

10.7 ∴ Power Series

a Find the series' radius & interval of convergence.

b For what values of x does the series converges absolutely?

c For what values of x does the series converges conditionally?

4 $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

Ratio test ∴ The series converges absolutely if $\Rightarrow$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$$

$$\lim_{n \rightarrow \infty} \left| \frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(3x-2)^n (3x-2)}{n+1} \cdot \frac{n}{(3x-2)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{n}{n+1} |3x-2|$$

$$= |3x-2| < 1$$

$$-1 < 3x-2 < 1$$

$$1 < 3x < 3$$

$$\boxed{\frac{1}{3} < x < 1}$$

★ for $x = \frac{1}{3}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, converges conditionally by A.S.T
($u_n = \frac{1}{n}$, 1) $u_n > 0$, 2) $u_n \downarrow$, 3) $\lim_{n \rightarrow \infty} u_n = 0$).

★ for $x = 1$ ∴ $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, diverges (harmonic series).

a The radius is $\frac{1}{3}$, the Interval of convergence is $\frac{1}{3} < x < 1$.

b The interval of absolute convergence is $\frac{1}{3} < x < 1$.

c The series converges conditionally at $x = \frac{1}{3}$.

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$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$$

Ratio test: The series converges absolutely if:

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1.$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{3^{n+1} x^{n+1}}{(n+1)!}}{\frac{3^n x^n}{n!}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\cancel{3} \cdot 3 \cdot \cancel{x^n} x}{(n+1) \cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{3^n} \cancel{x^n}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{3x}{n+1} \right|$$

$$= 3|x| \lim_{n \rightarrow \infty} \frac{1}{n+1}$$

$$= 0 < 1 \text{ for all } x.$$

a The radius is ∞ ; the series converges for all x .

b The series converges absolutely for all x .

c There are no values for which the series converges conditionally.

$$\boxed{14} \quad \sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n n^2}$$

Ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{3^{n+1} (n+1)^2} \cdot \frac{3^n n^2}{(x-1)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\cancel{(x-1)^n} \cdot (x-1)}{\cancel{3^n} \cdot 3 \cdot (n+1)^2} \cdot \frac{\cancel{3^n} n^2}{\cancel{(x-1)^n}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(x-1) n^2}{3(n+1)^2} \right|$$

$$= \frac{|x-1|}{3} \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2}$$

$$= \frac{|x-1|}{3} \cdot 1$$

$$= \frac{|x-1|}{3}$$

The series converges absolutely when $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$.

$$\Rightarrow \frac{|x-1|}{3} < 1$$

$$\Rightarrow |x-1| < 3$$

$$\Rightarrow -3 < x-1 < 3$$

$$\boxed{-2 < x < 4}$$

★ When $x = -2$ $\therefore$ we have $\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, an absolutely convergent series.

★ When $x = 4$ $\therefore$ we have $\sum_{n=1}^{\infty} \frac{3^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, an absolutely convergent series.

- a The radius is 3; the interval of convergence is $-2 \leq x \leq 4$.
- b The interval of absolute convergence is $-2 \leq x \leq 4$.
- c There are no values for which the series converges conditionally.

23 $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n x^n$

Ratio test: The series converges absolutely when $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$.

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\left(1 + \frac{1}{n+1}\right)^{n+1} x^{n+1}}{\left(1 + \frac{1}{n}\right)^n x^n} \right|$$

$$= |x| \frac{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n+1}\right)^{n+1}}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n}$$

$$= |x| \cdot \frac{e}{e}$$

$$= |x|$$

$$\Rightarrow |x| < 1$$

$$-1 < x < 1$$

★ When $x = -1$ we have $\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)^n \therefore$

a divergent series by n-th term test since

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n+1}\right)^{n+1} \\ & \text{let } h = n+1 \\ & n \rightarrow \infty, h \rightarrow \infty \\ & \lim_{h \rightarrow \infty} \left(1 + \frac{1}{h}\right)^h = e^1 = e \end{aligned}$$

★ When $x=1$ we have $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n \Rightarrow$

a divergent series by n-th term test.

