CHAPTER 6 APPLICATIONS OF DEFINITE INTEGRALS

6.1 VOLUMES USING CROSS-SECTIONS

1. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left(\sqrt{x} - \left(-\sqrt{x}\right)\right)^2}{2} = 2x;\ a = 0,\ b = 4;$
   $V = \int_a^b A(x)\,dx = \int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16$

2. $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi\left[(2 - x^2) - x^2\right]^2}{4} = \frac{\pi\left[2\left(1 - x^2\right)\right]^2}{4} = \pi\left(1 - 2x^2 + x^4\right);\ a = -1,\ b = 1;$
   $V = \int_a^b A(x)\,dx = \int_{-1}^1 \pi\left(1 - 2x^2 + x^4\right)dx = \pi\left[x - \frac{2}{3}x^3 + \frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16\pi}{15}$

3. $A(x) = (\text{edge})^2 = \left[\sqrt{1 - x^2} - \left(-\sqrt{1 - x^2}\right)\right]^2 = \left(2\sqrt{1 - x^2}\right)^2 = 4\left(1 - x^2\right);\ a = -1,\ b = 1;$
   $V = \int_a^b A(x)\,dx = \int_{-1}^1 4(1 - x^2)\,dx = 4\left[x - \frac{x^3}{3}\right]_{-1}^1 = 8\left(1 - \frac{1}{3}\right) = \frac{16}{3}$

4. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left[\sqrt{1 - x^2} - \left(-\sqrt{1 - x^2}\right)\right]^2}{2} = \frac{\left(2\sqrt{1 - x^2}\right)^2}{2} = 2\left(1 - x^2\right);\ a = -1,\ b = 1;$
   $V = \int_a^b A(x)\,dx = 2\int_{-1}^1 \left(1 - x^2\right)dx = 2\left[x - \frac{x^3}{3}\right]_{-1}^1 = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$

5. (a) STEP 1) $A(x) = \frac{1}{2}(\text{side})\cdot(\text{side})\cdot\left(\sin\frac{\pi}{3}\right) = \frac{1}{2}\cdot\left(2\sqrt{\sin x}\right)\cdot\left(2\sqrt{\sin x}\right)\left(\sin\frac{\pi}{3}\right) = \sqrt{3}\sin x$
   STEP 2) $a = 0,\ b = \pi$
   STEP 3) $V = \int_a^b A(x)\,dx = \sqrt{3}\int_0^\pi \sin x\,dx = \left[-\sqrt{3}\cos x\right]_0^\pi = \sqrt{3}(1 + 1) = 2\sqrt{3}$
   (b) STEP 1) $A(x) = (\text{side})^2 = \left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right) = 4\sin x$
   STEP 2) $a = 0,\ b = \pi$
   STEP 3) $V = \int_a^b A(x)\,dx = \int_0^\pi 4\sin x\,dx = \left[-4\cos x\right]_0^\pi = 8$

6. (a) STEP 1) $A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi}{4}(\sec x - \tan x)^2 = \frac{\pi}{4}\left(\sec^2 x + \tan^2 x - 2\sec x\tan x\right)$
   $= \frac{\pi}{4}\left[\sec^2 x + \left(\sec^2 x - 1\right) - 2\frac{\sin x}{\cos^2 x}\right]$
   STEP 2) $a = -\frac{\pi}{3},\ b = \frac{\pi}{3}$
   STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3} \frac{\pi}{4}\left(2\sec^2 x - 1 - \frac{2\sin x}{\cos^2 x}\right)dx = \frac{\pi}{4}\left[2\tan x - x + 2\left(-\frac{1}{\cos x}\right)\right]_{-\pi/3}^{\pi/3}$
   $= \frac{\pi}{4}\left[2\sqrt{3} - \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right) - \left(-2\sqrt{3} + \frac{\pi}{3} + 2\left(-\frac{1}{\left(\frac{1}{2}\right)}\right)\right)\right] = \frac{\pi}{4}\left(4\sqrt{3} - \frac{2\pi}{3}\right)$
   (b) STEP 1) $A(x) = (\text{edge})^2 = (\sec x - \tan x)^2 = \left(2\sec^2 x - 1 - 2\frac{\sin x}{\cos^2 x}\right)$
   STEP 2) $a = -\frac{\pi}{3},\ b = \frac{\pi}{3}$
   STEP 3) $V = \int_a^b A(x)\,dx = \int_{-\pi/3}^{\pi/3}\left(2\sec^2 x - 1 - \frac{2\sin x}{\cos^2 x}\right)dx = 2\left(2\sqrt{3} - \frac{\pi}{3}\right) = 4\sqrt{3} - \frac{2\pi}{3}$

7. (a) STEP 1) $A(x) = (\text{length})\cdot(\text{height}) = (6 - 3x)\cdot(10) = 60 - 30x$
   STEP 2) $a = 0,\ b = 2$
   STEP 3) $V = \int_a^b A(x)\,dx = \int_0^2 (60 - 30x)\,dx = \left[60x - 15x^2\right]_0^2 = (120 - 60) - 0 = 60$

(b) STEP 1) $A(x) = (\text{length}) \cdot (\text{height}) = (6-3x) \cdot \left(\frac{20-2(6-3x)}{2}\right) = (6-3x)(4+3x) = 24+6x-9x^2$

STEP 2) $a = 0, b = 2$

STEP 3) $V = \int_a^b A(x)\,dx = \int_0^2 (24+6x-9x^2)dx = [24x+3x^2-3x^3]_0^2 = (48+12-24)-0 = 36$

8. (a) STEP 1) $A(x) = \frac{1}{2}(\text{base}) \cdot (\text{height}) = \left(\sqrt{x} - \frac{x}{2}\right) \cdot (6) = 6\sqrt{x} - 3x$

STEP 2) $a = 0, b = 4$

STEP 3) $V = \int_a^b A(x)\,dx = \int_0^4 \left(6x^{1/2} - 3x\right) dx = \left[4x^{3/2} - \frac{3}{2}x^2\right]_0^4 = (32-24) - 0 = 8$

(b) STEP 1) $A(x) = \frac{1}{2} \cdot \pi\left(\frac{\text{diameter}}{2}\right)^2 = \frac{1}{2} \cdot \pi\left(\frac{\sqrt{x}-\frac{x}{2}}{2}\right)^2 = \frac{\pi}{2} \cdot \frac{x - x^{3/2} + \frac{1}{4}x^2}{4} = \frac{\pi}{8}\left(x - x^{3/2} + \frac{1}{4}x^2\right)$

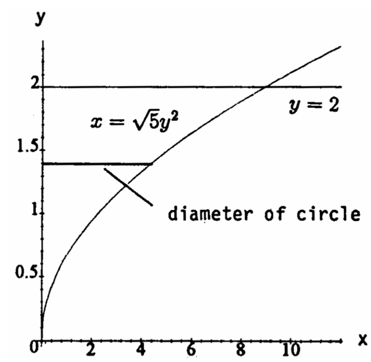
STEP 2) $a = 0, b = 4$

STEP 3) $V = \int_a^b A(x)\,dx = \frac{\pi}{8}\int_0^4 \left(x - x^{3/2} + \frac{1}{4}x^2\right) dx = \left[\frac{1}{2}x^2 - \frac{2}{5}x^{5/2} + \frac{1}{12}x^3\right]_0^4 = \frac{\pi}{8}\left(8 - \frac{64}{5} + \frac{16}{3}\right) - \frac{\pi}{8}(0) = \frac{\pi}{15}$

9. $A(y) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{5}y^2 - 0\right)^2 = \frac{5\pi}{4}y^4$;

$c = 0, d = 2$; $V = \int_c^d A(y)\,dy = \int_0^2 \frac{5\pi}{4} y^4\,dy$

$= \left[\left(\frac{5\pi}{4}\right)\left(\frac{y^5}{5}\right)\right]_0^2 = \frac{\pi}{4}(2^5 - 0) = 8\pi$



10. $A(y) = \frac{1}{2}(\text{leg})(\text{leg}) = \frac{1}{2}\left[\sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right)\right]^2 = \frac{1}{2}\left(2\sqrt{1-y^2}\right)^2 = 2\left(1-y^2\right)$; $c = -1, d = 1$;

$V = \int_c^d A(y)\,dy = \int_{-1}^1 2(1-y^2)\,dy = 2\left[y - \frac{y^3}{3}\right]_{-1}^1 = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$

11. The slices perpendicular to the edge labeled 5 are triangles, and by similar triangles we have $\frac{b}{h} = \frac{4}{3} \Rightarrow h = \frac{3}{4}b$. The equation of the line through $(5, 0)$ and $(0, 4)$ is $y = -\frac{4}{5}x + 4$, thus the length of the base $= -\frac{4}{5}x + 4$ and the height $= \frac{3}{4}\left(-\frac{4}{5}x + 4\right) = -\frac{3}{5}x + 3$.Thus $A(x) = \frac{1}{2}(\text{base}) \cdot (\text{height}) = \frac{1}{2}\left(-\frac{4}{5}x + 4\right) \cdot \left(-\frac{3}{5}x + 3\right) = \frac{6}{25}x^2 - \frac{12}{5}x + 6$ and $V = \int_a^b A(x)\,dx = \int_0^5 \left(\frac{6}{25}x^2 - \frac{12}{5}x + 6\right) dx = \left[\frac{2}{25}x^3 - \frac{6}{5}x^2 + 6x\right]_0^5 = (10 - 30 + 30) - 0 = 10$

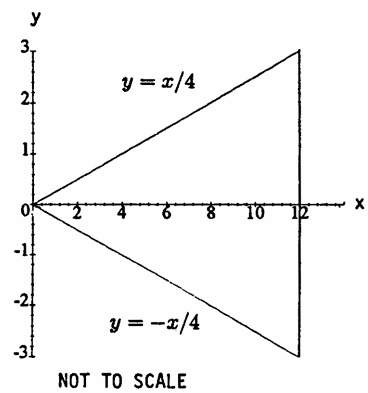
12. The slices parallel to the base are squares. The cross section of the pyramid is a triangle, and by similar triangles we have $\frac{b}{h} = \frac{3}{5} \Rightarrow b = \frac{3}{5}h$. Thus $A(y) = (\text{base})^2 = \left(\frac{3}{5}y\right)^2 = \frac{9}{25}y^2 \Rightarrow V = \int_c^d A(y)\,dy = \int_0^5 \frac{9}{25}y^2\,dy = \left[\frac{3}{25}y^3\right]_0^5 = 15 - 0 = 15$

13. (a) It follows from Cavalieri's Principle that the volume of a column is the same as the volume of a right prism with a square base of side length s and altitude h. Thus, STEP 1) $A(x) = (\text{side length})^2 = s^2$;

STEP 2) $a = 0, b = h$; STEP 3) $V = \int_a^b A(x)\,dx = \int_0^h s^2\,dx = s^2h$

(b) From Cavalieri's Principle we conclude that the volume of the column is the same as the volume of the prism described above, regardless of the number of turns $\Rightarrow V = s^2h$

14. 1) The solid and the cone have the same altitude of 12.
    2) The cross sections of the solid are disks of diameter $x - \left(\frac{x}{2}\right) = \frac{x}{2}$. If we place the vertex of the cone at the origin of the coordinate system and make its axis of symmetry coincide with the x-axis then the cone's cross sections will be circular disks of diameter $\frac{x}{4} - \left(-\frac{x}{4}\right) = \frac{x}{2}$ (see accompanying figure).
    3) The solid and the cone have equal altitudes and identical parallel cross sections. From Cavalieri's Principle we conclude that the solid and the cone have the same volume.



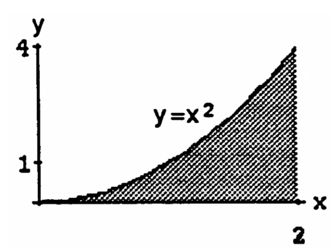
15. $R(x) = y = 1 - \frac{x}{2} \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2 \left(1 - \frac{x}{2}\right)^2 dx = \pi\int_0^2 \left(1 - x + \frac{x^2}{4}\right) dx = \pi\left[x - \frac{x^2}{2} + \frac{x^3}{12}\right]_0^2$
$= \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$

16. $R(y) = x = \frac{3y}{2} \Rightarrow V = \int_0^2 \pi[R(y)]^2\,dy = \pi\int_0^2 \left(\frac{3y}{2}\right)^2 dy = \pi\int_0^2 \frac{9}{4}y^2\,dy = \pi\left[\frac{3}{4}y^3\right]_0^2 = \pi \cdot \frac{3}{4} \cdot 8 = 6\pi$

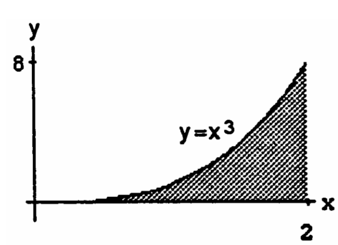
17. $R(y) = \tan\left(\frac{\pi}{4}y\right)$; $u = \frac{\pi}{4}y \Rightarrow du = \frac{\pi}{4}\,dy \Rightarrow 4\,du = \pi\,dy$; $y = 0 \Rightarrow u = 0$, $y = 1 \Rightarrow u = \frac{\pi}{4}$;
$V = \int_0^1 \pi[R(y)]^2\,dy = \pi\int_0^1 \left[\tan\left(\frac{\pi}{4}y\right)\right]^2 dy = 4\int_0^{\pi/4} \tan^2 u\,du = 4\int_0^{\pi/4} (-1 + \sec^2 u)\,du = 4[-u + \tan u]_0^{\pi/4}$
$= 4\left(-\frac{\pi}{4} + 1 - 0\right) = 4 - \pi$

18. $R(x) = \sin x \cos x$; $R(x) = 0 \Rightarrow a = 0$ and $b = \frac{\pi}{2}$ are the limits of integration; $V = \int_0^{\pi/2} \pi[R(x)]^2\,dx$
$= \pi\int_0^{\pi/2} (\sin x \cos x)^2\,dx = \pi\int_0^{\pi/2} \frac{(\sin 2x)^2}{4}\,dx$; $\left[u = 2x \Rightarrow du = 2\,dx \Rightarrow \frac{du}{8} = \frac{dx}{4}; x = 0 \Rightarrow u = 0,\right.$
$\left. x = \frac{\pi}{2} \Rightarrow u = \pi\right] \rightarrow V = \pi\int_0^{\pi} \frac{1}{8}\sin^2 u\,du = \frac{\pi}{8}\left[\frac{u}{2} - \frac{1}{4}\sin 2u\right]_0^{\pi} = \frac{\pi}{8}\left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{\pi^2}{16}$

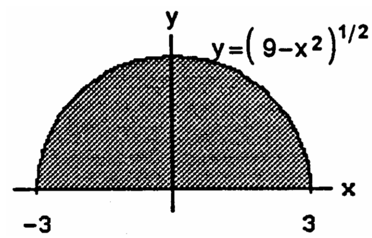
19. $R(x) = x^2 \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2 (x^2)^2\,dx$
$= \pi\int_0^2 x^4\,dx = \pi\left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$



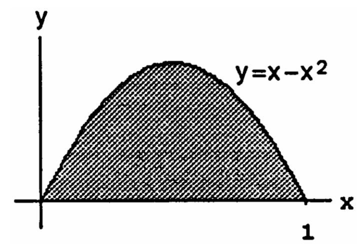
20. $R(x) = x^3 \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2 (x^3)^2\,dx$
$= \pi\int_0^2 x^6\,dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$



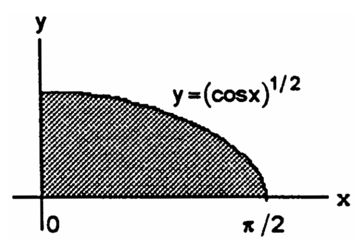
21. $R(x) = \sqrt{9 - x^2} \Rightarrow V = \int_{-3}^{3} \pi[R(x)]^2\, dx = \pi \int_{-3}^{3} (9 - x^2)\, dx$
$= \pi \left[9x - \frac{x^3}{3}\right]_{-3}^{3} = 2\pi\left[9(3) - \frac{27}{3}\right] = 2 \cdot \pi \cdot 18 = 36\pi$



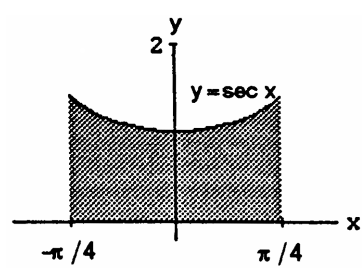
22. $R(x) = x - x^2 \Rightarrow V = \int_0^1 \pi[R(x)]^2\, dx = \pi \int_0^1 (x - x^2)^2\, dx$
$= \pi \int_0^1 (x^2 - 2x^3 + x^4)\, dx = \pi \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1$
$= \pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{30}(10 - 15 + 6) = \frac{\pi}{30}$



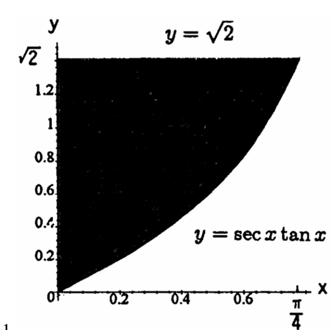
23. $R(x) = \sqrt{\cos x} \Rightarrow V = \int_0^{\pi/2} \pi[R(x)]^2\, dx = \pi \int_0^{\pi/2} \cos x\, dx$
$= \pi\left[\sin x\right]_0^{\pi/2} = \pi(1 - 0) = \pi$



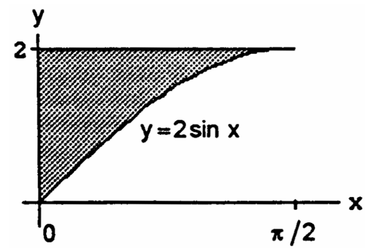
24. $R(x) = \sec x \Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi[R(x)]^2\, dx = \pi \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$
$= \pi\left[\tan x\right]_{-\pi/4}^{\pi/4} = \pi[1 - (-1)] = 2\pi$



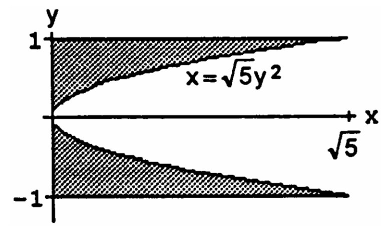
25. $R(x) = \sqrt{2} - \sec x \tan x \Rightarrow V = \int_0^{\pi/4} \pi[R(x)]^2\, dx$
$= \pi \int_0^{\pi/4} \left(\sqrt{2} - \sec x \tan x\right)^2 dx$
$= \pi \int_0^{\pi/4} \left(2 - 2\sqrt{2} \sec x \tan x + \sec^2 x \tan^2 x\right) dx$
$= \pi\left(\int_0^{\pi/4} 2\, dx - 2\sqrt{2} \int_0^{\pi/4} \sec x \tan x\, dx + \int_0^{\pi/4} (\tan x)^2 \sec^2 x\, dx\right)$
$= \pi\left([2x]_0^{\pi/4} - 2\sqrt{2}\,[\sec x]_0^{\pi/4} + \left[\frac{\tan^3 x}{3}\right]_0^{\pi/4}\right)$
$= \pi\left[\left(\frac{\pi}{2} - 0\right) - 2\sqrt{2}\left(\sqrt{2} - 1\right) + \frac{1}{3}\left(1^3 - 0\right)\right] = \pi\left(\frac{\pi}{2} + 2\sqrt{2} - \frac{11}{3}\right)$



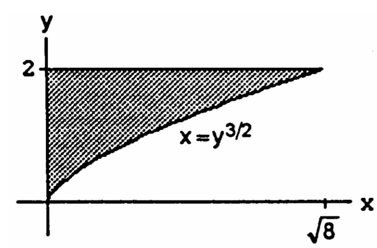
26. $R(x) = 2 - 2\sin x = 2(1 - \sin x) \Rightarrow V = \int_0^{\pi/2} \pi[R(x)]^2\,dx$
$= \pi \int_0^{\pi/2} 4(1 - \sin x)^2\,dx = 4\pi \int_0^{\pi/2} (1 + \sin^2 x - 2\sin x)\,dx$
$= 4\pi \int_0^{\pi/2} \left[1 + \frac{1}{2}(1 - \cos 2x) - 2\sin x\right] dx$
$= 4\pi \int_0^{\pi/2} \left(\frac{3}{2} - \frac{\cos 2x}{2} - 2\sin x\right)$
$= 4\pi \left[\frac{3}{2}x - \frac{\sin 2x}{4} + 2\cos x\right]_0^{\pi/2}$
$= 4\pi \left[\left(\frac{3\pi}{4} - 0 + 0\right) - (0 - 0 + 2)\right] = \pi(3\pi - 8)$



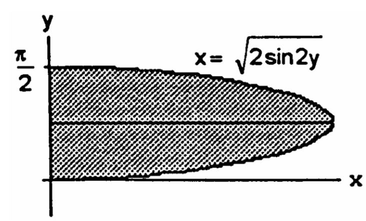
27. $R(y) = \sqrt{5}\,y^2 \Rightarrow V = \int_{-1}^{1} \pi[R(y)]^2\,dy = \pi \int_{-1}^{1} 5y^4\,dy$
$= \pi\left[y^5\right]_{-1}^{1} = \pi[1 - (-1)] = 2\pi$



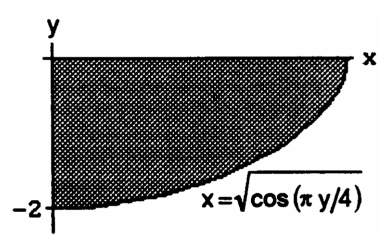
28. $R(y) = y^{3/2} \Rightarrow V = \int_0^2 \pi[R(y)]^2\,dy = \pi \int_0^2 y^3\,dy$
$= \pi\left[\frac{y^4}{4}\right]_0^2 = 4\pi$



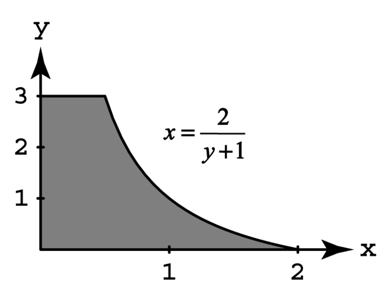
29. $R(y) = \sqrt{2\sin 2y} \Rightarrow V = \int_0^{\pi/2} \pi[R(y)]^2\,dy$
$= \pi \int_0^{\pi/2} 2\sin 2y\,dy = \pi\left[-\cos 2y\right]_0^{\pi/2}$
$= \pi[1 - (-1)] = 2\pi$



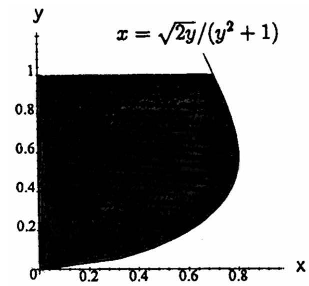
30. $R(y) = \sqrt{\cos \frac{\pi y}{4}} \Rightarrow V = \int_{-2}^{0} \pi[R(y)]^2\,dy$
$= \pi \int_{-2}^{0} \cos\left(\frac{\pi y}{4}\right) dy = 4\left[\sin \frac{\pi y}{4}\right]_{-2}^{0} = 4[0 - (-1)] = 4$



31. $R(y) = \frac{2}{y+1} \Rightarrow V = \int_0^3 \pi[R(y)]^2\,dy = 4\pi \int_0^3 \frac{1}{(y+1)^2}\,dy$
$= 4\pi\left[-\frac{1}{y+1}\right]_0^3 = 4\pi\left[-\frac{1}{4} - (-1)\right] = 3\pi$



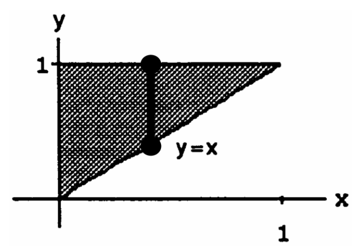
32. $R(y) = \frac{\sqrt{2y}}{y^2+1} \Rightarrow V = \int_0^1 \pi[R(y)]^2\,dy = \pi\int_0^1 2y\,(y^2+1)^{-2}\,dy$; $[u = y^2+1 \Rightarrow du = 2y\,dy;\ y = 0 \Rightarrow u = 1,\ y = 1 \Rightarrow u = 2]$ $\rightarrow V = \pi\int_1^2 u^{-2}\,du = \pi\left[-\frac{1}{u}\right]_1^2 = \pi\left[-\frac{1}{2} - (-1)\right] = \frac{\pi}{2}$



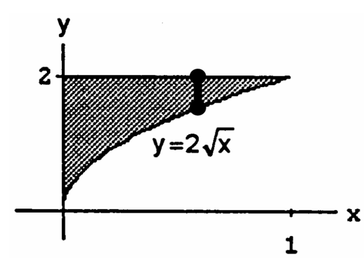
33. For the sketch given, $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$; $R(x) = 1$, $r(x) = \sqrt{\cos x}$; $V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right)dx$ $= \int_{-\pi/2}^{\pi/2} \pi(1 - \cos x)\,dx = 2\pi\int_0^{\pi/2}(1 - \cos x)\,dx = 2\pi[x - \sin x]_0^{\pi/2} = 2\pi\left(\frac{\pi}{2} - 1\right) = \pi^2 - 2\pi$

34. For the sketch given, $c = 0$, $d = \frac{\pi}{4}$; $R(y) = 1$, $r(y) = \tan y$; $V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right)dy$ $= \pi\int_0^{\pi/4}(1 - \tan^2 y)\,dy = \pi\int_0^{\pi/4}(2 - \sec^2 y)\,dy = \pi[2y - \tan y]_0^{\pi/4} = \pi\left(\frac{\pi}{2} - 1\right) = \frac{\pi^2}{2} - \pi$

35. $r(x) = x$ and $R(x) = 1 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right)dx$ $= \int_0^1 \pi\,(1 - x^2)\,dx = \pi\left[x - \frac{x^3}{3}\right]_0^1 = \pi\left[\left(1 - \frac{1}{3}\right) - 0\right] = \frac{2\pi}{3}$



36. $r(x) = 2\sqrt{x}$ and $R(x) = 2 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right)dx$ $= \pi\int_0^1 (4 - 4x)\,dx = 4\pi\left[x - \frac{x^2}{2}\right]_0^1 = 4\pi\left(1 - \frac{1}{2}\right) = 2\pi$



37. $r(x) = x^2 + 1$ and $R(x) = x + 3$

$\Rightarrow V = \int_{-1}^2 \pi\left([R(x)]^2 - [r(x)]^2\right)dx$

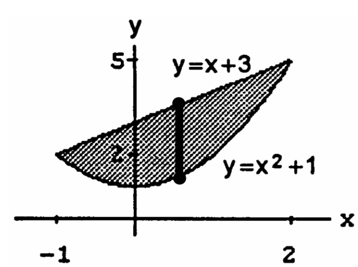
$= \pi\int_{-1}^2 \left[(x+3)^2 - (x^2+1)^2\right]dx$

$= \pi\int_{-1}^2 \left[(x^2 + 6x + 9) - (x^4 + 2x^2 + 1)\right]dx$

$= \pi\int_{-1}^2 (-x^4 - x^2 + 6x + 8)\,dx$

$= \pi\left[-\frac{x^5}{5} - \frac{x^3}{3} + \frac{6x^2}{2} + 8x\right]_{-1}^2$

$= \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + \frac{24}{2} + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + \frac{6}{2} - 8\right)\right] = \pi\left(-\frac{33}{5} - 3 + 28 - 3 + 8\right) = \pi\left(\frac{5\cdot 30 - 33}{5}\right) = \frac{117\pi}{5}$



38. $r(x) = 2 - x$ and $R(x) = 4 - x^2$

$\Rightarrow V = \int_{-1}^{2} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

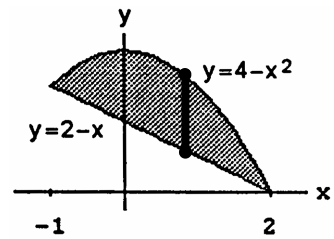
$= \pi\int_{-1}^{2} \left[(4 - x^2)^2 - (2 - x)^2\right] dx$

$= \pi \int_{-1}^{2} [(16 - 8x^2 + x^4) - (4 - 4x + x^2)]\, dx$

$= \pi\int_{-1}^{2} (12 + 4x - 9x^2 + x^4)\, dx$

$= \pi \left[12x + 2x^2 - 3x^3 + \frac{x^5}{5}\right]_{-1}^{2}$

$= \pi\left[\left(24 + 8 - 24 + \frac{32}{5}\right) - \left(-12 + 2 + 3 - \frac{1}{5}\right)\right] = \pi\left(15 + \frac{33}{5}\right) = \frac{108\pi}{5}$

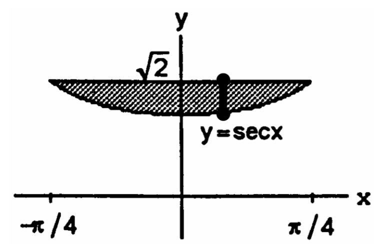


39. $r(x) = \sec x$ and $R(x) = \sqrt{2}$

$\Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_{-\pi/4}^{\pi/4} (2 - \sec^2 x)\, dx = \pi[2x - \tan x]_{-\pi/4}^{\pi/4}$

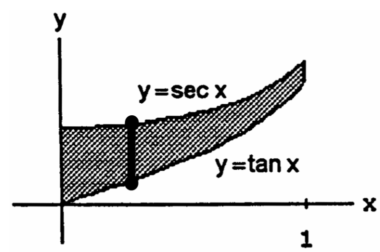
$= \pi\left[\left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right)\right] = \pi(\pi - 2)$



40. $R(x) = \sec x$ and $r(x) = \tan x$

$\Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_0^1 (\sec^2 x - \tan^2 x)\, dx = \pi \int_0^1 1\, dx = \pi[x]_0^1 = \pi$

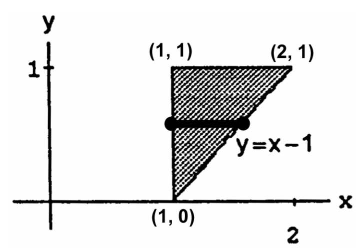


41. $r(y) = 1$ and $R(y) = 1 + y$

$\Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi\int_0^1 [(1 + y)^2 - 1]\, dy = \pi \int_0^1 (1 + 2y + y^2 - 1)\, dy$

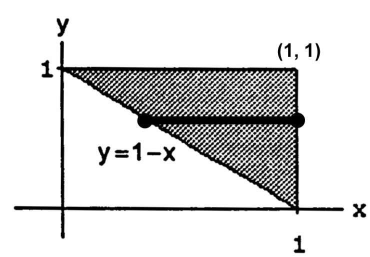
$= \pi \int_0^1 (2y + y^2)\, dy = \pi\left[y^2 + \frac{y^3}{3}\right]_0^1 = \pi\left(1 + \frac{1}{3}\right) = \frac{4\pi}{3}$



42. $R(y) = 1$ and $r(y) = 1 - y \Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi\int_0^1 [1 - (1 - y)^2]\, dy = \pi\int_0^1 [1 - (1 - 2y + y^2)]\, dy$

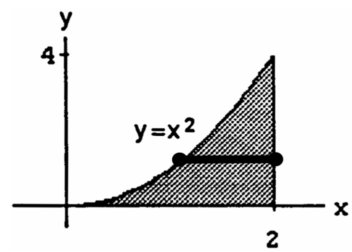
$= \pi\int_0^1 (2y - y^2)\, dy = \pi\left[y^2 - \frac{y^3}{3}\right]_0^1 = \pi\left(1 - \frac{1}{3}\right) = \frac{2\pi}{3}$



43. $R(y) = 2$ and $r(y) = \sqrt{y}$

$\Rightarrow V = \int_0^4 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^4 (4 - y)\, dy = \pi \left[4y - \frac{y^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$

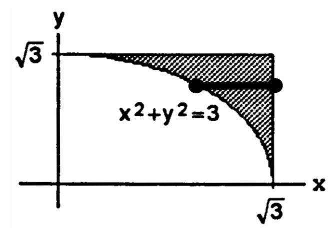


44. $R(y) = \sqrt{3}$ and $r(y) = \sqrt{3 - y^2}$

$\Rightarrow V = \int_0^{\sqrt{3}} \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^{\sqrt{3}} [3 - (3 - y^2)]\, dy = \pi \int_0^{\sqrt{3}} y^2\, dy$

$= \pi \left[\frac{y^3}{3}\right]_0^{\sqrt{3}} = \pi\sqrt{3}$



45. $R(y) = 2$ and $r(y) = 1 + \sqrt{y}$

$\Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

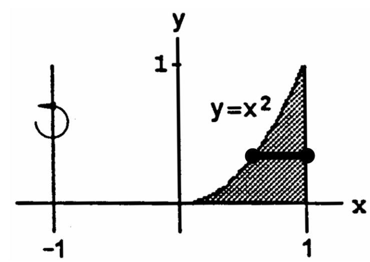
$= \pi \int_0^1 \left[4 - \left(1 + \sqrt{y}\right)^2\right] dy$

$= \pi \int_0^1 \left(4 - 1 - 2\sqrt{y} - y\right) dy$

$= \pi \int_0^1 \left(3 - 2\sqrt{y} - y\right) dy$

$= \pi \left[3y - \frac{4}{3}y^{3/2} - \frac{y^2}{2}\right]_0^1$

$= \pi\left(3 - \frac{4}{3} - \frac{1}{2}\right) = \pi\left(\frac{18-8-3}{6}\right) = \frac{7\pi}{6}$



46. $R(y) = 2 - y^{1/3}$ and $r(y) = 1$

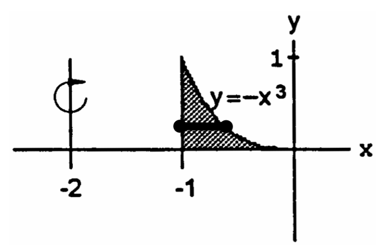
$\Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^1 \left[\left(2 - y^{1/3}\right)^2 - 1\right] dy$

$= \pi \int_0^1 \left(4 - 4y^{1/3} + y^{2/3} - 1\right) dy$

$= \pi \int_0^1 \left(3 - 4y^{1/3} + y^{2/3}\right) dy$

$= \pi \left[3y - 3y^{4/3} + \frac{3y^{5/3}}{5}\right]_0^1 = \pi\left(3 - 3 + \frac{3}{5}\right) = \frac{3\pi}{5}$



47. (a) $r(x) = \sqrt{x}$ and $R(x) = 2$

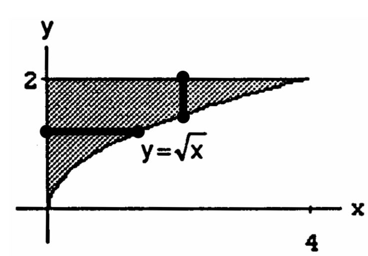
$\Rightarrow V = \int_0^4 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_0^4 (4 - x)\, dx = \pi \left[4x - \frac{x^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$

(b) $r(y) = 0$ and $R(y) = y^2$

$\Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^2 y^4\, dy = \pi \left[\frac{y^5}{5}\right]_0^2 = \frac{32\pi}{5}$

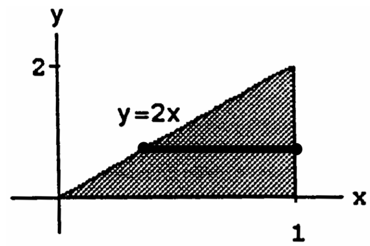


(c) $r(x) = 0$ and $R(x) = 2 - \sqrt{x} \Rightarrow V = \int_0^4 \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_0^4 \left(2 - \sqrt{x}\right)^2 dx$

$= \pi \int_0^4 \left(4 - 4\sqrt{x} + x\right) dx = \pi \left[4x - \frac{8x^{3/2}}{3} + \frac{x^2}{2}\right]_0^4 = \pi\left(16 - \frac{64}{3} + \frac{16}{2}\right) = \frac{8\pi}{3}$

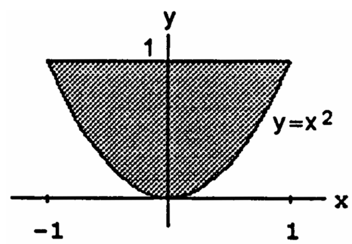
(d) $r(y) = 4 - y^2$ and $R(y) = 4 \Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi \int_0^2 \left[16 - \left(4 - y^2\right)^2\right] dy$
$= \pi \int_0^2 \left(16 - 16 + 8y^2 - y^4\right) dy = \pi \int_0^2 \left(8y^2 - y^4\right) dy = \pi \left[\frac{8}{3} y^3 - \frac{y^5}{5}\right]_0^2 = \pi\left(\frac{64}{3} - \frac{32}{5}\right) = \frac{224\pi}{15}$

48. (a) $r(y) = 0$ and $R(y) = 1 - \frac{y}{2}$
$\Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$
$= \pi \int_0^2 \left(1 - \frac{y}{2}\right)^2 dy = \pi \int_0^2 \left(1 - y + \frac{y^2}{4}\right) dy$
$= \pi \left[y - \frac{y^2}{2} + \frac{y^3}{12}\right]_0^2 = \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$



(b) $r(y) = 1$ and $R(y) = 2 - \frac{y}{2}$
$\Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi \int_0^2 \left[\left(2 - \frac{y}{2}\right)^2 - 1\right] dy = \pi \int_0^2 \left(4 - 2y + \frac{y^2}{4} - 1\right) dy$
$= \pi \int_0^2 \left(3 - 2y + \frac{y^2}{4}\right) dy = \pi \left[3y - y^2 + \frac{y^3}{12}\right]_0^2 = \pi\left(6 - 4 + \frac{8}{12}\right) = \pi\left(2 + \frac{2}{3}\right) = \frac{8\pi}{3}$

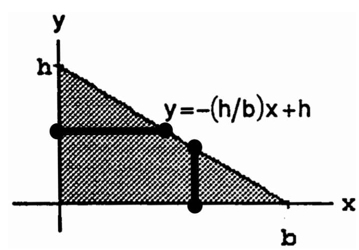
49. (a) $r(x) = 0$ and $R(x) = 1 - x^2$
$\Rightarrow V = \int_{-1}^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$
$= \pi \int_{-1}^1 \left(1 - x^2\right)^2 dx = \pi \int_{-1}^1 \left(1 - 2x^2 + x^4\right) dx$
$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right)$
$= 2\pi\left(\frac{15-10+3}{15}\right) = \frac{16\pi}{15}$



(b) $r(x) = 1$ and $R(x) = 2 - x^2 \Rightarrow V = \int_{-1}^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^1 \left[\left(2 - x^2\right)^2 - 1\right] dx$
$= \pi \int_{-1}^1 \left(4 - 4x^2 + x^4 - 1\right) dx = \pi \int_{-1}^1 \left(3 - 4x^2 + x^4\right) dx = \pi \left[3x - \frac{4}{3} x^3 + \frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(3 - \frac{4}{3} + \frac{1}{5}\right)$
$= \frac{2\pi}{15}(45 - 20 + 3) = \frac{56\pi}{15}$

(c) $r(x) = 1 + x^2$ and $R(x) = 2 \Rightarrow V = \int_{-1}^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^1 \left[4 - \left(1 + x^2\right)^2\right] dx$
$= \pi \int_{-1}^1 \left(4 - 1 - 2x^2 - x^4\right) dx = \pi \int_{-1}^1 \left(3 - 2x^2 - x^4\right) dx = \pi \left[3x - \frac{2}{3} x^3 - \frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(3 - \frac{2}{3} - \frac{1}{5}\right)$
$= \frac{2\pi}{15}(45 - 10 - 3) = \frac{64\pi}{15}$

50. (a) $r(x) = 0$ and $R(x) = -\frac{h}{b} x + h$
$\Rightarrow V = \int_0^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx$
$= \pi \int_0^b \left(-\frac{h}{b} x + h\right)^2 dx$
$= \pi \int_0^b \left(\frac{h^2}{b^2} x^2 - \frac{2h^2}{b} x + h^2\right) dx$
$= \pi h^2 \left[\frac{x^3}{3b^2} - \frac{x^2}{b} + x\right]_0^b = \pi h^2\left(\frac{b}{3} - b + b\right) = \frac{\pi h^2 b}{3}$



(b) $r(y) = 0$ and $R(y) = b\left(1 - \frac{y}{h}\right) \Rightarrow V = \int_0^h \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi b^2 \int_0^h \left(1 - \frac{y}{h}\right)^2 dy$
$= \pi b^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy = \pi b^2 \left[y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right]_0^h = \pi b^2\left(h - h + \frac{h}{3}\right) = \frac{\pi b^2 h}{3}$

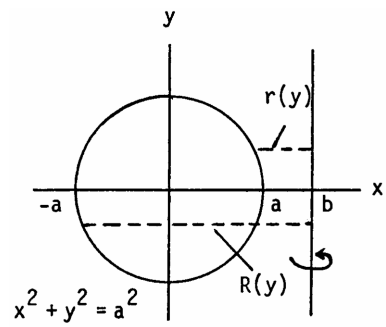
51. $R(y) = b + \sqrt{a^2 - y^2}$ and $r(y) = b - \sqrt{a^2 - y^2}$

$\Rightarrow\ V = \int_{-a}^{a} \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_{-a}^{a} \left[\left(b + \sqrt{a^2 - y^2}\right)^2 - \left(b - \sqrt{a^2 - y^2}\right)^2\right] dy$

$= \pi \int_{-a}^{a} 4b\sqrt{a^2 - y^2}\, dy = 4b\pi \int_{-a}^{a} \sqrt{a^2 - y^2}\, dy$

$= 4b\pi \cdot$ area of semicircle of radius $a = 4b\pi \cdot \frac{\pi a^2}{2} = 2a^2 b\pi^2$



52. (a) A cross section has radius $r = \sqrt{2y}$ and area $\pi r^2 = 2\pi y$. The volume is $\int_0^5 2\pi y dy = \pi\left[y^2\right]_0^5 = 25\pi$.

