1.1 System of Linear Equations

* A linear equation with n unknowns has the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where the coefficients $a_1, a_2, \dots, a_n$ are real numbers and $x_1, x_2, \dots, x_n$ are the variables.

* A linear system of m equations and n unknowns is denoted by $m \times n$ linear system given by:

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m \end{aligned}$$

a_{ij}'s $\in \mathbb{R}$

b_i's $\in \mathbb{R}$

$Ax = b$

Exp [a] $x_1 + 2x_2 = 5$
$2x_1 + 3x_2 = 8$

- is 2×2 linear system
- The solution set is $\{(1,2)\}$

Thus, the system is consistent "has at least one solution"

[b] $x_1 - x_2 + x_3 = 2$
$2x_1 + x_2 - x_3 = 4$

- 2×3 linear system
- The solution set is $\{(2, \alpha, \alpha) : \alpha \in \mathbb{R}\}$ "Infinitly many solution"

Thus, the system is consistent.

[c] $x_1 + x_2 = 2$
$x_1 - x_2 = 1$
$x_1 = 4$

- is 3×2 linear system
- The solution set is $\emptyset$ "The system has no solution".

Thus, the system is inconsistent.

Notes

* If a linear system has no solution, then the system is inconsistent.
* If a linear system has at least one solution, then the system is consistent.
* The set of solutions of a linear system is called the solution set.

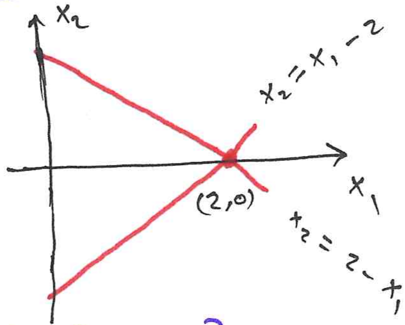
2x2 Systems

$$a_{11} x_1 + a_{12} x_2 = b_1$$
$$a_{21} x_1 + a_{22} x_2 = b_2$$

· Each equation is a line in the $x_1 x_2$-plane.
· The pair (x_1, x_2) will be a solution of this system iff (x_1, x_2) lies on both lines.

Exp $\boxed{a}$ $x_1 + x_2 = 2$

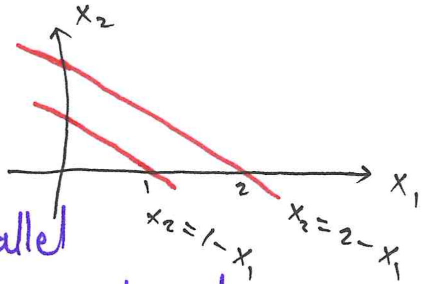
$x_1 - x_2 = 2$



* The solution set is $\{(2,0)\}$
* The two lines intersect at the point $(2,0)$
* The system is consistent.

$\boxed{b}$ $x_1 + x_2 = 2$

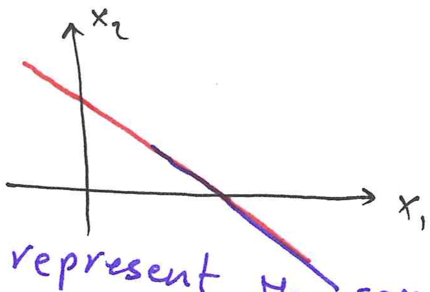
$x_1 + x_2 = 1$



* The two lines are parallel
* The solution set is the empty set.
* The system is inconsistent.

$\boxed{c}$ $x_1 + x_2 = 2$

$-x_1 - x_2 = -2$



- The two equations represent the same line.
- Any point on this line will be a solution.
- The solution set has infinitly many solutions.
- The system is consistence.

Equivalent Systems

Def Two systems of equations involving the same variables are equivalent if they have the same solution set.

