L(v_n).$$

Proof (3):-

$$L(-v) = L((-1) \cdot v) = -L(v)$$

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Remark:

* if $L: V \to W$ is a function, and we want to check if L is linear transformation or not, if $L(0_V) \neq 0_W$, then L is not linear transformation.

* but if $L(0) = 0 \to$ that doesn't mean that its linear combination.

Example: which of the following is not linear transformation.

(1) $L_1 : P_3 \to \mathbb{R}$, $L(P(x)) = 3P(1)$

(2) $L_2 : P_3 \to \mathbb{R}$, $L(P(x)) = 3P(0) + 5$

(3) $L_3 : P_3 \to \mathbb{R}$, $L(P(x)) = P(1) + P'(0)$.

(1) $L(0) = 3(0) = 0 \to$ maybe

(2) $L(0) = 3P(0) + 5 \neq 0 \to$ not L.T.

(3) $L_3(0) = 0 \to$ maybe.

Def: let $L: V \to W$ is L.T we define:

(1) Kernel of L as $\operatorname{Ker}(L) = \{ v \in V , L(v) = 0_W \}$.

(2) Image of L (Range of L) as $\operatorname{Im}(L) = \{ L(v) : v \in V \}$

$L(v) = w$

Properties:
1. for $\operatorname{Ker}(L)$ is subspace of V.
2. $\operatorname{Im}(L)$ is subspace of W.

$\operatorname{Ker}(L) \to$ كل العناصر اللي في domain اللي ال Imge تاعهم = صفر.

Theorem: let $T: V \to W$ be a linear transformation where V, W are vector spaces.
Suppose the dimension of V is n. Then

$$n = \dim(\operatorname{Ker}(T)) + \dim(\operatorname{Im}(T)).$$

Ex. $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

① Find a basis and dim (Ker(L)).

② Find a basis and dim (Im(L)).

① $\text{Ker}(L) = \{ v \in \mathbb{R}^2 : L(v) = 0_{\mathbb{R}^3} \}$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 , \; L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{array}{l} x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \\ x_1 + x_2 = 0 \end{array}$$

$$\boxed{x_1 = 0} , \boxed{x_2 = 0}$$

Sol. $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

basis for x $\Rightarrow$ Has no Basis.

dim (Ker (L))

هاي دائماً لما يكون الفيكتور صفر

- ما في إله basis

- الدايمنشن إله 0.

② $\text{Im}(L) = \left\{ L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}$

$$= \left\{ \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 + x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$

$$= \left\{ x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} , \; x_1, x_2 \in \mathbb{R} \right\}$$

sp. set for Im(L) is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

linearly Independent.

bases for Im(L) $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\therefore$ dim (Im(L)) = 2.

Example: let $L: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$L(x) = (x_1 + x_2,\ x_2 + x_3)^T$, $\mathbb{R}^3$ spanned by e_1 and e_3

Find ① Ker (L) :- $= \{ v \in \mathbb{R}^3 ,\ L(v) = 0_{\mathbb{R}^2} \}$

$$= \left\{ L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0$$
$$x_2 + x_3 = 0$$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \longrightarrow \begin{array}{l} x_3 = \alpha \\ x_2 = -\alpha \\ x_1 = \alpha \end{array}$$

$$x = \begin{bmatrix} \alpha \\ -\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad , \text{ basis } = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

dim = 1

② Im (L(s)) :-

$$\mathrm{Im}(L) = \left\{ L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \right\}$$

$$= \left\{ \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix} ;\ x_1, x_2, x_3 \in \mathbb{R} \right\}$$

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

span $\mathrm{Im}(L) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, they are not linear Independent.

so delet $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ basis for Im (L).

dim (Im(L)) = 2.

Ex. $L: P_2 \to \mathbb{R}^2$, $L(P(x)) = \begin{pmatrix} \int_0^1 P(x)\,dx \\ P(1) \end{pmatrix}$

$* \; L(2x+3) = \begin{pmatrix} \int_0^1 2x+3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$L(ax+b) = \begin{pmatrix} \int_0^1 ax+b \; dx \\ a+b \end{pmatrix} = \begin{pmatrix} \frac{a}{2}+b \\ a+b \end{pmatrix}$

① find basis and dim (Ker(L))
② // // // // (Im(L)).

① $Ker(L) = \{ V \in P_2 \, , \, L(V) = 0_{\mathbb{R}^2} \}$

$= \begin{bmatrix} \frac{a}{2}+b \\ a+b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= \frac{a}{2}+b = 0 \quad , \quad a+b=0$

$a = -b$

$\frac{-b}{2} + \frac{2b}{2} = 0 \rightarrow \boxed{b=0} \; , \; \boxed{a=0}$

Sol. $X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, basis (Ker(L)) = Has no basis
dim (Ker (L)) = 0.

② $Im(L) = \{ L(ax+b) : ax+b \in P_2 \}$

$\begin{bmatrix} \frac{a}{2}+b \\ a+b \end{bmatrix} = a \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ span Im(L).

and they are Independent. so they are basis.

dim (Im(L)) = 2.