(b) $V(h) = \int A(h)dh$, so $\frac{dV}{dh} = A(h)$. Therefore $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = A(h) \cdot \frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{1}{A(h)} \cdot \frac{dV}{dt}$.

For $h = 4$, the area is $2\pi(4) = 8\pi$, so $\frac{dh}{dt} = \frac{1}{8\pi} \cdot 3\,\frac{\text{units}^3}{\text{sec}} = \frac{3}{8\pi} \cdot \frac{\text{units}^3}{\text{sec}}$.

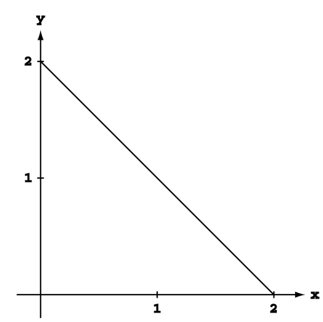
53. (a) $R(y) = \sqrt{a^2 - y^2} \Rightarrow V = \pi \int_{-a}^{h-a} \left(a^2 - y^2\right) dy = \pi\left[a^2 y - \frac{y^3}{3}\right]_{-a}^{h-a} = \pi\left[a^2 h - a^3 - \frac{(h-a)^3}{3} - \left(-a^3 + \frac{a^3}{3}\right)\right]$

$= \pi\left[a^2 h - \frac{1}{3}\left(h^3 - 3h^2 a + 3ha^2 - a^3\right) - \frac{a^3}{3}\right] = \pi\left(a^2 h - \frac{h^3}{3} + h^2 a - ha^2\right) = \frac{\pi h^2(3a - h)}{3}$

(b) Given $\frac{dV}{dt} = 0.2\ m^3/\text{sec}$ and $a = 5$ m, find $\left.\frac{dh}{dt}\right|_{h=4}$. From part (a), $V(h) = \frac{\pi h^2(15 - h)}{3} = 5\pi h^2 - \frac{\pi h^3}{3}$

$\Rightarrow \frac{dV}{dh} = 10\pi h - \pi h^2 \Rightarrow \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = \pi h(10 - h)\frac{dh}{dt} \Rightarrow \left.\frac{dh}{dt}\right|_{h=4} = \frac{0.2}{4\pi(10-4)} = \frac{1}{(20\pi)(6)} = \frac{1}{120\pi}$ m/sec.

54. Suppose the solid is produced by revolving $y = 2 - x$ about the y-axis. Cast a shadow of the solid on a plane parallel to the xy-plane.
Use an approximation such as the Trapezoid Rule, to estimate $\int_a^b \pi[R(y)]^2\, dy \approx \sum_{k=1}^{n} \pi\left(\frac{d_{\hat{k}}}{2}\right)^2 \triangle y$.



55. The cross section of a solid right circular cylinder with a cone removed is a disk with radius R from which a disk of radius h has been removed. Thus its area is $A_1 = \pi R^2 - \pi h^2 = \pi\left(R^2 - h^2\right)$. The cross section of the hemisphere is a disk of radius $\sqrt{R^2 - h^2}$. Therefore its area is $A_2 = \pi\left(\sqrt{R^2 - h^2}\right)^2 = \pi\left(R^2 - h^2\right)$. We can see that $A_1 = A_2$. The altitudes of both solids are R. Applying Cavalieri's Principle we find
Volume of Hemisphere = (Volume of Cylinder) − (Volume of Cone) $= \left(\pi R^2\right) R - \frac{1}{3}\pi\left(R^2\right) R = \frac{2}{3}\pi R^3$.

56. $R(x) = \frac{x}{12}\sqrt{36 - x^2} \Rightarrow V = \int_0^6 \pi[R(x)]^2\, dx = \pi \int_0^6 \frac{x^2}{144}\left(36 - x^2\right) dx = \frac{\pi}{144}\int_0^6 \left(36x^2 - x^4\right) dx$

$= \frac{\pi}{144}\left[12x^3 - \frac{x^5}{5}\right]_0^6 = \frac{\pi}{144}\left(12 \cdot 6^3 - \frac{6^5}{5}\right) = \frac{\pi \cdot 6^3}{144}\left(12 - \frac{36}{5}\right) = \left(\frac{196\pi}{144}\right)\left(\frac{60-36}{5}\right) = \frac{36\pi}{5}\ \text{cm}^3$. The plumb bob will weigh about $W = (8.5)\left(\frac{36\pi}{5}\right) \approx 192$ gm, to the nearest gram.

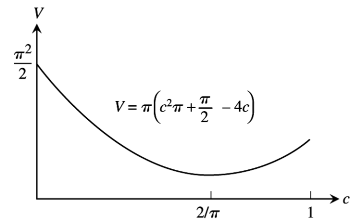
57. $R(y) = \sqrt{256 - y^2} \Rightarrow V = \int_{-16}^{-7} \pi[R(y)]^2\, dy = \pi \int_{-16}^{-7} \left(256 - y^2\right) dy = \pi\left[256y - \frac{y^3}{3}\right]_{-16}^{-7}$

$= \pi\left[(256)(-7) + \frac{7^3}{3} - \left((256)(-16) + \frac{16^3}{3}\right)\right] = \pi\left(\frac{7^3}{3} + 256(16 - 7) - \frac{16^3}{3}\right) = 1053\pi\ \text{cm}^3 \approx 3308\ \text{cm}^3$

58. (a) $R(x) = |c - \sin x|$, so $V = \pi\int_0^{\pi} [R(x)]^2\,dx = \pi\int_0^{\pi} (c - \sin x)^2\,dx = \pi\int_0^{\pi} (c^2 - 2c\sin x + \sin^2 x)\,dx$
$= \pi\int_0^{\pi} \left(c^2 - 2c\sin x + \frac{1-\cos 2x}{2}\right) dx = \pi\int_0^{\pi} \left(c^2 + \frac{1}{2} - 2c\sin x - \frac{\cos 2x}{2}\right) dx$
$= \pi\left[\left(c^2 + \frac{1}{2}\right)x + 2c\cos x - \frac{\sin 2x}{4}\right]_0^{\pi} = \pi\left[\left(c^2\pi + \frac{\pi}{2} - 2c - 0\right) - (0 + 2c - 0)\right] = \pi\left(c^2\pi + \frac{\pi}{2} - 4c\right)$. Let $V(c) = \pi\left(c^2\pi + \frac{\pi}{2} - 4c\right)$. We find the extreme values of V(c): $\frac{dV}{dc} = \pi(2c\pi - 4) = 0 \Rightarrow c = \frac{2}{\pi}$ is a critical point, and $V\left(\frac{2}{\pi}\right) = \pi\left(\frac{4}{\pi} + \frac{\pi}{2} - \frac{8}{\pi}\right) = \pi\left(\frac{\pi}{2} - \frac{4}{\pi}\right) = \frac{\pi^2}{2} - 4$; Evaluate V at the endpoints: $V(0) = \frac{\pi^2}{2}$ and $V(1) = \pi\left(\frac{3}{2}\pi - 4\right) = \frac{\pi^2}{2} - (4 - \pi)\pi$. Now we see that the function's absolute minimum value is $\frac{\pi^2}{2} - 4$, taken on at the critical point $c = \frac{2}{\pi}$. (See also the accompanying graph.)

(b) From the discussion in part (a) we conclude that the function's absolute maximum value is $\frac{\pi^2}{2}$, taken on at the endpoint $c = 0$.

(c) The graph of the solid's volume as a function of c for $0 \le c \le 1$ is given at the right. As c moves away from $[0, 1]$ the volume of the solid increases without bound. If we approximate the solid as a set of solid disks, we can see that the radius of a typical disk increases without bounds as c moves away from $[0, 1]$.



59. Volume of the solid generated by rotating the region bounded by the x-axis and $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is $V = \int_a^b \pi[f(x)]^2\,dx = 4\pi$, and the volume of the solid generated by rotating the same region about the line $y = -1$ is $V = \int_a^b \pi[f(x) + 1]^2\,dx = 8\pi$. Thus $\int_a^b \pi[f(x) + 1]^2\,dx - \int_a^b \pi[f(x)]^2\,dx = 8\pi - 4\pi$
$\Rightarrow \pi\int_a^b ([f(x)]^2 + 2f(x) + 1 - [f(x)]^2)\,dx = 4\pi \Rightarrow \int_a^b (2f(x) + 1)\,dx = 4 \Rightarrow 2\int_a^b f(x)\,dx + \int_a^b dx = 4$
$\Rightarrow \int_a^b f(x)\,dx + \frac{1}{2}(b - a) = 2 \Rightarrow \int_a^b f(x)\,dx = \frac{4 - b + a}{2}$

60. Volume of the solid generated by rotating the region bounded by the x-axis and $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is $V = \int_a^b \pi[f(x)]^2\,dx = 6\pi$, and the volume of the solid generated by rotating the same region about the line $y = -2$ is $V = \int_a^b \pi[f(x) + 2]^2\,dx = 10\pi$. Thus $\int_a^b \pi[f(x) + 2]^2\,dx - \int_a^b \pi[f(x)]^2\,dx = 10\pi - 6\pi$
$\Rightarrow \pi\int_a^b ([f(x)]^2 + 4f(x) + 4 - [f(x)]^2)\,dx = 4\pi \Rightarrow \int_a^b (4f(x) + 4)\,dx = 4 \Rightarrow 4\int_a^b f(x)\,dx + 4\int_a^b dx = 4$
$\Rightarrow \int_a^b f(x)\,dx + (b - a) = 1 \Rightarrow \int_a^b f(x)\,dx = 1 - b + a$

6.2 VOLUME USING CYLINDRICAL SHELLS

1. For the sketch given, $a = 0$, $b = 2$;
$V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^2 2\pi x\left(1 + \frac{x^2}{4}\right) dx = 2\pi\int_0^2 \left(x + \frac{x^3}{4}\right) dx = 2\pi\left[\frac{x^2}{2} + \frac{x^4}{16}\right]_0^2 = 2\pi\left(\frac{4}{2} + \frac{16}{16}\right)$
$= 2\pi \cdot 3 = 6\pi$

2. For the sketch given, $a = 0$, $b = 2$;
$V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^2 2\pi x\left(2 - \frac{x^2}{4}\right) dx = 2\pi\int_0^2 \left(2x - \frac{x^3}{4}\right) dx = 2\pi\left[x^2 - \frac{x^4}{16}\right]_0^2 = 2\pi(4 - 1) = 6\pi$

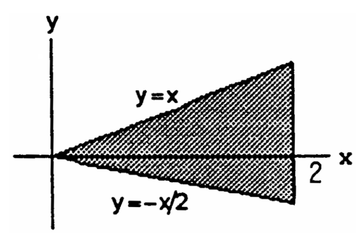
3. For the sketch given, $c = 0$, $d = \sqrt{2}$;
$V = \int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^{\sqrt{2}} 2\pi y \cdot (y^2)\,dy = 2\pi\int_0^{\sqrt{2}} y^3\,dy = 2\pi\left[\frac{y^4}{4}\right]_0^{\sqrt{2}} = 2\pi$

4. For the sketch given, $c = 0$, $d = \sqrt{3}$;
$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^{\sqrt{3}} 2\pi y \cdot [3 - (3 - y^2)]\, dy = 2\pi \int_0^{\sqrt{3}} y^3\, dy = 2\pi \left[\frac{y^4}{4}\right]_0^{\sqrt{3}} = \frac{9\pi}{2}$

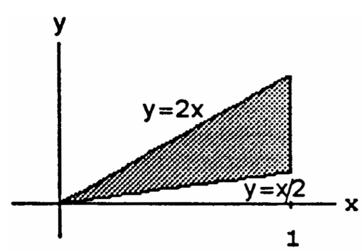
5. For the sketch given, $a = 0$, $b = \sqrt{3}$;
$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^{\sqrt{3}} 2\pi x \cdot \left(\sqrt{x^2+1}\right) dx$;
$\left[u = x^2 + 1 \Rightarrow du = 2x\, dx;\ x = 0 \Rightarrow u = 1,\ x = \sqrt{3} \Rightarrow u = 4\right]$
$\rightarrow V = \pi \int_1^4 u^{1/2}\, du = \pi \left[\frac{2}{3} u^{3/2}\right]_1^4 = \frac{2\pi}{3}\left(4^{3/2} - 1\right) = \left(\frac{2\pi}{3}\right)(8 - 1) = \frac{14\pi}{3}$

6. For the sketch given, $a = 0$, $b = 3$;
$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^3 2\pi x \left(\frac{9x}{\sqrt{x^3+9}}\right) dx$;
$[u = x^3 + 9 \Rightarrow du = 3x^2\, dx \Rightarrow 3\, du = 9x^2\, dx;\ x = 0 \Rightarrow u = 9,\ x = 3 \Rightarrow u = 36]$
$\rightarrow V = 2\pi \int_9^{36} 3u^{-1/2}\, du = 6\pi \left[2u^{1/2}\right]_9^{36} = 12\pi\left(\sqrt{36} - \sqrt{9}\right) = 36\pi$

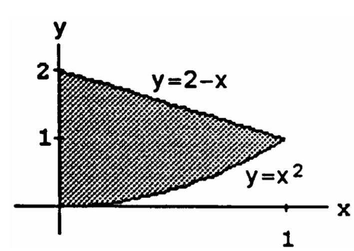
7. $a = 0$, $b = 2$;
$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^2 2\pi x \left[x - \left(-\frac{x}{2}\right)\right] dx$
$= \int_0^2 2\pi x^2 \cdot \frac{3}{2}\, dx = \pi \int_0^2 3x^2\, dx = \pi \left[x^3\right]_0^2 = 8\pi$



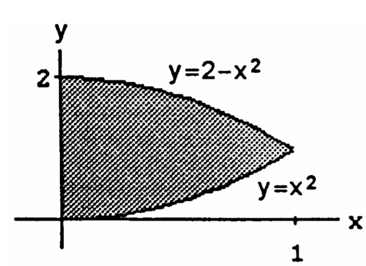
8. $a = 0$, $b = 1$;
$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x \left(2x - \frac{x}{2}\right) dx$
$= \pi \int_0^1 2\left(\frac{3x^2}{2}\right) dx = \pi \int_0^1 3x^2\, dx = \pi \left[x^3\right]_0^1 = \pi$



9. $a = 0$, $b = 1$;
$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x \left[(2 - x) - x^2\right] dx$
$= 2\pi \int_0^1 (2x - x^2 - x^3)\, dx = 2\pi \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$
$= 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = 2\pi\left(\frac{12 - 4 - 3}{12}\right) = \frac{10\pi}{12} = \frac{5\pi}{6}$

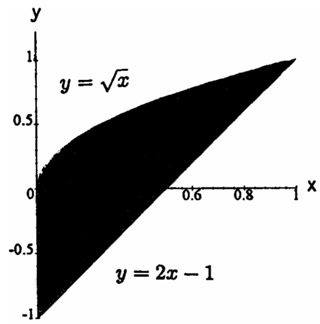


10. $a = 0$, $b = 1$;
$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x \left[(2 - x^2) - x^2\right] dx$
$= 2\pi \int_0^1 x\,(2 - 2x^2)\, dx = 4\pi \int_0^1 (x - x^3)\, dx$
$= 4\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 4\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \pi$



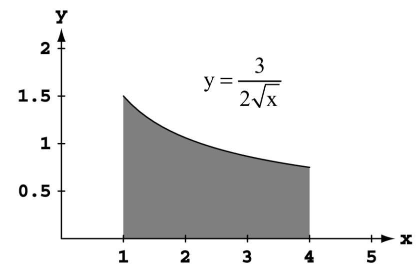
11. $a = 0, b = 1;$

$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right) \left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x \left[\sqrt{x} - (2x-1)\right] dx$
$= 2\pi \int_0^1 \left(x^{3/2} - 2x^2 + x\right) dx = 2\pi \left[\frac{2}{5} x^{5/2} - \frac{2}{3} x^3 + \frac{1}{2} x^2\right]_0^1$
$= 2\pi \left(\frac{2}{5} - \frac{2}{3} + \frac{1}{2}\right) = 2\pi \left(\frac{12-20+15}{30}\right) = \frac{7\pi}{15}$



12. $a = 1, b = 4;$

$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right) \left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_1^4 2\pi x \left(\frac{3}{2} x^{-1/2}\right) dx$
$= 3\pi \int_1^4 x^{1/2}\, dx = 3\pi \left[\frac{2}{3} x^{3/2}\right]_1^4 = 2\pi \left(4^{3/2} - 1\right)$
$= 2\pi(8-1) = 14\pi$



13. (a) $xf(x) = \begin{cases} x \cdot \frac{\sin x}{x}, & 0 < x \le \pi \\ x, & x = 0 \end{cases} \Rightarrow xf(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ 0, & x = 0 \end{cases}$; since $\sin 0 = 0$ we have

$xf(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ \sin x, & x = 0 \end{cases} \Rightarrow xf(x) = \sin x,\ 0 \le x \le \pi$

(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right) \left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^\pi 2\pi x \cdot f(x)\, dx$ and $x \cdot f(x) = \sin x,\ 0 \le x \le \pi$ by part (a)

$\Rightarrow V = 2\pi \int_0^\pi \sin x\, dx = 2\pi[-\cos x]_0^\pi = 2\pi(-\cos \pi + \cos 0) = 4\pi$

14. (a) $xg(x) = \begin{cases} x \cdot \frac{\tan^2 x}{x}, & 0 < x \le \frac{\pi}{4} \\ x \cdot 0, & x = 0 \end{cases} \Rightarrow xg(x) = \begin{cases} \tan^2 x, & 0 < x \le \pi/4 \\ 0, & x = 0 \end{cases}$; since $\tan 0 = 0$ we have

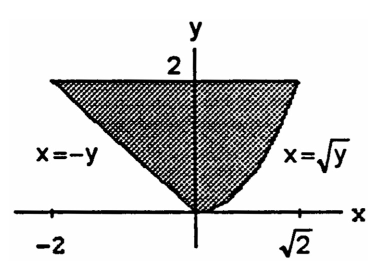
$xg(x) = \begin{cases} \tan^2 x, & 0 < x \le \pi/4 \\ \tan^2 x, & x = 0 \end{cases} \Rightarrow xg(x) = \tan^2 x,\ 0 \le x \le \pi/4$

(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right) \left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^{\pi/4} 2\pi x \cdot g(x)\, dx$ and $x \cdot g(x) = \tan^2 x,\ 0 \le x \le \pi/4$ by part (a)

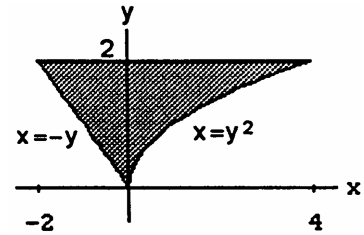
$\Rightarrow V = 2\pi \int_0^{\pi/4} \tan^2 x\, dx = 2\pi \int_0^{\pi/4} (\sec^2 x - 1)\, dx = 2\pi[\tan x - x]_0^{\pi/4} = 2\pi \left(1 - \frac{\pi}{4}\right) = \frac{4\pi - \pi^2}{2}$

15. $c = 0, d = 2;$

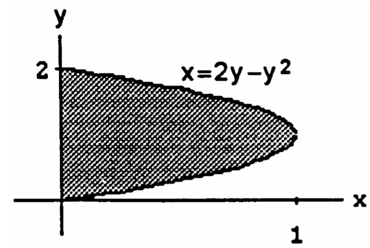
$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right) \left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y \left[\sqrt{y} - (-y)\right] dy$
$= 2\pi \int_0^2 \left(y^{3/2} + y^2\right) dy = 2\pi \left[\frac{2y^{5/2}}{5} + \frac{y^3}{3}\right]_0^2$
$= 2\pi \left[\frac{2}{5}\left(\sqrt{2}\right)^5 + \frac{2^3}{3}\right] = 2\pi \left(\frac{8\sqrt{2}}{5} + \frac{8}{3}\right) = 16\pi \left(\frac{\sqrt{2}}{5} + \frac{1}{3}\right)$
$= \frac{16\pi}{15}\left(3\sqrt{2} + 5\right)$



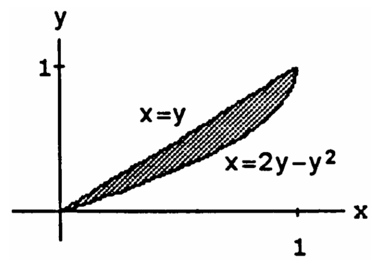
16. $c = 0, d = 2;$
$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y\,[y^2 - (-y)]dy$
$= 2\pi \int_0^2 (y^3 + y^2)\,dy = 2\pi \left[\frac{y^4}{4} + \frac{y^3}{3}\right]_0^2 = 16\pi\left(\frac{2}{4} + \frac{1}{3}\right)$
$= 16\pi\left(\frac{5}{6}\right) = \frac{40\pi}{3}$



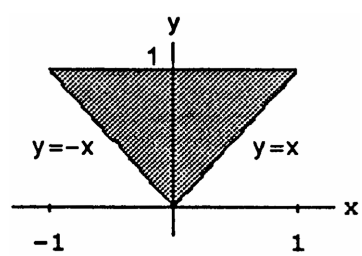
17. $c = 0, d = 2;$
$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y\,(2y - y^2)dy$
$= 2\pi \int_0^2 (2y^2 - y^3)\,dy = 2\pi \left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = 2\pi\left(\frac{16}{3} - \frac{16}{4}\right)$
$= 32\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{32\pi}{12} = \frac{8\pi}{3}$



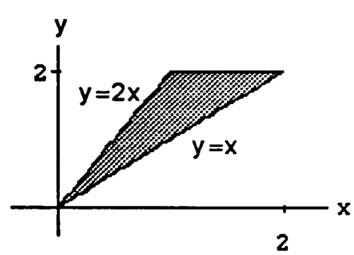
18. $c = 0, d = 1;$
$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^1 2\pi y\,(2y - y^2 - y)dy$
$= 2\pi \int_0^1 y\,(y - y^2)\,dy = 2\pi \int_0^1 (y^2 - y^3)\,dy$
$= 2\pi \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$



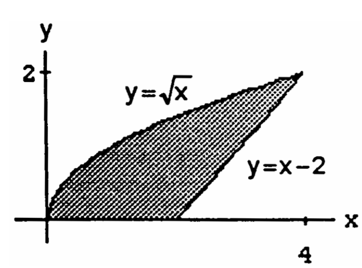
19. $c = 0, d = 1;$
$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = 2\pi \int_0^1 y[y - (-y)]dy$
$= 2\pi \int_0^1 2y^2\,dy = \frac{4\pi}{3}\left[y^3\right]_0^1 = \frac{4\pi}{3}$



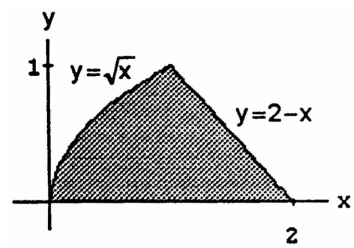
20. $c = 0, d = 2;$
$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi\, y\left(y - \frac{y}{2}\right)dy$
$= 2\pi \int_0^2 \frac{y^2}{2}dy = \frac{\pi}{3}\left[y^3\right]_0^2 = \frac{8\pi}{3}$



21. $c = 0, d = 2;$
$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y\,[(2 + y) - y^2]\,dy$
$= 2\pi \int_0^2 (2y + y^2 - y^3)\,dy = 2\pi \left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2$
$= 2\pi\left(4 + \frac{8}{3} - \frac{16}{4}\right) = \frac{\pi}{6}(48 + 32 - 48) = \frac{16\pi}{3}$



22. $c = 0, d = 1;$
$V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^1 2\pi y\,[(2-y)-y^2]\,dy$
$= 2\pi\int_0^1 (2y - y^2 - y^3)\,dy = 2\pi\left[y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$
$= 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}(12-4-3) = \frac{5\pi}{6}$



23. (a) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^2 2\pi x\,(3x)dx = 6\pi\int_0^2 x^2\,dx = 2\pi\left[x^3\right]_0^2 = 16\pi$

(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^2 2\pi(4-x)(3x)dx = 6\pi\int_0^2 (4x - x^2)dx = 6\pi\left[2x^2 - \frac{1}{3}x^3\right]_0^2 = 6\pi\left(8 - \frac{8}{3}\right) = 32\pi$

(c) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^2 2\pi(x+1)(3x)dx = 6\pi\int_0^2 (x^2 + x)dx = 6\pi\left[\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^2 = 6\pi\left(\frac{8}{3} + 2\right) = 28\pi$

(d) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^6 2\pi y\left(2 - \frac{1}{3}y\right)dy = 2\pi\int_0^6 \left(2y - \frac{1}{3}y^2\right)dy = 2\pi\left[y^2 - \frac{1}{9}y^3\right]_0^6 = 2\pi(36-24) = 24\pi$

(e) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^6 2\pi(7-y)\left(2 - \frac{1}{3}y\right)dy = 2\pi\int_0^6 \left(14 - \frac{13}{3}y + \frac{1}{3}y^2\right)dy = 2\pi\left[14y - \frac{13}{6}y^2 + \frac{1}{9}y^3\right]_0^6$
$= 2\pi(84 - 78 + 24) = 60\pi$

(f) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^6 2\pi(y+2)\left(2 - \frac{1}{3}y\right)dy = 2\pi\int_0^6 \left(4 + \frac{4}{3}y - \frac{1}{3}y^2\right)dy = 2\pi\left[4y + \frac{2}{3}y^2 - \frac{1}{9}y^3\right]_0^6$
$= 2\pi(24 + 24 - 24) = 48\pi$

24. (a) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^2 2\pi x\,(8 - x^3)dx = 2\pi\int_0^2 (8x - x^4)dx = 2\pi\left[4x^2 - \frac{1}{5}x^5\right]_0^2 = 2\pi\left(16 - \frac{32}{5}\right) = \frac{96\pi}{5}$

(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^2 2\pi(3-x)\,(8 - x^3)dx = 2\pi\int_0^2 (24 - 8x - 3x^3 + x^4)dx$
$= 2\pi\left[24x - 4x^2 - \frac{3}{4}x^4 + \frac{1}{5}x^5\right]_0^2 = 2\pi\left(48 - 16 - 12 + \frac{32}{5}\right) = \frac{264\pi}{5}$

(c) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^2 2\pi(x+2)\,(8 - x^3)dx = 2\pi\int_0^2 (16 + 8x - 2x^3 - x^4)dx$
$= 2\pi\left[16x + 4x^2 - \frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2 = 2\pi\left(32 + 16 - 8 - \frac{32}{5}\right) = \frac{336\pi}{5}$

(d) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^8 2\pi y \cdot y^{1/3}dy = 2\pi\int_0^8 y^{4/3}dy = \frac{6\pi}{7}\left[y^{7/3}\right]_0^8 = \frac{6\pi}{7}(128) = \frac{768\pi}{7}$

(e) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^8 2\pi(8-y)\,y^{1/3}dy = 2\pi\int_0^8 \left(8y^{1/3} - y^{4/3}\right)dy = 2\pi\left[6y^{4/3} - \frac{3}{7}y^{7/3}\right]_0^8$
$= 2\pi\left(96 - \frac{384}{7}\right) = \frac{576\pi}{7}$

(f) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^8 2\pi(y+1)\,y^{1/3}dx = 2\pi\int_0^8 \left(y^{4/3} + y^{1/3}\right)dy = 2\pi\left[\frac{3}{7}y^{7/3} + \frac{3}{4}y^{4/3}\right]_0^8$
$= 2\pi\left(\frac{384}{7} + 12\right) = \frac{936\pi}{7}$

25. (a) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_{-1}^2 2\pi(2-x)(x + 2 - x^2)dx = 2\pi\int_{-1}^2 (4 - 3x^2 + x^3)dx = 2\pi\left[4x - x^3 + \frac{1}{4}x^4\right]_{-1}^2$
$= 2\pi(8 - 8 + 4) - 2\pi\left(-4 + 1 + \frac{1}{4}\right) = \frac{27\pi}{2}$

. (b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_{-1}^2 2\pi(x+1)(x + 2 - x^2)dx = 2\pi\int_{-1}^2 (2 + 3x - x^3)dx = 2\pi\left[2x + \frac{3}{2}x^2 - \frac{1}{4}x^4\right]_{-1}^2$
$= 2\pi(4 + 6 - 4) - 2\pi\left(-2 + \frac{3}{2} - \frac{1}{4}\right) = \frac{27\pi}{2}$

(c) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^1 2\pi y\left(\sqrt{y} - \left(-\sqrt{y}\right)\right)dy + \int_1^4 2\pi y\left(\sqrt{y} - (y-2)\right)dy$
$= 4\pi\int_0^1 y^{3/2}dy + 2\pi\int_1^4 \left(y^{3/2} - y^2 + 2y\right)dy = \frac{8\pi}{5}\left[y^{5/2}\right]_0^1 + 2\pi\left[\frac{2}{5}y^{5/2} - \frac{1}{3}y^3 + y^2\right]_1^4$
$= \frac{8\pi}{5}(1) + 2\pi\left(\frac{64}{5} - \frac{64}{3} + 16\right) - 2\pi\left(\frac{2}{5} - \frac{1}{3} + 1\right) = \frac{72\pi}{5}$

(d) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi(4-y)\left(\sqrt{y} - \left(-\sqrt{y}\right)\right)dy + \int_1^4 2\pi(4-y)\left(\sqrt{y} - (y-2)\right)dy$

$= 4\pi \int_0^1 \left(4\sqrt{y} - y^{3/2}\right)dy + 2\pi \int_1^4 \left(y^2 - y^{3/2} - 6y + 4\sqrt{y} + 8\right)dy$

$= 4\pi \left[\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1 + 2\pi \left[\frac{1}{3}y^3 - \frac{2}{5}y^{5/2} - 3y^2 + \frac{8}{3}y^{3/2} + 8y\right]_1^4$

$= 4\pi\left(\frac{8}{3} - \frac{2}{5}\right) + 2\pi\left(\frac{64}{3} - \frac{64}{5} - 48 + \frac{64}{3} + 32\right) - 2\pi\left(\frac{1}{3} - \frac{2}{5} - 3 + \frac{8}{3} + 8\right) = \frac{108\pi}{5}$

26. (a) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = \int_{-1}^1 2\pi(1-x)(4 - 3x^2 - x^4)dx = 2\pi \int_{-1}^1 (x^5 - x^4 + 3x^3 - 3x^2 - 4x + 4)dx$

$= 2\pi\left[\frac{1}{6}x^6 - \frac{1}{5}x^5 + \frac{3}{4}x^4 - x^3 - 2x^2 + 4x\right]_{-1}^1 = 2\pi\left(\frac{1}{6} - \frac{1}{5} + \frac{3}{4} - 1 - 2 + 4\right) - 2\pi\left(\frac{1}{6} + \frac{1}{5} + \frac{3}{4} + 1 - 2 - 4\right) = \frac{56\pi}{5}$

(b) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi y\left(\sqrt[4]{y} - \left(-\sqrt[4]{y}\right)\right)dy + \int_1^4 2\pi y\left[\sqrt{\frac{4-y}{3}} - \left(-\sqrt{\frac{4-y}{3}}\right)\right]dy$

$= 4\pi \int_0^1 y^{5/4}dy + \frac{4\pi}{\sqrt{3}} \int_1^4 y\sqrt{4-y}\,dy$ $[u = 4 - y \Rightarrow y = 4 - u \Rightarrow du = -dy;\ y = 1 \Rightarrow u = 3,\ y = 4 \Rightarrow u = 0]$

$= \frac{16\pi}{9}\left[y^{9/4}\right]_0^1 - \frac{4\pi}{\sqrt{3}} \int_3^0 (4-u)\sqrt{u}\,du = \frac{16\pi}{9}(1) + \frac{4\pi}{\sqrt{3}} \int_0^3 \left(4\sqrt{u} - u^{3/2}\right)du = \frac{16\pi}{9} + \frac{4\pi}{\sqrt{3}}\left[\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^3$

$= \frac{16\pi}{9} + \frac{4\pi}{\sqrt{3}}\left(8\sqrt{3} - \frac{18}{5}\sqrt{3}\right) = \frac{16\pi}{9} + \frac{88\pi}{5} = \frac{872\pi}{45}$

27. (a) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi y \cdot 12(y^2 - y^3)\,dy = 24\pi \int_0^1 (y^3 - y^4)\,dy = 24\pi\left[\frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$

$= 24\pi\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{24\pi}{20} = \frac{6\pi}{5}$

(b) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi(1-y)\left[12(y^2 - y^3)\right]dy = 24\pi \int_0^1 (1-y)(y^2 - y^3)\,dy$

$= 24\pi \int_0^1 (y^2 - 2y^3 + y^4)\,dy = 24\pi\left[\frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5}\right]_0^1 = 24\pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = 24\pi\left(\frac{1}{30}\right) = \frac{4\pi}{5}$

(c) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi\left(\frac{8}{5} - y\right)\left[12(y^2 - y^3)\right]dy = 24\pi \int_0^1 \left(\frac{8}{5} - y\right)(y^2 - y^3)\,dy$

$= 24\pi \int_0^1 \left(\frac{8}{5}y^2 - \frac{13}{5}y^3 + y^4\right)dy = 24\pi\left[\frac{8}{15}y^3 - \frac{13}{20}y^4 + \frac{y^5}{5}\right]_0^1 = 24\pi\left(\frac{8}{15} - \frac{13}{20} + \frac{1}{5}\right) = \frac{24\pi}{60}(32 - 39 + 12)$

$= \frac{24\pi}{12} = 2\pi$

(d) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^1 2\pi\left(y + \frac{2}{5}\right)\left[12(y^2 - y^3)\right]dy = 24\pi \int_0^1 \left(y + \frac{2}{5}\right)(y^2 - y^3)\,dy$

$= 24\pi \int_0^1 \left(y^3 - y^4 + \frac{2}{5}y^2 - \frac{2}{5}y^3\right)dy = 24\pi \int_0^1 \left(\frac{2}{5}y^2 + \frac{3}{5}y^3 - y^4\right)dy = 24\pi\left[\frac{2}{15}y^3 + \frac{3}{20}y^4 - \frac{y^5}{5}\right]_0^1$

$= 24\pi\left(\frac{2}{15} + \frac{3}{20} - \frac{1}{5}\right) = \frac{24\pi}{60}(8 + 9 - 12) = \frac{24\pi}{12} = 2\pi$

28. (a) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi y\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right]dy = \int_0^2 2\pi y\left(y^2 - \frac{y^4}{4}\right)dy = 2\pi \int_0^2 \left(y^3 - \frac{y^5}{4}\right)dy$

$= 2\pi\left[\frac{y^4}{4} - \frac{y^6}{24}\right]_0^2 = 2\pi\left(\frac{2^4}{4} - \frac{2^6}{24}\right) = 32\pi\left(\frac{1}{4} - \frac{4}{24}\right) = 32\pi\left(\frac{1}{4} - \frac{1}{6}\right) = 32\pi\left(\frac{2}{24}\right) = \frac{8\pi}{3}$

(b) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi(2-y)\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right]dy = \int_0^2 2\pi(2-y)\left(y^2 - \frac{y^4}{4}\right)dy$

$= 2\pi \int_0^2 \left(2y^2 - \frac{y^4}{2} - y^3 + \frac{y^5}{4}\right)dy = 2\pi\left[\frac{2y^3}{3} - \frac{y^5}{10} - \frac{y^4}{4} + \frac{y^6}{24}\right]_0^2 = 2\pi\left(\frac{16}{3} - \frac{32}{10} - \frac{16}{4} + \frac{64}{24}\right) = \frac{8\pi}{5}$

(c) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi(5-y)\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right]dy = \int_0^2 2\pi(5-y)\left(y^2 - \frac{y^4}{4}\right)dy$

$= 2\pi \int_0^2 \left(5y^2 - \frac{5}{4}y^4 - y^3 + \frac{y^5}{4}\right)dy = 2\pi\left[\frac{5y^3}{3} - \frac{5y^5}{20} - \frac{y^4}{4} + \frac{y^6}{24}\right]_0^2 = 2\pi\left(\frac{40}{3} - \frac{160}{20} - \frac{16}{4} + \frac{64}{24}\right) = 8\pi$

(d) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi\left(y + \frac{5}{8}\right)\left[\frac{y^2}{2} - \left(\frac{y^4}{4} - \frac{y^2}{2}\right)\right]dy = \int_0^2 2\pi\left(y + \frac{5}{8}\right)\left(y^2 - \frac{y^4}{4}\right)dy$