Exp Consider the two systems

$\boxed{a}$

$$3x_1 + 2x_2 - x_3 = -2$$
$$x_2 = 3$$
$$2x_3 = 4$$

$\boxed{b}$

$$3x_1 + 2x_2 - x_3 = -2$$
$$-3x_1 - x_2 + x_3 = 5$$
$$3x_1 + 2x_2 + x_3 = 2$$

system $\boxed{a}$ $x_3 = 2$, $x_2 = 3$ $\Rightarrow 3x_1 + 2(3) - (2) = -2$

$\Rightarrow x_1 = -2$

The solution set is $\{(-2,3,2)\}$

system $\boxed{b}$ $R1 + R2 \Rightarrow x_2 = 3$

$R_1 - R_3 \Rightarrow -2x_3 = -4 \Rightarrow x_3 = 2$

$3x_1 + 2x_2 - x_3 = -2 \Leftrightarrow 3x_1 + 6 - 2 = -2 \Leftrightarrow x_1 = -2$

The solution set is $\{(-2,3,2)\}$

Hence, system $\boxed{a}$ and system $\boxed{b}$ are equivalent.

* There are $\underline{\underline{3}}$ operations that can be used on a system to obtain an equivalent system:

$\boxed{1}$ The order in which any two equations are written may be interchanged.

$\underline{\underline{\text{Exp}}}$

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 - x_2 &= 2 \\ 4x_1 + x_2 &= 6 \end{aligned} \quad \text{and} \quad \begin{aligned} 4x_1 + x_2 &= 6 \\ 3x_1 - x_2 &= 2 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

are equivalent.

$\boxed{2}$ Both sides of an equation may be multiplied by the same nonzero real number.

$\underline{\underline{\text{Exp}}}$

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ -2x_1 - x_2 + 4x_3 &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 6 \\ -2x_1 - x_2 + 4x_3 &= 1 \end{aligned}$$

are equivalent

$\boxed{3}$ A multiple of one equation may be added to (or subtracted from) another.

$\underline{\underline{\text{Exp}}}$

$$\begin{aligned} x_1 + x_2 &= 4 \\ 2x_1 - 3x_2 &= 3 \end{aligned} \quad \text{and} \quad \begin{aligned} x_1 + x_2 &= 4 \\ 4x_1 - x_2 &= 11 \end{aligned}$$

are equivalent. Because $2(x_1 + x_2 = 4)$ is

$$\begin{aligned} 2x_1 + 2x_2 &= 8 \\ 2x_1 - 3x_2 &= 3 \quad + \\ \hline 4x_1 - x_2 &= 11 \end{aligned}$$

$\{(3,1)\}$ is the solution set.

See the green page 6 "easier"

n x n Systems

Def A system is said to be in strict triangular form if in the k^{th} equation, the coefficients of the first $k-1$ variables are all zero and the coefficient of x_k is nonzero, where $k = 1, \dots, n$.

Exp The system

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 4 \end{aligned}$$

is in strict triangular form.

Because in 2^{nd} equation, the coefficients are $0, 1, -1$
in 3^{rd} equation, the coefficients are $0, 0, 2$

* If the system is in strict triangular form, then it is easy to solve by back substitution. That is,

$x_3 = 2 \Rightarrow x_2 - (2) = 2 \Rightarrow x_2 = 4 \Rightarrow 3x_1 + 2(4) + (2) = 1$
$\Rightarrow x_1 = -3$

$\Rightarrow$ The solution set is $\{(-3, 4, 2)\}$

Exp Solve the system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= 5 \\ 3x_2 + x_3 - 2x_4 &= 1 \\ -x_3 + 2x_4 &= -1 \\ 4x_4 &= 4 \end{aligned}$$

Using back substitution $\Rightarrow$

$$\begin{aligned} 4x_4 = 4 &\Rightarrow x_4 = 1 \\ -x_3 + 2(1) = -1 &\Rightarrow x_3 = 3 \\ 3x_2 + (3) - 2(1) = 1 &\Rightarrow x_2 = 0 \\ x_1 + 0 + 6 + 1 = 5 &\Rightarrow x_1 = -2 \end{aligned}$$

The solution is $(-2, 0, 3, 1)$

Exp Solve the system
$$x_1 + 2x_2 + x_3 = 3$$
$$3x_1 - x_2 - 3x_3 = -1$$
$$2x_1 + 3x_2 + x_3 = 4$$

- We can write the system in the form $\underset{3\times3}{A}\,\underset{3\times1}{X} = \underset{3\times1}{B}$

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}}_{\text{The Coefficient Matrix}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{variable Matrix}} = \underbrace{\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}}_{\text{constant Matrix}}$$

- The augmented matrix has the form $A|B$:

pivot $a_{11}=1$

entries to be eliminated $a_{21}=3$, $a_{31}=2$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right] \rightarrow \text{pivotal row}$$

$3R_1 - R_2$ $\quad \frac{3}{1} = 3$

$2R_1 - R_3$ $\quad \frac{2}{1} = 2$

quotients

- Now we apply the Elementary Row Operation on the augmented matrix to obtain an equivelent system:
  - I. Interchange two rows
  - II. Multiply a row by a non zero real number
  - III. Replace a row by its sum with a multiple of another row.