Ex $L: P_3 \to P_3$, $L(P_3(x)) = p'(x)$

Find a basis and dim for Ker(L) and Im(L).

$\text{Ker}(L) = \{ L(P_3) = 0_{P_3} \}$

$P_3 = ax^2 + bx + c$

$L(ax^2+bx+c) = 0$

$2ax + b = 0 \to \boxed{b=0}$

$2a=0 \to \boxed{a=0}$ بعوض

$0+0+c = c = c(1)$

span set for ker(L) = (1) indep. basis = {1}

dim = (1).

$\text{Im}(L) = \{ L(ax^2+bx+c) , a,b,c \in \mathbb{R} \}$

$= 2ax + b$

$= a(2x) + b(1) \to LI$

basis = {2x, 1}

Example:- Is $L: \mathbb{R}^3 \to \mathbb{R}^2$, $L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}$ a L.T ?

$$L\left(\alpha_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \alpha_2 \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$$

$$= L\left(\begin{bmatrix} \alpha_1 x_1 + \alpha_2 y_1 \\ \alpha_1 x_2 + \alpha_2 y_2 \\ \alpha_1 x_3 + \alpha_2 y_3 \end{bmatrix}\right) = \begin{bmatrix} \alpha_1 x_1 + \alpha_2 y_1 + \alpha_1 x_2 + \alpha_2 y_2 \\ \alpha_1 x_2 + \alpha_2 y_2 + \alpha_1 x_3 + \alpha_2 y_3 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 x_1 + \alpha_1 x_2 \\ \alpha_1 x_2 + \alpha_1 x_3 \end{bmatrix} + \begin{bmatrix} \alpha_2 y_1 + \alpha_2 y_2 \\ \alpha_2 y_2 + \alpha_2 y_3 \end{bmatrix}$$

$$= \alpha_1 \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix} + \alpha_2 \begin{bmatrix} y_1 + y_2 \\ y_2 + y_3 \end{bmatrix}$$

$$= \alpha_1 L(v_1) + \alpha_2 L(v_2) \quad \#$$

So yes it's Linear T.

Example:- $L: \mathbb{R}^2 \to \mathbb{R}^3$, $L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 + x_2 + 5 \end{bmatrix}$, is L.T ?

$$L\left(\alpha_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \alpha_2 \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = L\left(\begin{matrix} \alpha_1 x_1 + \alpha_2 y_1 \\ \alpha_1 x_2 + \alpha_2 y_2 \end{matrix}\right) = \begin{bmatrix} \alpha_1 x_1 + \alpha_2 y_1 + \alpha_1 x_2 + \alpha_2 y_2 \\ \alpha_1 x_1 + \alpha_2 y_1 - \alpha_1 x_2 - \alpha_2 y_2 \\ \alpha_1 x_1 + \alpha_2 y_1 + \alpha_1 x_2 + \alpha_2 y_2 + 5 \end{bmatrix}$$

$$\alpha_1 L(v_1) + \alpha_2 L(v_2) = \alpha_1 \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 + x_2 + 5 \end{bmatrix} + \alpha_2 \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \\ y_1 + y_2 + 5 \end{bmatrix}$$

$$\neq$$

$$= \begin{bmatrix} \alpha_1 x_1 + \alpha_1 x_2 + \alpha_2 y_1 + \alpha_2 y_2 \\ \alpha_1 x_1 - \alpha_1 x_2 + \alpha_2 y_1 - \alpha_2 y_2 \\ \alpha_1 x_1 + \alpha_1 x_2 + 5\alpha_1 + \alpha_2 y_1 + \alpha_2 y_2 + 5\alpha_2 \end{bmatrix}$$

So, it's not Linear transformation #

Example :- $L: P_2 \to \mathbb{R}^2$, $L(P(x)) = \begin{pmatrix} P(0) \\ P(1) \end{pmatrix}$

$$L(\alpha_1 p(x) + \alpha_2 g(x)) = \begin{bmatrix} \alpha_1 P(0) + \alpha_2 g(0) \\ \alpha_1 P(1) + \alpha_2 P(1) \end{bmatrix} = \alpha_1 \begin{bmatrix} P(0) \\ P(1) \end{bmatrix} + \alpha_2 \begin{bmatrix} g(0) \\ g(1) \end{bmatrix}$$

$$= \alpha_1 L(P(x)) + \alpha_2 L(g(x)).$$

So it's L.T #

Example :- if $A = (a_{ij})_{2\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & -5 \end{bmatrix}$

let $L_A : \mathbb{R}^{?3} \longrightarrow \mathbb{R}^{?2}$ is L L.T ?

$L(x) = \underbrace{\underset{2\times3}{A}\ \underset{3\times1}{x}}_{2\times1}$

check :- $L(\alpha_1 V_1 + \alpha_2 V_2) = A(\alpha_1 V_1 + \alpha_2 V_2)$
$= \alpha_1 A V_1 + \alpha_2 A V_2$
$= \alpha_1 L(V_1) + \alpha_2 L(V_2)$ #

so its LT.

رَبَّنَا تَقَبَّلْ مِنَّا ۖ إِنَّكَ أَنتَ السَّمِيعُ الْعَلِيمُ