$= 2\pi \int_0^2 \left(y^3 - \frac{y^5}{4} + \frac{5}{8}y^2 - \frac{5}{32}y^4\right)dy = 2\pi\left[\frac{y^4}{4} - \frac{y^6}{24} + \frac{5y^3}{24} - \frac{5y^5}{160}\right]_0^2 = 2\pi\left(\frac{16}{4} - \frac{64}{24} + \frac{40}{24} - \frac{160}{160}\right) = 4\pi$

29. (a) About x-axis: $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy$

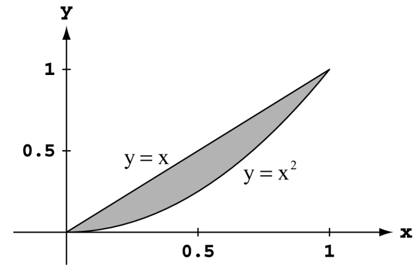
$= \int_0^1 2\pi y\left(\sqrt{y} - y\right) dy = 2\pi \int_0^1 \left(y^{3/2} - y^2\right) dy$

$= 2\pi \left[\frac{2}{5} y^{5/2} - \frac{1}{3} y^3\right]_0^1 = 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) = \frac{2\pi}{15}$

About y-axis: $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx$

$= \int_0^1 2\pi x(x - x^2) dx = 2\pi \int_0^1 (x^2 - x^3) dx$

$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$



(b) About x-axis: $R(x) = x$ and $r(x) = x^2 \Rightarrow V = \int_a^b \pi\left[R(x)^2 - r(x)^2\right] dx = \int_0^1 \pi[x^2 - x^4] dx$

$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$

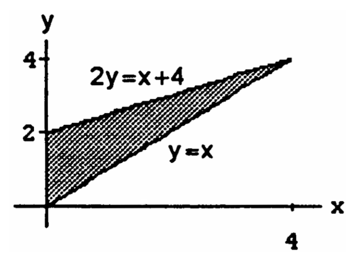
About y-axis: $R(y) = \sqrt{y}$ and $r(y) = y \Rightarrow V = \int_c^d \pi\left[R(y)^2 - r(y)^2\right] dy = \int_0^1 \pi[y - y^2] dy$

$= \pi \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$

30. (a) $V = \int_a^b \pi\left[R(x)^2 - r(x)^2\right] dx = \pi \int_0^4 \left[\left(\frac{x}{2} + 2\right)^2 - x^2\right] dx$

$= \pi \int_0^4 \left(-\frac{3}{4} x^2 + 2x + 4\right) dx = \pi \left[-\frac{x^3}{4} + x^2 + 4x\right]_0^4$

$= \pi(-16 + 16 + 16) = 16\pi$



(b) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi x\left(\frac{x}{2} + 2 - x\right) dx$

$= \int_0^4 2\pi x\left(2 - \frac{x}{2}\right) dx = 2\pi \int_0^4 \left(2x - \frac{x^2}{2}\right) dx$

$= 2\pi \left[x^2 - \frac{x^3}{6}\right]_0^4 = 2\pi\left(16 - \frac{64}{6}\right) = \frac{32\pi}{3}$

(c) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_0^4 2\pi(4 - x)\left(\frac{x}{2} + 2 - x\right) dx = \int_0^4 2\pi(4 - x)\left(2 - \frac{x}{2}\right) dx = 2\pi \int_0^4 \left(8 - 4x + \frac{x^2}{2}\right) dx$

$= 2\pi \left[8x - 2x^2 + \frac{x^3}{6}\right]_0^4 = 2\pi\left(32 - 32 + \frac{64}{6}\right) = \frac{64\pi}{3}$

(d) $V = \int_a^b \pi\left[R(x)^2 - r(x)^2\right] dx = \pi \int_0^4 \left[(8 - x)^2 - \left(6 - \frac{x}{2}\right)^2\right] dx = \pi \int_0^4 \left[(64 - 16x + x^2) - \left(36 - 6x + \frac{x^2}{4}\right)\right] dx$

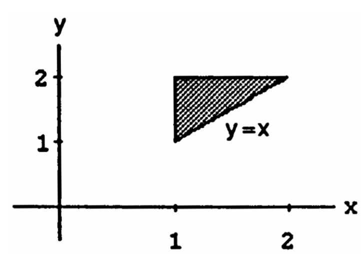
$\pi \int_0^4 \left(\frac{3}{4} x^2 - 10x + 28\right) dx = \pi \left[\frac{x^3}{4} - 5x^2 + 28x\right]_0^4 = \pi\left[16 - (5)(16) + (7)(16)\right] = \pi(3)(16) = 48\pi$

31. (a) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_1^2 2\pi y(y - 1)\, dy$

$= 2\pi \int_1^2 (y^2 - y)\, dy = 2\pi \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2$

$= 2\pi \left[\left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$

$= 2\pi \left(\frac{7}{3} - 2 + \frac{1}{2}\right) = \frac{\pi}{3}(14 - 12 + 3) = \frac{5\pi}{3}$



(b) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_1^2 2\pi x(2 - x)\, dx = 2\pi \int_1^2 (2x - x^2)\, dx = 2\pi \left[x^2 - \frac{x^3}{3}\right]_1^2$

$= 2\pi \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = 2\pi \left[\left(\frac{12 - 8}{3}\right) - \left(\frac{3 - 1}{3}\right)\right] = 2\pi \left(\frac{4}{3} - \frac{2}{3}\right) = \frac{4\pi}{3}$

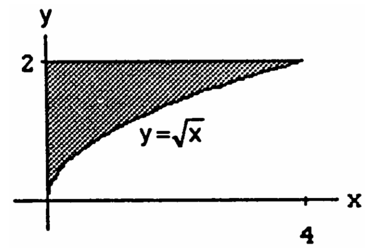
(c) $V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_1^2 2\pi \left(\frac{10}{3} - x\right)(2 - x)\, dx = 2\pi \int_1^2 \left(\frac{20}{3} - \frac{16}{3} x + x^2\right) dx$

$= 2\pi \left[\frac{20}{3} x - \frac{8}{3} x^2 + \frac{1}{3} x^3\right]_1^2 = 2\pi \left[\left(\frac{40}{3} - \frac{32}{3} + \frac{8}{3}\right) - \left(\frac{20}{3} - \frac{8}{3} + \frac{1}{3}\right)\right] = 2\pi \left(\frac{3}{3}\right) = 2\pi$

(d) $V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_1^2 2\pi(y - 1)(y - 1)\, dy = 2\pi \int_1^2 (y - 1)^2 = 2\pi \left[\frac{(y-1)^3}{3}\right]_1^2 = \frac{2\pi}{3}$

32. (a) $V = \int_c^d 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dy = \int_0^2 2\pi y(y^2 - 0)\, dy$

$= 2\pi \int_0^2 y^3\, dy = 2\pi \left[\frac{y^4}{4}\right]_0^2 = 2\pi\left(\frac{2^4}{4}\right) = 8\pi$



(b) $V = \int_a^b 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dx$

$= \int_0^4 2\pi x\left(2 - \sqrt{x}\right) dx = 2\pi \int_0^4 \left(2x - x^{3/2}\right) dx$

$= 2\pi\left[x^2 - \frac{2}{5}x^{5/2}\right]_0^4 = 2\pi\left(16 - \frac{2\cdot 2^5}{5}\right)$

$= 2\pi\left(16 - \frac{64}{5}\right) = \frac{2\pi}{5}(80 - 64) = \frac{32\pi}{5}$

(c) $V = \int_a^b 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dx = \int_0^4 2\pi(4 - x)\left(2 - \sqrt{x}\right) dx = 2\pi \int_0^4 \left(8 - 4x^{1/2} - 2x + x^{3/2}\right) dx$

$= 2\pi\left[8x - \frac{8}{3}x^{3/2} - x^2 + \frac{2}{5}x^{5/2}\right]_0^4 = 2\pi\left(32 - \frac{64}{3} - 16 + \frac{64}{5}\right) = \frac{2\pi}{15}(240 - 320 + 192) = \frac{2\pi}{15}(112) = \frac{224\pi}{15}$

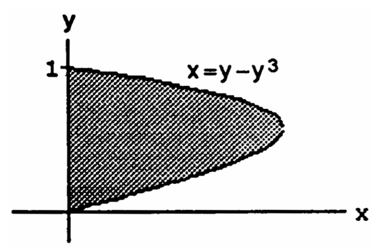
(d) $V = \int_c^d 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dy = \int_0^2 2\pi(2 - y)(y^2)\, dy = 2\pi \int_0^2 (2y^2 - y^3)\, dy = 2\pi\left[\frac{2}{3}y^3 - \frac{y^4}{4}\right]_0^2$

$= 2\pi\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{32\pi}{12}(4 - 3) = \frac{8\pi}{3}$

33. (a) $V = \int_c^d 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dy = \int_0^1 2\pi y(y - y^3)\, dy$

$= \int_0^1 2\pi\left(y^2 - y^4\right) dy = 2\pi\left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{5}\right)$

$= \frac{4\pi}{15}$



(b) $V = \int_c^d 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dy$

$= \int_0^1 2\pi(1 - y)(y - y^3)\, dy$

$= 2\pi \int_0^1 (y - y^2 - y^3 + y^4)\, dy = 2\pi\left[\frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5}\right]_0^1 = 2\pi\left(\frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\right) = \frac{2\pi}{60}(30 - 20 - 15 + 12) = \frac{7\pi}{30}$

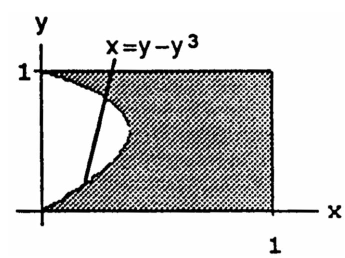
34. (a) $V = \int_c^d 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dy$

$= \int_0^1 2\pi y\left[1 - (y - y^3)\right] dy$

$= 2\pi \int_0^1 (y - y^2 + y^4)\, dy = 2\pi\left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{y^5}{5}\right]_0^1$

$= 2\pi\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{5}\right) = \frac{2\pi}{30}(15 - 10 + 6)$

$= \frac{11\pi}{15}$



(b) Use the washer method:

$V = \int_c^d \pi\left[R^2(y) - r^2(y)\right] dy = \int_0^1 \pi\left[1^2 - (y - y^3)^2\right] dy = \pi \int_0^1 (1 - y^2 - y^6 + 2y^4)\, dy = \pi\left[y - \frac{y^3}{3} - \frac{y^7}{7} + \frac{2y^5}{5}\right]_0^1$

$= \pi\left(1 - \frac{1}{3} - \frac{1}{7} + \frac{2}{5}\right) = \frac{\pi}{105}(105 - 35 - 15 + 42) = \frac{97\pi}{105}$

(c) Use the washer method:

$V = \int_c^d \pi\left[R^2(y) - r^2(y)\right] dy = \int_0^1 \pi\left[\left[1 - (y - y^3)\right]^2 - 0\right] dy = \pi \int_0^1 \left[1 - 2(y - y^3) + (y - y^3)^2\right] dy$

$= \pi \int_0^1 (1 + y^2 + y^6 - 2y + 2y^3 - 2y^4)\, dy = \pi\left[y + \frac{y^3}{3} + \frac{y^7}{7} - y^2 + \frac{y^4}{2} - \frac{2y^5}{5}\right]_0^1 = \pi\left(1 + \frac{1}{3} + \frac{1}{7} - 1 + \frac{1}{2} - \frac{2}{5}\right)$

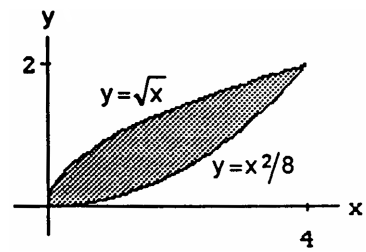
$= \frac{\pi}{210}(70 + 30 + 105 - 2\cdot 42) = \frac{121\pi}{210}$

(d) $V = \int_c^d 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dy = \int_0^1 2\pi(1 - y)\left[1 - (y - y^3)\right] dy = 2\pi \int_0^1 (1 - y)(1 - y + y^3)\, dy$

$= 2\pi \int_0^1 (1 - y + y^3 - y + y^2 - y^4)\, dy = 2\pi \int_0^1 (1 - 2y + y^2 + y^3 - y^4)\, dy = 2\pi\left[y - y^2 + \frac{y^3}{3} + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$

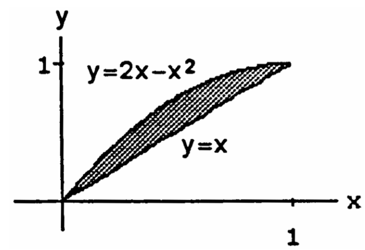
$= 2\pi\left(1 - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{5}\right) = \frac{2\pi}{60}(20 + 15 - 12) = \frac{23\pi}{30}$

35. (a) $V = \int_c^d 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_0^2 2\pi y \left(\sqrt{8y} - y^2\right) dy$
$= 2\pi \int_0^2 \left(2\sqrt{2}\, y^{3/2} - y^3\right) dy = 2\pi \left[\frac{4\sqrt{2}}{5} y^{5/2} - \frac{y^4}{4}\right]_0^2$
$= 2\pi \left(\frac{4\sqrt{2}\cdot\left(\sqrt{2}\right)^5}{5} - \frac{2^4}{4}\right) = 2\pi \left(\frac{4\cdot 2^3}{5} - \frac{4\cdot 4}{4}\right)$
$= 2\pi \cdot 4 \left(\frac{8}{5} - 1\right) = \frac{8\pi}{5}(8 - 5) = \frac{24\pi}{5}$



(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^4 2\pi x \left(\sqrt{x} - \frac{x^2}{8}\right) dx = 2\pi \int_0^4 \left(x^{3/2} - \frac{x^3}{8}\right) dx = 2\pi \left[\frac{2}{5} x^{5/2} - \frac{x^4}{32}\right]_0^4$
$= 2\pi \left(\frac{2\cdot 2^5}{5} - \frac{4^4}{32}\right) = 2\pi \left(\frac{2^6}{5} - \frac{2^8}{32}\right) = \frac{\pi\cdot 2^7}{160}(32 - 20) = \frac{\pi\cdot 2^9\cdot 3}{160} = \frac{\pi\cdot 2^4\cdot 3}{5} = \frac{48\pi}{5}$

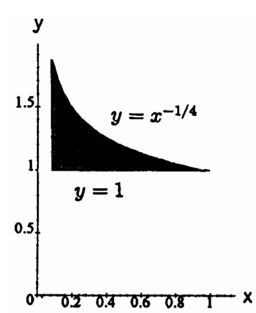
36. (a) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx$
$= \int_0^1 2\pi x \left[(2x - x^2) - x\right] dx$
$= 2\pi \int_0^1 x\,(x - x^2)\, dx = 2\pi \int_0^1 (x^2 - x^3)\, dx$
$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$



(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi(1 - x)\left[(2x - x^2) - x\right] dx = 2\pi \int_0^1 (1 - x)(x - x^2)\, dx$
$= 2\pi \int_0^1 (x - 2x^2 + x^3)\, dx = 2\pi \left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right]_0^1 = 2\pi \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{2\pi}{12}(6 - 8 + 3) = \frac{\pi}{6}$

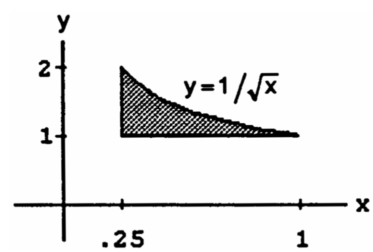
37. (a) $V = \int_a^b \pi \left[R^2(x) - r^2(x)\right] dx = \pi \int_{1/16}^1 \left(x^{-1/2} - 1\right) dx$
$= \pi \left[2x^{1/2} - x\right]_{1/16}^1 = \pi \left[(2 - 1) - \left(2\cdot\frac{1}{4} - \frac{1}{16}\right)\right]$
$= \pi \left(1 - \frac{7}{16}\right) = \frac{9\pi}{16}$

(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dy = \int_1^2 2\pi y \left(\frac{1}{y^4} - \frac{1}{16}\right) dy$
$= 2\pi \int_1^2 \left(y^{-3} - \frac{y}{16}\right) dy = 2\pi \left[-\frac{1}{2} y^{-2} - \frac{y^2}{32}\right]_1^2$
$= 2\pi \left[\left(-\frac{1}{8} - \frac{1}{8}\right) - \left(-\frac{1}{2} - \frac{1}{32}\right)\right] = 2\pi \left(\frac{1}{4} + \frac{1}{32}\right)$
$= \frac{2\pi}{32}(8 + 1) = \frac{9\pi}{16}$



38. (a) $V = \int_c^d \pi \left[R^2(y) - r^2(y)\right] dy = \int_1^2 \pi \left(\frac{1}{y^4} - \frac{1}{16}\right) dy$
$= \pi \left[-\frac{1}{3} y^{-3} - \frac{y}{16}\right]_1^2 = \pi \left[\left(-\frac{1}{24} - \frac{1}{8}\right) - \left(-\frac{1}{3} - \frac{1}{16}\right)\right]$
$= \frac{\pi}{48}(-2 - 6 + 16 + 3) = \frac{11\pi}{48}$

(b) $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_{1/4}^1 2\pi x \left(\frac{1}{\sqrt{x}} - 1\right) dx$
$= 2\pi \int_{1/4}^1 \left(x^{1/2} - x\right) dx = 2\pi \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2}\right]_{1/4}^1$
$= 2\pi \left[\left(\frac{2}{3} - \frac{1}{2}\right) - \left(\frac{2}{3}\cdot\frac{1}{8} - \frac{1}{32}\right)\right] = \pi \left(\frac{4}{3} - 1 - \frac{1}{6} + \frac{1}{16}\right) = \frac{\pi}{48}(4\cdot 16 - 48 - 8 + 3) = \frac{11\pi}{48}$



39. (a) *Disk*: $V = V_1 - V_2$
$V_1 = \int_{a_1}^{b_1} \pi[R_1(x)]^2\, dx$ and $V_2 = \int_{a_2}^{b_2} \pi[R_2(x)]^2$ with $R_1(x) = \sqrt{\frac{x+2}{3}}$ and $R_2(x) = \sqrt{x}$,
$a_1 = -2,\ b_1 = 1;\ a_2 = 0,\ b_2 = 1 \Rightarrow$ two integrals are required

(b) *Washer*: $V = V_1 + V_2$

$V_1 = \int_{a_1}^{b_1} \pi\left([R_1(x)]^2 - [r_1(x)]^2\right) dx$ with $R_1(x) = \sqrt{\frac{x+2}{3}}$ and $r_1(x) = 0$; $a_1 = -2$ and $b_1 = 0$;

$V_2 = \int_{a_2}^{b_2} \pi\left([R_2(x)]^2 - [r_2(x)]^2\right) dx$ with $R_2(x) = \sqrt{\frac{x+2}{3}}$ and $r_2(x) = \sqrt{x}$; $a_2 = 0$ and $b_2 = 1$

$\Rightarrow$ two integrals are required

(c) *Shell*: $V = \int_c^d 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dy = \int_c^d 2\pi y \left(\frac{\text{shell}}{\text{height}}\right) dy$ where shell height $= y^2 - (3y^2 - 2) = 2 - 2y^2$; $c = 0$ and $d = 1$. Only *one* integral is required. It is, therefore preferable to use the *shell* method. However, whichever method you use, you will get $V = \pi$.

40. (a) *Disk*: $V = V_1 - V_2 - V_3$

$V_i = \int_{c_i}^{d_i} \pi[R_i(y)]^2\, dy$, $i = 1, 2, 3$ with $R_1(y) = 1$ and $c_1 = -1$, $d_1 = 1$; $R_2(y) = \sqrt{y}$ and $c_2 = 0$ and $d_2 = 1$;

$R_3(y) = (-y)^{1/4}$ and $c_3 = -1$, $d_3 = 0$ $\Rightarrow$ three integrals are required

(b) *Washer*: $V = V_1 + V_2$

$V_i = \int_{c_i}^{d_i} \pi\left([R_i(y)]^2 - [r_i(y)]^2\right) dy$, $i = 1, 2$ with $R_1(y) = 1$, $r_1(y) = \sqrt{y}$, $c_1 = 0$ and $d_1 = 1$;

$R_2(y) = 1$, $r_2(y) = (-y)^{1/4}$, $c_2 = -1$ and $d_2 = 0$ $\Rightarrow$ two integrals are required

(c) *Shell*: $V = \int_a^b 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dx = \int_a^b 2\pi x \left(\frac{\text{shell}}{\text{height}}\right) dx$, where shell height $= x^2 - (-x^4) = x^2 + x^4$, $a = 0$ and $b = 1$ $\Rightarrow$ only one integral is required. It is, therefore preferable to use the *shell* method. However, whichever method you use, you will get $V = \frac{5\pi}{6}$.

41. (a) $V = \int_a^b \pi\left[R^2(x) - r^2(x)\right] dx = \int_{-4}^4 \pi\left[\left(\sqrt{25 - x^2}\right)^2 - (3)^2\right] dx = \pi \int_{-4}^4 \left[25 - x^2 - 9\right] dx = \pi \int_{-4}^4 (16 - x^2) dx$

$= \pi\left[16x - \frac{1}{3}x^3\right]_{-4}^{4} = \pi\left(64 - \frac{64}{3}\right) - \pi\left(-64 + \frac{64}{3}\right) = \frac{256\pi}{3}$

(b) Volume of sphere $= \frac{4}{3}\pi(5)^3 = \frac{500\pi}{3} \Rightarrow$ Volume of portion removed $= \frac{500\pi}{3} - \frac{256\pi}{3} = \frac{244\pi}{3}$

42. $V = \int_a^b 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dx = \int_1^{\sqrt{1+\pi}} 2\pi x \sin(x^2 - 1)\, dx$; $[u = x^2 - 1 \Rightarrow du = 2x\, dx;\ x = 1 \Rightarrow u = 0,$ $x = \sqrt{1+\pi} \Rightarrow u = \pi] \to \pi \int_0^\pi \sin u\, du = -\pi\left[\cos u\right]_0^\pi = -\pi(-1 - 1) = 2\pi$

43. $V = \int_a^b 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dx = \int_0^r 2\pi x\left(-\frac{h}{r}x + h\right) dx = 2\pi \int_0^r \left(-\frac{h}{r}x^2 + hx\right) dx = 2\pi\left[-\frac{h}{3r}x^3 + \frac{h}{2}x^2\right]_0^r$

$= 2\pi\left(-\frac{r^2 h}{3} + \frac{r^2 h}{2}\right) = \frac{1}{3}\pi r^2 h$

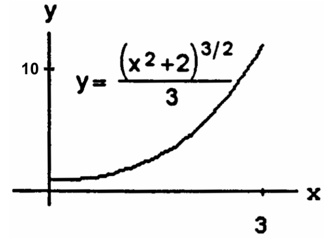
44. $V = \int_c^d 2\pi \left(\frac{\text{shell}}{\text{radius}}\right)\left(\frac{\text{shell}}{\text{height}}\right) dy = \int_0^r 2\pi y\left[\sqrt{r^2 - y^2} - \left(-\sqrt{r^2 - y^2}\right)\right] dy = 4\pi \int_0^r y\sqrt{r^2 - y^2}\, dy$

$[u = r^2 - y^2 \Rightarrow du = -2y\, dy;\ y = 0 \Rightarrow u = r^2,\ y = r \Rightarrow u = 0] \to -2\pi \int_{r^2}^0 \sqrt{u}\, du = 2\pi \int_0^{r^2} u^{1/2}\, du$

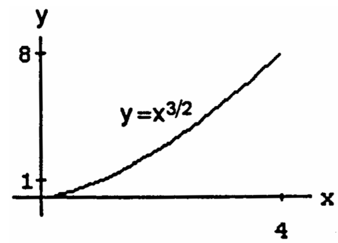
$= \frac{4\pi}{3}\left[u^{3/2}\right]_0^{r^2} = \frac{4\pi}{3} r^3$

6.3 ARC LENGTHS

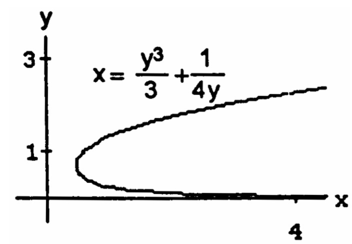
1. $\frac{dy}{dx} = \frac{1}{3} \cdot \frac{3}{2}\left(x^2+2\right)^{1/2} \cdot 2x = \sqrt{(x^2+2)} \cdot x$
$\Rightarrow L = \int_0^3 \sqrt{1+(x^2+2)\,x^2}\,dx = \int_0^3 \sqrt{1+2x^2+x^4}\,dx$
$= \int_0^3 \sqrt{(1+x^2)^2}\,dx = \int_0^3 (1+x^2)\,dx = \left[x+\frac{x^3}{3}\right]_0^3$
$= 3 + \frac{27}{3} = 12$



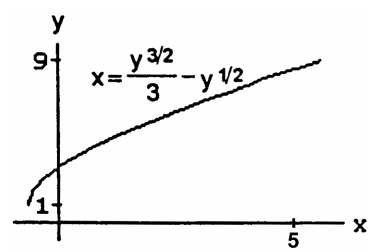
2. $\frac{dy}{dx} = \frac{3}{2}\sqrt{x} \Rightarrow L = \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx$; $\left[u = 1+\frac{9}{4}x\right.$
$\Rightarrow du = \frac{9}{4}\,dx \Rightarrow \frac{4}{9}\,du = dx;\ x=0 \Rightarrow u=1;\ x=4$
$\left.\Rightarrow u = 10\right] \rightarrow L = \int_1^{10} u^{1/2}\left(\frac{4}{9}\,du\right) = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10}$
$= \frac{8}{27}\left(10\sqrt{10}-1\right)$



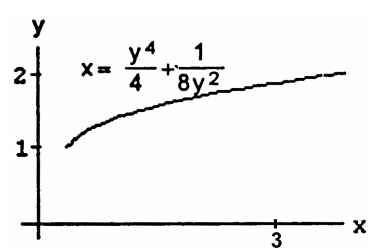
3. $\frac{dx}{dy} = y^2 - \frac{1}{4y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^4 - \frac{1}{2} + \frac{1}{16y^4}$
$\Rightarrow L = \int_1^3 \sqrt{1+y^4-\frac{1}{2}+\frac{1}{16y^4}}\,dy$
$= \int_1^3 \sqrt{y^4+\frac{1}{2}+\frac{1}{16y^4}}\,dy$
$= \int_1^3 \sqrt{\left(y^2+\frac{1}{4y^2}\right)^2}\,dy = \int_1^3 \left(y^2+\frac{1}{4y^2}\right)dy$
$= \left[\frac{y^3}{3} - \frac{y^{-1}}{4}\right]_1^3 = \left(\frac{27}{3}-\frac{1}{12}\right) - \left(\frac{1}{3}-\frac{1}{4}\right) = 9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4} = 9 + \frac{(-1-4+3)}{12} = 9 + \frac{(-2)}{12} = \frac{53}{6}$



4. $\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)$
$\Rightarrow L = \int_1^9 \sqrt{1+\frac{1}{4}\left(y-2+\frac{1}{y}\right)}\,dy$
$= \int_1^9 \sqrt{\frac{1}{4}\left(y+2+\frac{1}{y}\right)}\,dy = \int_1^9 \frac{1}{2}\sqrt{\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right)^2}\,dy$
$= \frac{1}{2}\int_1^9 \left(y^{1/2}+y^{-1/2}\right)dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2}+2y^{1/2}\right]_1^9$
$= \left[\frac{y^{3/2}}{3} + y^{1/2}\right]_1^9 = \left(\frac{3^3}{3}+3\right) - \left(\frac{1}{3}+1\right) = 11 - \frac{1}{3} = \frac{32}{3}$



5. $\frac{dx}{dy} = y^3 - \frac{1}{4y^3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^6 - \frac{1}{2} + \frac{1}{16y^6}$
$\Rightarrow L = \int_1^2 \sqrt{1+y^6-\frac{1}{2}+\frac{1}{16y^6}}\,dy$
$= \int_1^2 \sqrt{y^6+\frac{1}{2}+\frac{1}{16y^6}}\,dy = \int_1^2 \sqrt{\left(y^3+\frac{y^{-3}}{4}\right)^2}\,dy$
$= \int_1^2 \left(y^3+\frac{y^{-3}}{4}\right)dy = \left[\frac{y^4}{4} - \frac{y^{-2}}{8}\right]_1^2$
$= \left(\frac{16}{4} - \frac{1}{(16)(2)}\right) - \left(\frac{1}{4}-\frac{1}{8}\right) = 4 - \frac{1}{32} - \frac{1}{4} + \frac{1}{8} = \frac{128-1-8+4}{32} = \frac{123}{32}$



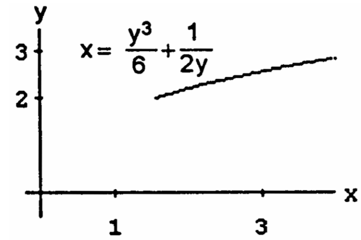
6. $\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y^4 - 2 + y^{-4}\right)$

$\Rightarrow L = \int_2^3 \sqrt{1 + \frac{1}{4}\left(y^4 - 2 + y^{-4}\right)}\, dy$

$= \int_2^3 \sqrt{\frac{1}{4}\left(y^4 + 2 + y^{-4}\right)}\, dy$

$= \frac{1}{2}\int_2^3 \sqrt{\left(y^2 + y^{-2}\right)^2}\, dy = \frac{1}{2}\int_2^3 \left(y^2 + y^{-2}\right) dy$

$= \frac{1}{2}\left[\frac{y^3}{3} - y^{-1}\right]_2^3 = \frac{1}{2}\left[\left(\frac{27}{3} - \frac{1}{3}\right) - \left(\frac{8}{3} - \frac{1}{2}\right)\right] = \frac{1}{2}\left(\frac{26}{3} - \frac{8}{3} + \frac{1}{2}\right) = \frac{1}{2}\left(6 + \frac{1}{2}\right) = \frac{13}{4}$



7. $\frac{dy}{dx} = x^{1/3} - \frac{1}{4}x^{-1/3} \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^{2/3} - \frac{1}{2} + \frac{x^{-2/3}}{16}$

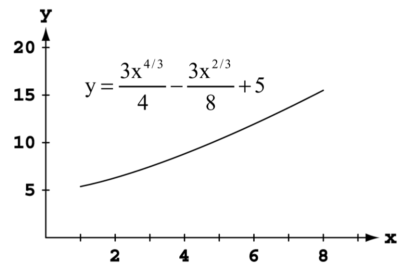
$\Rightarrow L = \int_1^8 \sqrt{1 + x^{2/3} - \frac{1}{2} + \frac{x^{-2/3}}{16}}\, dx$

$= \int_1^8 \sqrt{x^{2/3} + \frac{1}{2} + \frac{x^{-2/3}}{16}}\, dx$

$= \int_1^8 \sqrt{\left(x^{1/3} + \frac{1}{4}x^{-1/3}\right)^2}\, dx = \int_1^8 \left(x^{1/3} + \frac{1}{4}x^{-1/3}\right) dx$

$= \left[\frac{3}{4}x^{4/3} + \frac{3}{8}x^{2/3}\right]_1^8 = \frac{3}{8}\left[2x^{4/3} + x^{2/3}\right]_1^8$

$= \frac{3}{8}\left[\left(2 \cdot 2^4 + 2^2\right) - (2+1)\right] = \frac{3}{8}(32 + 4 - 3) = \frac{99}{8}$



8. $\frac{dy}{dx} = x^2 + 2x + 1 - \frac{4}{(4x+4)^2} = x^2 + 2x + 1 - \frac{1}{4}\,\frac{1}{(1+x)^2}$

$= (1+x)^2 - \frac{1}{4}\,\frac{1}{(1+x)^2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = (1+x)^4 - \frac{1}{2} + \frac{1}{16(1+x)^4}$

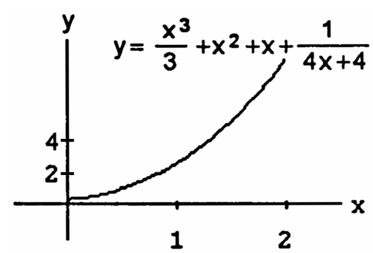
$\Rightarrow L = \int_0^2 \sqrt{1 + (1+x)^4 - \frac{1}{2} + \frac{(1+x)^{-4}}{16}}\, dx$

$= \int_0^2 \sqrt{(1+x)^4 + \frac{1}{2} + \frac{(1+x)^{-4}}{16}}\, dx$

$= \int_0^2 \sqrt{\left[(1+x)^2 + \frac{(1+x)^{-2}}{4}\right]^2}\, dx$

$= \int_0^2 \left[(1+x)^2 + \frac{(1+x)^{-2}}{4}\right] dx$; $[u = 1 + x \Rightarrow du = dx;\ x = 0 \Rightarrow u = 1,\ x = 2 \Rightarrow u = 3]$

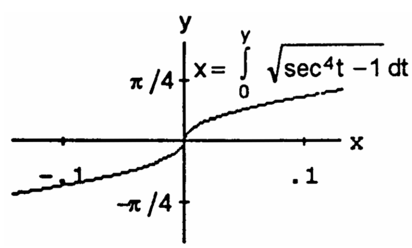
$\rightarrow L = \int_1^3 \left(u^2 + \frac{1}{4}u^{-2}\right) du = \left[\frac{u^3}{3} - \frac{1}{4}u^{-1}\right]_1^3 = \left(9 - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{108-1-4+3}{12} = \frac{106}{12} = \frac{53}{6}$



9. $\frac{dx}{dy} = \sqrt{\sec^4 y - 1} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \sec^4 y - 1$

$\Rightarrow L = \int_{-\pi/4}^{\pi/4} \sqrt{1 + \left(\sec^4 y - 1\right)}\, dy = \int_{-\pi/4}^{\pi/4} \sec^2 y\, dy$

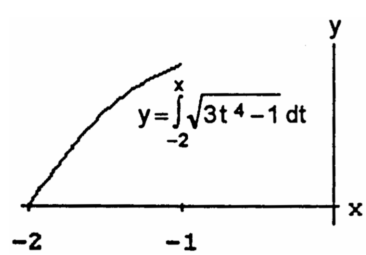
$= [\tan y]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$



10. $\frac{dy}{dx} = \sqrt{3x^4 - 1} \Rightarrow \left(\frac{dy}{dx}\right)^2 = 3x^4 - 1$

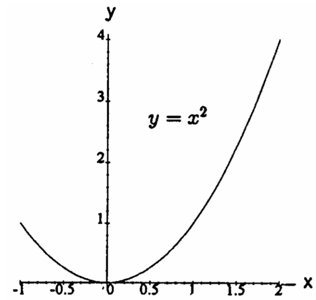
$\Rightarrow L = \int_{-2}^{-1} \sqrt{1 + \left(3x^4 - 1\right)}\, dx = \int_{-2}^{-1} \sqrt{3}\, x^2\, dx$

$= \sqrt{3}\left[\frac{x^3}{3}\right]_{-2}^{-1} = \frac{\sqrt{3}}{3}\left[-1 - (-2)^3\right] = \frac{\sqrt{3}}{3}(-1 + 8) = \frac{7\sqrt{3}}{3}$



11. (a) $\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)^2 = 4x^2$
$\Rightarrow L = \int_{-1}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
$= \int_{-1}^{2} \sqrt{1 + 4x^2}\, dx$

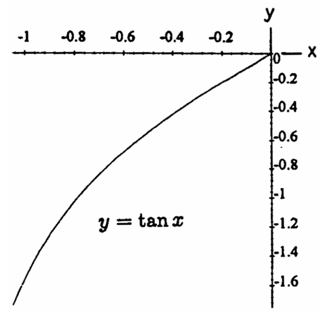
(b)



(c) $L \approx 6.13$

12. (a) $\frac{dy}{dx} = \sec^2 x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \sec^4 x$
$\Rightarrow L = \int_{-\pi/3}^{0} \sqrt{1 + \sec^4 x}\, dx$

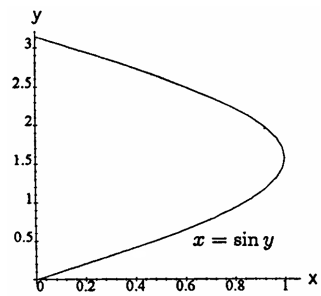
(b)



(c) $L \approx 2.06$

13. (a) $\frac{dx}{dy} = \cos y \Rightarrow \left(\frac{dx}{dy}\right)^2 = \cos^2 y$
$\Rightarrow L = \int_{0}^{\pi} \sqrt{1 + \cos^2 y}\, dy$

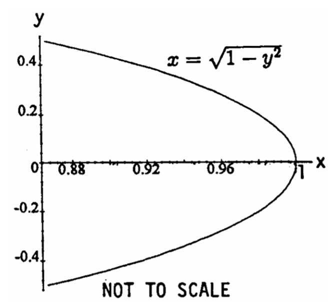
(b)



(c) $L \approx 3.82$

14. (a) $\frac{dx}{dy} = -\frac{y}{\sqrt{1-y^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{1-y^2}$
$\Rightarrow L = \int_{-1/2}^{1/2} \sqrt{1 + \frac{y^2}{(1-y^2)}}\, dy = \int_{-1/2}^{1/2} \sqrt{\frac{1}{1-y^2}}\, dy$
$= \int_{-1/2}^{1/2} (1-y^2)^{-1/2}\, dy$

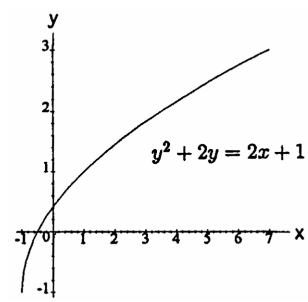
(b)



(c) $L \approx 1.05$

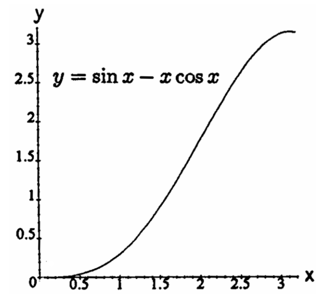
15. (a) $2y + 2 = 2\frac{dx}{dy} \Rightarrow \left(\frac{dx}{dy}\right)^2 = (y+1)^2$
$\Rightarrow L = \int_{-1}^{3} \sqrt{1 + (y+1)^2}\, dy$

(b)

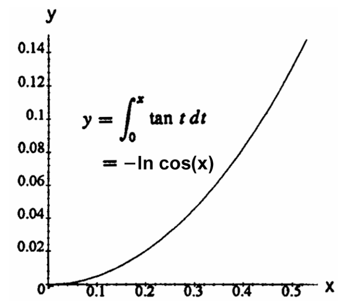


(c) $L \approx 9.29$

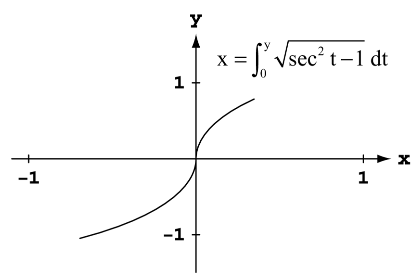
16. (a) $\frac{dy}{dx} = \cos x - \cos x + x \sin x \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^2 \sin^2 x$
$\Rightarrow L = \int_0^{\pi} \sqrt{1 + x^2 \sin^2 x}\, dx$
(c) $L \approx 4.70$

(b)


17. (a) $\frac{dy}{dx} = \tan x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x$
$\Rightarrow L = \int_0^{\pi/6} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/6} \sqrt{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}}\, dx$
$= \int_0^{\pi/6} \frac{dx}{\cos x} = \int_0^{\pi/6} \sec x\, dx$
(c) $L \approx 0.55$

(b)


18. (a) $\frac{dx}{dy} = \sqrt{\sec^2 y - 1} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \sec^2 y - 1$
$\Rightarrow L = \int_{-\pi/3}^{\pi/4} \sqrt{1 + (\sec^2 y - 1)}\, dy$
$= \int_{-\pi/3}^{\pi/4} |\sec y|\, dy = \int_{-\pi/3}^{\pi/4} \sec y\, dy$
(c) $L \approx 2.20$

(b)


19. (a) $\left(\frac{dy}{dx}\right)^2$ corresponds to $\frac{1}{4x}$ here, so take $\frac{dy}{dx}$ as $\frac{1}{2\sqrt{x}}$. Then $y = \sqrt{x} + C$ and since $(1, 1)$ lies on the curve, $C = 0$. So $y = \sqrt{x}$ from $(1, 1)$ to $(4, 2)$.
(b) Only one. We know the derivative of the function and the value of the function at one value of x.

