

Calculus with Parametric Curves

If you have $x = f(t)$, $y = g(t)$ $t \in I$

Then the **derivative** will be:-

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

← نشتق بالمتغير t

The **second derivative** is:-

$$y'' = \frac{d y'}{dx} = \frac{d y' / dt}{dx / dt}$$

Length of Curve :- (Parametric)

• Reminder

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$x = f(t)$$

$$y = g(t)$$

$$a \leq t \leq b$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

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How to Solve Problems (Examples)

• finding first & second derivatives

If $x = t - t^2$ and $y = t - t^3$:-

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$y'' = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{(1-2t)(-6t) - (1-3t^2)(-2)}{(1-2t)^2}$$

$$= \frac{6t^2 - 6t + 2}{(1-2t)^3}$$

• finding Tangent (obtaining $\frac{dy}{dx}$)

Ex: $x^3 + 2t^2 = 9$

$$2y^3 - 3t^2 = 4$$

at $t=2$

differentiate x and y to t :-

$$3x^2 \frac{dx}{dt} + 4t = 0$$

$$6y^2 \frac{dy}{dt} - 6t = 0$$

$$\left. \frac{dx}{dt} \right|_{t=2} = \frac{-4t}{3x^2}$$

$$\left. \frac{dy}{dt} \right|_{t=2} = \frac{6t}{6y^2} = \frac{t}{y^2}$$

at $t=2$:-

$$x^3 + 2t^2 = 9$$

$$x^3 + 8 = 9 \rightarrow \boxed{x=1}$$

$$\text{and } 2y^3 - 3t^2 = 4$$

$$2y^3 = 16 \Rightarrow \boxed{y=2}$$

Now :-

$$\left. \frac{dx}{dt} \right|_{\substack{t=2 \\ x=1}} = -\frac{4x^2}{3} = -\frac{8}{3} \quad \text{and} \quad \left. \frac{dy}{dt} \right|_{\substack{t=2 \\ y=2}} = \frac{2}{4} = \frac{1}{2}$$

→ Knowing that :- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\left. \frac{dy}{dx} \right|_{t=2} = \frac{\frac{1}{2}}{-\frac{8}{3}} = -\frac{3}{16}$$

• Area → Find Area under $y = x^3$ over $[0, 1]$

Ex :- $x = t^2 \quad y = t^6 \quad 0 \leq t \leq 1$ By substituting

$$\begin{aligned} A &= \int_0^1 y \, dx = \int_0^1 t^6 (2t \, dt) \\ &= \int_0^1 2t^7 \, dt = \left[\frac{2t^8}{8} \right]_0^1 \\ &= \frac{2}{8} = \frac{1}{4} \end{aligned}$$

• length of Curves

$x = \cos t$ and $y = \sin t + t \quad 0 \leq t \leq \pi$

find L :-

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t + 1)^2} \, dt \\ &= \int_0^\pi \sqrt{\sin^2 t + \cos^2 t + 1 + 2 \cos t} \, dt \\ &\quad \text{Alaa Elbawi} \end{aligned}$$

$$= \sqrt{2} \int_0^{\pi} \sqrt{1 + \cos \theta} \, d\theta$$

$$\cos \theta = 2 \cos^2 \frac{\theta}{2} - 1$$

$$= \sqrt{2} \int_0^{\pi} \sqrt{2 \cos^2 \frac{\theta}{2}} \, d\theta$$

$$= 2 \int_0^{\pi} |\cos \frac{\theta}{2}| \, d\theta$$

$$0 \leq \cos \frac{\theta}{2} \quad \text{for } 0 \leq \theta \leq \pi$$

$$= 2 \int_0^{\pi} \cos \frac{\theta}{2} \, d\theta$$

$$= 4 \sin \frac{\theta}{2} \Big|_0^{\pi} = 4 \sin \frac{\pi}{2} - 0 = 4$$

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