Exercises 1.3 :

Q.41 write a nontrivial negation of the following :-

a) All cows eat grass ⇒ Some cows does not eat grass.

b) There is a horse that does not eat grass ⇒ All horses eat grass.

c) There is a car that is blue and weighs less than 4000 pounds

⇒ All cars are not blue or weighs greater than or equal 4000 pounds.

d) Every math book is either blue or hard to read.

⇒ Some math book is not blue and easy to read.

e) Some cows are spotted ⇒ All cows are not spotted.

f) No car has 15 cylinders ⇒ ~~All cars has 15 cylinders.~~
"There is a car with 15 cylinders"

g) Some cars are old but are still in good running conditions:

=> All cars are not old or are not still in good ....

[Q. 42] An integer x has property P provided that for all integers a and b, whenever x divides ab, x divides a or x divides b. Explain what it means to say that x does not have property P :-

" for some integers a and b, for some x divides ab, x not divides a and x not divides b "

(Q.44) write a useful denial of the statement "For each pair of real numbers a and b with $a<b$, there is a rational number r such that $a<r<b$": -

"For some pair of real numbers a and b with $a<b$, All numbers are rational number r such that $r\leq a$ or $r\geq b$"

(Q.47) A sequence $\{X_n\}$ is a Cauchy sequence provided that for each $\varepsilon>0$, there is a natural number N such that if $m,n>N$ then $|X_n-X_m|<\varepsilon$. write what it means to say that $\{X_n\}$ is not a Cauchy sequence: -

" $\forall\varepsilon>0,\ \exists N\in\mathbb{N}:\forall(m,n>N\longrightarrow |X_n-X_m|<\varepsilon)$ "

$\Rightarrow$ " $\exists\varepsilon>0\ \forall N\in\mathbb{N}:\exists\ m,n>N\wedge|X_n-X_m|\geq\varepsilon$ "

" for some $\varepsilon>0$, for All natural number N such that there are $m,n>N$ and $|X_n-X_m|\geq\varepsilon$

1.1(b);
$\neg(P \rightarrow Q)$ and

(Q.48) write the negation and the contrapositive of the following statment, "if x is a positive number, then there is an $\varepsilon > 0$ such that $x < \varepsilon$ and $\frac{1}{\varepsilon} < x$"

* The negation : "x is a positive number and for every $\varepsilon > 0$ ~~such that~~ $x \geq \varepsilon$ or $\frac{1}{\varepsilon} \geq x$"

* The contrapositive : "if every $\varepsilon > 0$, $x \geq \varepsilon$ or $\frac{1}{\varepsilon} \geq x$ then x is a negative number".

(Q.54) Both of the following statments have the set of positive real numbers as their set of meanings, which statment is true ?

a) $(\forall x)(\exists y)(x < y^2)$ $\Rightarrow$ True.

b) $(\exists y)(\forall x)(x < y^2)$ $\Rightarrow$ False.

Exercises 1.4 :⇒

Q.57 Prove that the sum of two even integers is even :-

using Direct method ⇒ Suppose a, b are even integers

$\Rightarrow \exists k_1, k_2 \in \mathbb{Z}$; $a = 2k_1$ and $b = 2k_2$

$\Rightarrow$ Sum $= a + b = 2k_1 + 2k_2$

$\Rightarrow a + b = 2(k_1 + k_2) = 2k'$ where $k' = k_1 + k_2 \in \mathbb{Z}$

$\Rightarrow a+b$ is even

Q.58 Let A, B and C be integers. Prove that if A divides B and B divides C, then A divides C :⇒

using Direct method ⇒ Let A, B and C be integers, suppose A|B and B|C

$\Rightarrow \exists x \in \mathbb{Z}$; $Ax = B$ and

$\Rightarrow \exists y \in \mathbb{Z}$; $By = C \Rightarrow B = \frac{C}{y}$

$\Rightarrow Ax = B \Rightarrow Ax = \frac{C}{y} \Rightarrow Axy = C$

$\Rightarrow \exists z \in \mathbb{Z}$; $z = xy$

$\Rightarrow Az = C$, where $z \ni \mathbb{Z}$

$\Rightarrow$ A divides C

Q.61) Prove that no integer is both even and odd :⇒

using Contradiction :⇒

Assume not ⇒ There exists an integer n that is both even and odd

⇒ Since n is odd $n = 2k_1 + 1$ for an integer k_1.