20. (a) $\left(\frac{dx}{dy}\right)^2$ corresponds to $\frac{1}{y^4}$ here, so take $\frac{dy}{dx}$ as $\frac{1}{y^2}$. Then $x = -\frac{1}{y} + C$ and, since $(0, 1)$ lies on the curve, $C = 1$ So $y = \frac{1}{1-x}$.
(b) Only one. We know the derivative of the function and the value of the function at one value of x.

21. $y = \int_0^x \sqrt{\cos 2t}\, dt \Rightarrow \frac{dy}{dx} = \sqrt{\cos 2x} \Rightarrow L = \int_0^{\pi/4} \sqrt{1 + \left[\sqrt{\cos 2x}\right]^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \cos 2x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 x}\, dx$
$= \int_0^{\pi/4} \sqrt{2} \cos x\, dx = \sqrt{2}[\sin x]_0^{\pi/4} = \sqrt{2}\sin\left(\frac{\pi}{4}\right) - \sqrt{2}\sin(0) = 1$

22. $y = \left(1 - x^{2/3}\right)^{3/2},\ \frac{\sqrt{2}}{4} \le x \le 1 \Rightarrow \frac{dy}{dx} = \frac{3}{2}\left(1 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -\frac{\left(1 - x^{2/3}\right)^{1/2}}{x^{1/3}} \Rightarrow L = \int_{\sqrt{2}/4}^{1} \sqrt{1 + \left[-\frac{\left(1 - x^{2/3}\right)^{1/2}}{x^{1/3}}\right]^2}\, dx$
$= \int_{\sqrt{2}/4}^{1} \sqrt{1 + \frac{1 - x^{2/3}}{x^{2/3}}}\, dx = \int_{\sqrt{2}/4}^{1} \sqrt{1 + \frac{1}{x^{2/3}} - 1}\, dx = \int_{\sqrt{2}/4}^{1} \sqrt{\frac{1}{x^{2/3}}}\, dx = \int_{\sqrt{2}/4}^{1} \frac{1}{x^{1/3}}\, dx = \int_{\sqrt{2}/4}^{1} x^{-1/3}\, dx = \frac{3}{2}\left[x^{2/3}\right]_{\sqrt{2}/4}^{1}$
$= \frac{3}{2}(1)^{2/3} - \frac{3}{2}\left(\frac{\sqrt{2}}{4}\right)^{2/3} = \frac{3}{2} - \frac{3}{2}\left(\frac{1}{2}\right) = \frac{3}{4} \Rightarrow \text{total length} = 8\left(\frac{3}{4}\right) = 6$

23. $y = 3 - 2x, 0 \le x \le 2 \Rightarrow \frac{dy}{dx} = -2 \Rightarrow L = \int_0^2 \sqrt{1 + (-2)^2}\, dx = \int_0^2 \sqrt{5}\, dx = \left[\sqrt{5}\, x\right]_0^2 = 2\sqrt{5}.$
$d = \sqrt{(2-0)^2 + (3-(-1))^2} = 2\sqrt{5}$

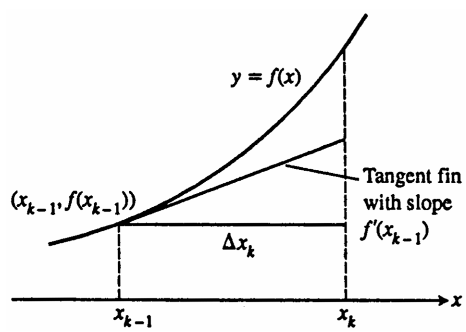
24. Consider the circle $x^2 + y^2 = r^2$, we will find the length of the portion in the first quadrant, and multiply our result by 4.
$y = \sqrt{r^2 - x^2}, 0 \le x \le r \Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}} \Rightarrow L = 4\int_0^r \sqrt{1 + \left[\frac{-x}{\sqrt{r^2 - x^2}}\right]^2}\, dx = 4\int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = 4\int_0^r \sqrt{\frac{r^2}{r^2 - x^2}}\, dx$
$= 4\int_0^r \frac{r}{\sqrt{r^2 - x^2}}\, dx = 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}}$

25. $9x^2 = y(y-3)^2 \Rightarrow \frac{d}{dy}\left[9x^2\right] = \frac{d}{dy}\left[y(y-3)^2\right] \Rightarrow 18x\frac{dx}{dy} = 2y(y-3) + (y-3)^2 = 3(y-3)(y-1) \Rightarrow \frac{dx}{dy} = \frac{(y-3)(y-1)}{6x}$
$\Rightarrow dx = \frac{(y-3)(y-1)}{6x}dy;\ ds^2 = dx^2 + dy^2 = \left[\frac{(y-3)(y-1)}{6x}dy\right]^2 + dy^2 = \frac{(y-3)^2(y-1)^2}{36x^2}dy^2 + dy^2 = \frac{(y-3)^2(y-1)^2}{4y(y-3)^2}dy^2 + dy^2$
$= \left[\frac{(y-1)^2}{4y} + 1\right]dy^2 = \frac{y^2 - 2y + 1 + 4y}{4y}dy^2 = \frac{(y+1)^2}{4y}dy^2$

26. $4x^2 - y^2 = 64 \Rightarrow \frac{d}{dx}\left[4x^2 - y^2\right] = \frac{d}{dx}\left[64\right] \Rightarrow 8x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{4x}{y} \Rightarrow dy = \frac{4x}{y}dx;\ ds^2 = dx^2 + dy^2$
$= dx^2 + \left[\frac{4x}{y}dx\right]^2 = dx^2 + \frac{16x^2}{y^2}dx^2 = \left(1 + \frac{16x^2}{y^2}\right)dx^2 = \frac{y^2 + 16x^2}{y^2}dx^2 = \frac{4x^2 - 64 + 16x^2}{y^2}dx^2 = \frac{20x^2 - 64}{y^2}dx^2 = \frac{4}{y^2}(5x^2 - 16)dx^2$

27. $\sqrt{2}\, x = \int_0^x \sqrt{1 + \left(\frac{dy}{dt}\right)^2}\, dt, x \ge 0 \Rightarrow \sqrt{2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Rightarrow \frac{dy}{dx} = \pm 1 \Rightarrow y = f(x) = \pm x + C$ where C is any real number.

28. (a) From the accompanying figure and definition of the differential (change along the tangent line) we see that $dy = f'(x_{k-1}) \triangle x_k \Rightarrow$ length of kth tangent fin is $\sqrt{(\triangle x_k)^2 + (dy)^2} = \sqrt{(\triangle x_k)^2 + [f'(x_{k-1}) \triangle x_k]^2}$.



(b) Length of curve $= \lim_{n\to\infty} \sum_{k=1}^{n}$ (length of kth tangent fin) $= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(\triangle x_k)^2 + [f'(x_{k-1}) \triangle x_k]^2}$
$= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(x_{k-1})]^2}\, \triangle x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

29. $x^2 + y^2 = 1 \Rightarrow y = \sqrt{1 - x^2};\ P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\} \Rightarrow L \approx \sum_{k=1}^{4} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\left(\frac{1}{4} - 0\right)^2 + \left(\frac{\sqrt{15}}{4} - 1\right)^2}$
$+ \sqrt{\left(\frac{1}{2} - \frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{15}}{4}\right)^2} + \sqrt{\left(\frac{3}{4} - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{4} - \frac{\sqrt{3}}{2}\right)^2} + \sqrt{\left(1 - \frac{3}{4}\right)^2 + \left(0 - \frac{\sqrt{7}}{4}\right)^2} \approx 1.55225$

30. Let (x_1, y_1) and (x_2, y_2), with $x_2 > x_1$, lie on $y = mx + b$, where $m = \frac{y_2 - y_1}{x_2 - x_1}$, then $\frac{dy}{dx} = m \Rightarrow L = \int_{x_1}^{x_2} \sqrt{1 + m^2}\, dx$
$= \sqrt{1 + m^2}\, [x]_{x_1}^{x_2} = \sqrt{1 + m^2}(x_2 - x_1) = \sqrt{1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2}(x_2 - x_1) = \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{(x_2 - x_1)^2}}(x_2 - x_1)$
$\frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{(x_2 - x_1)}(x_2 - x_1) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$

31. $y = 2x^{3/2} \Rightarrow \frac{dy}{dx} = 3x^{1/2}$; $L(x) = \int_0^x \sqrt{1 + \left[3t^{1/2}\right]^2}\,dt = \int_0^x \sqrt{1+9t}\,dt$; $[u = 1 + 9t \Rightarrow du = 9dt, t = 0 \Rightarrow u = 1,$ $t = x \Rightarrow u = 1 + 9x] \to \frac{1}{9}\int_1^{1+9x} \sqrt{u}\,du = \frac{2}{27}\left[u^{3/2}\right]_1^{1+9x} = \frac{2}{27}(1+9x)^{3/2} - \frac{2}{27}$; $L(1) = \frac{2}{27}(10)^{3/2} - \frac{2}{27} = \frac{2\left(10\sqrt{10}-1\right)}{27}$

32. $y = \frac{x^3}{3} + x^2 + x + \frac{1}{4x+4} \Rightarrow \frac{dy}{dx} = x^2 + 2x + 1 - \frac{1}{4(x+1)^2} = (x+1)^2 - \frac{1}{4(x+1)^2}$;

$L(x) = \int_0^x \sqrt{1 + \left[(t+1)^2 - \frac{1}{4(t+1)^2}\right]^2}\,dt = \int_0^x \sqrt{1 + \left[\frac{4(t+1)^4 - 1}{4(t+1)^2}\right]^2}\,dt = \int_0^x \sqrt{1 + \frac{\left[4(t+1)^4 - 1\right]^2}{16(t+1)^4}}\,dt$

$= \int_0^x \sqrt{\frac{16(t+1)^4 + 16(t+1)^8 - 8(t+1)^4 + 1}{16(t+1)^4}}\,dt = \int_0^x \sqrt{\frac{16(t+1)^8 + 8(t+1)^4 + 1}{16(t+1)^4}}\,dt = \int_0^x \sqrt{\frac{\left[4(t+1)^4 + 1\right]^2}{16(t+1)^4}}\,dt$

$= \int_0^x \frac{4(t+1)^4 + 1}{4(t+1)^2}\,dt = \int_0^x \left[(t+1)^2 + \frac{1}{4(t+1)^2}\right]dt$; $[u = t + 1 \Rightarrow du = dt, t = 0 \Rightarrow u = 1, t = x \Rightarrow u = x + 1]$

$\to \int_1^{x+1} \left[u^2 + \frac{1}{4}u^{-2}\right]du = \left[\frac{1}{3}u^3 - \frac{1}{4}u^{-1}\right]_1^{x+1} = \left(\frac{1}{3}(x+1)^3 - \frac{1}{4(x+1)}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{3}(x+1)^3 - \frac{1}{4(x+1)} - \frac{1}{12}$;

$L(1) = \frac{8}{3} - \frac{1}{8} - \frac{1}{12} = \frac{59}{24}$

33-38. Example CAS commands:

Maple:

```
with( plots );
with( Student[Calculus1] );
with( student );
f := x -> sqrt(1-x^2);a := -1;
b := 1;
N := [2, 4, 8 ];
for n in N do
  xx := [seq( a+i*(b-a)/n, i=0..n )];
  pts := [seq([x,f(x)],x=xx)];
  L := simplify(add( distance(pts[i+1],pts[i]), i=1..n ));            # (b)
  T := sprintf("#33(a) (Section 6.3)\nn=%3d  L=%8.5f\n", n, L );
  P[n] := plot( [f(x),pts], x=a..b, title=T ):                        # (a)
end do:
display( [seq(P[n],n=N)], insequence=true, scaling=constrained );
L := ArcLength( f(x), x=a..b, output=integral ):
L = evalf( L );                                                       # (c)
```

33-38. Example CAS commands:

Mathematica: (assigned function and values for a, b, and n may vary)

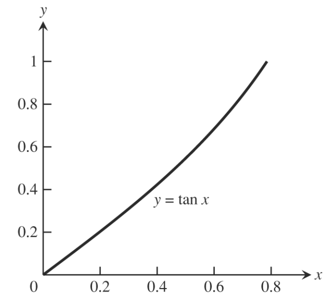
```
Clear[x, f]
{a, b} = {−1, 1}; f[x_] = Sqrt[1 − x^2]
p1 = Plot[f[x], {x, a, b}]
n = 8;
pts = Table[{xn, f[xn]}, {xn, a, b, (b − a)/n}]/ / N
Show[{p1,Graphics[{Line[pts]}]}]
Sum[ Sqrt[ (pts[[i + 1, 1]] − pts[[i, 1]])^2 + (pts[[i + 1, 2]] − pts[[i, 2]])^2], {i, 1, n}]
NIntegrate[ Sqrt[ 1 + f'[x]^2],{x, a, b}]
```

6.4 AREAS OF SURFACES OF REVOLUTION

1. (a) $\frac{dy}{dx} = \sec^2 x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \sec^4 x$

$\Rightarrow S = 2\pi \int_0^{\pi/4} (\tan x) \sqrt{1 + \sec^4 x}\, dx$

(b)

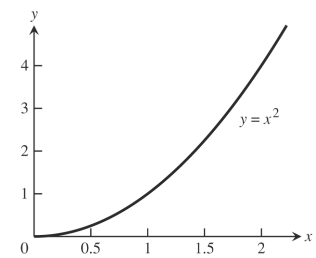


(c) $S \approx 3.84$

2. (a) $\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)^2 = 4x^2$

$\Rightarrow S = 2\pi \int_0^{2} x^2 \sqrt{1 + 4x^2}\, dx$

(b)

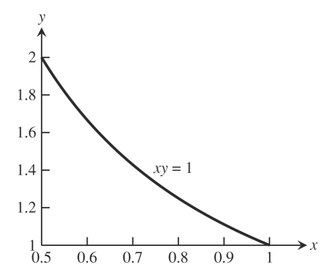


(c) $S \approx 53.23$

3. (a) $xy = 1 \Rightarrow x = \frac{1}{y} \Rightarrow \frac{dx}{dy} = -\frac{1}{y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{y^4}$

$\Rightarrow S = 2\pi \int_1^{2} \frac{1}{y} \sqrt{1 + y^{-4}}\, dy$

(b)

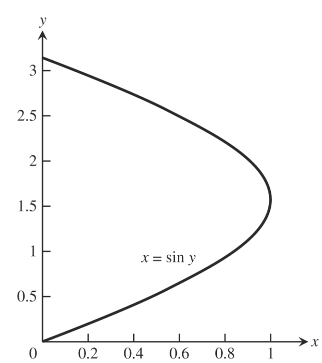


(c) $S \approx 5.02$

4. (a) $\frac{dx}{dy} = \cos y \Rightarrow \left(\frac{dx}{dy}\right)^2 = \cos^2 y$

$\Rightarrow S = 2\pi \int_0^{\pi} (\sin y) \sqrt{1 + \cos^2 y}\, dy$

(b)



(c) $S \approx 14.42$

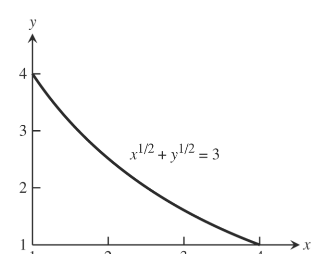
5. (a) $x^{1/2} + y^{1/2} = 3 \Rightarrow y = \left(3 - x^{1/2}\right)^2$

$\Rightarrow \frac{dy}{dx} = 2\left(3 - x^{1/2}\right)\left(-\frac{1}{2} x^{-1/2}\right)$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(1 - 3x^{-1/2}\right)^2$

$\Rightarrow S = 2\pi \int_1^{4} \left(3 - x^{1/2}\right)^2 \sqrt{1 + \left(1 - 3x^{-1/2}\right)^2}\, dx$

(b)

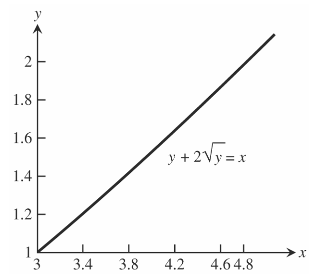


(c) $S \approx 63.37$

6. (a) $\frac{dx}{dy} = 1 + y^{-1/2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \left(1 + y^{-1/2}\right)^2$

$\Rightarrow S = 2\pi \int_1^2 \left(y + 2\sqrt{y}\right) \sqrt{1 + \left(1 + y^{-1/2}\right)^2}\, dx$

(b)



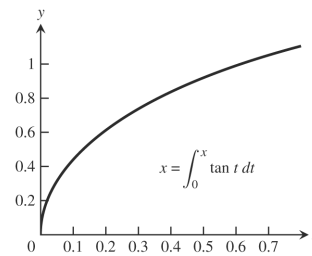
(c) $S \approx 51.33$

7. (a) $\frac{dx}{dy} = \tan y \Rightarrow \left(\frac{dx}{dy}\right)^2 = \tan^2 y$

$\Rightarrow S = 2\pi \int_0^{\pi/3} \left(\int_0^y \tan t\, dt\right) \sqrt{1 + \tan^2 y}\, dy$

$= 2\pi \int_0^{\pi/3} \left(\int_0^y \tan t\, dt\right) \sec y\, dy$

(b)



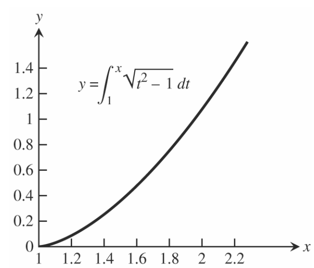
(c) $S \approx 2.08$

8. (a) $\frac{dy}{dx} = \sqrt{x^2 - 1} \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^2 - 1$

$\Rightarrow S = 2\pi \int_1^{\sqrt{5}} \left(\int_1^x \sqrt{t^2 - 1}\, dt\right) \sqrt{1 + (x^2 - 1)}\, dx$

$= 2\pi \int_1^{\sqrt{5}} \left(\int_1^x \sqrt{t^2 - 1}\, dt\right) x\, dx$

(b)



(c) $S \approx 8.55$

9. $y = \frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}$; $S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \Rightarrow S = \int_0^4 2\pi \left(\frac{x}{2}\right) \sqrt{1 + \frac{1}{4}}\, dx = \frac{\pi\sqrt{5}}{2} \int_0^4 x\, dx$

$= \frac{\pi\sqrt{5}}{2} \left[\frac{x^2}{2}\right]_0^4 = 4\pi\sqrt{5}$; Geometry formula: base circumference $= 2\pi(2)$, slant height $= \sqrt{4^2 + 2^2} = 2\sqrt{5}$

$\Rightarrow$ Lateral surface area $= \frac{1}{2}(4\pi)\left(2\sqrt{5}\right) = 4\pi\sqrt{5}$ in agreement with the integral value

10. $y = \frac{x}{2} \Rightarrow x = 2y \Rightarrow \frac{dx}{dy} = 2$; $S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^2 2\pi \cdot 2y\sqrt{1 + 2^2}\, dy = 4\pi\sqrt{5} \int_0^2 y\, dy = 2\pi\sqrt{5}\left[y^2\right]_0^2$

$= 2\pi\sqrt{5} \cdot 4 = 8\pi\sqrt{5}$; Geometry formula: base circumference $= 2\pi(4)$, slant height $= \sqrt{4^2 + 2^2} = 2\sqrt{5}$

$\Rightarrow$ Lateral surface area $= \frac{1}{2}(8\pi)\left(2\sqrt{5}\right) = 8\pi\sqrt{5}$ in agreement with the integral value

11. $\frac{dy}{dx} = \frac{1}{2}$; $S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^3 2\pi \frac{(x+1)}{2} \sqrt{1 + \left(\frac{1}{2}\right)^2}\, dx = \frac{\pi\sqrt{5}}{2} \int_1^3 (x+1)\, dx = \frac{\pi\sqrt{5}}{2}\left[\frac{x^2}{2} + x\right]_1^3$

$= \frac{\pi\sqrt{5}}{2}\left[\left(\frac{9}{2} + 3\right) - \left(\frac{1}{2} + 1\right)\right] = \frac{\pi\sqrt{5}}{2}(4 + 2) = 3\pi\sqrt{5}$; Geometry formula: $r_1 = \frac{1}{2} + \frac{1}{2} = 1$, $r_2 = \frac{3}{2} + \frac{1}{2} = 2$,

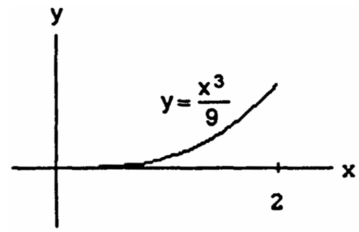
slant height $= \sqrt{(2-1)^2 + (3-1)^2} = \sqrt{5} \Rightarrow$ Frustum surface area $= \pi(r_1 + r_2) \times$ slant height $= \pi(1+2)\sqrt{5}$

$= 3\pi\sqrt{5}$ in agreement with the integral value

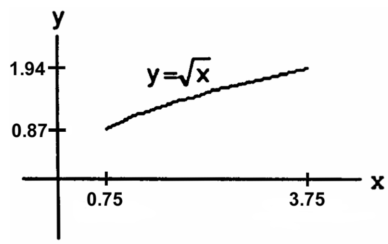
12. $y = \frac{x}{2} + \frac{1}{2} \Rightarrow x = 2y - 1 \Rightarrow \frac{dx}{dy} = 2$; $S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^2 2\pi(2y - 1)\sqrt{1 + 4}\, dy = 2\pi\sqrt{5} \int_1^2 (2y - 1)\, dy$

$= 2\pi\sqrt{5}\left[y^2 - y\right]_1^2 = 2\pi\sqrt{5}\left[(4 - 2) - (1 - 1)\right] = 4\pi\sqrt{5}$; Geometry formula: $r_1 = 1$, $r_2 = 3$, slant height $= \sqrt{(2-1)^2+(3-1)^2} = \sqrt{5} \Rightarrow$ Frustum surface area $= \pi(1+3)\sqrt{5} = 4\pi\sqrt{5}$ in agreement with the integral value

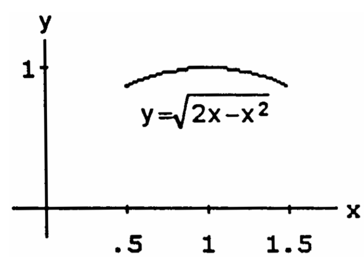
13. $\frac{dy}{dx} = \frac{x^2}{3} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^4}{9} \Rightarrow S = \int_0^2 \frac{2\pi x^3}{9}\sqrt{1+\frac{x^4}{9}}\,dx;$
$\left[u = 1+\frac{x^4}{9} \Rightarrow du = \frac{4}{9}x^3\,dx \Rightarrow \frac{1}{4}\,du = \frac{x^3}{9}\,dx;\right.$
$\left. x = 0 \Rightarrow u = 1, x = 2 \Rightarrow u = \frac{25}{9}\right]$
$\rightarrow S = 2\pi\int_1^{25/9} u^{1/2}\cdot\frac{1}{4}\,du = \frac{\pi}{2}\left[\frac{2}{3}u^{3/2}\right]_1^{25/9}$
$= \frac{\pi}{3}\left(\frac{125}{27}-1\right) = \frac{\pi}{3}\left(\frac{125-27}{27}\right) = \frac{98\pi}{81}$



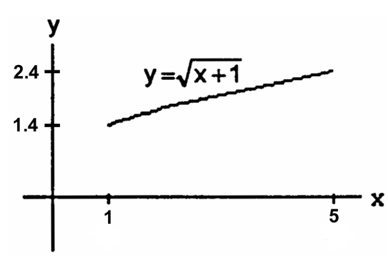
14. $\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4x}$
$\Rightarrow S = \int_{3/4}^{15/4} 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx$
$= 2\pi\int_{3/4}^{15/4}\sqrt{x+\frac{1}{4}}\,dx = 2\pi\left[\frac{2}{3}\left(x+\frac{1}{4}\right)^{3/2}\right]_{3/4}^{15/4}$
$= \frac{4\pi}{3}\left[\left(\frac{15}{4}+\frac{1}{4}\right)^{3/2} - \left(\frac{3}{4}+\frac{1}{4}\right)^{3/2}\right] = \frac{4\pi}{3}\left[\left(\frac{4}{2}\right)^3 - 1\right]$
$= \frac{4\pi}{3}(8-1) = \frac{28\pi}{3}$



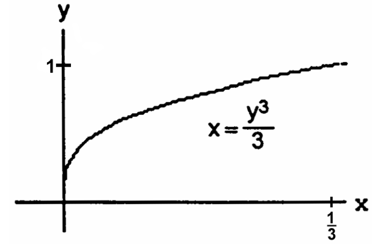
15. $\frac{dy}{dx} = \frac{1}{2}\frac{(2-2x)}{\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{2x-x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(1-x)^2}{2x-x^2}$
$\Rightarrow S = \int_{0.5}^{1.5} 2\pi\sqrt{2x-x^2}\sqrt{1+\frac{(1-x)^2}{2x-x^2}}\,dx$
$= 2\pi\int_{0.5}^{1.5}\sqrt{2x-x^2}\,\frac{\sqrt{2x-x^2+1-2x+x^2}}{\sqrt{2x-x^2}}\,dx$
$= 2\pi\int_{0.5}^{1.5} dx = 2\pi[x]_{0.5}^{1.5} = 2\pi$



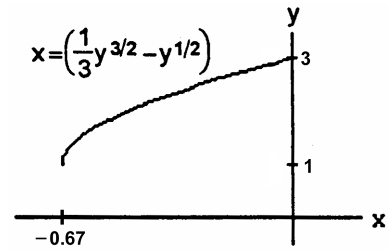
16. $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4(x+1)}$
$\Rightarrow S = \int_1^5 2\pi\sqrt{x+1}\sqrt{1+\frac{1}{4(x+1)}}\,dx$
$= 2\pi\int_1^5\sqrt{(x+1)+\frac{1}{4}}\,dx = 2\pi\int_1^5\sqrt{x+\frac{5}{4}}\,dx$
$= 2\pi\left[\frac{2}{3}\left(x+\frac{5}{4}\right)^{3/2}\right]_1^5 = \frac{4\pi}{3}\left[\left(5+\frac{5}{4}\right)^{3/2} - \left(1+\frac{5}{4}\right)^{3/2}\right]$
$= \frac{4\pi}{3}\left[\left(\frac{25}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2}\right] = \frac{4\pi}{3}\left(\frac{5^3}{2^3} - \frac{3^3}{2^3}\right)$
$= \frac{\pi}{6}(125-27) = \frac{98\pi}{6} = \frac{49\pi}{3}$



17. $\frac{dx}{dy} = y^2 \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^4 \Rightarrow S = \int_0^1 \frac{2\pi y^3}{3}\sqrt{1+y^4}\,dy;$
$\left[u = 1+y^4 \Rightarrow du = 4y^3\,dy \Rightarrow \frac{1}{4}\,du = y^3\,dy;\ y = 0\right.$
$\left.\Rightarrow u = 1, y = 1 \Rightarrow u = 2\right] \rightarrow S = \int_1^2 2\pi\left(\frac{1}{3}\right)u^{1/2}\left(\frac{1}{4}\,du\right)$
$= \frac{\pi}{6}\int_1^2 u^{1/2}\,du = \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]_1^2 = \frac{\pi}{9}\left(\sqrt{8}-1\right)$



18. $x = \left(\frac{1}{3}y^{3/2} - y^{1/2}\right) \le 0$, when $1 \le y \le 3$. To get positive area, we take $x = -\left(\frac{1}{3}y^{3/2} - y^{1/2}\right)$



$\Rightarrow \frac{dx}{dy} = -\frac{1}{2}\left(y^{1/2} - y^{-1/2}\right) \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y - 2 + y^{-1}\right)$

$\Rightarrow S = -\int_1^3 2\pi\left(\frac{1}{3}y^{3/2} - y^{1/2}\right)\sqrt{1 + \frac{1}{4}\left(y - 2 + y^{-1}\right)}\,dy$

$= -2\pi\int_1^3\left(\frac{1}{3}y^{3/2} - y^{1/2}\right)\sqrt{\frac{1}{4}\left(y + 2 + y^{-1}\right)}\,dy$

$= -2\pi\int_1^3\left(\frac{1}{3}y^{3/2} - y^{1/2}\right)\frac{\sqrt{\left(y^{1/2} + y^{-1/2}\right)^2}}{2}\,dy = -\pi\int_1^3 y^{1/2}\left(\frac{1}{3}y - 1\right)\left(y^{1/2} + \frac{1}{y^{1/2}}\right)dy = -\pi\int_1^3\left(\frac{1}{3}y - 1\right)(y+1)\,dy$

$= -\pi\int_1^3\left(\frac{1}{3}y^2 - \frac{2}{3}y - 1\right)dy = -\pi\left[\frac{y^3}{9} - \frac{y^2}{3} - y\right]_1^3 = -\pi\left[\left(\frac{27}{9} - \frac{9}{3} - 3\right) - \left(\frac{1}{9} - \frac{1}{3} - 1\right)\right] = -\pi\left(-3 - \frac{1}{9} + \frac{1}{3} + 1\right)$

$= -\frac{\pi}{9}(-18 - 1 + 3) = \frac{16\pi}{9}$

19. $\frac{dx}{dy} = \frac{-1}{\sqrt{4-y}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4-y} \Rightarrow S = \int_0^{15/4} 2\pi \cdot 2\sqrt{4-y}\sqrt{1 + \frac{1}{4-y}}\,dy = 4\pi\int_0^{15/4}\sqrt{(4-y)+1}\,dy$

$= 4\pi\int_0^{15/4}\sqrt{5-y}\,dy = -4\pi\left[\frac{2}{3}(5-y)^{3/2}\right]_0^{15/4} = -\frac{8\pi}{3}\left[\left(5 - \frac{15}{4}\right)^{3/2} - 5^{3/2}\right] = -\frac{8\pi}{3}\left[\left(\frac{5}{4}\right)^{3/2} - 5^{3/2}\right]$

$= \frac{8\pi}{3}\left(5\sqrt{5} - \frac{5\sqrt{5}}{8}\right) = \frac{8\pi}{3}\left(\frac{40\sqrt{5} - 5\sqrt{5}}{8}\right) = \frac{35\pi\sqrt{5}}{3}$

20. $\frac{dx}{dy} = \frac{1}{\sqrt{2y-1}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{2y-1} \Rightarrow S = \int_{5/8}^1 2\pi\sqrt{2y-1}\sqrt{1 + \frac{1}{2y-1}}\,dy = 2\pi\int_{5/8}^1\sqrt{(2y-1)+1}\,dy = 2\pi\int_{5/8}^1\sqrt{2}\,y^{1/2}\,dy$

$= 2\pi\sqrt{2}\left[\frac{2}{3}y^{3/2}\right]_{5/8}^1 = \frac{4\pi\sqrt{2}}{3}\left[1^{3/2} - \left(\frac{5}{8}\right)^{3/2}\right] = \frac{4\pi\sqrt{2}}{3}\left(1 - \frac{5\sqrt{5}}{8\sqrt{8}}\right) = \frac{4\pi\sqrt{2}}{3}\left(\frac{8\cdot 2\sqrt{2} - 5\sqrt{5}}{8\cdot 2\sqrt{2}}\right) = \frac{\pi}{12}\left(16\sqrt{2} - 5\sqrt{5}\right)$

21. $S = 2\pi\int_{1/2}^1\sqrt{2y-1}\sqrt{1 + \left(\frac{1}{\sqrt{2y-1}}\right)^2}\,dy = 2\pi\int_{1/2}^1\sqrt{2y-1}\sqrt{1 + \frac{1}{2y-1}}\,dy = 2\pi\int_{1/2}^1\sqrt{2y-1}\sqrt{\frac{2y}{2y-1}}\,dy$

$= 2\pi\int_{1/2}^1\sqrt{2y}\,dy = 2\sqrt{2}\pi\int_{1/2}^1\sqrt{y}\,dy = 2\sqrt{2}\pi\left[\frac{2}{3}y^{3/2}\right]_{1/2}^1 = 2\sqrt{2}\pi\left[\left(\frac{2}{3}\sqrt{1^3}\right) - \left(\frac{2}{3}\sqrt{\left(\frac{1}{2}\right)^3}\right)\right] = 2\sqrt{2}\pi\left(\frac{2}{3} - \frac{1}{3\sqrt{2}}\right)$

$= 2\sqrt{2}\pi\left(\frac{2\sqrt{2}-1}{3\sqrt{2}}\right) = \frac{2\pi}{3}\left(2\sqrt{2} - 1\right)$

22. $y = \frac{1}{3}\left(x^2+2\right)^{3/2} \Rightarrow dy = x\sqrt{x^2+2}\,dx \Rightarrow ds = \sqrt{1 + (2x^2 + x^4)}\,dx \Rightarrow S = 2\pi\int_0^{\sqrt{2}} x\sqrt{1 + 2x^2 + x^4}\,dx$

$= 2\pi\int_0^{\sqrt{2}} x\sqrt{\left(x^2+1\right)^2}\,dx = 2\pi\int_0^{\sqrt{2}} x\left(x^2+1\right)dx = 2\pi\int_0^{\sqrt{2}}\left(x^3 + x\right)dx = 2\pi\left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^{\sqrt{2}} = 2\pi\left(\frac{4}{4} + \frac{2}{2}\right) = 4\pi$

23. $ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(y^3 - \frac{1}{4y^3}\right)^2 + 1}\,dy = \sqrt{\left(y^6 - \frac{1}{2} + \frac{1}{16y^6}\right) + 1}\,dy = \sqrt{\left(y^6 + \frac{1}{2} + \frac{1}{16y^6}\right)}\,dy$

$= \sqrt{\left(y^3 + \frac{1}{4y^3}\right)^2}\,dy = \left(y^3 + \frac{1}{4y^3}\right)dy$; $S = \int_1^2 2\pi y\,ds = 2\pi\int_1^2 y\left(y^3 + \frac{1}{4y^3}\right)dy = 2\pi\int_1^2\left(y^4 + \frac{1}{4}y^{-2}\right)dy$

$= 2\pi\left[\frac{y^5}{5} - \frac{1}{4}y^{-1}\right]_1^2 = 2\pi\left[\left(\frac{32}{5} - \frac{1}{8}\right) - \left(\frac{1}{5} - \frac{1}{4}\right)\right] = 2\pi\left(\frac{31}{5} + \frac{1}{8}\right) = \frac{2\pi}{40}(8\cdot 31 + 5) = \frac{253\pi}{20}$

24. $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \sin^2 x \Rightarrow S = 2\pi\int_{-\pi/2}^{\pi/2}(\cos x)\sqrt{1 + \sin^2 x}\,dx$

25. $y = \sqrt{a^2 - x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(a^2 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{a^2 - x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{\left(a^2 - x^2\right)}$

$\Rightarrow S = 2\pi\int_{-a}^{a}\sqrt{a^2 - x^2}\sqrt{1 + \frac{x^2}{\left(a^2 - x^2\right)}}\,dx = 2\pi\int_{-a}^{a}\sqrt{\left(a^2 - x^2\right) + x^2}\,dx = 2\pi\int_{-a}^{a} a\,dx = 2\pi a[x]_{-a}^{a}$

$= 2\pi a[a - (-a)] = (2\pi a)(2a) = 4\pi a^2$

26. $y = \frac{r}{h}x \Rightarrow \frac{dy}{dx} = \frac{r}{h} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{r^2}{h^2} \Rightarrow S = 2\pi\int_0^h \frac{r}{h}x\sqrt{1+\frac{r^2}{h^2}}\,dx = 2\pi\int_0^h \frac{r}{h}x\sqrt{\frac{h^2+r^2}{h^2}}\,dx$
$= \frac{2\pi r}{h}\sqrt{\frac{h^2+r^2}{h^2}}\int_0^h x\,dx = \frac{2\pi r}{h^2}\sqrt{h^2+r^2}\left[\frac{x^2}{2}\right]_0^h = \frac{2\pi r}{h^2}\sqrt{h^2+r^2}\left(\frac{h^2}{2}\right) = \pi r\sqrt{h^2+r^2}$

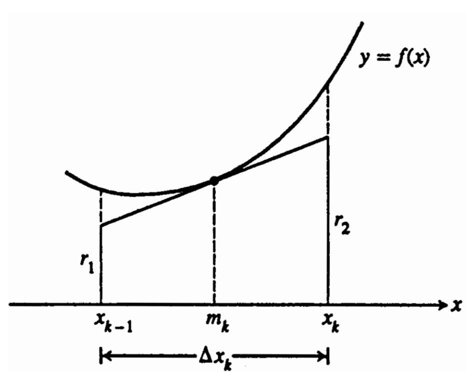
27. The area of the surface of one wok is $S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$. Now, $x^2+y^2 = 16^2 \Rightarrow x = \sqrt{16^2-y^2}$
$\Rightarrow \frac{dx}{dy} = \frac{-y}{\sqrt{16^2-y^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{16^2-y^2}$; $S = \int_{-16}^{-7} 2\pi\sqrt{16^2-y^2}\sqrt{1+\frac{y^2}{16^2-y^2}}\,dy = 2\pi\int_{-16}^{-7}\sqrt{(16^2-y^2)+y^2}\,dy$
$= 2\pi\int_{-16}^{-7} 16\,dy = 32\pi \cdot 9 = 288\pi \approx 904.78\text{ cm}^2$. The enamel needed to cover one surface of one wok is
$V = S \cdot 0.5\text{ mm} = S \cdot 0.05\text{ cm} = (904.78)(0.05)\text{ cm}^3 = 45.24\text{ cm}^3$. For 5000 woks, we need
$5000 \cdot V = 5000 \cdot 45.24\text{ cm}^3 = (5)(45.24)\text{L} = 226.2\text{L} \Rightarrow$ 226.2 liters of each color are needed.

28. $y = \sqrt{r^2-x^2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}\frac{2x}{\sqrt{r^2-x^2}} = \frac{-x}{\sqrt{r^2-x^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{x^2}{r^2-x^2}$; $S = 2\pi\int_a^{a+h}\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$
$= 2\pi\int_a^{a+h}\sqrt{(r^2-x^2)+x^2}\,dx = 2\pi r\int_a^{a+h}dx = 2\pi rh$, which is independent of a.