- $$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 7 & 6 & 10 \\ 0 & 1 & 1 & 2 \end{array}\right] \rightarrow \text{pivotal row}$$

$R_2 - 7R_3$

pivot $= 7$

quotient $= -\frac{1}{7}$

- $$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 7 & 6 & 10 \\ 0 & 0 & -1 & -4 \end{array}\right]$$

Applying back substitution to this eq. triangular system $x_3 = 4$

$7x_2 + 6(4) = 10 \Rightarrow x_2 = -2$

$x_1 + 2(-2) + (4) = 3 \Rightarrow x_1 = 3$

Exp Solve this system:

$$\begin{aligned} x_2 + x_3 - x_4 &= -3 \\ x_1 + x_2 + x_3 + x_4 &= -1 \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= -5 \\ 3x_1 + x_2 - 2x_3 + 2x_4 &= 3 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & -3 \\ 1 & 1 & 1 & 1 & -1 \\ 2 & 4 & 1 & -2 & -5 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right]$$

- The pivot $= 0$, so it is not possible to eliminate any entries. So we interchange the first two rows.
- If the first column is zeros, then this method fails.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & -3 \\ 2 & 4 & 1 & -2 & -5 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_1 \\ R_4 - 3R_1 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & -3 \\ 0 & 2 & -1 & -4 & -3 \\ 0 & -2 & -5 & -1 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 + 2R_2 \end{array} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & -3 \\ 0 & 0 & -3 & -2 & 3 \\ 0 & 0 & -3 & -3 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - R_3 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & -3 \\ 0 & 0 & -3 & -2 & 3 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right]$$

$$\boxed{x_4 = 3} \qquad -3x_3 - \boxed{6} = 3$$

$$\boxed{x_3 = -3} \qquad x_2 - \boxed{3} - \boxed{3} = -3$$

$$\boxed{x_2 = 3} \qquad x_1 + \boxed{3} - \boxed{3} + \boxed{3} = -1$$

$$\boxed{x_1 = -4}$$

The solution set is $\{(-4, 3, -3, 3)\}$

7 old

Exp Solve the system

$$\begin{aligned} x_2 + x_3 - x_4 &= 4 \\ x_1 + x_2 + x_3 + x_4 &= 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 &= 3 \end{aligned}$$

- The augmented matrix is

if the first column is 0 then, this method fails *

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right]$$

The pivot = 0, so it is not possible to eliminate any entries. We then interchange the first two rows:

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -1 & 4 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] \xrightarrow[R_4 - 3R_1]{R_3 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & \boxed{1} & 1 & -1 & 4 \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{array}\right]$$

$$\xrightarrow[R_4 + 2R_2]{R_3 - 2R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -1 & 4 \\ 0 & 0 & \boxed{-3} & -2 & -21 \\ 0 & 0 & -3 & -3 & -7 \end{array}\right] \xrightarrow{R_4 - R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -1 & 4 \\ 0 & 0 & -3 & -2 & -21 \\ 0 & 0 & 0 & -1 & 14 \end{array}\right]$$

- Apply back substitution to this eq. triangular system:

$x_4 = -14 \Rightarrow -3x_3 - 2(-14) = -21 \Rightarrow x_3 = \frac{49}{3}$

$\Rightarrow x_2 + \frac{49}{3} + 14 = 4 \Rightarrow x_2 = -\frac{79}{3}$

$\Rightarrow x_1 - \frac{79}{3} + \frac{49}{3} - 14 = 6 \Rightarrow x_1 = 30$

The solution is $\left(30, -\frac{79}{3}, \frac{49}{3}, -14\right)$