[a] The radius is 1, the interval of convergence $-1 < x < 1$.

[b] The interval of absolute convergence is $-1 < x < 1$.

[c] There are no values for which the series converges conditionally.

[29] $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$

Ratio test: The series converges absolutely when:

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1.$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1(\ln(n+1))^2} \cdot \frac{n(\ln n)^2}{x^n} \right| < 1.$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x n(\ln n)^2}{n+1(\ln(n+1))^2} \right| < 1.$$

$$\Rightarrow |x| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) \left(\lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} \right)^2 < 1.$$

$$\Rightarrow |x| \cdot (1) \left(\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)} \right)^2 < 1.$$

$$\Rightarrow |x| \left(\lim_{n \rightarrow \infty} \frac{n+1}{n} \right)^2 < 1.$$

$$\Rightarrow |x| (1)^2 < 1.$$

$$\Rightarrow |x| < 1$$

$$\Rightarrow -1 < x < 1$$

★ When $x=-1$, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$

which converges absolutely by the integral test.

$$\Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2} = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

have positive terms

$$\Rightarrow \text{let } f(x) = \frac{1}{x(\ln x)^2} \rightarrow \text{cont, positive \& decreasing for } x \geq 2.$$

$$\Rightarrow \int_2^{\infty} \frac{1}{x(\ln x)^2} dx \quad \left\{ \begin{array}{l} \text{let } u = \ln x \\ du = \frac{dx}{x} \end{array} \right.$$

$$\Rightarrow \lim_{b \rightarrow \infty} \int_2^b u^{-2} du = \lim_{b \rightarrow \infty} \left. -\frac{1}{u} \right|_2^b$$

$$= \lim_{b \rightarrow \infty} -\frac{1}{b} + \frac{1}{\ln 2} = \frac{1}{\ln 2} \quad (\text{converges})$$

★ when $x=1$, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, Converges by the integral test.

[a] the radius is 1, the interval of convergence is $-1 \leq x \leq 1$

[b] the interval of absolute convergence is $-1 \leq x \leq 1$.

[c] There are no values for which the series converges conditionally.

[32] $\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$

by ratio test, the series converges absolutely if:

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1.$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(3x+1)^{n+2}}{2n+4} \cdot \frac{2n+2}{(3x+1)^{n+1}} \right| < 1.$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(3x+1) \cdot 2n+2}{2n+4} \right| < 1.$$

$$\Rightarrow |3x+1| \lim_{n \rightarrow \infty} \frac{2n+2}{2n+4} < 1.$$

$$\Rightarrow |3x+1| \cdot (1) < 1.$$

$$\Rightarrow |3x+1| < 1.$$

$$\Rightarrow -1 < 3x+1 < 1$$

$$\Rightarrow -2 < 3x < 0$$

$$\Rightarrow \boxed{-\frac{2}{3} < x < 0}$$

★ when $x = -\frac{2}{3}$, we have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+2}$,

a conditionally convergent series by A.S.T.

★ when $x=0$, we have $\sum_{n=1}^{\infty} \frac{1}{2n+2}$

a divergent series.

★ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+2}$, $u_n = \frac{1}{2n+2}$

A.S.T. $\Rightarrow$

1) $u_n > 0$. ✓

2) $u_n > u_{n+1}$. ✓

3) $\lim_{n \rightarrow \infty} u_n = 0$. ✓

★ $\sum_{n=1}^{\infty} \frac{1}{2n+2}$ use L.C.T & compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ (div harmonic series)

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{2n+2}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{2n+2} = \frac{1}{2} \neq 0 \text{ (both div)}$$

a The radius is $\frac{1}{3}$, the interval of convergence is $-\frac{2}{3} \leq x < 0$.

b The interval of absolute convergence is $-\frac{2}{3} < x < 0$.

c The series converges conditionally at $x = -\frac{2}{3}$.