29. $y = \sqrt{R^2-x^2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}\frac{2x}{\sqrt{R^2-x^2}} = \frac{-x}{\sqrt{R^2-x^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{x^2}{R^2-x^2}$; $S = 2\pi\int_a^{a+h}\sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx$
$= 2\pi\int_a^{a+h}\sqrt{(R^2-x^2)+x^2}\,dx = 2\pi R\int_a^{a+h}dx = 2\pi Rh$

30. (a) $x^2+y^2 = 45^2 \Rightarrow x = \sqrt{45^2-y^2} \Rightarrow \frac{dx}{dy} = \frac{-y}{\sqrt{45^2-y^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{45^2-y^2}$;
$S = \int_{-22.5}^{45} 2\pi\sqrt{45^2-y^2}\sqrt{1+\frac{y^2}{45^2-y^2}}\,dy = 2\pi\int_{-22.5}^{45}\sqrt{(45^2-y^2)+y^2}\,dy = 2\pi \cdot 45\int_{-22.5}^{45}dy$
$= (2\pi)(45)(67.5) = 6075\pi$ square feet
(b) 19,085 square feet

31. (a) An equation of the tangent line segment is (see figure) $y = f(m_k) + f'(m_k)(x-m_k)$.
When $x = x_{k-1}$ we have
$r_1 = f(m_k) + f'(m_k)(x_{k-1}-m_k)$
$= f(m_k) + f'(m_k)\left(-\frac{\Delta x_k}{2}\right) = f(m_k) - f'(m_k)\frac{\Delta x_k}{2}$;
when $x = x_k$ we have
$r_2 = f(m_k) + f'(m_k)(x_k - m_k)$
$= f(m_k) + f'(m_k)\frac{\Delta x_k}{2}$;



(b) $L_k^2 = (\Delta x_k)^2 + (r_2-r_1)^2$
$= (\Delta x_k)^2 + \left[f'(m_k)\frac{\Delta x_k}{2} - \left(-f'(m_k)\frac{\Delta x_k}{2}\right)\right]^2$
$= (\Delta x_k)^2 + [f'(m_k)\Delta x_k]^2 \Rightarrow L_k = \sqrt{(\Delta x_k)^2 + [f'(m_k)\Delta x_k]^2}$, as claimed
(c) From geometry it is a fact that the lateral surface area of the frustum obtained by revolving the tangent line segment about the x-axis is given by $\Delta S_k = \pi(r_1+r_2)L_k = \pi[2f(m_k)]\sqrt{(\Delta x_k)^2 + [f'(m_k)\Delta x_k]^2}$ using parts (a) and (b) above. Thus, $\Delta S_k = 2\pi f(m_k)\sqrt{1+[f'(m_k)]^2}\,\Delta x_k$.
(d) $S = \lim_{n\to\infty}\sum_{k=1}^{n}\Delta S_k = \lim_{n\to\infty}\sum_{k=1}^{n} 2\pi f(m_k)\sqrt{1+[f'(m_k)]^2}\,\Delta x_k = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

32. $y = \left(1-x^{2/3}\right)^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2}\left(1-x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -\frac{\left(1-x^{2/3}\right)^{1/2}}{x^{1/3}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1-x^{2/3}}{x^{2/3}} = \frac{1}{x^{2/3}} - 1$
$\Rightarrow S = 2\int_0^1 2\pi\left(1-x^{2/3}\right)^{3/2}\sqrt{1+\left(\frac{1}{x^{2/3}}-1\right)}\,dx = 4\pi\int_0^1\left(1-x^{2/3}\right)^{3/2}\sqrt{x^{-2/3}}\,dx$

$= 4\pi\int_0^1 \left(1 - x^{2/3}\right)^{3/2} x^{-1/3}\,dx$; $\left[u = 1 - x^{2/3} \Rightarrow du = -\frac{2}{3}x^{-1/3}\,dx \Rightarrow -\frac{3}{2}\,du = x^{-1/3}\,dx;\right.$
$\left. x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 0\right] \rightarrow S = 4\pi\int_1^0 u^{3/2}\left(-\frac{3}{2}\,du\right) = -6\pi\left[\frac{2}{5}u^{5/2}\right]_1^0 = -6\pi\left(0 - \frac{2}{5}\right) = \frac{12\pi}{5}$

6.5 WORK AND FLUID FORCES

1. The force required to stretch the spring from its natural length of 2 m to a length of 5 m is $F(x) = kx$. The work done by F is $W = \int_0^3 F(x)\,dx = k\int_0^3 x\,dx = \frac{k}{2}\left[x^2\right]_0^3 = \frac{9k}{2}$. This work is equal to 1800 J $\Rightarrow \frac{9}{2}k = 1800 \Rightarrow k = 400$ N/m

2. (a) We find the force constant from Hooke's Law: $F = kx \Rightarrow k = \frac{F}{x} \Rightarrow k = \frac{800}{4} = 200$ lb/in.
   (b) The work done to stretch the spring 2 inches beyond its natural length is $W = \int_0^2 kx\,dx = 200\int_0^2 x\,dx = 200\left[\frac{x^2}{2}\right]_0^2$ $= 200(2 - 0) = 400\text{ in}\cdot\text{lb} = 33.3\text{ ft}\cdot\text{lb}$
   (c) We substitute $F = 1600$ into the equation $F = 200x$ to find $1600 = 200x \Rightarrow x = 8$ in.

3. We find the force constant from Hooke's law: $F = kx$. A force of 2 N stretches the spring to 0.02 m $\Rightarrow 2 = k\cdot(0.02)$ $\Rightarrow k = 100\,\frac{N}{m}$. The force of 4 N will stretch the rubber band y m, where $F = ky \Rightarrow y = \frac{F}{k} \Rightarrow y = \frac{4N}{100\,\frac{N}{m}} \Rightarrow y = 0.04$ m $= 4$ cm. The work done to stretch the rubber band 0.04 m is $W = \int_0^{0.04} kx\,dx = 100\int_0^{0.04} x\,dx = 100\left[\frac{x^2}{2}\right]_0^{0.04}$ $= \frac{(100)(0.04)^2}{2} = 0.08$ J

4. We find the force constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} \Rightarrow k = \frac{90}{1} \Rightarrow k = 90\,\frac{N}{m}$. The work done to stretch the spring 5 m beyond its natural length is $W = \int_0^5 kx\,dx = 90\int_0^5 x\,dx = 90\left[\frac{x^2}{2}\right]_0^5 = (90)\left(\frac{25}{2}\right) = 1125$ J

5. (a) We find the spring's constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} = \frac{21{,}714}{8-5} = \frac{21{,}714}{3} \Rightarrow k = 7238\,\frac{\text{lb}}{\text{in}}$
   (b) The work done to compress the assembly the first half inch is $W = \int_0^{0.5} kx\,dx = 7238\int_0^{0.5} x\,dx = 7238\left[\frac{x^2}{2}\right]_0^{0.5}$ $= (7238)\,\frac{(0.5)^2}{2} = \frac{(7238)(0.25)}{2} \approx 905\text{ in}\cdot\text{lb}$. The work done to compress the assembly the second half inch is: $W = \int_{0.5}^{1.0} kx\,dx = 7238\int_{0.5}^{1.0} x\,dx = 7238\left[\frac{x^2}{2}\right]_{0.5}^{1.0} = \frac{7238}{2}\left[1 - (0.5)^2\right] = \frac{(7238)(0.75)}{2} \approx 2714\text{ in}\cdot\text{lb}$

6. First, we find the force constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} = \frac{150}{\left(\frac{1}{16}\right)} = 16\cdot 150 = 2{,}400\,\frac{\text{lb}}{\text{in}}$. If someone compresses the scale $x = \frac{1}{8}$ in, he/she must weigh $F = kx = 2{,}400\left(\frac{1}{8}\right) = 300$ lb. The work done to compress the scale this far is $W = \int_0^{1/8} kx\,dx = 2400\left[\frac{x^2}{2}\right]_0^{1/8} = \frac{2400}{2\cdot 64} = 18.75\text{ lb}\cdot\text{in.} = \frac{25}{16}\text{ ft}\cdot\text{lb}$

7. The force required to haul up the rope is equal to the rope's weight, which varies steadily and is proportional to x, the length of the rope still hanging: $F(x) = 0.624x$. The work done is: $W = \int_0^{50} F(x)\,dx = \int_0^{50} 0.624x\,dx = 0.624\left[\frac{x^2}{2}\right]_0^{50}$ $= 780$ J

8. The weight of sand decreases steadily by 72 lb over the 18 ft, at 4 lb/ft. So the weight of sand when the bag is x ft off the ground is $F(x) = 144 - 4x$. The work done is: $W = \int_a^b F(x)\,dx = \int_0^{18}(144 - 4x)dx = \left[144x - 2x^2\right]_0^{18} = 1944\text{ ft}\cdot\text{lb}$

9. The force required to lift the cable is equal to the weight of the cable paid out: $F(x) = (4.5)(180 - x)$ where x is the position of the car off the first floor. The work done is: $W = \int_0^{180} F(x)\,dx = 4.5\int_0^{180}(180 - x)\,dx$

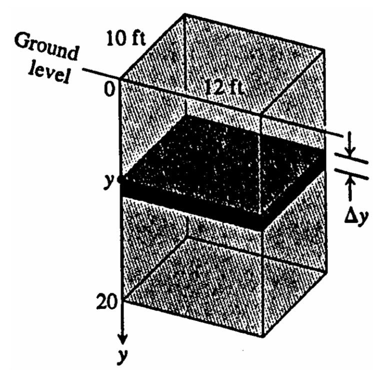
$= 4.5\left[180x - \frac{x^2}{2}\right]_0^{180} = 4.5\left(180^2 - \frac{180^2}{2}\right) = \frac{4.5\cdot 180^2}{2} = 72{,}900$ ft · lb

10. Since the force is acting toward the origin, it acts opposite to the positive x-direction. Thus $F(x) = -\frac{k}{x^2}$. The work done is $W = \int_a^b -\frac{k}{x^2}\,dx = k\int_a^b -\frac{1}{x^2}\,dx = k\left[\frac{1}{x}\right]_a^b = k\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{k(a-b)}{ab}$

11. Let r = the constant rate of leakage. Since the bucket is leaking at a constant rate and the bucket is rising at a constant rate, the amount of water in the bucket is proportional to $(20 - x)$, the distance the bucket is being raised. The leakage rate of the water is 0.8 lb/ft raised and the weight of the water in the bucket is $F = 0.8(20 - x)$. So:
$W = \int_0^{20} 0.8(20 - x)\,dx = 0.8\left[20x - \frac{x^2}{2}\right]_0^{20} = 160$ ft · lb.

12. Let r = the constant rate of leakage. Since the bucket is leaking at a constant rate and the bucket is rising at a constant rate, the amount of water in the bucket is proportional to $(20 - x)$, the distance the bucket is being raised. The leakage rate of the water is 2 lb/ft raised and the weight of the water in the bucket is $F = 2(20 - x)$. So:
$W = \int_0^{20} 2(20 - x)\,dx = 2\left[20x - \frac{x^2}{2}\right]_0^{20} = 400$ ft · lb.
Note that since the force in Exercise 12 is 2.5 times the force in Exercise 11 at each elevation, the total work is also 2.5 times as great.

13. We will use the coordinate system given.



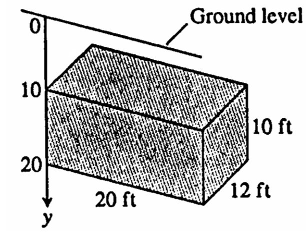
(a) The typical slab between the planes at y and $y + \Delta y$ has a volume of $\Delta V = (10)(12)\,\Delta y = 120\,\Delta y$ ft^3. The force F required to lift the slab is equal to its weight: $F = 62.4\,\Delta V = 62.4 \cdot 120\,\Delta y$ lb. The distance through which F must act is about y ft, so the work done lifting the slab is about ΔW = force × distance $= 62.4 \cdot 120 \cdot y \cdot \Delta y$ ft · lb. The work it takes to lift all the water is approximately $W \approx \sum_0^{20} \Delta W$ $= \sum_0^{20} 62.4 \cdot 120y \cdot \Delta y$ ft · lb. This is a Riemann sum for the function $62.4 \cdot 120y$ over the interval $0 \le y \le 20$. The work of pumping the tank empty is the limit of these sums:
$W = \int_0^{20} 62.4 \cdot 120y\,dy = (62.4)(120)\left[\frac{y^2}{2}\right]_0^{20} = (62.4)(120)\left(\frac{400}{2}\right) = (62.4)(120)(200) = 1{,}497{,}600$ ft · lb

(b) The time t it takes to empty the full tank with $\left(\frac{5}{11}\right)$–hp motor is $t = \frac{W}{250\,\frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = \frac{1{,}497{,}600\ \text{ft}\cdot\text{lb}}{250\,\frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = 5990.4$ sec $= 1.664$ hr
$\Rightarrow t \approx 1$ hr and 40 min

(c) Following all the steps of part (a), we find that the work it takes to lower the water level 10 ft is
$W = \int_0^{10} 62.4 \cdot 120y\,dy = (62.4)(120)\left[\frac{y^2}{2}\right]_0^{10} = (62.4)(120)\left(\frac{100}{2}\right) = 374{,}400$ ft · lb and the time is $t = \frac{W}{250\,\frac{\text{ft}\cdot\text{lb}}{\text{sec}}}$
$= 1497.6$ sec $= 0.416$ hr ≈ 25 min

(d) In a location where water weighs 62.26 $\frac{\text{lb}}{\text{ft}^3}$:
a) $W = (62.26)(24{,}000) = 1{,}494{,}240$ ft · lb.
b) $t = \frac{1{,}494{,}240}{250} = 5976.96$ sec ≈ 1.660 hr $\Rightarrow t \approx 1$ hr and 40 min
In a location where water weighs 62.59 $\frac{\text{lb}}{\text{ft}^3}$
a) $W = (62.59)(24{,}000) = 1{,}502{,}160$ ft · lb
b) $t = \frac{1{,}502{,}160}{250} = 6008.64$ sec ≈ 1.669 hr $\Rightarrow t \approx 1$ hr and 40.1 min

14. We will use the coordinate system given.



(a) The typical slab between the planes at y and $y + \Delta y$ has a volume of $\Delta V = (20)(12)\,\Delta y = 240\,\Delta y\ \text{ft}^3$. The force F required to lift the slab is equal to its weight: $F = 62.4\,\Delta V = 62.4 \cdot 240\,\Delta y$ lb. The distance through which F must act is about y ft, so the work done lifting the slab is about $\Delta W =$ force $\times$ distance $= 62.4 \cdot 240 \cdot y \cdot \Delta y$ ft $\cdot$ lb. The work it takes to lift all the water is approximately $W \approx \sum_{10}^{20} \Delta W = \sum_{10}^{20} 62.4 \cdot 240y \cdot \Delta y$ ft $\cdot$ lb. This is a Riemann sum for the function $62.4 \cdot 240y$ over the interval $10 \le y \le 20$. The work it takes to empty the cistern is the limit of these sums: $W = \int_{10}^{20} 62.4 \cdot 240y\,dy$ $= (62.4)(240)\left[\frac{y^2}{2}\right]_{10}^{20} = (62.4)(240)(200 - 50) = (62.4)(240)(150) = 2{,}246{,}400$ ft $\cdot$ lb

(b) $t = \frac{W}{275\,\frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = \frac{2{,}246{,}400\ \text{ft}\cdot\text{lb}}{275} \approx 8168.73$ sec ≈ 2.27 hours ≈ 2 hr and 16.1 min

(c) Following all the steps of part (a), we find that the work it takes to empty the tank halfway is $W = \int_{10}^{15} 62.4 \cdot 240y\,dy = (62.4)(240)\left[\frac{y^2}{2}\right]_{10}^{15} = (62.4)(240)\left(\frac{225}{2} - \frac{100}{2}\right) = (62.4)(240)\left(\frac{125}{2}\right) = 936{,}000$ ft. Then the time is $t = \frac{W}{275\,\frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = \frac{936{,}000}{275} \approx 3403.64$ sec ≈ 56.7 min

(d) In a location where water weighs $62.26\ \frac{\text{lb}}{\text{ft}^3}$:
a) $W = (62.26)(240)(150) = 2{,}241{,}360$ ft $\cdot$ lb.
b) $t = \frac{2{,}241{,}360}{275} = 8150.40$ sec $= 2.264$ hours ≈ 2 hr and 15.8 min
c) $W = (62.26)(240)\left(\frac{125}{2}\right) = 933{,}900$ ft $\cdot$ lb; $t = \frac{933{,}900}{275} = 3396$ sec ≈ 0.94 hours ≈ 56.6 min
In a location where water weighs $62.59\ \frac{\text{lb}}{\text{ft}^3}$
a) $W = (62.59)(240)(150) = 2{,}253{,}240$ ft $\cdot$ lb.
b) $t = \frac{2{,}253{,}240}{275} = 8193.60$ sec $= 2.276$ hours ≈ 2 hr and 16.56 min
c) $W = (62.59)(240)\left(\frac{125}{2}\right) = 938{,}850$ ft $\cdot$ lb; $t = \frac{938{,}850}{275} \approx 3414$ sec ≈ 0.95 hours ≈ 56.9 min

15. The slab is a disk of area $\pi x^2 = \pi\left(\frac{y}{2}\right)^2$, thickness $\triangle y$, and height below the top of the tank $(10 - y)$. So the work to pump the oil in this slab, $\triangle W$, is $57(10 - y)\pi\left(\frac{y}{2}\right)^2$. The work to pump all the oil to the top of the tank is $W = \int_0^{10} \frac{57\pi}{4}(10y^2 - y^3)dy = \frac{57\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^{10} = 11{,}875\pi$ ft $\cdot$ lb $\approx 37{,}306$ ft $\cdot$ lb.

16. Each slab of oil is to be pumped to a height of 14 ft. So the work to pump a slab is $(14 - y)(\pi)\left(\frac{y}{2}\right)^2$ and since the tank is half full and the volume of the original cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(5^2)(10) = \frac{250\pi}{3}$ ft^3, half the volume $= \frac{250\pi}{6}$ ft^3, and with half the volume the cone is filled to a height y, $\frac{250\pi}{6} = \frac{1}{3}\pi\frac{y^2}{4}y \Rightarrow y = \sqrt[3]{500}$ ft. So $W = \int_0^{\sqrt[3]{500}} \frac{57\pi}{4}(14y^2 - y^3)\,dy$ $= \frac{57\pi}{4}\left[\frac{14y^3}{3} - \frac{y^4}{4}\right]_0^{\sqrt[3]{500}} \approx 60{,}042$ ft $\cdot$ lb.

17. The typical slab between the planes at y and and $y + \Delta y$ has a volume of $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{20}{2}\right)^2 \Delta y$ $= \pi \cdot 100\ \Delta y$ ft^3. The force F required to lift the slab is equal to its weight: $F = 51.2\,\Delta V = 51.2 \cdot 100\pi\,\Delta y$ lb $\Rightarrow F = 5120\pi\,\Delta y$ lb. The distance through which F must act is about $(30 - y)$ ft. The work it takes to lift all the kerosene is approximately $W \approx \sum_0^{30} \Delta W = \sum_0^{30} 5120\pi(30 - y)\,\Delta y$ ft $\cdot$ lb which is a Riemann sum. The work to pump the tank dry is the limit of these sums: $W = \int_0^{30} 5120\pi(30 - y)\,dy = 5120\pi\left[30y - \frac{y^2}{2}\right]_0^{30} = 5120\pi\left(\frac{900}{2}\right) = (5120)(450\pi)$ $\approx 7{,}238{,}229.48$ ft $\cdot$ lb

18. (a) Follow all the steps of Example 5 but make the substitution of 64.5 $\frac{\text{lb}}{\text{ft}^3}$ for 57 $\frac{\text{lb}}{\text{ft}^3}$. Then,

$W = \int_0^8 \frac{64.5\pi}{4}(10-y)y^2\,dy = \frac{64.5\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8 = \frac{64.5\pi}{4}\left(\frac{10\cdot 8^3}{3} - \frac{8^4}{4}\right) = \left(\frac{64.5\pi}{4}\right)(8^3)\left(\frac{10}{3} - 2\right)$

$= \frac{64.5\pi\cdot 8^3}{3} = 21.5\pi\cdot 8^3 \approx 34{,}582.65 \text{ ft}\cdot\text{lb}$

(b) Exactly as done in Example 5 but change the distance through which F acts to distance $\approx (13-y)$ ft. Then

$W = \int_0^8 \frac{57\pi}{4}(13-y)y^2\,dy = \frac{57\pi}{4}\left[\frac{13y^3}{3} - \frac{y^4}{4}\right]_0^8 = \frac{57\pi}{4}\left(\frac{13\cdot 8^3}{3} - \frac{8^4}{4}\right) = \left(\frac{57\pi}{4}\right)(8^3)\left(\frac{13}{3} - 2\right) = \frac{57\pi\cdot 8^3\cdot 7}{3\cdot 4}$

$= (19\pi)(8^2)(7)(2) \approx 53{,}482.5 \text{ ft}\cdot\text{lb}$

19. The typical slab between the planes at y and y+Δy has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\sqrt{y}\right)^2 \Delta y \text{ ft}^3$. The force F(y) required to lift this slab is equal to its weight: $F(y) = 73\cdot\Delta V = 73\pi\left(\sqrt{y}\right)^2\Delta y = 73\pi\, y\,\Delta y$ lb. The distance through which F(y) must act to lift the slab to the top of the reservoir is about $(4-y)$ ft, so the work done is approximately $\Delta W \approx 73\pi\, y\,(4-y)\Delta y$ ft $\cdot$ lb. The work done lifting all the slabs from $y = 0$ ft to $y = 4$ ft is approximately $W \approx \sum_{k=0}^{n} 73\pi\, y_k\,(4-y_k)\Delta y$ ft $\cdot$ lb. Taking the limit of these Riemann sums as $n \to \infty$, we get

$W = \int_0^4 73\pi\, y\,(4-y)dy = 73\pi\int_0^4 (4y - y^2)dy = 73\pi\left[2y^2 - \frac{1}{3}y^3\right]_0^4 = 73\pi\left(32 - \frac{64}{3}\right) = \frac{2336\pi}{3}$ ft $\cdot$ lb.

20. The typical slab between the planes at y and y+Δy has a volume of about $\Delta V = (\text{length})(\text{width})(\text{thickness})$ $= \left(2\sqrt{25-y^2}\right)(10)\,\Delta y \text{ ft}^3$. The force F(y) required to lift this slab is equal to its weight: $F(y) = 53\cdot\Delta V$ $= 53\left(2\sqrt{25-y^2}\right)(10)\,\Delta y = 1060\sqrt{25-y^2}\Delta y$ lb. The distance through which F(y) must act to lift the slab to the level of 15 m above the top of the reservoir is about $(20-y)$ ft, so the work done is approximately $\Delta W \approx 1060\sqrt{25-y^2}(20-y)\Delta y$ ft $\cdot$ lb. The work done lifting all the slabs from $y = -5$ ft to $y = 5$ ft is approximately $W \approx \sum_{k=0}^{n} 1060\sqrt{25-y_k^2}\,(20-y_k)\Delta y$ ft $\cdot$ lb. Taking the limit of these Riemann sums as $n\to\infty$, we get

$W = \int_{-5}^{5} 1060\sqrt{25-y^2}(20-y)dy = 1060\int_{-5}^{5}(20-y)\sqrt{25-y^2}dy = 1060\left[\int_{-5}^{5} 20\sqrt{25-y^2}dy - \int_{-5}^{5} y\sqrt{25-y^2}dy\right]$

To evaluate the first integral, we use we can interpret $\int_{-5}^{5}\sqrt{25-y^2}dy$ as the area of the semicircle whose radius is 5, thus $\int_{-5}^{5} 20\sqrt{25-y^2}dy = 20\int_{-5}^{5}\sqrt{25-y^2}dy = 20\left[\frac{1}{2}\pi(5)^2\right] = 250\pi$. To evaluate the second integral let $u = 25 - y^2$ $\Rightarrow du = -2y\,dy$; $y = -5 \Rightarrow u = 0$, $y = 5 \Rightarrow u = 0$, thus $\int_{-5}^{5} y\sqrt{25-y^2}dy = -\frac{1}{2}\int_0^0 \sqrt{u}\,du = 0$. Thus,

$1060\left[\int_{-5}^{5} 20\sqrt{25-y^2}dy - \int_{-5}^{5} y\sqrt{25-y^2}dy\right] = 1060(250\pi - 0) = 265000\pi \approx 832522$ ft $\cdot$ lb.

21. The typical slab between the planes at y and y+Δy has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\sqrt{25-y^2}\right)^2\Delta y \text{ m}^3$. The force F(y) required to lift this slab is equal to its weight: $F(y) = 9800\cdot\Delta V$ $= 9800\pi\left(\sqrt{25-y^2}\right)^2\Delta y = 9800\pi\,(25-y^2)\,\Delta y$ N. The distance through which F(y) must act to lift the slab to the level of 4 m above the top of the reservoir is about $(4-y)$ m, so the work done is approximately $\Delta W \approx 9800\pi\,(25-y^2)\,(4-y)\,\Delta y$ N $\cdot$ m. The work done lifting all the slabs from $y = -5$ m to $y = 0$ m is approximately $W \approx \sum_{-5}^{0} 9800\pi\,(25-y^2)\,(4-y)\,\Delta y$ N $\cdot$ m. Taking the limit of these Riemann sums, we get

$W = \int_{-5}^{0} 9800\pi\,(25-y^2)\,(4-y)\,dy = 9800\pi\int_{-5}^{0}(100 - 25y - 4y^2 + y^3)\,dy = 9800\pi\left[100y - \frac{25}{2}y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_{-5}^{0}$

$= -9800\pi\left(-500 - \frac{25\cdot 25}{2} + \frac{4}{3}\cdot 125 + \frac{625}{4}\right) \approx 15{,}073{,}099.75$ J

22. The typical slab between the planes at y and y+Δy has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\sqrt{100-y^2}\right)^2\Delta y = \pi\,(100-y^2)\,\Delta y \text{ ft}^3$. The force is $F(y) = \frac{56\text{ lb}}{\text{ft}^3}\cdot\Delta V = 56\pi\,(100-y^2)\,\Delta y$ lb. The distance through which F(y) must act to lift the slab to the level of 2 ft above the top of the tank is about

$(12 - y)$ ft, so the work done is $\Delta W \approx 56\pi\,(100 - y^2)\,(12 - y)\,\Delta y$ lb · ft. The work done lifting all the slabs from $y = 0$ ft to $y = 10$ ft is approximately $W \approx \sum_0^{10} 56\pi\,(100 - y^2)\,(12 - y)\,\Delta y$ lb · ft. Taking the limit of these Riemann sums, we get $W = \int_0^{10} 56\pi\,(100 - y^2)\,(12 - y)\,dy = 56\pi\int_0^{10} (100 - y^2)\,(12 - y)\,dy$

$= 56\pi\int_0^{10} (1200 - 100y - 12y^2 + y^3)\,dy = 56\pi\left[1200y - \frac{100y^2}{2} - \frac{12y^3}{3} + \frac{y^4}{4}\right]_0^{10}$

$= 56\pi\left(12{,}000 - \frac{10{,}000}{2} - 4 \cdot 1000 + \frac{10{,}000}{4}\right) = (56\pi)\left(12 - 5 - 4 + \frac{5}{2}\right)(1000) \approx 967{,}611$ ft · lb.

It would cost (0.5)(967,611) = 483,805¢ = \$4838.05. Yes, you can afford to hire the firm.

23. $F = m\,\frac{dv}{dt} = mv\,\frac{dv}{dx}$ by the chain rule $\Rightarrow$ $W = \int_{x_1}^{x_2} mv\,\frac{dv}{dx}\,dx = m\int_{x_1}^{x_2}\left(v\,\frac{dv}{dx}\right)dx = m\left[\frac{1}{2}\,v^2(x)\right]_{x_1}^{x_2}$

$= \frac{1}{2}\,m\left[v^2(x_2) - v^2(x_1)\right] = \frac{1}{2}\,mv_2^2 - \frac{1}{2}\,mv_1^2$, as claimed.

24. weight $= 2$ oz $= \frac{2}{16}$ lb; mass $= \frac{\text{weight}}{32} = \frac{\frac{1}{8}}{32} = \frac{1}{256}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{1}{256}\text{ slugs}\right)(160\text{ ft/sec})^2 \approx 50$ ft · lb

25. 90 mph $= \frac{90\text{ mi}}{1\text{ hr}} \cdot \frac{1\text{ hr}}{60\text{ min}} \cdot \frac{1\text{ min}}{60\text{ sec}} \cdot \frac{5280\text{ ft}}{1\text{ mi}} = 132$ ft/sec; $m = \frac{0.3125\text{ lb}}{32\text{ ft/sec}^2} = \frac{0.3125}{32}$ slugs;
$W = \left(\frac{1}{2}\right)\left(\frac{0.3125\text{ lb}}{32\text{ ft/sec}^2}\right)(132\text{ ft/sec})^2 \approx 85.1$ ft · lb

26. weight $= 1.6$ oz $= 0.1$ lb $\Rightarrow$ $m = \frac{0.1\text{ lb}}{32\text{ ft/sec}^2} = \frac{1}{320}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{1}{320}\text{ slugs}\right)(280\text{ ft/sec})^2 = 122.5$ ft · lb

27. $v_1 = 0$ mph $= 0\,\frac{\text{ft}}{\text{sec}}$, $v_2 = 153$ mph $= 224.4\,\frac{\text{ft}}{\text{sec}}$; 2 oz $= 0.125$ lb $\Rightarrow m = \frac{0.125\text{ lb}}{32\text{ ft/sec}^2} = \frac{1}{256}$ slugs;
$W = \int_{x_1}^{x_2} F(x)\,dx = \frac{1}{2}\,mv_2^2 - \frac{1}{2}\,mv_1^2 = \frac{1}{2}\left(\frac{1}{256}\right)(224.4)^2 - \frac{1}{2}\left(\frac{1}{256}\right)(0)^2 = 98.35$ ft-lb.

28. weight $= 6.5$ oz $= \frac{6.5}{16}$ lb $\Rightarrow$ $m = \frac{6.5}{(16)(32)}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{6.5}{(16)(32)}\text{ slugs}\right)(132\text{ ft/sec})^2 \approx 110.6$ ft · lb

29. We imagine the milkshake divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval $[0, 7]$. The typical slab between the planes at y and $y + \Delta y$ has a volume of about$\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\frac{y + 17.5}{14}\right)^2 \Delta y$ in^3. The force $F(y)$ required to lift this slab is equal to its weight: $F(y) = \frac{4}{9}\ \Delta V = \frac{4\pi}{9}\left(\frac{y + 17.5}{14}\right)^2 \Delta y$ oz. The distance through which $F(y)$ must act to lift this slab to the level of 1 inch above the top is about $(8 - y)$ in. The work done lifting the slab is about $\Delta W = \left(\frac{4\pi}{9}\right)\frac{(y + 17.5)^2}{14^2}(8 - y)\Delta y$ in · oz. The work done lifting all the slabs from $y = 0$ to $y = 7$ is approximately $W = \sum_0^7 \frac{4\pi}{9 \cdot 14^2}(y + 17.5)^2(8 - y)\,\Delta y$ in · oz which is a Riemann sum. The work is the limit of these sums as the norm of the partition goes to zero: $W = \int_0^7 \frac{4\pi}{9 \cdot 14^2}(y + 17.5)^2(8 - y)dy$

$= \frac{4\pi}{9 \cdot 14^2}\int_0^7 (2450 - 26.25y - 27y^2 - y^3)dy = \frac{4\pi}{9 \cdot 14^2}\left[-\frac{y^4}{4} - 9y^3 - \frac{26.25}{2}\ y^2 + 2450y\right]_0^7$

$= \frac{4\pi}{9 \cdot 14^2}\left[-\frac{7^4}{4} - 9 \cdot 7^3 - \frac{26.25}{2} \cdot 7^2 + 2450 \cdot 7\right] \approx 91.32$ in · oz

30. Work $= \int_{6{,}370{,}000}^{35{,}780{,}000} \frac{1000\,MG}{r^2}\,dr = 1000\,MG\int_{6{,}370{,}000}^{35{,}780{,}000} \frac{dr}{r^2} = 1000\,MG\left[-\frac{1}{r}\right]_{6{,}370{,}000}^{35{,}780{,}000}$

$= (1000)\,(5.975 \cdot 10^{24})\,(6.672 \cdot 10^{-11})\left(\frac{1}{6{,}370{,}000} - \frac{1}{35{,}780{,}000}\right) \approx 5.144 \times 10^{10}$ J

31. To find the width of the plate at a typical depth y, we first find an equation for the line of the plate's right-hand edge: $y = x - 5$. If we let x denote the width of the right-hand half of the triangle at depth y, then $x = 5 + y$ and the total width is $L(y) = 2x = 2(5 + y)$. The depth of the strip is $(-y)$. The force exerted by the water against one side of the plate is therefore $F = \int_{-5}^{-2} w(-y) \cdot L(y)\,dy = \int_{-5}^{-2} 62.4 \cdot (-y) \cdot 2(5 + y)\,dy$

$= 124.8 \int_{-5}^{-2} (-5y - y^2)\, dy = 124.8 \left[-\frac{5}{2} y^2 - \frac{1}{3} y^3 \right]_{-5}^{-2} = 124.8 \left[\left(-\frac{5}{2} \cdot 4 + \frac{1}{3} \cdot 8 \right) - \left(-\frac{5}{2} \cdot 25 + \frac{1}{3} \cdot 125 \right) \right]$
$= (124.8) \left(\frac{105}{2} - \frac{117}{3} \right) = (124.8) \left(\frac{315 - 234}{6} \right) = 1684.8 \text{ lb}$

32. An equation for the line of the plate's right-hand edge is $y = x - 3 \Rightarrow x = y + 3$. Thus the total width is $L(y) = 2x = 2(y + 3)$. The depth of the strip is $(2 - y)$. The force exerted by the water is
$F = \int_{-3}^{0} w(2 - y)L(y)\, dy = \int_{-3}^{0} 62.4 \cdot (2 - y) \cdot 2(3 + y)\, dy = 124.8 \int_{-3}^{0} (6 - y - y^2)\, dy = 124.8 \left[6y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{-3}^{0}$
$= (-124.8) \left(-18 - \frac{9}{2} + 9 \right) = (-124.8) \left(-\frac{27}{2} \right) = 1684.8 \text{ lb}$

33. (a) The width of the strip is $L(y) = 4$, the depth of the strip is $(10 - y) \Rightarrow F = \int_a^b w \cdot \left(\begin{smallmatrix} \text{strip} \\ \text{depth} \end{smallmatrix} \right) F(y) dy$
$= \int_0^3 62.4(10 - y)(4) dy = 249.6 \int_0^3 (10 - y) dy = 249.6 \left[10y - \frac{y^2}{2} \right]_0^3 = 249.6 \left(30 - \frac{9}{2} \right) = 6364.8 \text{ lb}$

(b) The width of the strip is $L(y) = 3$, the depth of the strip is $(10 - y) \Rightarrow F = \int_a^b w \cdot \left(\begin{smallmatrix} \text{strip} \\ \text{depth} \end{smallmatrix} \right) F(y) dy$
$= \int_0^4 62.4(10 - y)(3) dy = 187.2 \int_0^4 (10 - y) dy = 187.2 \left[10y - \frac{y^2}{2} \right]_0^4 = 187.2(40 - 8) = 5990.4 \text{ lb}$

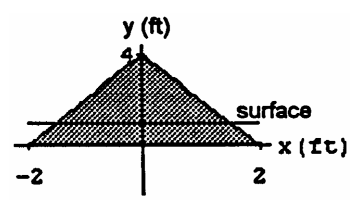
34. The width of the strip is $L(y) = 2\sqrt{25 - y^2}$, the depth of the strip is $(6 - y) \Rightarrow F = \int_a^b w \cdot \left(\begin{smallmatrix} \text{strip} \\ \text{depth} \end{smallmatrix} \right) F(y) dy$
$= \int_0^5 62.4(6 - y)\left(2\sqrt{25 - y^2}\right) dy = 124.8 \int_0^5 (6 - y)\sqrt{25 - y^2} dy = 124.8 \left[\int_0^5 6\sqrt{25 - y^2} dy - \int_0^5 y\sqrt{25 - y^2} dy \right]$
To evaluate the first integral, we use we can interpret $\int_0^5 \sqrt{25 - y^2} dy$ as the area of a quarter circle whose radius is 5, thus $\int_0^5 6\sqrt{25 - y^2} dy = 6 \int_0^5 \sqrt{25 - y^2} dy = 6\left[\frac{1}{4}\pi(5)^2\right] = \frac{75\pi}{2}$. To evaluate the second integral let $u = 25 - y^2$ $\Rightarrow du = -2y\, dy$; $y = 0 \Rightarrow u = 25$, $y = 5 \Rightarrow u = 0$, thus $\int_0^5 y\sqrt{25 - y^2} dy = -\frac{1}{2} \int_{25}^{0} \sqrt{u}\, du = \frac{1}{2} \int_0^{25} u^{1/2}\, du$
$= \frac{1}{3}\left[u^{3/2}\right]_0^{25} = \frac{125}{3}$. Thus, $124.8 \left[\int_0^5 6\sqrt{25 - y^2} dy - \int_0^5 y\sqrt{25 - y^2} dy \right] = 124.8\left(\frac{75\pi}{2} - \frac{125}{3}\right) \approx 9502.7$ lb.

35. Using the coordinate system of Exercise 32, we find the equation for the line of the plate's right-hand edge to be $y = 2x - 4 \Rightarrow x = \frac{y + 4}{2}$ and $L(y) = 2x = y + 4$. The depth of the strip is $(1 - y)$.