⇒ Since n is even $n = 2k_2$ for an integer k_2

so $2k_1 + 1 = 2k_2$. Subtracting, $2(k_2 - k_1) = 1$

that is $k_2 - k_1 = 1/2$ but this is impossible for integers k_1, k_2. # Contradiction

Q.66) let x be a real number. Prove that $x = -1$ if and only if $x^3 + x^2 + x + 1 = 0$:⇒

we must prove two things :⇒

① if $x = -1$ then $x^3 + x^2 + x + 1 = 0$ and

② if $x^3 + x^2 + x + 1 = 0$ then $x = -1$

* First suppose $x = -1$. Then
$x^3 + x^2 + x + 1 = (-1)^3 + (-1)^2 + (-1) + 1 = 0$
$\Rightarrow$ This proves the first part of the theorem

* Now prove the seconed part. Assume that $x^3 + x^2 + x + 1 = 0$
Since $x^3 + x^2 + x + 1 = (x+1)(x^2+1) = 0$
either $x + 1 = 0$ or $x^2 + 1 = 0$, if $x^2 + 1 = 0$
the quadratic formula yields $x = \pm i$, this contradicts
the hypothesis that x is real. Therefore $x^2 + 1 \neq 0$
Hence $x + 1 = 0$ and thus, $x = -1$.

$\Longrightarrow$

$\boxed{Q.71}$ Prove that for any natural number n, either n is a prime or a perfect square, or n divides $(n-1)!$

* Suppose n is neither prime nor a perfect square then we wish to show that n divides $(n-1)!$
* since n is not prime, there exist two natural number $p, q > 1$ so that $n = p \cdot q$
* since n is not a perfect square, $p \neq q$ and thus p and q are distinct divisor of $(n-1)!$

$\Rightarrow$ Hence, their product n divides $(n-1)!$

Exercises 1.5 :

Q.75 Prove by contradiction that the sum of two even integers is even :

Assume not : The sum of two even integers is odd

$\Rightarrow$ Suppose a, b are even integers

$\Rightarrow a = 2k_1 \ , \ b = 2k_2$ where $k_1, k_2 \in \mathbb{Z}$

$\Rightarrow a + b = 2k_1 + 2k_2 = 2(k_1 + k_2)$

$\Rightarrow a + b = 2k'$ where $k' = (k_1 + k_2)$, $k' \in \mathbb{Z}$

$\Rightarrow a + b$ is even , a contradiction $\neq$

Q.76 Either prove or ~~a~~ give a counterexample to the Converse of "if x and y are even, then xy is even":?

* Converse : "if xy is even then x and y are even"

* if xy is even, this only implies that x or y is even or both, so there are many counterexamples :-

take $x = 2, y = 3, xy = 6$

$x = 3, y = 8, xy = 24$

$x = 2, y = 4, xy = 8$

Q.18 Prove that if x is a positive real number, then $x/(x+1) < (x+1)/(x+2)$:-

* using contradiction :-

Suppose x is a positive number and $x/(x+1) \geq (x+1)/(x+2)$

$\Rightarrow$ Subtracting $x/(x+1)$ from both sides

$$\Rightarrow 0 \geq \frac{(x+1)}{(x+2)} - \frac{x}{(x+1)} \Rightarrow \frac{\cancel{x^2} + \cancel{2x} + 1 - \cancel{x^2} - \cancel{2x}}{(x+1)(x+2)}$$

$$0 \geq \frac{1}{(x+1)(x+2)}$$, but this is not true

for x is a positive real number

$\Rightarrow$ Therefore, $x/(x+1) < (x+1)/(x+2)$ is true.

$\boxed{Q\#9}$ Prove that if x and y are positive real numbers and $x \neq y$, then $x+y > 4xy/(x+y)$:-

* using Contradiction :-

Suppose x and y are positive real numbers and $x \neq y$, and $x+y \leq 4xy/(x+y)$

$$\Rightarrow x+y \leq \frac{4xy}{x+y} \Rightarrow (x+y)^2 \leq 4xy$$

$$x^2 + 2xy + y^2 \leq 4xy$$

$$x^2 + y^2 \leq 2xy$$

$$x^2 - 2xy + y^2 \leq 0 \Rightarrow (x-y)^2 \leq 0$$

~~a~~ a Contradiction

$\Rightarrow$ $\cancel{x-y=0 \Rightarrow x=y}$ ~~a contradiction~~

Therefore, $x+y > 4xy/(x+y)$ is true.

Q.82) let $P_1, P_2, P_3, \dots P_n$ be primes. Prove that for each $i = 1, 2, \dots n$, $(P_1 \times P_2 \times \dots \times P_n) + 1$ is not divisible by P_i :=

using Contradiction : suppose $(P_1 \times P_2 \times \dots P_n) + 1$ is divisible by P_i

$\Rightarrow$ let $m = P_1 P_2 \dots P_n$, the product of these prime number. this number m is evenly divisible by each of $P_1, P_2, \dots P_n$ (P_i)

$\Rightarrow$ m divisble by P_i

$\Rightarrow m = k P_i$

$\Rightarrow (m+1)$ divisble by P_i

$\Rightarrow m+1 = d P_i$

$k P_i + 1 = d P_i$

$(d-k) P_i = 1$, $P_i > 1$

$\Rightarrow$ impossible for there product to be 1.