40 Find the series' radius of convergence.

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2} x^n$$

Root test: The series converges absolutely if

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{n}{n+1} \right)^{n^2} x^n \right|} < 1.$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \left(\frac{n}{n+1} \right)^{\frac{n^2}{n}} x^{\frac{n}{n}} \right| < 1.$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n |x| < 1.$$

$$\Rightarrow |x| \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{n+1}{n} \right)^n} < 1.$$

$$\Rightarrow |x| \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n} \right)^n} < 1.$$

$$\Rightarrow |x| \cdot \frac{1}{e} < 1.$$

$$\Rightarrow |x| < e$$

$$\Rightarrow -e < x < e.$$

So, The radius of convergence $R = e$.

46 Use Theorem (20) - in the text book - to find the series' interval of convergence & within this interval, the sum of the series as a function

$$\sum_{n=0}^{\infty} (-\ln x)^n$$

Theorem (20) $\Rightarrow$ If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function f on $|f(x)| < R$.

by Ratio test $\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$.
 $\Rightarrow$ The series converges absolutely if

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(\ln x)^{n+1}}{(\ln x)^n} \right| < 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} |\ln x| < 1$$

$$\Rightarrow |\ln x| < 1$$

$$\Rightarrow -1 < \ln x < 1$$

$$\Rightarrow e^{-1} < x < e$$

$\star$ when $x = e^{-1}$ we have $\sum_{n=0}^{\infty} (\ln(e^{-1}))^n = \sum_{n=0}^{\infty} (-1)^n$ which is a divergent series.

$\star$ when $x = e$ we have $\sum_{n=0}^{\infty} (\ln e)^n = \sum_{n=0}^{\infty} 1^n$ which is a divergent series.

$\Rightarrow$ the interval of convergence is $e^{-1} < x < e$.

$\Rightarrow$ the sum of the series is $\frac{1}{1 - \ln x}$ when $e^{-1} < x < e$

(since $\sum_{n=0}^{\infty} (\ln x)^n$ is a geometric series with $r = \ln x$ & $a = 1$).

49 For what values of x does the series

$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \dots + (-\frac{1}{2})^n(x-3)^n + \dots$ converge? what is its sum?

what series do you get if you differentiate the given series term by term?

$\sum_{n=0}^{\infty} (-\frac{1}{2})^n (x-3)^n$ converges absolutely if $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$ (Ratio test).

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(-\frac{1}{2})^{n+1} (x-3)^{n+1}}{(-\frac{1}{2})^n (x-3)^n} \right| < 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{1}{2} (x-3) \right| < 1$$

$$\Rightarrow \frac{1}{2} |x-3| < 1$$

$$\Rightarrow |x-3| < 2$$

$$\Rightarrow -2 < x-3 < 2$$

$$\Rightarrow \boxed{1 < x < 5}$$

★ when $x=1$, we have $\sum_{n=0}^{\infty} (-\frac{1}{2})^n (-2)^n = \sum_{n=0}^{\infty} 1^n$ which diverges.

★ when $x=5$, we have $\sum_{n=0}^{\infty} (-\frac{1}{2})^n (2)^n = \sum_{n=0}^{\infty} (-1)^n$ which diverges.

⇒ The interval of convergence is $1 < x < 5$.

⇒ The sum of this convergent geometric series is $\frac{1}{1 + \frac{(x-3)}{2}} = \frac{1}{\frac{2+x-3}{2}} = \boxed{\frac{2}{x-1}}$

$$\text{Now, } f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \dots + (-\frac{1}{2})^n (x-3)^n + \dots = \frac{2}{x-1}$$

$$f'(x) = 0 - \frac{1}{2} + \frac{1}{2}(x-3) + \dots + (-\frac{1}{2})^n \cdot n(x-3)^{n-1} + \dots = -\frac{2}{(x-1)^2}$$

↳ is convergent when $1 < x < 5$, & diverges when $x=1$ or 5 .

Note that: The sum for $f'(x)$ is $-\frac{2}{(x-1)^2}$ the derivative of $\frac{2}{x-1}$.