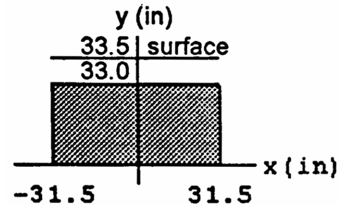
(a) $F = \int_{-4}^{0} w(1 - y)L(y)\, dy = \int_{-4}^{0} 62.4 \cdot (1 - y)(y + 4)\, dy = 62.4 \int_{-4}^{0} (4 - 3y - y^2)\, dy = 62.4 \left[4y - \frac{3y^2}{2} - \frac{y^3}{3} \right]_{-4}^{0}$
$= (-62.4) \left[(-4)(4) - \frac{(3)(16)}{2} + \frac{64}{3} \right] = (-62.4) \left(-16 - 24 + \frac{64}{3} \right) = \frac{(-62.4)(-120 + 64)}{3} = 1164.8 \text{ lb}$

(b) $F = (-64.0) \left[(-4)(4) - \frac{(3)(16)}{2} + \frac{64}{3} \right] = \frac{(-64.0)(-120 + 64)}{3} \approx 1194.7 \text{ lb}$

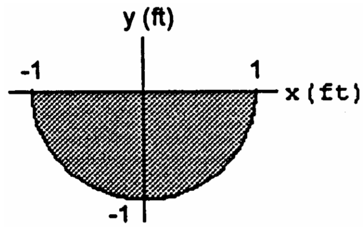
36. Using the coordinate system given, we find an equation for the line of the plate's right-hand edge to be $y = -2x + 4$ $\Rightarrow x = \frac{4 - y}{2}$ and $L(y) = 2x = 4 - y$. The depth of the strip is $(1 - y) \Rightarrow F = \int_0^1 w(1 - y)(4 - y)\, dy$
$= 62.4 \int_0^1 (y^2 - 5y + 4)\, dy = 62.4 \left[\frac{y^3}{3} - \frac{5y^2}{2} + 4y \right]_0^1$
$= (62.4) \left(\frac{1}{3} - \frac{5}{2} + 4 \right) = (62.4) \left(\frac{2 - 15 + 24}{6} \right) = \frac{(62.4)(11)}{6} = 114.4 \text{ lb}$



37. Using the coordinate system given in the accompanying figure, we see that the total width is $L(y) = 63$ and the depth of the strip is $(33.5 - y) \Rightarrow F = \int_0^{33} w(33.5 - y)L(y)\,dy$
$= \int_0^{33} \frac{64}{12^3} \cdot (33.5 - y) \cdot 63\,dy = \left(\frac{64}{12^3}\right)(63)\int_0^{33} (33.5 - y)\,dy$
$= \left(\frac{64}{12^3}\right)(63)\left[33.5y - \frac{y^2}{2}\right]_0^{33} = \left(\frac{64\cdot 63}{12^3}\right)\left[(33.5)(33) - \frac{33^2}{2}\right]$
$= \frac{(64)(63)(33)(67-33)}{(2)(12^3)} = 1309$ lb



38. Using the coordinate system given in the accompanying figure, we see that the right-hand edge is $x = \sqrt{1-y^2}$ so the total width is $L(y) = 2x = 2\sqrt{1-y^2}$ and the depth of the strip is $(-y)$. The force exerted by the water is therefore $F = \int_{-1}^{0} w \cdot (-y) \cdot 2\sqrt{1-y^2}\,dy$



$= 62.4\int_{-1}^{0} \sqrt{1-y^2}\,d\left(1-y^2\right) = 62.4\left[\frac{2}{3}\left(1-y^2\right)^{3/2}\right]_{-1}^{0} = (62.4)\left(\frac{2}{3}\right)(1-0) = 41.6$ lb

39. (a) $F = \left(62.4\,\frac{\text{lb}}{\text{ft}^3}\right)(8\text{ ft})(25\text{ ft}^2) = 12480$ lb

(b) The width of the strip is $L(y) = 5$, the depth of the strip is $(8-y) \Rightarrow F = \int_a^b w \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right)F(y)dy$
$= \int_0^5 62.4(8-y)(5)dy = 312\int_0^5 (8-y)dy = 312\left[8y - \frac{y^2}{2}\right]_0^5 = 312\left(40 - \frac{25}{2}\right) = 8580$ lb

(c) The width of the strip is $L(y) = 5$, the depth of the strip is $(8-y)$, the height of the strip is $\sqrt{2}\,dy$
$\Rightarrow F = \int_a^b w \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right)F(y)dy = \int_0^{5/\sqrt{2}} 62.4(8-y)(5)\sqrt{2}\,dy = 312\sqrt{2}\int_0^{5/\sqrt{2}} (8-y)dy = 312\sqrt{2}\left[8y - \frac{y^2}{2}\right]_0^{5/\sqrt{2}}$
$= 312\sqrt{2}\left(\frac{40}{\sqrt{2}} - \frac{25}{4}\right) = 9722.3$

40. The width of the strip is $L(y) = \frac{3}{4}\left(2\sqrt{3} - y\right)$, the depth of the strip is $(6-y)$, the height of the strip is $\frac{2}{\sqrt{3}}dy$
$\Rightarrow F = \int_a^b w \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right)F(y)dy = \int_0^{2\sqrt{3}} 62.4(6-y)\cdot\frac{3}{4}\left(2\sqrt{3}-y\right)\frac{2}{\sqrt{3}}dy = \frac{93.6}{\sqrt{3}}\int_0^{2\sqrt{3}}\left(12\sqrt{3} - 6y - 2y\sqrt{3} + y^2\right)dy$
$= \frac{93.6}{\sqrt{3}}\left[12y\sqrt{3} - 3y^2 - y^2\sqrt{3} + \frac{y^3}{3}\right]_0^{2\sqrt{3}} = \frac{93.6}{\sqrt{3}}\left(72 - 36 - 12\sqrt{3} + 8\sqrt{3}\right) \approx 1571.04$ lb

41. The coordinate system is given in the text. The right-hand edge is $x = \sqrt{y}$ and the total width is $L(y) = 2x = 2\sqrt{y}$.

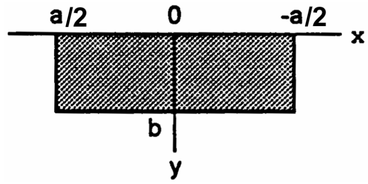
(a) The depth of the strip is $(2-y)$ so the force exerted by the liquid on the gate is $F = \int_0^1 w(2-y)L(y)\,dy$
$= \int_0^1 50(2-y)\cdot 2\sqrt{y}\,dy = 100\int_0^1 (2-y)\sqrt{y}\,dy = 100\int_0^1 \left(2y^{1/2} - y^{3/2}\right)dy = 100\left[\frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1$
$= 100\left(\frac{4}{3} - \frac{2}{5}\right) = \left(\frac{100}{15}\right)(20-6) = 93.33$ lb

(b) We need to solve $160 = \int_0^1 w(H-y)\cdot 2\sqrt{y}\,dy$ for h. $160 = 100\left(\frac{2H}{3} - \frac{2}{5}\right) \Rightarrow H = 3$ ft.

42. Suppose that h is the maximum height. Using the coordinate system given in the text, we find an equation for the line of the end plate's right-hand edge is $y = \frac{5}{2}x \Rightarrow x = \frac{2}{5}y$. The total width is $L(y) = 2x = \frac{4}{5}y$ and the depth of the typical horizontal strip at level y is $(h-y)$. Then the force is $F = \int_0^h w(h-y)L(y)\,dy = F_{max}$, where $F_{max} = 6667$ lb. Hence, $F_{max} = w\int_0^h (h-y)\cdot\frac{4}{5}y\,dy = (62.4)\left(\frac{4}{5}\right)\int_0^h \left(hy - y^2\right)dy$
$= (62.4)\left(\frac{4}{5}\right)\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h = (62.4)\left(\frac{4}{5}\right)\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = (62.4)\left(\frac{4}{5}\right)\left(\frac{1}{6}\right)h^3 = (10.4)\left(\frac{4}{5}\right)h^3 \Rightarrow h = \sqrt[3]{\left(\frac{5}{4}\right)\left(\frac{F_{max}}{10.4}\right)}$

$= \sqrt[3]{\left(\frac{5}{4}\right)\left(\frac{6667}{10.4}\right)} \approx 9.288$ ft. The volume of water which the tank can hold is $V = \frac{1}{2}$ (Base)(Height) $\cdot$ 30, where Height = h and $\frac{1}{2}$ (Base) $= \frac{2}{5}h \Rightarrow V = \left(\frac{2}{5}h^2\right)(30) = 12h^2 \approx 12(9.288)^2 \approx 1035\ \text{ft}^3$.

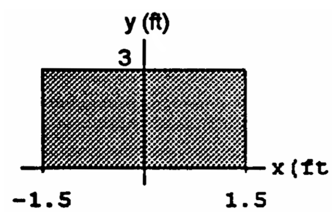
43. The pressure at level y is $p(y) = w \cdot y \Rightarrow$ the average pressure is $\overline{p} = \frac{1}{b}\int_0^b p(y)\,dy = \frac{1}{b}\int_0^b w \cdot y\,dy = \frac{1}{b} w \left[\frac{y^2}{2}\right]_0^b = \left(\frac{w}{b}\right)\left(\frac{b^2}{2}\right) = \frac{wb}{2}$. This is the pressure at level $\frac{b}{2}$, which is the pressure at the middle of the plate.



44. The force exerted by the fluid is $F = \int_0^b w(\text{depth})(\text{length})\,dy = \int_0^b w \cdot y \cdot a\,dy = (w \cdot a)\int_0^b y\,dy = (w \cdot a)\left[\frac{y^2}{2}\right]_0^b = w\left(\frac{ab^2}{2}\right) = \left(\frac{wb}{2}\right)(ab) = \overline{p} \cdot \text{Area}$, where $\overline{p}$ is the average value of the pressure.

45. When the water reaches the top of the tank the force on the movable side is $\int_{-2}^{0} (62.4)\left(2\sqrt{4-y^2}\right)(-y)\,dy = (62.4)\int_{-2}^{0}(4-y^2)^{1/2}(-2y)\,dy = (62.4)\left[\frac{2}{3}(4-y^2)^{3/2}\right]_{-2}^{0} = (62.4)\left(\frac{2}{3}\right)\left(4^{3/2}\right) = 332.8\ \text{ft}\cdot\text{lb}$. The force compressing the spring is $F = 100x$, so when the tank is full we have $332.8 = 100x \Rightarrow x \approx 3.33$ ft. Therefore the movable end does not reach the required 5 ft to allow drainage $\Rightarrow$ the tank will overflow.

46. (a) Using the given coordinate system we see that the total width is $L(y) = 3$ and the depth of the strip is $(3 - y)$. Thus, $F = \int_0^3 w(3-y)L(y)\,dy = \int_0^3 (62.4)(3-y)\cdot 3\,dy = (62.4)(3)\int_0^3 (3-y)\,dy = (62.4)(3)\left[3y - \frac{y^2}{2}\right]_0^3 = (62.4)(3)\left(9 - \frac{9}{2}\right) = (62.4)(3)\left(\frac{9}{2}\right) = 842.4$ lb



(b) Find a new water level Y such that $F_Y = (0.75)(842.4\ \text{lb}) = 631.8$ lb. The new depth of the strip is $(Y - y)$ and Y is the new upper limit of integration. Thus, $F_Y = \int_0^Y w(Y-y)L(y)\,dy = 62.4\int_0^Y (Y-y)\cdot 3\,dy = (62.4)(3)\int_0^Y (Y-y)\,dy = (62.4)(3)\left[Yy - \frac{y^2}{2}\right]_0^Y = (62.4)(3)\left(Y^2 - \frac{Y^2}{2}\right) = (62.4)(3)\left(\frac{Y^2}{2}\right)$. Therefore, $Y = \sqrt{\frac{2F_Y}{(62.4)(3)}} = \sqrt{\frac{1263.6}{187.2}} = \sqrt{6.75} \approx 2.598$ ft. So, $\Delta Y = 3 - Y \approx 3 - 2.598 \approx 0.402\ \text{ft} \approx 4.8$ in

6.6 MOMENTS AND CENTERS OF MASS

1. Since the plate is symmetric about the y-axis and its density is constant, the distribution of mass is symmetric about the y-axis and the center of mass lies on the y-axis. This means that $\overline{x} = 0$. It remains to find $\overline{y} = \frac{M_x}{M}$. We model the distribution of mass with *vertical* strips. The typical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{x^2+4}{2}\right)$, length: $4 - x^2$, width: dx, area: $dA = (4 - x^2)\,dx$, mass: $dm = \delta\,dA = \delta(4 - x^2)\,dx$. The moment of the strip about the x-axis is $\tilde{y}\,dm = \left(\frac{x^2+4}{2}\right)\delta(4-x^2)\,dx = \frac{\delta}{2}(16 - x^4)\,dx$. The moment of the plate about the x-axis is $M_x = \int \tilde{y}\,dm = \int_{-2}^{2}\frac{\delta}{2}(16-x^4)\,dx = \frac{\delta}{2}\left[16x - \frac{x^5}{5}\right]_{-2}^{2} = \frac{\delta}{2}\left[\left(16\cdot 2 - \frac{2^5}{5}\right) - \left(-16\cdot 2 + \frac{2^5}{5}\right)\right] = \frac{\delta\cdot 2}{2}\left(32 - \frac{32}{5}\right) = \frac{128\delta}{5}$. The mass of the

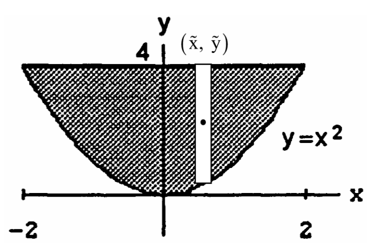
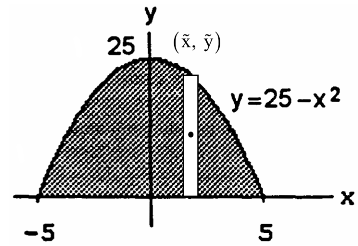


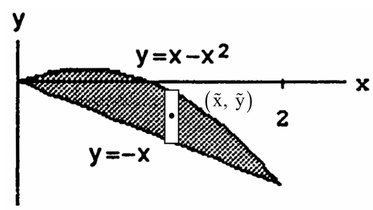
plate is $M = \int \delta(4 - x^2)\,dx = \delta\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 2\delta\left(8 - \frac{8}{3}\right) = \frac{32\delta}{3}$. Therefore $\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{128\delta}{5}\right)}{\left(\frac{32\delta}{3}\right)} = \frac{12}{5}$. The plate's center of mass is the point $(\bar{x}, \bar{y}) = \left(0, \frac{12}{5}\right)$.

2. Applying the symmetry argument analogous to the one in Exercise 1, we find $\bar{x} = 0$. To find $\bar{y} = \frac{M_x}{M}$, we use the *vertical* strips technique. The typical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{25 - x^2}{2}\right)$, length: $25 - x^2$, width: dx, area: $dA = (25 - x^2)dx$, mass: $dm = \delta\,dA = \delta(25 - x^2)\,dx$. The moment of the strip about the x-axis is



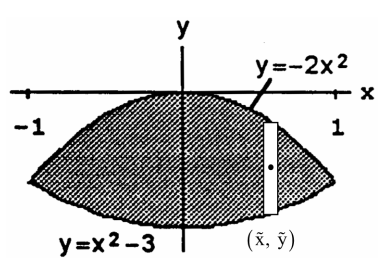
$\tilde{y}\,dm = \left(\frac{25 - x^2}{2}\right)\delta(25 - x^2)\,dx = \frac{\delta}{2}(25 - x^2)^2\,dx$. The moment of the plate about the x-axis is $M_x = \int \tilde{y}\,dm$
$= \int_{-5}^{5} \frac{\delta}{2}(25 - x^2)^2\,dx = \frac{\delta}{2}\int_{-5}^{5}(625 - 50x^2 + x^4)\,dx = \frac{\delta}{2}\left[625x - \frac{50}{3}x^3 + \frac{x^5}{5}\right]_{-5}^{5} = 2 \cdot \frac{\delta}{2}\left(625 \cdot 5 - \frac{50}{3} \cdot 5^3 + \frac{5^5}{5}\right)$
$= \delta \cdot 625\left(5 - \frac{10}{3} + 1\right) = \delta \cdot 625 \cdot \left(\frac{8}{3}\right)$. The mass of the plate is $M = \int dm = \int_{-5}^{5} \delta(25 - x^2)\,dx = \delta\left[25x - \frac{x^3}{3}\right]_{-5}^{5}$
$= 2\delta\left(5^3 - \frac{5^3}{3}\right) = \frac{4}{3}\delta \cdot 5^3$. Therefore $\bar{y} = \frac{M_x}{M} = \frac{\delta \cdot 5^4 \cdot \left(\frac{8}{3}\right)}{\delta \cdot 5^3 \cdot \left(\frac{4}{3}\right)} = 10$. The plate's center of mass is the point $(\bar{x}, \bar{y}) = (0, 10)$.

3. Intersection points: $x - x^2 = -x \Rightarrow 2x - x^2 = 0$ $\Rightarrow x(2 - x) = 0 \Rightarrow x = 0$ or $x = 2$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{(x - x^2) + (-x)}{2}\right)$ $= \left(x, -\frac{x^2}{2}\right)$, length: $(x - x^2) - (-x) = 2x - x^2$, width: dx, area: $dA = (2x - x^2)\,dx$, mass: $dm = \delta\,dA = \delta(2x - x^2)\,dx$. The moment of the strip about the x-axis is



$\tilde{y}\,dm = \left(-\frac{x^2}{2}\right)\delta(2x - x^2)\,dx$; about the y-axis it is $\tilde{x}\,dm = x \cdot \delta(2x - x^2)\,dx$. Thus, $M_x = \int \tilde{y}\,dm$
$= -\int_0^2 \left(\frac{\delta}{2}x^2\right)(2x - x^2)\,dx = -\frac{\delta}{2}\int_0^2 (2x^3 - x^4)\,dx = -\frac{\delta}{2}\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = -\frac{\delta}{2}\left(2^3 - \frac{2^5}{5}\right) = -\frac{\delta}{2} \cdot 2^3\left(1 - \frac{4}{5}\right)$
$= -\frac{4\delta}{5}$; $M_y = \int \tilde{x}\,dm = \int_0^2 x \cdot \delta(2x - x^2)\,dx = \delta\int_0^2 (2x^2 - x^3) = \delta\left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_0^2 = \delta\left(2 \cdot \frac{2^3}{3} - \frac{2^4}{4}\right) = \frac{\delta \cdot 2^4}{12} = \frac{4\delta}{3}$;
$M = \int dm = \int_0^2 \delta(2x - x^2)\,dx = \delta\int_0^2 (2x - x^2)\,dx = \delta\left[x^2 - \frac{x^3}{3}\right]_0^2 = \delta\left(4 - \frac{8}{3}\right) = \frac{4\delta}{3}$. Therefore, $\bar{x} = \frac{M_y}{M}$
$= \left(\frac{4\delta}{3}\right)\left(\frac{3}{4\delta}\right) = 1$ and $\bar{y} = \frac{M_x}{M} = \left(-\frac{4\delta}{5}\right)\left(\frac{3}{4\delta}\right) = -\frac{3}{5} \Rightarrow (\bar{x}, \bar{y}) = \left(1, -\frac{3}{5}\right)$ is the center of mass.

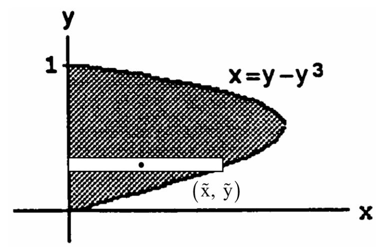
4. Intersection points: $x^2 - 3 = -2x^2 \Rightarrow 3x^2 - 3 = 0$ $\Rightarrow 3(x - 1)(x + 1) = 0 \Rightarrow x = -1$ or $x = 1$. Applying the symmetry argument analogous to the one in Exercise 1, we find $\bar{x} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{-2x^2 + (x^2 - 3)}{2}\right) = \left(x, \frac{-x^2 - 3}{2}\right)$, length: $-2x^2 - (x^2 - 3) = 3(1 - x^2)$, width: dx, area: $dA = 3(1 - x^2)\,dx$, mass: $dm = \delta\,dA = 3\delta(1 - x^2)\,dx$. The moment of the strip about the x-axis is



$\tilde{y}\,dm = \frac{3}{2}\delta(-x^2 - 3)(1 - x^2)\,dx = \frac{3}{2}\delta(x^4 + 3x^2 - x^2 - 3)\,dx = \frac{3}{2}\delta(x^4 + 2x^2 - 3)\,dx$; $M_x = \int \tilde{y}\,dm$
$= \frac{3}{2}\delta\int_{-1}^{1}(x^4 + 2x^2 - 3)\,dx = \frac{3}{2}\delta\left[\frac{x^5}{5} + \frac{2x^3}{3} - 3x\right]_{-1}^{1} = \frac{3}{2} \cdot \delta \cdot 2\left(\frac{1}{5} + \frac{2}{3} - 3\right) = 3\delta\left(\frac{3+10-45}{15}\right) = -\frac{32\delta}{5}$;

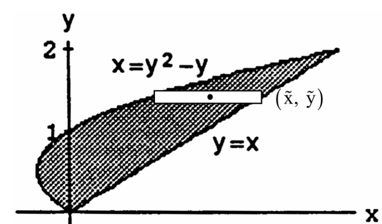
$M = \int dm = 3\delta \int_{-1}^{1} (1 - x^2)\, dx = 3\delta \left[x - \frac{x^3}{3}\right]_{-1}^{1} = 3\delta \cdot 2\left(1 - \frac{1}{3}\right) = 4\delta$. Therefore, $\overline{y} = \frac{M_x}{M} = -\frac{\delta \cdot 32}{5 \cdot \delta \cdot 4} = -\frac{8}{5}$
$\Rightarrow (\overline{x}, \overline{y}) = \left(0, -\frac{8}{5}\right)$ is the center of mass.

5. The typical *horizontal* strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(\frac{y - y^3}{2}, y\right)$, length: $y - y^3$, width: dy, area: $dA = (y - y^3)\, dy$, mass: $dm = \delta\, dA = \delta\,(y - y^3)\, dy$. The moment of the strip about the y-axis is $\widetilde{x}\, dm = \delta\left(\frac{y - y^3}{2}\right)(y - y^3)\, dy = \frac{\delta}{2}(y - y^3)^2\, dy$ $= \frac{\delta}{2}(y^2 - 2y^4 + y^6)\, dy$; the moment about the x-axis is



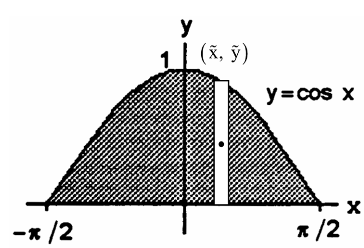
$\widetilde{y}\, dm = \delta y\,(y - y^3)\, dy = \delta\,(y^2 - y^4)\, dy$. Thus, $M_x = \int \widetilde{y}\, dm = \delta \int_0^1 (y^2 - y^4)\, dy = \delta\left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = \delta\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\delta}{15}$;
$M_y = \int \widetilde{x}\, dm = \frac{\delta}{2}\int_0^1 (y^2 - 2y^4 + y^6)\, dy = \frac{\delta}{2}\left[\frac{y^3}{3} - \frac{2y^5}{5} + \frac{y^7}{7}\right]_0^1 = \frac{\delta}{2}\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) = \frac{\delta}{2}\left(\frac{35 - 42 + 15}{3 \cdot 5 \cdot 7}\right) = \frac{4\delta}{105}$; $M = \int dm$
$= \delta \int_0^1 (y - y^3)\, dy = \delta\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \delta\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\delta}{4}$. Therefore, $\overline{x} = \frac{M_y}{M} = \left(\frac{4\delta}{105}\right)\left(\frac{4}{\delta}\right) = \frac{16}{105}$ and $\overline{y} = \frac{M_x}{M} = \left(\frac{2\delta}{15}\right)\left(\frac{4}{\delta}\right)$
$= \frac{8}{15} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{16}{105}, \frac{8}{15}\right)$ is the center of mass.

6. Intersection points: $y = y^2 - y \Rightarrow y^2 - 2y = 0$ $\Rightarrow y(y - 2) = 0 \Rightarrow y = 0$ or $y = 2$. The typical *horizontal* strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(\frac{(y^2 - y) + y}{2}, y\right) = \left(\frac{y^2}{2}, y\right)$, length: $y - (y^2 - y) = 2y - y^2$, width: dy, area: $dA = (2y - y^2)\, dy$, mass: $dm = \delta\, dA = \delta\,(2y - y^2)\, dy$. The moment about the y-axis is $\widetilde{x}\, dm = \frac{\delta}{2} \cdot y^2\,(2y - y^2)\, dy$



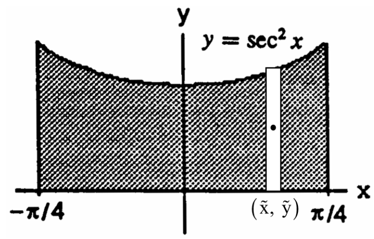
$= \frac{\delta}{2}(2y^3 - y^4)\, dy$; the moment about the x-axis is $\widetilde{y}\, dm = \delta y\,(2y - y^2)\, dy = \delta\,(2y^2 - y^3)\, dy$. Thus,
$M_x = \int \widetilde{y}\, dm = \int_0^2 \delta\,(2y^2 - y^3)\, dy = \delta\left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = \delta\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{16\delta}{12}(4 - 3) = \frac{4\delta}{3}$; $M_y = \int \widetilde{x}\, dm$
$= \int_0^2 \frac{\delta}{2}(2y^3 - y^4)\, dy = \frac{\delta}{2}\left[\frac{y^4}{2} - \frac{y^5}{5}\right]_0^2 = \frac{\delta}{2}\left(8 - \frac{32}{5}\right) = \frac{\delta}{2}\left(\frac{40-32}{5}\right) = \frac{4\delta}{5}$; $M = \int dm = \int_0^2 \delta\,(2y - y^2)\, dy$
$= \delta\left[y^2 - \frac{y^3}{3}\right]_0^2 = \delta\left(4 - \frac{8}{3}\right) = \frac{4\delta}{3}$. Therefore, $\overline{x} = \frac{M_y}{M} = \left(\frac{4\delta}{5}\right)\left(\frac{3}{4\delta}\right) = \frac{3}{5}$ and $\overline{y} = \frac{M_x}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{3}{4\delta}\right) = 1$
$\Rightarrow (\overline{x}, \overline{y}) = \left(\frac{3}{5}, 1\right)$ is the center of mass.

7. Applying the symmetry argument analogous to the one used in Exercise 1, we find $\overline{x} = 0$. The typical *vertical* strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{\cos x}{2}\right)$, length: $\cos x$, width: dx, area: $dA = \cos x\, dx$, mass: $dm = \delta\, dA = \delta \cos x\, dx$. The moment of the strip about the x-axis is $\widetilde{y}\, dm = \delta \cdot \frac{\cos x}{2} \cdot \cos x\, dx$ $= \frac{\delta}{2}\cos^2 x\, dx = \frac{\delta}{2}\left(\frac{1 + \cos 2x}{2}\right) dx = \frac{\delta}{4}(1 + \cos 2x)\, dx$; thus,



$M_x = \int \widetilde{y}\, dm = \int_{-\pi/2}^{\pi/2} \frac{\delta}{4}(1 + \cos 2x)\, dx = \frac{\delta}{4}\left[x + \frac{\sin 2x}{2}\right]_{-\pi/2}^{\pi/2} = \frac{\delta}{4}\left[\left(\frac{\pi}{2} + 0\right) - \left(-\frac{\pi}{2}\right)\right] = \frac{\delta\pi}{4}$; $M = \int dm = \delta \int_{-\pi/2}^{\pi/2} \cos x\, dx$
$= \delta[\sin x]_{-\pi/2}^{\pi/2} = 2\delta$. Therefore, $\overline{y} = \frac{M_x}{M} = \frac{\delta\pi}{4 \cdot 2\delta} = \frac{\pi}{8} \Rightarrow (\overline{x}, \overline{y}) = \left(0, \frac{\pi}{8}\right)$ is the center of mass.

8. Applying the symmetry argument analogous to the one used in Exercise 1, we find $\overline{x} = 0$. The typical vertical strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{\sec^2 x}{2}\right)$, length: $\sec^2 x$, width: dx, area: $dA = \sec^2 x\, dx$, mass: $dm = \delta\, dA = \delta \sec^2 x\, dx$. The moment about the x-axis is $\widetilde{y}\, dm = \left(\frac{\sec^2 x}{2}\right)(\delta \sec^2 x)\, dx$ $= \frac{\delta}{2}\sec^4 x\, dx$. $M_x = \int_{-\pi/4}^{\pi/4} \widetilde{y}\, dm = \frac{\delta}{2}\int_{-\pi/4}^{\pi/4} \sec^4 x\, dx$

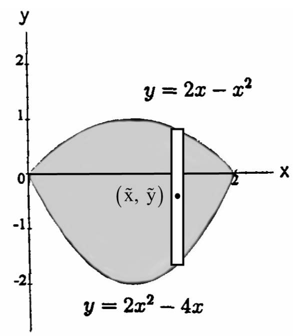


$= \frac{\delta}{2}\int_{-\pi/4}^{\pi/4} (\tan^2 x + 1)(\sec^2 x)\, dx = \frac{\delta}{2}\int_{-\pi/4}^{\pi/4} (\tan x)^2 (\sec^2 x)\, dx + \frac{\delta}{2}\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \frac{\delta}{2}\left[\frac{(\tan x)^3}{3}\right]_{-\pi/4}^{\pi/4} + \frac{\delta}{2}[\tan x]_{-\pi/4}^{\pi/4}$

$= \frac{\delta}{2}\left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] + \frac{\delta}{2}[1 - (-1)] = \frac{\delta}{3} + \delta = \frac{4\delta}{3}$; $M = \int dm = \delta \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \delta[\tan x]_{-\pi/4}^{\pi/4} = \delta[1 - (-1)] = 2\delta$.

Therefore, $\overline{y} = \frac{M_x}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{1}{2\delta}\right) = \frac{2}{3} \Rightarrow (\overline{x}, \overline{y}) = \left(0, \frac{2}{3}\right)$ is the center of mass.

9. Since the plate is symmetric about the line $x = 1$ and its density is constant, the distribution of mass is symmetric about this line and the center of mass lies on it. This means that $\overline{x} = 1$. The typical *vertical* strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{(2x - x^2) + (2x^2 - 4x)}{2}\right) = \left(x, \frac{x^2 - 2x}{2}\right)$, length: $(2x - x^2) - (2x^2 - 4x) = -3x^2 + 6x = 3(2x - x^2)$, width: dx, area: $dA = 3(2x - x^2)\, dx$, mass: $dm = \delta\, dA$ $= 3\delta(2x - x^2)\, dx$. The moment about the x-axis is $\widetilde{y}\, dm = \frac{3}{2}\delta(x^2 - 2x)(2x - x^2)\, dx = -\frac{3}{2}\delta(x^2 - 2x)^2\, dx$



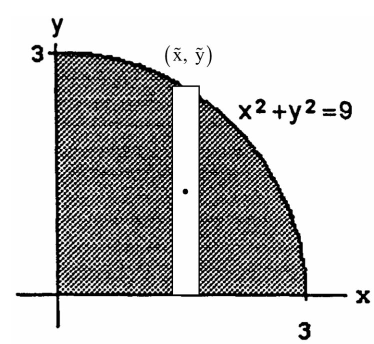
$= -\frac{3}{2}\delta(x^4 - 4x^3 + 4x^2)\, dx$. Thus, $M_x = \int \widetilde{y}\, dm = -\int_0^2 \frac{3}{2}\delta(x^4 - 4x^3 + 4x^2)\, dx = -\frac{3}{2}\delta\left[\frac{x^5}{5} - x^4 + \frac{4}{3}x^3\right]_0^2$

$= -\frac{3}{2}\delta\left(\frac{2^5}{5} - 2^4 + \frac{4}{3}\cdot 2^3\right) = -\frac{3}{2}\delta \cdot 2^4\left(\frac{2}{5} - 1 + \frac{2}{3}\right) = -\frac{3}{2}\delta \cdot 2^4\left(\frac{6 - 15 + 10}{15}\right) = -\frac{8\delta}{5}$; $M = \int dm$

$= \int_0^2 3\delta(2x - x^2)\, dx = 3\delta\left[x^2 - \frac{x^3}{3}\right]_0^2 = 3\delta\left(4 - \frac{8}{3}\right) = 4\delta$. Therefore, $\overline{y} = \frac{M_x}{M} = \left(-\frac{8\delta}{5}\right)\left(\frac{1}{4\delta}\right) = -\frac{2}{5}$

$\Rightarrow (\overline{x}, \overline{y}) = \left(1, -\frac{2}{5}\right)$ is the center of mass.

10. (a) Since the plate is symmetric about the line $x = y$ and its density is constant, the distribution of mass is symmetric about this line. This means that $\overline{x} = \overline{y}$. The typical *vertical* strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{\sqrt{9 - x^2}}{2}\right)$, length: $\sqrt{9 - x^2}$, width: dx, area: $dA = \sqrt{9 - x^2}\, dx$, mass: $dm = \delta\, dA = \delta\sqrt{9 - x^2}\, dx$. The moment about the x-axis is

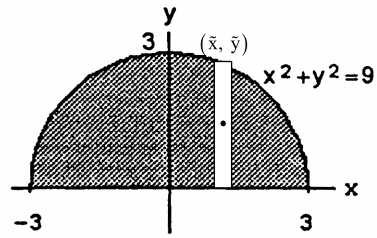


$\widetilde{y}\, dm = \delta\left(\frac{\sqrt{9 - x^2}}{2}\right)\sqrt{9 - x^2}\, dx = \frac{\delta}{2}(9 - x^2)\, dx$. Thus, $M_x = \int \widetilde{y}\, dm = \int_0^3 \frac{\delta}{2}(9 - x^2)\, dx = \frac{\delta}{2}\left[9x - \frac{x^3}{3}\right]_0^3$

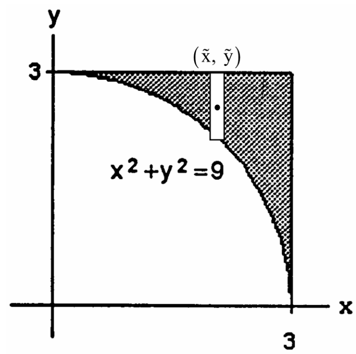
$= \frac{\delta}{2}(27 - 9) = 9\delta$; $M = \int dm = \int \delta\, dA = \delta \int dA = \delta$(Area of a quarter of a circle of radius 3) $= \delta\left(\frac{9\pi}{4}\right) = \frac{9\pi\delta}{4}$.

Therefore, $\overline{y} = \frac{M_x}{M} = (9\delta)\left(\frac{4}{9\pi\delta}\right) = \frac{4}{\pi} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{4}{\pi}, \frac{4}{\pi}\right)$ is the center of mass.

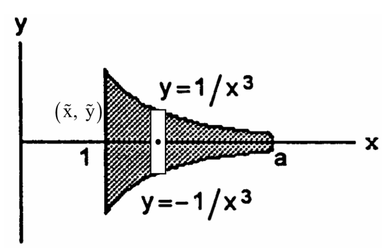
(b) Applying the symmetry argument analogous to the one used in Exercise 1, we find that $\overline{x} = 0$. The typical vertical strip has the same parameters as in part (a). Thus, $M_x = \int \tilde{y}\, dm = \int_{-3}^{3} \frac{\delta}{2}\left(9 - x^2\right) dx$ $= 2\int_0^3 \frac{\delta}{2}\left(9 - x^2\right) dx = 2(9\delta) = 18\delta$; $M = \int dm = \int \delta\, dA = \delta \int dA$ $= \delta(\text{Area of a semi-circle of radius } 3) = \delta\left(\frac{9\pi}{2}\right) = \frac{9\pi\delta}{2}$. Therefore, $\overline{y} = \frac{M_x}{M} = (18\delta)\left(\frac{2}{9\pi\delta}\right) = \frac{4}{\pi}$, the same $\overline{y}$ as in part (a) $\Rightarrow$ $(\overline{x}, \overline{y}) = \left(0, \frac{4}{\pi}\right)$ is the center of mass.



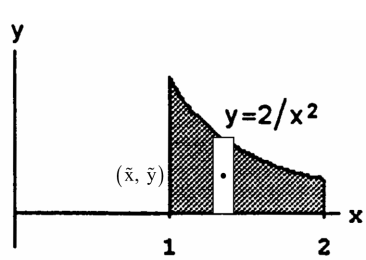
11. Since the plate is symmetric about the line $x = y$ and its density is constant, the distribution of mass is symmetric about this line. This means that $\overline{x} = \overline{y}$. The typical *vertical* strip has
center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{3+\sqrt{9-x^2}}{2}\right)$,
length: $3 - \sqrt{9 - x^2}$, width: dx,
area: $dA = \left(3 - \sqrt{9 - x^2}\right) dx$,
mass: $dm = \delta\, dA = \delta\left(3 - \sqrt{9 - x^2}\right) dx$.
The moment about the x-axis is
$\tilde{y}\, dm = \delta\, \frac{\left(3+\sqrt{9-x^2}\right)\left(3-\sqrt{9-x^2}\right)}{2}\, dx = \frac{\delta}{2}\left[9 - \left(9 - x^2\right)\right] dx = \frac{\delta x^2}{2}\, dx$. Thus, $M_x = \int_0^3 \frac{\delta x^2}{2}\, dx = \frac{\delta}{6}\left[x^3\right]_0^3 = \frac{9\delta}{2}$. The area equals the area of a square with side length 3 minus one quarter the area of a disk with radius 3 $\Rightarrow$ $A = 3^2 - \frac{\pi 9}{4}$ $= \frac{9}{4}(4 - \pi)$ $\Rightarrow$ $M = \delta A = \frac{9\delta}{4}(4 - \pi)$. Therefore, $\overline{y} = \frac{M_x}{M} = \left(\frac{9\delta}{2}\right)\left[\frac{4}{9\delta(4-\pi)}\right] = \frac{2}{4-\pi}$ $\Rightarrow$ $(\overline{x}, \overline{y}) = \left(\frac{2}{4-\pi}, \frac{2}{4-\pi}\right)$ is the center of mass.



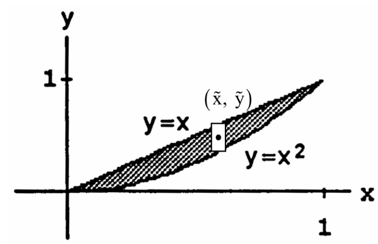
12. Applying the symmetry argument analogous to the one used in Exercise 1, we find that $\overline{y} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\frac{1}{x^3} - \frac{1}{x^3}}{2}\right) = (x, 0)$,
length: $\frac{1}{x^3} - \left(-\frac{1}{x^3}\right) = \frac{2}{x^3}$, width: dx, area: $dA = \frac{2}{x^3}\, dx$,
mass: $dm = \delta\, dA = \frac{2\delta}{x^3}\, dx$. The moment about the y-axis is
$\tilde{x}\, dm = x \cdot \frac{2\delta}{x^3}\, dx = \frac{2\delta}{x^2}\, dx$. Thus, $M_y = \int \tilde{x}\, dm = \int_1^a \frac{2\delta}{x^2}\, dx$ $= 2\delta\left[-\frac{1}{x}\right]_1^a = 2\delta\left(-\frac{1}{a} + 1\right) = \frac{2\delta(a-1)}{a}$; $M = \int dm = \int_1^a \frac{2\delta}{x^3}\, dx = \delta\left[-\frac{1}{x^2}\right]_1^a = \delta\left(-\frac{1}{a^2} + 1\right) = \frac{\delta\left(a^2-1\right)}{a^2}$. Therefore, $\overline{x} = \frac{M_y}{M} = \left[\frac{2\delta(a-1)}{a}\right]\left[\frac{a^2}{\delta\left(a^2-1\right)}\right] = \frac{2a}{a+1}$ $\Rightarrow$ $(\overline{x}, \overline{y}) = \left(\frac{2a}{a+1}, 0\right)$. Also, $\lim_{a \to \infty} \overline{x} = 2$.