$\Rightarrow$ $m+1$ is a new prime number, thus $m+1$ is only divisble by 1 and itself

so $m+1$ is not divisible by P_i

$\Rightarrow$ a Contradiction #

Q.87 Prove that the equation $x^5+6x^4+17x^3+3x^2+7x+4=0$ has no positive real solution :—

* using Contradiction : Suppose $x^5+6x^4+17x^3+3x^2+7x+4=0$ has positive real solution

$\Rightarrow$ $x>0$ (all positive are greater than 0)

$\Rightarrow$ $x^5>0$, $x^4>0$, $x^3>0$, $x^2>0$, $x>0$

$\Rightarrow$ The sum of all positive real numbers are positive (or >0), and multiple number by a positive is >0

$\Rightarrow$ $x^5+6x^4+17x^3+3x^2+7x+4$ is much more greater than 0

$\Rightarrow$ a contradiction

$\Rightarrow$ The equation has no positive real roots.

Exercises 1.6 :

(Q.92) Prove that if n is natural number, then n^2+n is even :

* there are two types of natural numbers ⇒ even and odd
* if a, b are natural numbers, a is odd and b is even then ab is always even :-

proof using Direct method ⇒ suppose $a = 2k_1+1$, $b = 2k_2$ where $k_1, k_2 \in \mathbb{Z}$ then

$$\Rightarrow ab = (2k_1+1)(2k_2)$$
$$= 4k_1k_2 + 2k_2$$
$$= 2(2k_1k_2 + k_2)$$
$$ab = 2k' \text{ where } k' = 2k_1k_2+k_2, \in \mathbb{Z}$$

⇒ ab is even

⇒ using Contradiction : n is natural and $n(n+1)$ is odd

- Case 1 : if n is even then $(n+1)$ will be odd

Thus $n(n+1)$ = even x odd = even

- Case 2 : if n is odd then $(n+1)$ will be even

Thus $n(n+1)$ = odd x even = even a contradiction.

Q.95 Prove that if n is a natural number greater than 1, then $n!+1$ is odd :—

using contradiction : $n>1$ and $n!+1$ is even

$\Rightarrow$ for any $n>1$, $n! = n(n-1)(n-2) \cdots \times 2 \times 1$

Since 2 is a factor of $n!$

then $n!$ is even

$\Rightarrow n! = 2k, \; k \in \mathbb{Z}$

$\Rightarrow n!+1 = 2k+1$ is odd

$\Rightarrow$ a contradiction

Q.94 Prove that if a and b are rational numbers with $a<b$, then there exists a rational number r such that $a<r<b$: ⇒ Direct method

* let $a = \frac{m}{n}$, $b = \frac{s}{t} \in \mathbb{Q}$

$\Rightarrow \exists\, r \in \mathbb{Q}$, take $r = \frac{a+b}{2} = \frac{\frac{m}{n} + \frac{s}{t}}{2}$

$\Rightarrow r = \frac{mt + sn}{2nt}$, then $r \in \mathbb{Q}$ since

$mt + sn$, $2nt \in \mathbb{Z}$.

and $a<r$ since $r-a = \frac{a+b}{2} - a = \frac{b-a}{2} > 0$

$\Rightarrow a<r$

also $r<b$ since $b-r = b - \frac{a+b}{2} = \frac{b-a}{2} > 0$

$\Rightarrow r<b$.

(Q. 101) Prove that if x is any real number greater than 2, then there is a negative real number y such that $x = 2y/(1+y)$:-

* let $x \in \mathbb{R}$, $x > 2$, take $y = \dfrac{-x}{x-2}$ negative / positive

then $y < 0$ and $\dfrac{2y}{1+y} = \dfrac{2\left(\frac{-x}{x-2}\right)}{1+\frac{-x}{x-2}} = \dfrac{\frac{-2x}{\cancel{x-2}}}{\frac{-2}{\cancel{x-2}}} = x$ ✓

or by Contradiction :-

Assume $x \in \mathbb{R}$, $x > 2$ and $y \geq 0$, $y \in \mathbb{R}$

$\Rightarrow x = 2y/(1+y)$

$x + xy = 2y$

$xy - 2y = -x$

$y(x-2) = -x$

$y = \dfrac{-x}{x-2}$, that is y is negative

a Contradiction