

5.2 Sigma Notation and Limits of Finite Sums

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Finite Sum:

$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n$$

The index k ends at $k=n$
 The index k starts at $k=1$
 is the formula for the k^{th} term.

Sigma "Sum"

Example ① $\sum_{k=1}^4 k = 1 + 2 + 3 + 4 = 10$

② write $\sum_{k=0}^2 \frac{6k}{k+1}$ without sigma notation

$$\sum_{k=0}^2 \frac{6k}{k+1} = 0 + \frac{6}{2} + \frac{12}{3} = 3 + 4 = 7$$

③ $\sum_{k=1}^2 (-1)^{k+1} \sin \frac{\pi}{k} = \sin \pi - \sin \frac{\pi}{2} = 0 - 1 = -1$

Example: Express the sum $\frac{2}{3} + \frac{4}{3} + \frac{6}{3} + \frac{8}{3} + \frac{10}{3}$ in sigma notation

$$\frac{1}{3} [2 + 4 + 6 + 8 + 10] = \frac{1}{3} \sum_{k=1}^5 2k$$

Algebra Rules for Finite Sums:

① Sum Rule: $\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$

② Difference Rule: $\sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n a_k - \sum_{k=1}^n b_k$

③ Constant Multiple Rule: $\sum_{k=1}^n c a_k = c \sum_{k=1}^n a_k$

④ Constant Value Rule: $\sum_{k=1}^n c = nc$

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$$\boxed{5} \quad \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

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$$\boxed{6} \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\boxed{7} \quad \sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Example: Evaluate:

$$\textcircled{1} \quad \sum_{k=1}^7 (-2k) = -2 \sum_{k=1}^7 k = -2 \frac{7(7+1)}{2} = -7(8) = -56$$

$$\textcircled{2} \quad \sum_{k=1}^6 (3 - k^2) = \sum_{k=1}^6 3 - \sum_{k=1}^6 k^2 = (6)(3) - \frac{6(7)(13)}{6} = 18 - 91 = -73$$

Riemann Sums

* Consider a bounded function f defined on $[a, b]$

* Divide $[a, b]$ into n closed sub intervals

by choosing $n-1$ points
 $\{x_1, x_2, \dots, x_{n-1}\}$ between a
 and b .

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

* The set $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$
 is called a partition of $[a, b]$.

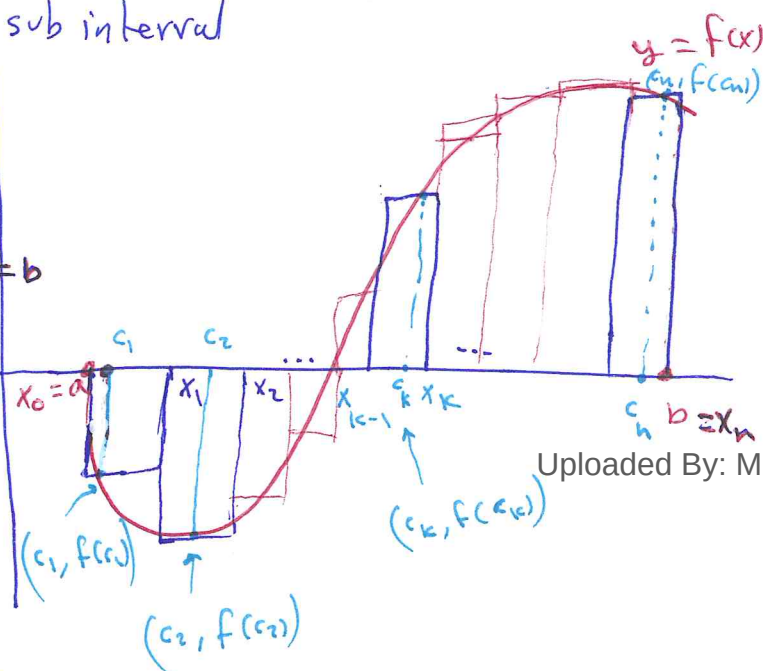
* The partition P divides $[a, b]$
 into n closed subintervals:

$$[x_0, x_1], [x_1, x_2], \dots, [x_{k-1}, x_k], \dots, [x_{n-1}, x_n]$$

k^{th} subinterval

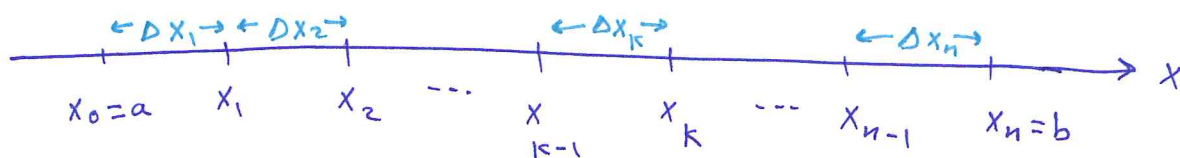
* $\Delta x_k = x_k - x_{k-1}$ is the width of the k^{th} subinterval.

* $\max |\Delta x_k|$ is the norm of the partition P denoted by $\|P\|$.



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- * If all subintervals have equal width, then the width of any subinterval is $\Delta x = \frac{b-a}{n}$



- * select a point c_k in Δx_k for all $k=1, 2, \dots, n$

- * On each subinterval we find $f(c_k) \Delta x_k$. This product is
- positive when $f(c_k)$ is positive. This gives area above x
 - negative when $f(c_k)$ is negative. This gives area below x

- * The Riemann sum S_p for f on the interval $[a, b]$ is

$$S_p = \sum_{k=1}^n f(c_k) \Delta x_k.$$

- * If we choose n subintervals all having equal width, $\Delta x = \frac{b-a}{n}$, and we choose the point c_k to be the right hand endpoint of each subinterval, Then Riemann Sum formula becomes
- $$S_n = \sum_{k=1}^n f\left(a + k \frac{b-a}{n}\right) \left(\frac{b-a}{n}\right)$$

Example Let $f(x) = 1 - x^2$ on $[0, 1]$

Find formula for the Riemann sum by dividing $[0, 1]$ into n equal subintervals and using c_k to be the right hand endpoint. $a=0, b=1$

$\Delta x_k = \frac{1-0}{n} = \frac{1}{n}$ the width of the k^{th} subinterval $k=1, 2, \dots, n$

$$\begin{aligned} S_n &= \sum_{k=1}^n f\left(\frac{k}{n}\right) \frac{1}{n} = \sum_{k=1}^n \left[1 - \left(\frac{k}{n}\right)^2\right] \frac{1}{n} = \sum_{k=1}^n \frac{1}{n} - \sum_{k=1}^n \frac{k^2}{n^3} \\ &= \frac{n}{n} - \frac{1}{n^3} \sum_{k=1}^n k^2 = 1 - \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

$$S_n = 1 - \frac{2n^3 + 3n^2 + n}{6n^3} = 1 - \left[\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right] \quad (98)$$

$$S_n = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$$

[b] Calculate the area under the curve over $[0, 1]$
 "Take limit for S_n as $n \rightarrow \infty$ "

$$\lim_{n \rightarrow \infty} S_n = \left[\frac{2}{3} \right]$$

[c] Find $\int_0^1 (1 - x^2) dx = x - \frac{x^3}{3} \Big|_0^1 = 1 - \frac{1}{3} = \left[\frac{2}{3} \right]$

Example: Find the norm of the partition

$$P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$$

The subintervals are: $[0, 1.2]$, $[1.2, 1.5]$, $[1.5, 2.3]$, $[2.3, 2.6]$, $[2.6, 3]$
 $\Delta x_1 = 1.2$ $\Delta x_2 = 0.3$ $\Delta x_3 = 0.8$ $\Delta x_4 = 0.3$ $\Delta x_5 = 0.4$

The norm of the partition P is $\|P\| = 1.2$