13. $M_x = \int \tilde{y}\, dm = \int_1^2 \frac{\left(\frac{2}{x^2}\right)}{2} \cdot \delta \cdot \left(\frac{2}{x^2}\right) dx$
$= \int_1^2 \left(\frac{1}{x^2}\right)\left(x^2\right)\left(\frac{2}{x^2}\right) dx = \int_1^2 \frac{2}{x^2}\, dx = 2\int_1^2 x^{-2}\, dx$
$= 2\left[-x^{-1}\right]_1^2 = 2\left[\left(-\frac{1}{2}\right) - (-1)\right] = 2\left(\frac{1}{2}\right) = 1$;
$M_y = \int \tilde{x}\, dm = \int_1^2 x \cdot \delta \cdot \left(\frac{2}{x^2}\right) dx$
$= \int_1^2 x\left(x^2\right)\left(\frac{2}{x^2}\right) dx = 2\int_1^2 x\, dx = 2\left[\frac{x^2}{2}\right]_1^2$
$= 2\left(2 - \frac{1}{2}\right) = 4 - 1 = 3$; $M = \int dm = \int_1^2 \delta\left(\frac{2}{x^2}\right) dx = \int_1^2 x^2\left(\frac{2}{x^2}\right) dx = 2\int_1^2 dx = 2[x]_1^2 = 2(2-1) = 2$. So $\overline{x} = \frac{M_y}{M} = \frac{3}{2}$ and $\overline{y} = \frac{M_x}{M} = \frac{1}{2}$ $\Rightarrow$ $(\overline{x}, \overline{y}) = \left(\frac{3}{2}, \frac{1}{2}\right)$ is the center of mass.



14. We use the *vertical* strip approach:



$M_x = \int \tilde{y}\ dm = \int_0^1 \frac{(x+x^2)}{2}(x - x^2) \cdot \delta\ dx$

$= \frac{1}{2}\int_0^1 (x^2 - x^4) \cdot 12x\ dx$

$= 6\int_0^1 (x^3 - x^5)\ dx = 6\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$

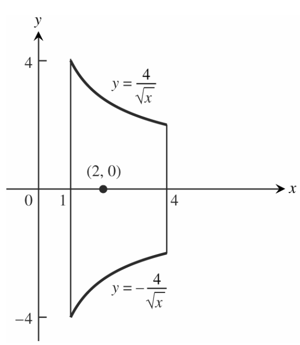
$= 6\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{6}{4} - 1 = \frac{1}{2}$;

$M_y = \int \tilde{x}\ dm = \int_0^1 x(x - x^2) \cdot \delta\ dx = \int_0^1 (x^2 - x^3) \cdot 12x\ dx = 12\int_0^1 (x^3 - x^4)\ dx = 12\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 12\left(\frac{1}{4} - \frac{1}{5}\right)$
$= \frac{12}{20} = \frac{3}{5}$; $M = \int dm = \int_0^1 (x - x^2) \cdot \delta\ dx = 12\int_0^1 (x^2 - x^3)\ dx = 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 12\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{12}{12} = 1$. So $\overline{x} = \frac{M_y}{M} = \frac{3}{5}$ and $\overline{y} = \frac{M_x}{M} = \frac{1}{2} \Rightarrow \left(\frac{3}{5}, \frac{1}{2}\right)$ is the center of mass.

15. (a) We use the shell method: $V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_1^4 2\pi x\left[\frac{4}{\sqrt{x}} - \left(-\frac{4}{\sqrt{x}}\right)\right] dx = 16\pi\int_1^4 \frac{x}{\sqrt{x}}\ dx$
$= 16\pi\int_1^4 x^{1/2}\ dx = 16\pi\left[\frac{2}{3}x^{3/2}\right]_1^4 = 16\pi\left(\frac{2}{3} \cdot 8 - \frac{2}{3}\right) = \frac{32\pi}{3}(8 - 1) = \frac{224\pi}{3}$

(b) Since the plate is symmetric about the x-axis and its density $\delta(x) = \frac{1}{x}$ is a function of x alone, the distribution of its mass is symmetric about the x-axis. This means that $\overline{y} = 0$. We use the vertical strip approach to find $\overline{x}$:
$M_y = \int \tilde{x}\ dm = \int_1^4 x \cdot \left[\frac{4}{\sqrt{x}} - \left(-\frac{4}{\sqrt{x}}\right)\right] \cdot \delta\ dx = \int_1^4 x \cdot \frac{8}{\sqrt{x}} \cdot \frac{1}{x}\ dx = 8\int_1^4 x^{-1/2}\ dx = 8\left[2x^{1/2}\right]_1^4 = 8(2 \cdot 2 - 2) = 16$;
$M = \int dm = \int_1^4 \left[\frac{4}{\sqrt{x}} - \left(\frac{-4}{\sqrt{x}}\right)\right] \cdot \delta\ dx = 8\int_1^4 \left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{x}\right) dx = 8\int_1^4 x^{-3/2}\ dx = 8\left[-2x^{-1/2}\right]_1^4 = 8[-1 - (-2)] = 8$.
So $\overline{x} = \frac{M_y}{M} = \frac{16}{8} = 2 \Rightarrow (\overline{x}, \overline{y}) = (2, 0)$ is the center of mass.

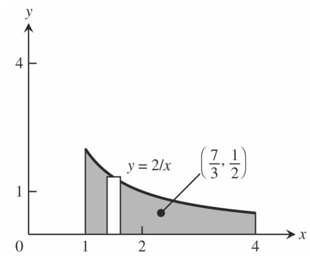
(c)



16. (a) We use the disk method: $V = \int_a^b \pi R^2(x)\ dx = \int_1^4 \pi\left(\frac{4}{x^2}\right) dx = 4\pi\int_1^4 x^{-2}\ dx = 4\pi\left[-\frac{1}{x}\right]_1^4 = 4\pi\left[\frac{-1}{4} - (-1)\right]$
$= \pi[-1 + 4] = 3\pi$

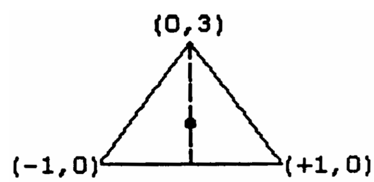
(b) We model the distribution of mass with vertical strips: $M_x = \int \tilde{y}\ dm = \int_1^4 \frac{\left(\frac{2}{x}\right)}{2} \cdot \left(\frac{2}{x}\right) \cdot \delta\ dx = \int_1^4 \frac{2}{x^2} \cdot \sqrt{x}\ dx$
$= 2\int_1^4 x^{-3/2}\ dx = 2\left[\frac{-2}{\sqrt{x}}\right]_1^4 = 2[-1 - (-2)] = 2$; $M_y = \int \tilde{x}\ dm = \int_1^4 x \cdot \frac{2}{x} \cdot \delta\ dx = 2\int_1^4 x^{1/2}\ dx = 2\left[\frac{2x^{3/2}}{3}\right]_1^4 =$
$2\left[\frac{16}{3} - \frac{2}{3}\right] = \frac{28}{3}$; $M = \int dm = \int_1^4 \frac{2}{x} \cdot \delta\ dx = 2\int_1^4 \frac{\sqrt{x}}{x}\ dx = 2\int_1^4 x^{-1/2}\ dx = 2\left[2x^{1/2}\right]_1^4 = 2(4 - 2) = 4$. So
$\overline{x} = \frac{M_y}{M} = \frac{\left(\frac{28}{3}\right)}{4} = \frac{7}{3}$ and $\overline{y} = \frac{M_x}{M} = \frac{2}{4} = \frac{1}{2} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{7}{3}, \frac{1}{2}\right)$ is the center of mass.

(c)

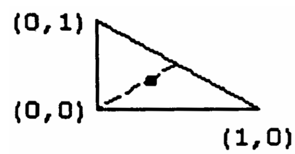


17. The mass of a horizontal strip is $dm = \delta\, dA = \delta L\, dy$, where L is the width of the triangle at a distance of y above its base on the x-axis as shown in the figure in the text. Also, by similar triangles we have $\frac{L}{b} = \frac{h-y}{h}$ $\Rightarrow L = \frac{b}{h}(h-y)$. Thus, $M_x = \int \tilde{y}\, dm = \int_0^h \delta y \left(\frac{b}{h}\right)(h-y)\, dy = \frac{\delta b}{h}\int_0^h (hy - y^2)\, dy = \frac{\delta b}{h}\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h$ $= \frac{\delta b}{h}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = \delta bh^2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\delta bh^2}{6}$; $M = \int dm = \int_0^h \delta\left(\frac{b}{h}\right)(h-y)\, dy = \frac{\delta b}{h}\int_0^h (h-y)\, dy = \frac{\delta b}{h}\left[hy - \frac{y^2}{2}\right]_0^h$ $= \frac{\delta b}{h}\left(h^2 - \frac{h^2}{2}\right) = \frac{\delta bh}{2}$. So $\bar{y} = \frac{M_x}{M} = \left(\frac{\delta bh^2}{6}\right)\left(\frac{2}{\delta bh}\right) = \frac{h}{3} \Rightarrow$ the center of mass lies above the base of the triangle one-third of the way toward the opposite vertex. Similarly the other two sides of the triangle can be placed on the x-axis and the same results will occur. Therefore the centroid does lie at the intersection of the medians, as claimed.

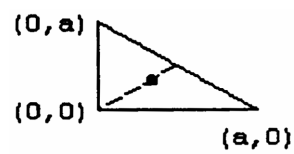
18. From the symmetry about the y-axis it follows that $\bar{x} = 0$. It also follows that the line through the points $(0, 0)$ and $(0, 3)$ is a median $\Rightarrow \bar{y} = \frac{1}{3}(3-0) = 1 \Rightarrow (\bar{x}, \bar{y}) = (0, 1)$.



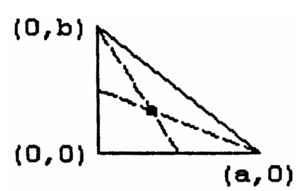
19. From the symmetry about the line $x = y$ it follows that $\bar{x} = \bar{y}$. It also follows that the line through the points $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$ is a median $\Rightarrow \bar{y} = \bar{x} = \frac{2}{3}\cdot\left(\frac{1}{2} - 0\right) = \frac{1}{3}$ $\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{1}{3}, \frac{1}{3}\right)$.



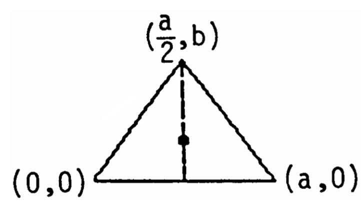
20. From the symmetry about the line $x = y$ it follows that $\bar{x} = \bar{y}$. It also follows that the line through the point $(0, 0)$ and $\left(\frac{a}{2}, \frac{a}{2}\right)$ is a median $\Rightarrow \bar{y} = \bar{x} = \frac{2}{3}\left(\frac{a}{2} - 0\right) = \frac{1}{3}a$ $\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{a}{3}, \frac{a}{3}\right)$.



21. The point of intersection of the median from the vertex $(0, b)$ to the opposite side has coordinates $\left(0, \frac{a}{2}\right)$ $\Rightarrow \bar{y} = (b-0)\cdot\frac{1}{3} = \frac{b}{3}$ and $\bar{x} = \left(\frac{a}{2} - 0\right)\cdot\frac{2}{3} = \frac{a}{3}$ $\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{a}{3}, \frac{b}{3}\right)$.



22. From the symmetry about the line $x = \frac{a}{2}$ it follows that $\bar{x} = \frac{a}{2}$. It also follows that the line through the points $\left(\frac{a}{2}, 0\right)$ and $\left(\frac{a}{2}, b\right)$ is a median $\Rightarrow \bar{y} = \frac{1}{3}(b-0) = \frac{b}{3}$ $\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{b}{3}\right)$.



23. $y = x^{1/2} \Rightarrow dy = \frac{1}{2} x^{-1/2}\, dx$

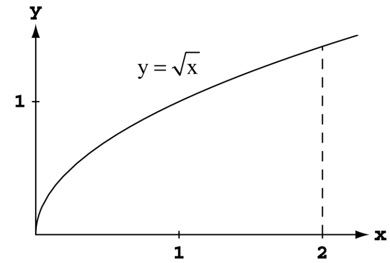
$\Rightarrow ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \frac{1}{4x}}\, dx$;

$M_x = \delta \int_0^2 \sqrt{x} \sqrt{1 + \frac{1}{4x}}\, dx$

$= \delta \int_0^2 \sqrt{x + \frac{1}{4}}\, dx = \frac{2\delta}{3} \left[\left(x + \frac{1}{4}\right)^{3/2} \right]_0^2$

$= \frac{2\delta}{3} \left[\left(2 + \frac{1}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2} \right]$

$= \frac{2\delta}{3} \left[\left(\frac{9}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2} \right] = \frac{2\delta}{3} \left(\frac{27}{8} - \frac{1}{8}\right) = \frac{13\delta}{6}$



24. $y = x^3 \Rightarrow dy = 3x^2\, dx$

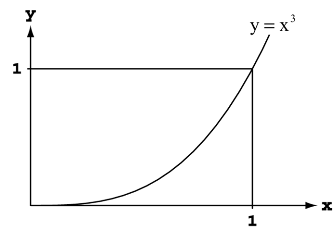
$\Rightarrow dx = \sqrt{(dx)^2 + \left(3x^2\, dx\right)^2} = \sqrt{1 + 9x^4}\, dx$;

$M_x = \delta \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx$;

$[u = 1 + 9x^4 \Rightarrow du = 36x^3\, dx \Rightarrow \frac{1}{36}\, du = x^3\, dx$;

$x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 10]$

$\to M_x = \delta \int_1^{10} \frac{1}{36} u^{1/2}\, du = \frac{\delta}{36} \left[\frac{2}{3} u^{3/2} \right]_1^{10} = \frac{\delta}{54} \left(10^{3/2} - 1\right)$



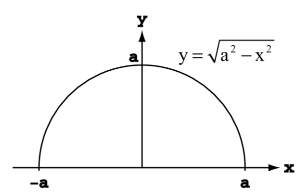
25. From Example 4 we have $M_x = \int_0^\pi a(a \sin\theta)(k \sin\theta)\, d\theta = a^2 k \int_0^\pi \sin^2\theta\, d\theta = \frac{a^2 k}{2} \int_0^\pi (1 - \cos 2\theta)\, d\theta = \frac{a^2 k}{2} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi$ $= \frac{a^2 k\pi}{2}$; $M_y = \int_0^\pi a(a \cos\theta)(k \sin\theta)\, d\theta = a^2 k \int_0^\pi \sin\theta \cos\theta\, d\theta = \frac{a^2 k}{2} \left[\sin^2\theta\right]_0^\pi = 0$; $M = \int_0^\pi ak \sin\theta\, d\theta = ak[-\cos\theta]_0^\pi$ $= 2ak$. Therefore, $\bar{x} = \frac{M_y}{M} = 0$ and $\bar{y} = \frac{M_x}{M} = \left(\frac{a^2 k\pi}{2}\right)\left(\frac{1}{2ak}\right) = \frac{a\pi}{4} \Rightarrow \left(0, \frac{a\pi}{4}\right)$ is the center of mass.

26. $M_x = \int \tilde{y}\, dm = \int_0^\pi (a \sin\theta) \cdot \delta \cdot a\, d\theta$

$= \int_0^\pi \left(a^2 \sin\theta\right)(1 + k\,|\cos\theta|)\, d\theta$

$= a^2 \int_0^{\pi/2} (\sin\theta)(1 + k \cos\theta)\, d\theta$

$+ a^2 \int_{\pi/2}^{\pi} (\sin\theta)(1 - k \cos\theta)\, d\theta$



$= a^2 \int_0^{\pi/2} \sin\theta\, d\theta + a^2 k \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta + a^2 \int_{\pi/2}^{\pi} \sin\theta\, d\theta - a^2 k \int_{\pi/2}^{\pi} \sin\theta \cos\theta\, d\theta$

$= a^2 [-\cos\theta]_0^{\pi/2} + a^2 k \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} + a^2[-\cos\theta]_{\pi/2}^{\pi} - a^2 k \left[\frac{\sin^2\theta}{2}\right]_{\pi/2}^{\pi}$

$= a^2[0 - (-1)] + a^2 k \left(\frac{1}{2} - 0\right) + a^2[-(-1) - 0] - a^2 k \left(0 - \frac{1}{2}\right) = a^2 + \frac{a^2 k}{2} + a^2 + \frac{a^2 k}{2} = 2a^2 + a^2 k = a^2(2 + k)$;

$M_y = \int \tilde{x}\, dm = \int_0^\pi (a \cos\theta) \cdot \delta \cdot a\, d\theta = \int_0^\pi \left(a^2 \cos\theta\right)(1 + k\,|\cos\theta|)\, d\theta$

$= a^2 \int_0^{\pi/2} (\cos\theta)(1 + k \cos\theta)\, d\theta + a^2 \int_{\pi/2}^{\pi} (\cos\theta)(1 - k \cos\theta)\, d\theta$

$= a^2 \int_0^{\pi/2} \cos\theta\, d\theta + a^2 k \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta + a^2 \int_{\pi/2}^{\pi} \cos\theta\, d\theta - a^2 k \int_{\pi/2}^{\pi} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$

$= a^2[\sin\theta]_0^{\pi/2} + \frac{a^2 k}{2} \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} + a^2[\sin\theta]_{\pi/2}^{\pi} - \frac{a^2 k}{2} \left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/2}^{\pi}$

$= a^2(1 - 0) + \frac{a^2 k}{2} \left[\left(\frac{\pi}{2} - 0\right) - (0 + 0)\right] + a^2(0 - 1) - \frac{a^2 k}{2} \left[(\pi + 0) - \left(\frac{\pi}{2} + 0\right)\right] = a^2 + \frac{a^2 k\pi}{4} - a^2 - \frac{a^2 k\pi}{4} = 0$;

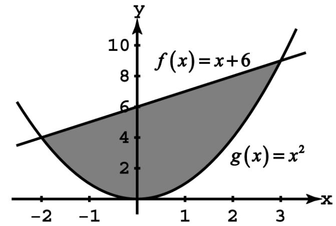
$M = \int_0^\pi \delta \cdot a\, d\theta = a \int_0^\pi (1 + k\,|\cos\theta|)\, d\theta = a \int_0^{\pi/2} (1 + k \cos\theta)\, d\theta + a \int_{\pi/2}^{\pi} (1 - k \cos\theta)\, d\theta$

$= a[\theta + k \sin\theta]_0^{\pi/2} + a[\theta - k \sin\theta]_{\pi/2}^{\pi} = a\left[\left(\frac{\pi}{2} + k\right) - 0\right] + a\left[(\pi + 0) - \left(\frac{\pi}{2} - k\right)\right]$

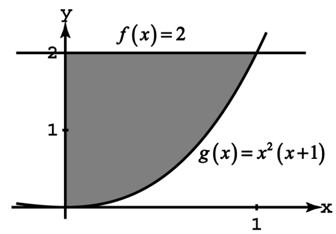
$= \frac{a\pi}{2} + ak + a\left(\frac{\pi}{2} + k\right) = a\pi + 2ak = a(\pi + 2k)$. So $\bar{x} = \frac{M_y}{M} = 0$ and $\bar{y} = \frac{M_x}{M} = \frac{a^2(2 + k)}{a(\pi + 2k)} = \frac{a(2 + k)}{\pi + 2k}$

$\Rightarrow \left(0, \frac{2a + ka}{\pi + 2k}\right)$ is the center of mass.

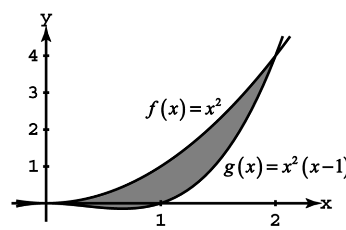
27. $f(x) = x + 6$, $g(x) = x^2$, $f(x) = g(x) \Rightarrow x + 6 = x^2$
$\Rightarrow x^2 - x - 6 = 0 \Rightarrow x = 3, x = -2; \delta = 1$
$M = \int_{-2}^{3}[(x+6) - x^2]dx = \left[\frac{1}{2}x^2 + 6x - \frac{1}{3}x^3\right]_{-2}^{3}$
$= \left(\frac{9}{2} + 18 - 9\right) - \left(2 - 12 + \frac{8}{3}\right) = \frac{125}{6}$
$\overline{x} = \frac{1}{125/6}\int_{-2}^{3} x[(x+6) - x^2]dx = \frac{6}{125}\int_{-2}^{3}[x^2 + 6x - x^3]dx$
$= \frac{6}{125}\left[\frac{1}{3}x^3 + 3x^2 - \frac{1}{4}x^4\right]_{-2}^{3}$
$= \frac{6}{125}\left(9 + 27 - \frac{81}{4}\right) - \frac{6}{125}\left(-\frac{8}{3} + 12 - 4\right) = \frac{1}{2}$; $\overline{y} = \frac{1}{125/6}\int_{-2}^{3}\frac{1}{2}\left[(x+6)^2 - (x^2)^2\right]dx = \frac{3}{125}\int_{-2}^{3}[x^2 + 12x + 36 - x^4]dx$
$= \frac{3}{125}\left[\frac{1}{3}x^3 + 6x^2 + 36x - \frac{1}{5}x^5\right]_{-2}^{3} = \frac{3}{125}\left(9 + 54 + 108 - \frac{243}{5}\right) - \frac{3}{125}\left(-\frac{8}{3} + 24 - 72 + \frac{32}{5}\right) = 4$
$\Rightarrow \left(\frac{1}{2}, 4\right)$ is the center of mass.



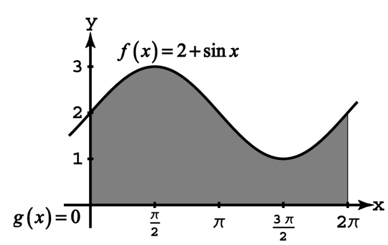
28. $f(x) = 2$, $g(x) = x^2(x+1)$, $f(x) = g(x) \Rightarrow 2 = x^2(x+1)$
$\Rightarrow x^3 + x^2 - 2 = 0 \Rightarrow x = 1; \delta = 1$
$M = \int_0^1 [2 - x^2(x+1)]\,dx = \int_0^1 [2 - x^3 - x^2]\,dx$
$= \left[2x - \frac{1}{4}x^4 - \frac{1}{3}x^3\right]_0^1 = \left(2 - \frac{1}{4} - \frac{1}{3}\right) - 0 = \frac{17}{12}$
$\overline{x} = \frac{1}{17/12}\int_0^1 x[2 - x^2(x+1)]dx = \frac{12}{17}\int_0^1 [2x - x^4 - x^3]dx$
$= \frac{12}{17}\left[x^2 - \frac{1}{5}x^5 - \frac{1}{4}x^4\right]_0^1$
$= \frac{12}{17}\left(1 - \frac{1}{5} - \frac{1}{4}\right) - 0 = \frac{33}{85}$; $\overline{y} = \frac{1}{17/12}\int_0^1 \frac{1}{2}\left[2^2 - (x^2(x+1))^2\right]dx = \frac{6}{17}\int_0^1 [4 - x^6 - 2x^5 - x^4]dx$
$= \frac{6}{17}\left[4x - \frac{1}{7}x^7 - \frac{1}{3}x^6 - \frac{1}{5}x^5\right]_0^1 = \frac{6}{17}\left(4 - \frac{1}{7} - \frac{1}{3} - \frac{1}{5}\right) - 0 = \frac{698}{595} \Rightarrow \left(\frac{33}{85}, \frac{698}{595}\right)$ is the center of mass.



29. $f(x) = x^2$, $g(x) = x^2(x-1)$, $f(x) = g(x) \Rightarrow x^2 = x^2(x-1)$
$\Rightarrow x^3 - 2x^2 = 0 \Rightarrow x = 0, x = 2; \delta = 1$
$M = \int_0^2 [x^2 - x^2(x-1)]dx = \int_0^2 [2x^2 - x^3]dx$
$= \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \left(\frac{16}{3} - 4\right) - 0 = \frac{4}{3}$
$\overline{x} = \frac{1}{4/3}\int_0^2 x[x^2 - x^2(x-1)]dx = \frac{3}{4}\int_0^2 [2x^3 - x^4]dx$
$= \frac{3}{4}\left[\frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2 = \frac{3}{4}\left(8 - \frac{32}{5}\right) - 0 = \frac{6}{5}$;
$\overline{y} = \frac{1}{4/3}\int_0^2 \frac{1}{2}\left[(x^2)^2 - (x^2(x-1))^2\right]dx = \frac{3}{8}\int_0^2 [2x^5 - x^6]dx = \frac{3}{8}\left[\frac{1}{3}x^6 - \frac{1}{7}x^7\right]_0^2 = \frac{3}{8}\left(\frac{64}{3} - \frac{128}{7}\right) - 0 = \frac{8}{7}$
$\Rightarrow \left(\frac{6}{5}, \frac{8}{7}\right)$ is the center of mass.



30. $f(x) = 2 + \sin x$, $g(x) = 0$, $x = 0$, $x = 2\pi$; $\delta = 1$;
$M = \int_0^{2\pi}[2 + \sin x]dx = [2x - \cos x]_0^{2\pi}$
$= (4\pi - 1) - (0 - 1) = 4\pi$
$\overline{x} = \frac{1}{4\pi}\int_0^{2\pi} x[2 + \sin x - 0]dx = \frac{1}{4\pi}\int_0^{2\pi}[2x + x\sin x]dx$
$= \frac{1}{4\pi}\int_0^{2\pi} 2x\,dx + \frac{1}{4\pi}\int_0^{2\pi} x\sin x dx$
$= \frac{1}{4\pi}[x^2]_0^{2\pi} + \frac{1}{4\pi}[\sin x - x\cos x]_0^{2\pi}$
$= \frac{1}{4\pi}(4\pi^2) - 0 + \frac{1}{4\pi}(0 - 2\pi) - 0 = \frac{2\pi - 1}{2}$; $\overline{y} = \frac{1}{4\pi}\int_0^{2\pi}\frac{1}{2}\left[(2 + \sin x)^2 - (0)^2\right]dx = \frac{1}{8\pi}\int_0^{2\pi}[4 + 4\sin x + \sin^2 x]dx$
$= \frac{1}{8\pi}\int_0^{2\pi}[4 + 4\sin x]dx + \frac{1}{8\pi}\int_0^{2\pi}[\sin^2 x]dx = \frac{1}{8\pi}\int_0^{2\pi}[4 + 4\sin x]dx + \frac{1}{8\pi}\int_0^{2\pi}\left[\frac{1 - \cos 2x}{2}\right]dx$
$= \frac{1}{8\pi}[4x - 4\cos x +]_0^{2\pi} + \frac{1}{16\pi}\int_0^{2\pi}dx - \frac{1}{16\pi}\int_0^{2\pi}\cos 2x\,dx$ $[u = 2x \Rightarrow du = 2dx, x = 0 \Rightarrow u = 0, x = 2\pi \Rightarrow u = 4\pi]$



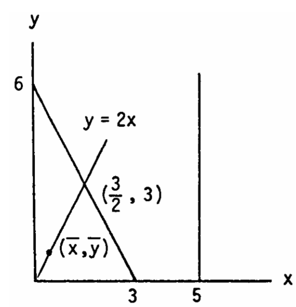
$\rightarrow \frac{1}{8\pi}[4x - 4\cos x]_0^{2\pi} + \frac{1}{16\pi}[x]_0^{2\pi} - \frac{1}{32\pi}\int_0^{4\pi} \cos u\, du = \frac{1}{8\pi}[4x - 4\cos x]_0^{2\pi} + \frac{1}{16\pi}[x]_0^{2\pi} - \frac{1}{32\pi}[\sin u]_0^{4\pi}$
$= \frac{1}{8\pi}(8\pi - 4) - \frac{1}{8\pi}(0 - 4) + \frac{1}{16\pi}(2\pi) - 0 - 0 = \frac{9}{8} \Rightarrow \left(\frac{2\pi - 1}{2}, \frac{9}{8}\right)$ is the center of mass.

31. Consider the curve as an infinite number of line segments joined together. From the derivation of arc length we have that the length of a particular segment is $ds = \sqrt{(dx)^2 + (dy)^2}$. This implies that $M_x = \int \delta y\, ds$, $M_y = \int \delta x\, ds$ and $M = \int \delta\, ds$. If δ is constant, then $\overline{x} = \frac{M_y}{M} = \frac{\int x\, ds}{\int ds} = \frac{\int x\, ds}{\text{length}}$ and $\overline{y} = \frac{M_x}{M} = \frac{\int y\, ds}{\int ds} = \frac{\int y\, ds}{\text{length}}$.

32. Applying the symmetry argument analogous to the one used in Exercise 1, we find that $\overline{x} = 0$. The typical vertical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{a + \frac{x^2}{4p}}{2}\right)$, length: $a - \frac{x^2}{4p}$, width: dx, area: $dA = \left(a - \frac{x^2}{4p}\right) dx$, mass: $dm = \delta\, dA$
$= \delta\left(a - \frac{x^2}{4p}\right) dx$. Thus, $M_x = \int \tilde{y}\, dm = \int_{-2\sqrt{pa}}^{2\sqrt{pa}} \frac{1}{2}\left(a + \frac{x^2}{4p}\right)\left(a - \frac{x^2}{4p}\right) \delta\, dx = \frac{\delta}{2}\int_{-2\sqrt{pa}}^{2\sqrt{pa}}\left(a^2 - \frac{x^4}{16p^2}\right) dx$
$= \frac{\delta}{2}\left[a^2 x - \frac{x^5}{80p^2}\right]_{-2\sqrt{pa}}^{2\sqrt{pa}} = 2 \cdot \frac{\delta}{2}\left[a^2 x - \frac{x^5}{80p^2}\right]_0^{2\sqrt{pa}} = \delta\left(2a^2\sqrt{pa} - \frac{2^5 p^2 a^2 \sqrt{pa}}{80p^2}\right) = 2a^2\delta\sqrt{pa}\left(1 - \frac{16}{80}\right) = 2a^2\delta\sqrt{pa}\left(\frac{80-16}{80}\right)$
$= 2a^2\delta\sqrt{pa}\left(\frac{64}{80}\right) = \frac{8a^2\delta\sqrt{pa}}{5}$; $M = \int dm = \delta\int_{-2\sqrt{pa}}^{2\sqrt{pa}}\left(a - \frac{x^2}{4p}\right) dx = \delta\left[ax - \frac{x^3}{12p}\right]_{-2\sqrt{pa}}^{2\sqrt{pa}} = 2 \cdot \delta\left[ax - \frac{x^3}{12p}\right]_0^{2\sqrt{pa}}$
$= 2\delta\left(2a\sqrt{pa} - \frac{2^3 pa\sqrt{pa}}{12p}\right) = 4a\delta\sqrt{pa}\left(1 - \frac{4}{12}\right) = 4a\delta\sqrt{pa}\left(\frac{12-4}{12}\right) = \frac{8a\delta\sqrt{pa}}{3}$. So $\overline{y} = \frac{M_x}{M} = \left(\frac{8a^2\delta\sqrt{pa}}{5}\right)\left(\frac{3}{8a\delta\sqrt{pa}}\right)$
$= \frac{3}{5}a$, as claimed.

33. The centroid of the square is located at $(2, 2)$. The volume is $V = (2\pi)(\overline{y})(A) = (2\pi)(2)(8) = 32\pi$ and the surface area is $S = (2\pi)(\overline{y})(L) = (2\pi)(2)\left(4\sqrt{8}\right) = 32\sqrt{2}\pi$ (where $\sqrt{8}$ is the length of a side).

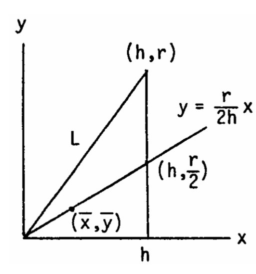
34. The midpoint of the hypotenuse of the triangle is $\left(\frac{3}{2}, 3\right)$ $\Rightarrow$ $y = 2x$ is an equation of the median $\Rightarrow$ the line $y = 2x$ contains the centroid. The point $\left(\frac{3}{2}, 3\right)$ is $\frac{3\sqrt{5}}{2}$ units from the origin $\Rightarrow$ the x-coordinate of the centroid solves the equation $\sqrt{\left(x - \frac{3}{2}\right)^2 + (2x - 3)^2}$
$= \frac{\sqrt{5}}{2} \Rightarrow \left(x^2 - 3x + \frac{9}{4}\right) + (4x^2 - 12x + 9) = \frac{5}{4}$
$\Rightarrow 5x^2 - 15x + 9 = -1$



$\Rightarrow x^2 - 3x + 2 = (x - 2)(x - 1) = 0 \Rightarrow \overline{x} = 1$ since the centroid must lie inside the triangle $\Rightarrow \overline{y} = 2$. By the Theorem of Pappus, the volume is $V =$ (distance traveled by the centroid)(area of the region) $= 2\pi(5 - \overline{x})\left[\frac{1}{2}(3)(6)\right]$
$= (2\pi)(4)(9) = 72\pi$

35. The centroid is located at $(2, 0) \Rightarrow V = (2\pi)(\overline{x})(A) = (2\pi)(2)(\pi) = 4\pi^2$

36. We create the cone by revolving the triangle with vertices $(0, 0)$, (h, r) and $(h, 0)$ about the x-axis (see the accompanying figure). Thus, the cone has height h and base radius r. By Theorem of Pappus, the lateral surface area swept out by the hypotenuse L is given by $S = 2\pi\overline{y}L = 2\pi\left(\frac{r}{2}\right)\sqrt{h^2 + r^2}$ $= \pi r\sqrt{r^2 + h^2}$. To calculate the volume we need the position of the centroid of the triangle. From the diagram we see that



the centroid lies on the line $y = \frac{r}{2h}x$. The x-coordinate of the centroid solves the equation $\sqrt{(x - h)^2 + \left(\frac{r}{2h}x - \frac{r}{2}\right)^2}$

$= \frac{1}{3}\sqrt{h^2 + \frac{r^2}{4}} \Rightarrow \left(\frac{4h^2 + r^2}{4h^2}\right)x^2 - \left(\frac{4h^2 + r^2}{2h}\right)x + \frac{r^2}{4} + \frac{2(r^2 + 4h^2)}{9} = 0 \Rightarrow x = \frac{2h}{3}$ or $\frac{4h}{3} \Rightarrow \bar{x} = \frac{2h}{3}$, since the centroid must lie inside the triangle $\Rightarrow \bar{y} = \frac{r}{2h}\bar{x} = \frac{r}{3}$. By the Theorem of Pappus, $V = \left[2\pi\left(\frac{r}{3}\right)\right]\left(\frac{1}{2}hr\right) = \frac{1}{3}\pi r^2 h$.

37. $S = 2\pi\bar{y}L \Rightarrow 4\pi a^2 = (2\pi\bar{y})(\pi a) \Rightarrow \bar{y} = \frac{2a}{\pi}$, and by symmetry $\bar{x} = 0$

38. $S = 2\pi\rho L \Rightarrow \left[2\pi\left(a - \frac{2a}{\pi}\right)\right](\pi a) = 2\pi a^2(\pi - 2)$

39. $V = 2\pi\bar{y}A \Rightarrow \frac{4}{3}\pi ab^2 = (2\pi\bar{y})\left(\frac{\pi ab}{2}\right) \Rightarrow \bar{y} = \frac{4b}{3\pi}$ and by symmetry $\bar{x} = 0$

40. $V = 2\pi\rho A \Rightarrow V = \left[2\pi\left(a + \frac{4a}{3\pi}\right)\right]\left(\frac{\pi a^2}{2}\right) = \frac{\pi a^3(3\pi + 4)}{3}$

41. $V = 2\pi\rho A = (2\pi)$(area of the region) $\cdot$ (distance from the centroid to the line $y = x - a$). We must find the distance from $\left(0, \frac{4a}{3\pi}\right)$ to $y = x - a$. The line containing the centroid and perpendicular to $y = x - a$ has slope -1 and contains the point $\left(0, \frac{4a}{3\pi}\right)$. This line is $y = -x + \frac{4a}{3\pi}$. The intersection of $y = x - a$ and $y = -x + \frac{4a}{3\pi}$ is the point $\left(\frac{4a + 3a\pi}{6\pi}, \frac{4a - 3a\pi}{6\pi}\right)$. Thus, the distance from the centroid to the line $y = x - a$ is $\sqrt{\left(\frac{4a + 3a\pi}{6\pi}\right)^2 + \left(\frac{4a}{3\pi} - \frac{4a}{6\pi} + \frac{3a\pi}{6\pi}\right)^2} = \frac{\sqrt{2}(4a + 3a\pi)}{6\pi}$
$\Rightarrow V = (2\pi)\left(\frac{\sqrt{2}(4a + 3a\pi)}{6\pi}\right)\left(\frac{\pi a^2}{2}\right) = \frac{\sqrt{2}\pi a^3(4 + 3\pi)}{6}$

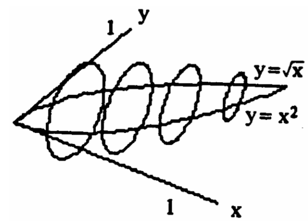
42. The line perpendicular to $y = x - a$ and passing through the centroid $\left(0, \frac{2a}{\pi}\right)$ has equation $y = -x + \frac{2a}{\pi}$. The intersection of the two perpendicular lines occurs when $x - a = -x + \frac{2a}{\pi} \Rightarrow x = \frac{2a + a\pi}{2\pi} \Rightarrow y = \frac{2a - a\pi}{2\pi}$. Thus the distance from the centroid to the line $y = x - a$ is $\sqrt{\left(\frac{2a + \pi a}{2} - 0\right)^2 + \left(\frac{2a - \pi a}{2} - \frac{2a}{2}\right)^2} = \frac{a(2+\pi)}{\sqrt{2}\pi}$. Therefore, by the Theorem of Pappus the surface area is $S = 2\pi\left[\frac{a(2+\pi)}{\sqrt{2}\pi}\right](\pi a) = \sqrt{2}\pi a^2(2 + \pi)$.

43. If we revolve the region about the y-axis: $r = a$, $h = b \Rightarrow A = \frac{1}{2}ab$, $V = \frac{1}{3}\pi a^2 b$, and $\rho = \bar{x}$. By the Theorem of Pappus: $\frac{1}{3}\pi a^2 b = 2\pi\bar{x}\left(\frac{1}{2}ab\right) \Rightarrow \bar{x} = \frac{a}{3}$; If we revolve the region about the x-axis: $r = b$, $h = a \Rightarrow A = \frac{1}{2}ab$, $V = \frac{1}{3}\pi b^2 a$, and $\rho = \bar{y}$. By the Theorem of Pappus: $\frac{1}{3}\pi b^2 a = 2\pi\bar{y}\left(\frac{1}{2}ab\right) \Rightarrow \bar{y} = \frac{b}{3} \Rightarrow \left(\frac{a}{3}, \frac{b}{3}\right)$ is the center of mass.

