8.1 Integration By Parts:

If u and v are diff function of x, then

$$\int u \, dv = u \cdot v - \int v \, du$$

Evaluate the following integral :-

1 $\int x e^{3x} \, dx$

$u = x$ $\quad dv = e^{3x} \, dx$

$du = dx$ $\quad v = \frac{e^{3x}}{3}$

$$\int x e^{3x} dx = \frac{x e^{3x}}{3} - \int \frac{e^{3x}}{3} dx$$

$$= \frac{x e^{3x}}{3} - \frac{e^{3x}}{9} + C$$

2 $\int x^7 \ln x \, dx$

$u = \ln x$ $\quad dv = x^7 \, dx$

$du = \frac{1}{x} dx$ $\quad v = \frac{x^8}{8}$

$$\int x^7 \ln x \, dx = \frac{x^8}{8} \ln x - \int \frac{x^7}{8} dx$$

$$= \frac{x^8}{8} \ln x - \frac{1}{8} \frac{x^8}{8} + C.$$

$\boxed{3}$ $\int \ln x \, dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx$$

$$\int \ln x \, dx = x \ln x - x + C$$

$\boxed{4}$ $$\int \frac{\ln x}{\sqrt{x}} dx = \int \ln x \, (x)^{-1/2} \, dx$$

$u = \ln x \qquad dv = x^{-1/2} dx$

$du = \frac{1}{x} dx \qquad v = 2x^{1/2}$

$$= 2x^{1/2} \ln x - \int 2x^{1/2} \cdot \frac{1}{x} dx$$

$$= 2x^{1/2} \ln x - 4x^{1/2} + C$$

$\boxed{5}$ $\int 3x^2 \tan^{-1} x \, dx$

$u = \tan^{-1} x \qquad dv = 3x^2 dx$

$du = \frac{1}{1+x^2} dx \qquad v = \frac{3x^3}{3}$

$$\int 3x^2 \tan^{-1} x \, dx = x^3 \tan^{-1} x - \int \frac{x^3}{1+x^2} dx$$

$$\begin{array}{r} x \\ 1+x^2 \overline{) \; x^3 \quad} \\ x^3 + x \\ \hline -x \end{array}$$

$$= x^3 \tan^{-1} x - \int x - \frac{1}{2} \frac{2x}{1+x^2} dx$$

$$= x^3 \tan^{-1} x - \frac{x^2}{2} + \frac{1}{2} \ln|1+x^2| + C$$

$\boxed{6}$ $\int \sin^{-1} x \, dx$

$u = \sin^{-1} x$ $\qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} dx$ $\qquad v = x$

$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$

$u = 1 - x^2$
$du = -2x \, dx$

$= x \sin^{-1} x + \frac{1}{2} \int \frac{1}{\sqrt{u}} du$

$= x \sin^{-1} x + \frac{2}{2} u^{1/2} + C$

$= x \sin^{-1} x + \sqrt{1-x^2} + C$

$\boxed{7}$ $\int x^2 \cosh 2x \, dx$

$u = x^2$ $\qquad dv = \cosh 2x \, dx$

$du = 2x \, dx$ $\qquad v = \frac{\sinh(2x)}{2}$

$\int x^2 \cosh 2x \, dx = \frac{x^2 \sinh(2x)}{2} - \int x \sinh(2x) \, dx$

$u = x$ $\qquad dv = \sinh(2x) \, dx$

$du = dx$ $\qquad v = \frac{\cosh(2x)}{2}$

$= \frac{x^2 \sinh(2x)}{2} - \frac{x \cosh(2x)}{2} + \int \frac{\cosh(2x)}{2} dx$

$= \frac{1}{2} x^2 \sinh(2x) - \frac{1}{2} x \cosh(2x) + \frac{1}{4} \sinh(2x) + C$

Remark:

If n is positive integer. We use the tabular integration in the following cases:-

$$\int x^n \cosh(ax)\, dx \qquad \int x^n \sinh(ax)\, dx$$

$$\int x^n \cos(ax)\, dx \qquad \int x^n \sin(ax)\, dx$$

$$\int x^n e^{ax}\, dx \qquad \int x^n a^x\, dx$$

Exp:- $\boxed{7}$ $\int \cosh(2x)\ x^2\ dx$

Derivative		Integral
		$\cosh(2x)$
x^2	$+$	$\frac{\sinh(2x)}{2}$
$2x$	$-$	$\frac{\cosh(2x)}{4}$
2	$+$	$\frac{\sinh(2x)}{8}$
0		

$$\int x^2 \cosh(2x)\, dx = \frac{1}{2}x^2\sinh(2x) - \frac{1}{2}x\cosh(2x) + \frac{1}{4}\sinh(2x) + C$$

8 $\int x^2 3^x \, dx$

Derivative	Integral
x^2	3^x
$2x$	$\frac{3^x}{\ln 3}$
2	$\frac{3^x}{(\ln 3)^2}$
0	$\frac{3^x}{(\ln 3)^3}$

$+$, $-$, $+$

$$\int x^2 3^x \, dx = \frac{3^x x^2}{\ln 3} - \frac{2x\, 3^x}{(\ln 3)^2} + \frac{2 \cdot 3^x}{(\ln 3)^3}$$

Remark

Some times we use the Substitution and then we use integration by Parts.

9 $\int x (\ln x)^2 \, dx$

$u = \ln x$

$du = \frac{1}{x} dx$

$$\int x^2 u^2 \, du = \int e^{2u} u^2 \, du$$

Derivative	Integral
u^2	e^{2u}
$2u$	$\frac{e^{2u}}{2}$
2	$\frac{e^{2u}}{4}$
0	$\frac{e^{2u}}{8}$

$+$, $-$, $+$

$$= \frac{1}{2} u^2 e^{2u} - \frac{1}{2} u e^{2u} + \frac{1}{4} e^{2u} + C$$

$$= \frac{1}{2} x^2 (\ln x)^2 - \frac{1}{2} x^2 \ln x + \frac{1}{4} x^2 + C.$$

$\boxed{10}$ $\int \cos\left(\sqrt[3]{x}\right) dx$

$y = \sqrt[3]{x}$

$dy = \frac{1}{3} x^{-2/3} dx$

$3 \int \cos(y)\, y^2\, dy$

$3 \int y^2 \cos(y)\, dy$

Derivative		Integral
y^2	$+$	$\cos(y)$
$2y$	$-$	$\sin(y)$
2	$+$	$-\cos(y)$
0		$-\sin(y)$

$= 3y^2 \sin(y) + 6y\cos(y) - 6\sin(y)$

$= 3x^{2/3} \sin\left(x^{1/3}\right) + 6x^{1/3} \cos\left(x^{1/3}\right) - 6\sin\left(x^{1/3}\right)$

$\boxed{11}$ $\int \left(\sin^{-1} x\right)^2 dx$

$y = \sin^{-1} x$

$\sin(y) = x$

$\cos(y)\, dy = dx$

$\int y^2 \cos(y)\, dy$

Derivative		Integral
y^2	$+$	$\cos(y)$
$2y$	$-$	$\sin(y)$
2	$+$	$-\cos(y)$
0		$-\sin(y)$

$= y^2 \sin y + 2y\cos y - 2\sin y + C$

$= \left(\sin^{-1} x\right)^2 x + 2\sin^{-1} x \cos\left(\sin^{-1} x\right) - 2x + C$

Outline Exercises $\boxed{11}$

$$\int \tan^{-1} y \, dy$$

$u = \tan^{-1}(y)$ $\quad$ $dv = dy$

$du = \dfrac{1}{1+y^2}\, dy$ $\quad$ $v = y$

$$\int \tan^{-1} y \, dy = y \tan^{-1}(y) - \frac{1}{2}\int \frac{2y}{1+y^2}\, dy$$

$$= y \tan^{-1}(y) - \frac{1}{2} \ln|1+y^2| + C.$$

Outline Exercises $\boxed{25}$

$$\int e^{\sqrt{3s+9}}\, ds$$

$u = \sqrt{3s+9}$

$du = \dfrac{3}{2\sqrt{3s+9}}\, ds$

$$\frac{2}{3}\int e^{u}\, u \, du$$

Derivative		Integral
u	$+$	e^{u}
1	$-$	e^{u}
0		e^{u}

$$\frac{2}{3}\left[u e^{u} - e^{u}\right] + C$$

$$\frac{2}{3}\sqrt{3s+9}\; e^{\sqrt{3s+9}} - \frac{2}{3} e^{\sqrt{3s+9}} + C$$