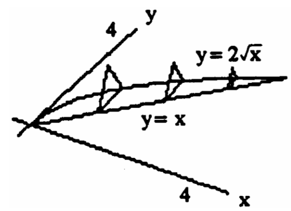
44. Let $O(0, 0)$, $P(a, c)$, and $Q(a, b)$ be the vertices of the given triangle. If we revolve the region about the x-axis: Let R be the point $R(a, 0)$. The volume is given by the volume of the outer cone, radius $= RP = c$, minus the volume of the inner cone, radius $= RQ = b$, thus $V = \frac{1}{3}\pi c^2 a - \frac{1}{3}\pi b^2 a = \frac{1}{3}\pi a(c^2 - b^2)$, the area is given by the area of triangle OPR minus area of triangle OQR, $A = \frac{1}{2}ac - \frac{1}{2}ab = \frac{1}{2}a(c - b)$, and $\rho = \bar{y}$. By the Theorem of Pappus: $\frac{1}{3}\pi a(c^2 - b^2) = 2\pi\bar{y}\left[\frac{1}{2}a(c - b)\right] \Rightarrow \bar{y} = \frac{c + b}{3}$; If we revolve the region about the y-axis: Let S and T be the points $S(0, c)$ and $T(0, b)$, respectively. Then the volume is the volume of the cylinder with radius $OR = a$ and height $RP = c$, minus the sum of the volumes of the cone with radius $= SP = a$ and height $= OS = c$ and the portion of the cylinder with height $= OT = b$ and radius $= TQ = a$ with a cone of height $= OT = b$ and radius $= TQ = a$ removed. Thus
$V = \pi a^2 c - \left[\frac{1}{3}\pi a^2 c + \left(\pi a^2 b - \frac{1}{3}\pi a^2 b\right)\right] = \frac{2}{3}\pi a^2 c - \frac{2}{3}\pi a^2 b = \frac{2}{3}\pi a^2(a - b)$. The area of the triangle is the same as before, $A = \frac{1}{2}ac - \frac{1}{2}ab = \frac{1}{2}a(c - b)$, and $\rho = \bar{x}$. By the Theorem of Pappus: $\frac{2}{3}\pi a^2(a - b) = 2\pi\bar{x}\left[\frac{1}{2}a(c - b)\right]$
$\Rightarrow \bar{x} = \frac{2a(a - b)}{3(c - b)} \Rightarrow \left(\frac{2a(a - b)}{3(c - b)}, \frac{c + b}{2}\right)$ is the center of mass.

CHAPTER 6 PRACTICE EXERCISES

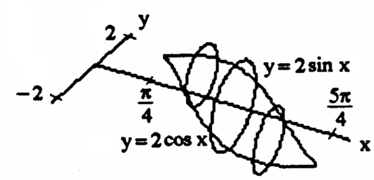
1. $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\sqrt{x} - x^2\right)^2$
$= \frac{\pi}{4}\left(x - 2\sqrt{x}\cdot x^2 + x^4\right)$; $a = 0, b = 1$
$\Rightarrow V = \int_a^b A(x)\,dx = \frac{\pi}{4}\int_0^1 \left(x - 2x^{5/2} + x^4\right) dx$
$= \frac{\pi}{4}\left[\frac{x^2}{2} - \frac{4}{7}x^{7/2} + \frac{x^5}{5}\right]_0^1 = \frac{\pi}{4}\left(\frac{1}{2} - \frac{4}{7} + \frac{1}{5}\right)$
$= \frac{\pi}{4\cdot 70}(35 - 40 + 14) = \frac{9\pi}{280}$



2. $A(x) = \frac{1}{2}(\text{side})^2\left(\sin\frac{\pi}{3}\right) = \frac{\sqrt{3}}{4}\left(2\sqrt{x} - x\right)^2$
$= \frac{\sqrt{3}}{4}\left(4x - 4x\sqrt{x} + x^2\right)$; $a = 0, b = 4$
$\Rightarrow V = \int_a^b A(x)\,dx = \frac{\sqrt{3}}{4}\int_0^4 \left(4x - 4x^{3/2} + x^2\right) dx$
$= \frac{\sqrt{3}}{4}\left[2x^2 - \frac{8}{5}x^{5/2} + \frac{x^3}{3}\right]_0^4 = \frac{\sqrt{3}}{4}\left(32 - \frac{8\cdot 32}{5} + \frac{64}{3}\right)$
$= \frac{32\sqrt{3}}{4}\left(1 - \frac{8}{5} + \frac{2}{3}\right) = \frac{8\sqrt{3}}{15}(15 - 24 + 10) = \frac{8\sqrt{3}}{15}$



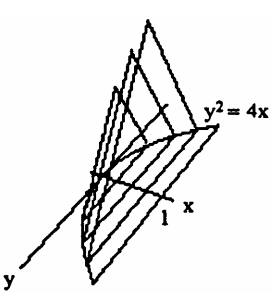
3. $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}(2\sin x - 2\cos x)^2$
$= \frac{\pi}{4}\cdot 4\left(\sin^2 x - 2\sin x\cos x + \cos^2 x\right)$
$= \pi(1 - \sin 2x)$; $a = \frac{\pi}{4}$, $b = \frac{5\pi}{4}$
$\Rightarrow V = \int_a^b A(x)\,dx = \pi\int_{\pi/4}^{5\pi/4}(1 - \sin 2x)\,dx$
$= \pi\left[x + \frac{\cos 2x}{2}\right]_{\pi/4}^{5\pi/4}$
$= \pi\left[\left(\frac{5\pi}{4} + \frac{\cos\frac{5\pi}{2}}{2}\right) - \left(\frac{\pi}{4} - \frac{\cos\frac{\pi}{2}}{2}\right)\right] = \pi^2$



4. $A(x) = (\text{edge})^2 = \left(\left(\sqrt{6} - \sqrt{x}\right)^2 - 0\right)^2 = \left(\sqrt{6} - \sqrt{x}\right)^4 = 36 - 24\sqrt{6}\sqrt{x} + 36x - 4\sqrt{6}\,x^{3/2} + x^2$;
$a = 0, b = 6 \Rightarrow V = \int_a^b A(x)\,dx = \int_0^6 \left(36 - 24\sqrt{6}\sqrt{x} + 36x - 4\sqrt{6}\,x^{3/2} + x^2\right) dx$
$= \left[36x - 24\sqrt{6}\cdot\frac{2}{3}x^{3/2} + 18x^2 - 4\sqrt{6}\cdot\frac{2}{5}x^{5/2} + \frac{x^3}{3}\right]_0^6 = 216 - 16\cdot\sqrt{6}\sqrt{6}\cdot 6 + 18\cdot 6^2 - \frac{8}{5}\sqrt{6}\sqrt{6}\cdot 6^2 + \frac{6^3}{3}$
$= 216 - 576 + 648 - \frac{1728}{5} + 72 = 360 - \frac{1728}{5} = \frac{1800 - 1728}{5} = \frac{72}{5}$

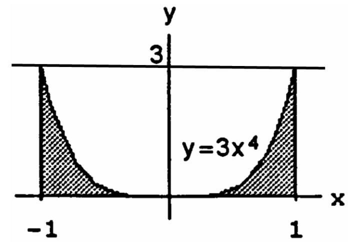
5. $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(2\sqrt{x} - \frac{x^2}{4}\right)^2 = \frac{\pi}{4}\left(4x - x^{5/2} + \frac{x^4}{16}\right)$; $a = 0, b = 4 \Rightarrow V = \int_a^b A(x)\,dx$
$= \frac{\pi}{4}\int_0^4 \left(4x - x^{5/2} + \frac{x^4}{16}\right) dx = \frac{\pi}{4}\left[2x^2 - \frac{2}{7}x^{7/2} + \frac{x^5}{5\cdot 16}\right]_0^4 = \frac{\pi}{4}\left(32 - 32\cdot\frac{8}{7} + \frac{2}{5}\cdot 32\right)$
$= \frac{32\pi}{4}\left(1 - \frac{8}{7} + \frac{2}{5}\right) = \frac{8\pi}{35}(35 - 40 + 14) = \frac{72\pi}{35}$

6. $A(x) = \frac{1}{2}(\text{edge})^2\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{4}\left[2\sqrt{x} - \left(-2\sqrt{x}\right)\right]^2$
$= \frac{\sqrt{3}}{4}\left(4\sqrt{x}\right)^2 = 4\sqrt{3}\,x$; $a = 0, b = 1$
$\Rightarrow V = \int_a^b A(x)\,dx = \int_0^1 4\sqrt{3}\,x\,dx = \left[2\sqrt{3}\,x^2\right]_0^1$
$= 2\sqrt{3}$



7. (a) *disk method*:

$V = \int_a^b \pi R^2(x)\,dx = \int_{-1}^{1} \pi\,(3x^4)^2\,dx = \pi \int_{-1}^{1} 9x^8\,dx$
$= \pi\,[x^9]_{-1}^{1} = 2\pi$



(b) *shell method*:

$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x\,(3x^4)\,dx = 2\pi \cdot 3\int_0^1 x^5\,dx = 2\pi \cdot 3\left[\frac{x^6}{6}\right]_0^1 = \pi$

Note: The lower limit of integration is 0 rather than -1.

(c) *shell method*:

$V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = 2\pi\int_{-1}^{1}(1-x)\,(3x^4)\,dx = 2\pi\left[\frac{3x^5}{5} - \frac{x^6}{2}\right]_{-1}^{1} = 2\pi\left[\left(\frac{3}{5}-\frac{1}{2}\right) - \left(-\frac{3}{5}-\frac{1}{2}\right)\right] = \frac{12\pi}{5}$

(d) *washer method*:

$R(x) = 3,\ r(x) = 3 - 3x^4 = 3\,(1-x^4) \Rightarrow V = \int_a^b \pi\,[R^2(x) - r^2(x)]\,dx = \int_{-1}^{1} \pi\left[9 - 9\,(1-x^4)^2\right] dx$

$= 9\pi \int_{-1}^{1}[1 - (1 - 2x^4 + x^8)]\,dx = 9\pi \int_{-1}^{1}(2x^4 - x^8)\,dx = 9\pi\left[\frac{2x^5}{5} - \frac{x^9}{9}\right]_{-1}^{1} = 18\pi\left[\frac{2}{5} - \frac{1}{9}\right] = \frac{2\pi\cdot 13}{5} = \frac{26\pi}{5}$

8. (a) *washer method*:

$R(x) = \frac{4}{x^3},\ r(x) = \frac{1}{2} \Rightarrow V = \int_a^b \pi[R^2(x) - r^2(x)]\,dx = \int_1^2 \pi\left[\left(\frac{4}{x^3}\right)^2 - \left(\frac{1}{2}\right)^2\right] dx = \pi\left[-\frac{16}{5}x^{-5} - \frac{x}{4}\right]_1^2$

$= \pi\left[\left(\frac{-16}{5\cdot 32} - \frac{1}{2}\right) - \left(-\frac{16}{5} - \frac{1}{4}\right)\right] = \pi\left(-\frac{1}{10} - \frac{1}{2} + \frac{16}{5} + \frac{1}{4}\right) = \frac{\pi}{20}(-2 - 10 + 64 + 5) = \frac{57\pi}{20}$

(b) *shell method*:

$V = 2\pi\int_1^2 x\left(\frac{4}{x^3} - \frac{1}{2}\right) dx = 2\pi\left[-4x^{-1} - \frac{x^2}{4}\right]_1^2 = 2\pi\left[\left(-\frac{4}{2} - 1\right) - \left(-4 - \frac{1}{4}\right)\right] = 2\pi\left(\frac{5}{4}\right) = \frac{5\pi}{2}$

(c) *shell method*:

$V = 2\pi\int_a^b \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = 2\pi\int_1^2 (2-x)\left(\frac{4}{x^3} - \frac{1}{2}\right) dx = 2\pi\int_1^2 \left(\frac{8}{x^3} - \frac{4}{x^2} - 1 + \frac{x}{2}\right) dx$

$= 2\pi\left[-\frac{4}{x^2} + \frac{4}{x} - x + \frac{x^2}{4}\right]_1^2 = 2\pi\left[(-1 + 2 - 2 + 1) - \left(-4 + 4 - 1 + \frac{1}{4}\right)\right] = \frac{3\pi}{2}$

(d) *washer method*:

$V = \int_a^b \pi[R^2(x) - r^2(x)]\,dx$

$= \pi \int_1^2 \left[\left(\frac{7}{2}\right)^2 - \left(4 - \frac{4}{x^3}\right)^2\right] dx$

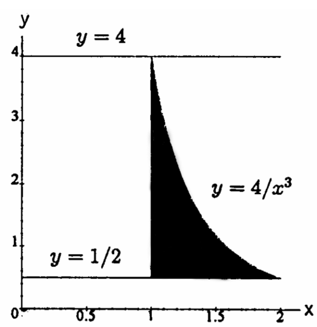
$= \frac{49\pi}{4} - 16\pi\int_1^2 (1 - 2x^{-3} + x^{-6})\,dx$

$= \frac{49\pi}{4} - 16\pi\left[x + x^{-2} - \frac{x^{-5}}{5}\right]_1^2$

$= \frac{49\pi}{4} - 16\pi\left[\left(2 + \frac{1}{4} - \frac{1}{5\cdot 32}\right) - \left(1 + 1 - \frac{1}{5}\right)\right]$

$= \frac{49\pi}{4} - 16\pi\left(\frac{1}{4} - \frac{1}{160} + \frac{1}{5}\right)$

$= \frac{49\pi}{4} - \frac{16\pi}{160}(40 - 1 + 32) = \frac{49\pi}{4} - \frac{71\pi}{10} = \frac{103\pi}{20}$



9. (a) *disk method*:

$V = \pi \int_1^5 \left(\sqrt{x-1}\right)^2 dx = \pi\int_1^5 (x-1)\,dx = \pi\left[\frac{x^2}{2} - x\right]_1^5$

$= \pi\left[\left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right)\right] = \pi\left(\frac{24}{2} - 4\right) = 8\pi$

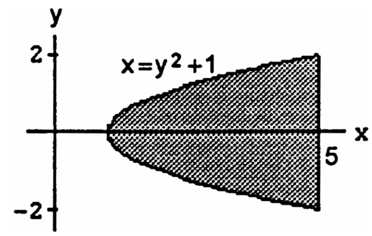
(b) *washer method*:

$R(y) = 5,\ r(y) = y^2 + 1 \Rightarrow V = \int_c^d \pi\,[R^2(y) - r^2(y)]\,dy = \pi \int_{-2}^{2}\left[25 - (y^2+1)^2\right] dy$

$= \pi\int_{-2}^{2}(25 - y^4 - 2y^2 - 1)\,dy = \pi \int_{-2}^{2}(24 - y^4 - 2y^2)\,dy = \pi\left[24y - \frac{y^5}{5} - \frac{2}{3}y^3\right]_{-2}^{2} = 2\pi\left(24\cdot 2 - \frac{32}{5} - \frac{2}{3}\cdot 8\right)$

$= 32\pi\left(3 - \frac{2}{5} - \frac{1}{3}\right) = \frac{32\pi}{15}(45 - 6 - 5) = \frac{1088\pi}{15}$

(c) *disk method*:



$R(y) = 5 - (y^2 + 1) = 4 - y^2$

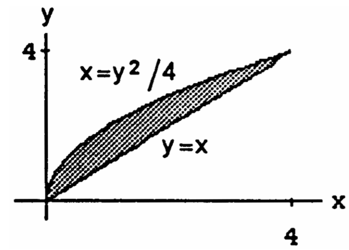
$\Rightarrow\ V = \int_c^d \pi R^2(y)\, dy = \int_{-2}^{2} \pi\left(4 - y^2\right)^2 dy$

$= \pi \int_{-2}^{2} (16 - 8y^2 + y^4)\, dy$

$= \pi\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_{-2}^{2} = 2\pi\left(32 - \frac{64}{3} + \frac{32}{5}\right)$

$= 64\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{64\pi}{15}(15 - 10 + 3) = \frac{512\pi}{15}$

10. (a) *shell method*:



$V = \int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\\text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\\text{height}\end{smallmatrix}\right) dy = \int_0^4 2\pi y\left(y - \frac{y^2}{4}\right) dy$

$= 2\pi \int_0^4 \left(y^2 - \frac{y^3}{4}\right) dy = 2\pi\left[\frac{y^3}{3} - \frac{y^4}{16}\right]_0^4 = 2\pi\left(\frac{64}{3} - \frac{64}{4}\right)$

$= \frac{2\pi}{12} \cdot 64 = \frac{32\pi}{3}$

(b) *shell method*:

$V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\\text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\\text{height}\end{smallmatrix}\right) dx = \int_0^4 2\pi x\left(2\sqrt{x} - x\right) dx = 2\pi \int_0^4 \left(2x^{3/2} - x^2\right) dx = 2\pi\left[\frac{4}{5}x^{5/2} - \frac{x^3}{3}\right]_0^4$

$= 2\pi\left(\frac{4}{5} \cdot 32 - \frac{64}{3}\right) = \frac{128\pi}{15}$

(c) *shell method*:

$V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\\text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\\text{height}\end{smallmatrix}\right) dx = \int_0^4 2\pi(4 - x)\left(2\sqrt{x} - x\right) dx = 2\pi \int_0^4 \left(8x^{1/2} - 4x - 2x^{3/2} + x^2\right) dx$

$= 2\pi\left[\frac{16}{3}x^{3/2} - 2x^2 - \frac{4}{5}x^{5/2} + \frac{x^3}{3}\right]_0^4 = 2\pi\left(\frac{16}{3} \cdot 8 - 32 - \frac{4}{5} \cdot 32 + \frac{64}{3}\right) = 64\pi\left(\frac{4}{3} - 1 - \frac{4}{5} + \frac{2}{3}\right)$

$= 64\pi\left(1 - \frac{4}{5}\right) = \frac{64\pi}{5}$

(d) *shell method*:

$V = \int_c^d 2\pi\left(\begin{smallmatrix}\text{shell}\\\text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\\text{height}\end{smallmatrix}\right) dy = \int_0^4 2\pi(4 - y)\left(y - \frac{y^2}{4}\right) dy = 2\pi \int_0^4 \left(4y - y^2 - y^2 + \frac{y^3}{4}\right) dy$

$= 2\pi \int_0^4 \left(4y - 2y^2 + \frac{y^3}{4}\right) dy = 2\pi\left[2y^2 - \frac{2}{3}y^3 + \frac{y^4}{16}\right]_0^4 = 2\pi\left(32 - \frac{2}{3} \cdot 64 + 16\right) = 32\pi\left(2 - \frac{8}{3} + 1\right) = \frac{32\pi}{3}$

11. *disk method*:

$R(x) = \tan x,\ a = 0,\ b = \frac{\pi}{3}\ \Rightarrow\ V = \pi \int_0^{\pi/3} \tan^2 x\, dx = \pi \int_0^{\pi/3} (\sec^2 x - 1)\, dx = \pi[\tan x - x]_0^{\pi/3} = \frac{\pi\left(3\sqrt{3} - \pi\right)}{3}$

12. *disk method*:

$V = \pi \int_0^{\pi} (2 - \sin x)^2\, dx = \pi \int_0^{\pi} (4 - 4\sin x + \sin^2 x)\, dx = \pi \int_0^{\pi} \left(4 - 4\sin x + \frac{1 - \cos 2x}{2}\right) dx$

$= \pi\left[4x + 4\cos x + \frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi} = \pi\left[\left(4\pi - 4 + \frac{\pi}{2} - 0\right) - (0 + 4 + 0 - 0)\right] = \pi\left(\frac{9\pi}{2} - 8\right) = \frac{\pi}{2}(9\pi - 16)$

13. (a) *disk method*:

$V = \pi \int_0^2 \left(x^2 - 2x\right)^2 dx = \pi \int_0^2 (x^4 - 4x^3 + 4x^2)\, dx = \pi\left[\frac{x^5}{5} - x^4 + \frac{4}{3}x^3\right]_0^2 = \pi\left(\frac{32}{5} - 16 + \frac{32}{3}\right)$

$= \frac{16\pi}{15}(6 - 15 + 10) = \frac{16\pi}{15}$

(b) *washer method*:

$V = \int_0^2 \pi\left[1^2 - \left(x^2 - 2x + 1\right)^2\right] dx = \int_0^2 \pi\, dx - \int_0^2 \pi\,(x - 1)^4\, dx = 2\pi - \left[\pi\,\frac{(x-1)^5}{5}\right]_0^2 = 2\pi - \pi \cdot \frac{2}{5} = \frac{8\pi}{5}$

(c) *shell method*:

$V = \int_a^b 2\pi\left(\begin{smallmatrix}\text{shell}\\\text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\\text{height}\end{smallmatrix}\right) dx = 2\pi \int_0^2 (2 - x)\left[-\left(x^2 - 2x\right)\right] dx = 2\pi \int_0^2 (2 - x)\left(2x - x^2\right) dx$

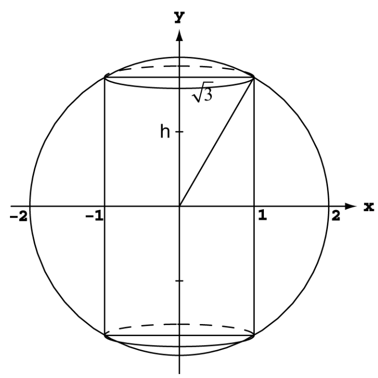
$= 2\pi\int_0^2 (4x - 2x^2 - 2x^2 + x^3)\,dx = 2\pi\int_0^2 (x^3 - 4x^2 + 4x)\,dx = 2\pi\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2 = 2\pi\left(4 - \frac{32}{3} + 8\right)$
$= \frac{2\pi}{3}(36 - 32) = \frac{8\pi}{3}$

(d) *washer method*:

$V = \pi\int_0^2 [2 - (x^2 - 2x)]^2\,dx - \pi\int_0^2 2^2\,dx = \pi\int_0^2 \left[4 - 4(x^2 - 2x) + (x^2 - 2x)^2\right]dx - 8\pi$
$= \pi\int_0^2 (4 - 4x^2 + 8x + x^4 - 4x^3 + 4x^2)\,dx - 8\pi = \pi\int_0^2 (x^4 - 4x^3 + 8x + 4)\,dx - 8\pi$
$= \pi\left[\frac{x^5}{5} - x^4 + 4x^2 + 4x\right]_0^2 - 8\pi = \pi\left(\frac{32}{5} - 16 + 16 + 8\right) - 8\pi = \frac{\pi}{5}(32 + 40) - 8\pi = \frac{72\pi}{5} - \frac{40\pi}{5} = \frac{32\pi}{5}$

14. *disk method*:
$V = 2\pi\int_0^{\pi/4} 4\tan^2 x\,dx = 8\pi\int_0^{\pi/4} (\sec^2 x - 1)\,dx = 8\pi[\tan x - x]_0^{\pi/4} = 2\pi(4 - \pi)$

15. The material removed from the sphere consists of a cylinder and two "caps." From the diagram, the height of the cylinder is 2h, where $h^2 + \left(\sqrt{3}\right)^2 = 2^2$, i.e. $h = 1$. Thus $V_{cyl} = (2h)\pi\left(\sqrt{3}\right)^2 = 6\pi\ \text{ft}^3$. To get the volume of a cap, use the disk method and $x^2 + y^2 = 2^2$: $V_{cap} = \int_1^2 \pi x^2 dy$
$= \int_1^2 \pi(4 - y^2)dy = \pi\left[4y - \frac{y^3}{3}\right]_1^2$
$= \pi\left[\left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right)\right] = \frac{5\pi}{3}\ \text{ft}^3$. Therefore,
$V_{removed} = V_{cyl} + 2V_{cap} = 6\pi + \frac{10\pi}{3} = \frac{28\pi}{3}\ \text{ft}^3$.



16. We rotate the region enclosed by the curve $y = \sqrt{12\left(1 - \frac{4x^2}{121}\right)}$ and the x-axis around the x-axis. To find the volume we use the *disk* method: $V = \int_a^b \pi R^2(x)\,dx = \int_{-11/2}^{11/2} \pi\left(\sqrt{12\left(1 - \frac{4x^2}{121}\right)}\right)^2 dx = \pi\int_{-11/2}^{11/2} 12\left(1 - \frac{4x^2}{121}\right)dx$
$= 12\pi\int_{-11/2}^{11/2}\left(1 - \frac{4x^2}{121}\right)dx = 12\pi\left[x - \frac{4x^3}{363}\right]_{-11/2}^{11/2} = 24\pi\left[\frac{11}{2} - \left(\frac{4}{363}\right)\left(\frac{11}{2}\right)^3\right] = 132\pi\left[1 - \left(\frac{4}{363}\right)\left(\frac{11^2}{4}\right)\right]$
$= 132\pi\left(1 - \frac{1}{3}\right) = \frac{264\pi}{3} = 88\pi \approx 276\ \text{in}^3$

17. $y = x^{1/2} - \frac{x^{3/2}}{3} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(\frac{1}{x} - 2 + x\right) \Rightarrow L = \int_1^4 \sqrt{1 + \frac{1}{4}\left(\frac{1}{x} - 2 + x\right)}\,dx$
$\Rightarrow L = \int_1^4 \sqrt{\frac{1}{4}\left(\frac{1}{x} + 2 + x\right)}\,dx = \int_1^4 \sqrt{\frac{1}{4}\left(x^{-1/2} + x^{1/2}\right)^2}\,dx = \int_1^4 \frac{1}{2}\left(x^{-1/2} + x^{1/2}\right)dx = \frac{1}{2}\left[2x^{1/2} + \frac{2}{3}x^{3/2}\right]_1^4$
$= \frac{1}{2}\left[\left(4 + \frac{2}{3}\cdot 8\right) - \left(2 + \frac{2}{3}\right)\right] = \frac{1}{2}\left(2 + \frac{14}{3}\right) = \frac{10}{3}$

18. $x = y^{2/3} \Rightarrow \frac{dx}{dy} = \frac{2}{3}y^{-1/3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{4y^{-2/3}}{9} \Rightarrow L = \int_1^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_1^8 \sqrt{1 + \frac{4}{9y^{2/3}}}\,dy$
$= \int_1^8 \frac{\sqrt{9y^{2/3} + 4}}{3y^{1/3}}\,dy = \frac{1}{3}\int_1^8 \sqrt{9y^{2/3} + 4}\,\left(y^{-1/3}\right)dy$; $[u = 9y^{2/3} + 4 \Rightarrow du = 6y^{-1/3}\,dy;\ y = 1 \Rightarrow u = 13,$
$y = 8 \Rightarrow u = 40] \rightarrow L = \frac{1}{18}\int_{13}^{40} u^{1/2}\,du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} = \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \approx 7.634$

19. $y = \frac{5}{12}x^{6/5} - \frac{5}{8}x^{4/5} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{1/5} - \frac{1}{2}x^{-1/5} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)$
$\Rightarrow L = \int_1^{32} \sqrt{1 + \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)}\,dx \Rightarrow L = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{2/5} + 2 + x^{-2/5}\right)}\,dx = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{1/5} + x^{-1/5}\right)^2}\,dx$

$= \int_1^{32} \frac{1}{2}\left(x^{1/5} + x^{-1/5}\right) dx = \frac{1}{2}\left[\frac{5}{6}x^{6/5} + \frac{5}{4}x^{4/5}\right]_1^{32} = \frac{1}{2}\left[\left(\frac{5}{6}\cdot 2^6 + \frac{5}{4}\cdot 2^4\right) - \left(\frac{5}{6} + \frac{5}{4}\right)\right] = \frac{1}{2}\left(\frac{315}{6} + \frac{75}{4}\right)$
$= \frac{1}{48}(1260 + 450) = \frac{1710}{48} = \frac{285}{8}$

20. $x = \frac{1}{12}y^3 + \frac{1}{y} \Rightarrow \frac{dx}{dy} = \frac{1}{4}y^2 - \frac{1}{y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4} \Rightarrow L = \int_1^2 \sqrt{1 + \left(\frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4}\right)}\, dy$
$= \int_1^2 \sqrt{\frac{1}{16}y^4 + \frac{1}{2} + \frac{1}{y^4}}\, dy = \int_1^2 \sqrt{\left(\frac{1}{4}y^2 + \frac{1}{y^2}\right)^2}\, dy = \int_1^2 \left(\frac{1}{4}y^2 + \frac{1}{y^2}\right) dy = \left[\frac{1}{12}y^3 - \frac{1}{y}\right]_1^2$
$= \left(\frac{8}{12} - \frac{1}{2}\right) - \left(\frac{1}{12} - 1\right) = \frac{7}{12} + \frac{1}{2} = \frac{13}{12}$

21. $S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx;\ \frac{dy}{dx} = \frac{1}{\sqrt{2x+1}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{2x+1} \Rightarrow S = \int_0^3 2\pi\sqrt{2x+1}\sqrt{1 + \frac{1}{2x+1}}\, dx$
$= 2\pi\int_0^3 \sqrt{2x+1}\sqrt{\frac{2x+2}{2x+1}}\, dx = 2\sqrt{2}\pi\int_0^3 \sqrt{x+1}\, dx = 2\sqrt{2}\pi\left[\frac{2}{3}(x+1)^{3/2}\right]_0^3 = 2\sqrt{2}\pi\cdot\frac{2}{3}(8-1) = \frac{28\pi\sqrt{2}}{3}$

22. $S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx;\ \frac{dy}{dx} = x^2 \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^4 \Rightarrow S = \int_0^1 2\pi\cdot\frac{x^3}{3}\sqrt{1+x^4}\, dx = \frac{\pi}{6}\int_0^1 \sqrt{1+x^4}\,(4x^3)\, dx$
$= \frac{\pi}{6}\int_0^1 \sqrt{1+x^4}\, d(1+x^4) = \frac{\pi}{6}\left[\frac{2}{3}(1+x^4)^{3/2}\right]_0^1 = \frac{\pi}{9}\left[2\sqrt{2} - 1\right]$

23. $S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy;\ \frac{dx}{dy} = \frac{\left(\frac{1}{2}\right)(4-2y)}{\sqrt{4y-y^2}} = \frac{2-y}{\sqrt{4y-y^2}} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = \frac{4y - y^2 + 4 - 4y + y^2}{4y - y^2} = \frac{4}{4y-y^2}$
$\Rightarrow S = \int_1^2 2\pi\sqrt{4y-y^2}\sqrt{\frac{4}{4y-y^2}}\, dy = 4\pi\int_1^2 dx = 4\pi$

24. $S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy;\ \frac{dx}{dy} = \frac{1}{2\sqrt{y}} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4y} = \frac{4y+1}{4y} \Rightarrow S = \int_2^6 2\pi\sqrt{y}\cdot\frac{\sqrt{4y+1}}{\sqrt{4y}}\, dy$
$= \pi\int_2^6 \sqrt{4y+1}\, dy = \frac{\pi}{4}\left[\frac{2}{3}(4y+1)^{3/2}\right]_2^6 = \frac{\pi}{6}(125 - 27) = \frac{\pi}{6}(98) = \frac{49\pi}{3}$

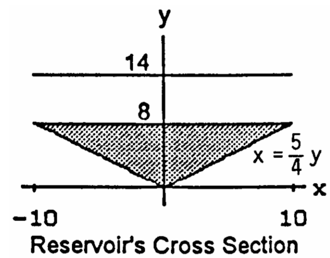
25. The equipment alone: the force required to lift the equipment is equal to its weight $\Rightarrow F_1(x) = 100$ N. The work done is $W_1 = \int_a^b F_1(x)\, dx = \int_0^{40} 100\, dx = [100x]_0^{40} = 4000$ J; the rope alone: the force required to lift the rope is equal to the weight of the rope paid out at elevation x $\Rightarrow F_2(x) = 0.8(40 - x)$. The work done is $W_2 = \int_a^b F_2(x)\, dx = \int_0^{40} 0.8(40-x)\, dx = 0.8\left[40x - \frac{x^2}{2}\right]_0^{40} = 0.8\left(40^2 - \frac{40^2}{2}\right) = \frac{(0.8)(1600)}{2} = 640$ J; the total work is $W = W_1 + W_2 = 4000 + 640 = 4640$ J

26. The force required to lift the water is equal to the water's weight, which varies steadily from $8\cdot 800$ lb to $8\cdot 400$ lb over the 4750 ft elevation. When the truck is x ft off the base of Mt. Washington, the water weight is $F(x) = 8\cdot 800\cdot\left(\frac{2\cdot 4750 - x}{2\cdot 4750}\right) = (6400)\left(1 - \frac{x}{9500}\right)$ lb. The work done is $W = \int_a^b F(x)\, dx$
$= \int_0^{4750} 6400\left(1 - \frac{x}{9500}\right) dx = 6400\left[x - \frac{x^2}{2\cdot 9500}\right]_0^{4750} = 6400\left(4750 - \frac{4750^2}{4\cdot 4750}\right) = \left(\frac{3}{4}\right)(6400)(4750)$
$= 22{,}800{,}000$ ft · lb

27. Force constant: $F = kx \Rightarrow 20 = k\cdot 1 \Rightarrow k = 20$ lb/ft; the work to stretch the spring 1 ft is $W = \int_0^1 kx\, dx = k\int_0^1 x\, dx = \left[20\,\frac{x^2}{2}\right]_0^1 = 10$ ft · lb; the work to stretch the spring an additional foot is $W = \int_1^2 kx\, dx = k\int_1^2 x\, dx = 20\left[\frac{x^2}{2}\right]_1^2 = 20\left(\frac{4}{2} - \frac{1}{2}\right) = 20\left(\frac{3}{2}\right) = 30$ ft · lb

28. Force constant: $F = kx \Rightarrow 200 = k(0.8) \Rightarrow k = 250$ N/m; the 300 N force stretches the spring $x = \frac{F}{k}$
$= \frac{300}{250} = 1.2$ m; the work required to stretch the spring that far is then $W = \int_0^{1.2} F(x)\,dx = \int_0^{1.2} 250x\,dx$
$= [125x^2]_0^{1.2} = 125(1.2)^2 = 180$ J

29. We imagine the water divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval $[0, 8]$. The typical slab between the planes at y and $y + \Delta y$ has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\frac{5}{4}y\right)^2 \Delta y = \frac{25\pi}{16} y^2 \Delta y$ ft³. The force F(y) required to lift this slab is equal to its weight: $F(y) = 62.4\,\Delta V$ $= \frac{(62.4)(25)}{16}\pi y^2 \Delta y$ lb. The distance through which F(y) must act to lift this slab to the level 6 ft above the top is about $(6 + 8 - y)$ ft, so the work done lifting the slab is about $\Delta W = \frac{(62.4)(25)}{16}\pi y^2(14 - y)\,\Delta y$ ft · lb. The work done lifting all the slabs from $y = 0$ to $y = 8$ to the level 6 ft above the top is approximately



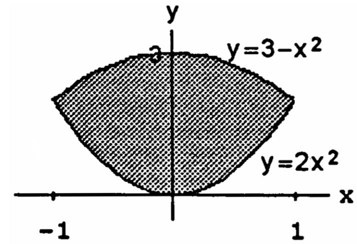
$W \approx \sum_0^8 \frac{(62.4)(25)}{16}\pi y^2 (14 - y)\,\Delta y$ ft · lb so the work to pump the water is the limit of these Riemann sums as the norm of the partition goes to zero: $W = \int_0^8 \frac{(62.4)(25)}{(16)}\pi y^2(14 - y)\,dy = \frac{(62.4)(25)\pi}{16}\int_0^8 (14y^2 - y^3)\,dy = (62.4)\left(\frac{25\pi}{16}\right)\left[\frac{14}{3}y^3 - \frac{y^4}{4}\right]_0^8$
$= (62.4)\left(\frac{25\pi}{16}\right)\left(\frac{14}{3}\cdot 8^3 - \frac{8^4}{4}\right) \approx 418{,}208.81$ ft · lb

30. The same as in Exercise 29, but change the distance through which F(y) must act to $(8 - y)$ rather than $(6 + 8 - y)$. Also change the upper limit of integration from 8 to 5. The integral is:$W = \int_0^5 \frac{(62.4)(25)\pi}{16} y^2(8 - y)\,dy$
$= (62.4)\left(\frac{25\pi}{16}\right)\int_0^5 (8y^2 - y^3)\,dy = (62.4)\left(\frac{25\pi}{16}\right)\left[\frac{8}{3}y^3 - \frac{y^4}{4}\right]_0^5 = (62.4)\left(\frac{25\pi}{16}\right)\left(\frac{8}{3}\cdot 5^3 - \frac{5^4}{4}\right) \approx 54{,}241.56$ ft · lb

31. The tank's cross section looks like the figure in Exercise 29 with right edge given by $x = \frac{5}{10}y = \frac{y}{2}$. A typical horizontal slab has volume $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{y}{2}\right)^2 \Delta y = \frac{\pi}{4}y^2\,\Delta y$. The force required to lift thisslab is its weight: $F(y) = 60 \cdot \frac{\pi}{4}y^2\,\Delta y$. The distance through which F(y) must act is $(2 + 10 - y)$ ft, so the work to pump the liquid is $W = 60\int_0^{10} \pi(12 - y)\left(\frac{y^2}{4}\right)dy = 15\pi\left[\frac{12y^3}{3} - \frac{y^4}{4}\right]_0^{10} = 22{,}500\pi$ ft · lb; the time needed to empty the tank is $\frac{22{,}500\pi \text{ ft·lb}}{275 \text{ ft·lb/sec}} \approx 257$ sec

32. A typical horizontal slab has volume about $\Delta V = (20)(2x)\Delta y = (20)\left(2\sqrt{16 - y^2}\right)\Delta y$ and the force required to lift this slab is its weight $F(y) = (57)(20)\left(2\sqrt{16 - y^2}\right)\Delta y$. The distance through which F(y) must act is $(6 + 4 - y)$ ft, so the work to pump the olive oil from the half-full tank is $W = 57\int_{-4}^0 (10 - y)(20)\left(2\sqrt{16 - y^2}\right)dy$
$= 2880\int_{-4}^0 10\sqrt{16 - y^2}\,dy + 1140\int_{-4}^0 (16 - y^2)^{1/2}(-2y)\,dy$
$= 22{,}800 \cdot (\text{area of a quarter circle having radius 4}) + \frac{2}{3}(1140)\left[(16 - y^2)^{3/2}\right]_{-4}^0 = (22{,}800)(4\pi) + 48{,}640$
$= 335{,}153.25$ ft · lb

33. Intersection points: $3 - x^2 = 2x^2 \Rightarrow 3x^2 - 3 = 0$ $\Rightarrow 3(x-1)(x+1) = 0 \Rightarrow x = -1$ or $x = 1$. Symmetry suggests that $\overline{x} = 0$. The typical *vertical* strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{2x^2 + (3 - x^2)}{2}\right) = \left(x, \frac{x^2+3}{2}\right)$, length: $(3 - x^2) - 2x^2 = 3(1 - x^2)$, width: dx, area: $dA = 3(1 - x^2)\,dx$, and mass: $dm = \delta \cdot dA$ $= 3\delta(1 - x^2)\,dx \Rightarrow$ the moment about the x-axis is

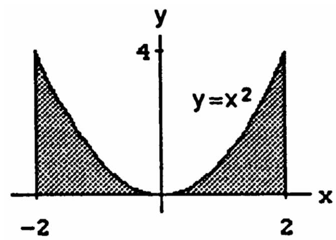


$\widetilde{y}\,dm = \frac{3}{2}\,\delta\,(x^2+3)(1-x^2)\,dx = \frac{3}{2}\,\delta\,(-x^4 - 2x^2 + 3)\,dx \Rightarrow M_x = \int \widetilde{y}\,dm = \frac{3}{2}\,\delta \int_{-1}^{1} (-x^4 - 2x^2 + 3)\,dx$

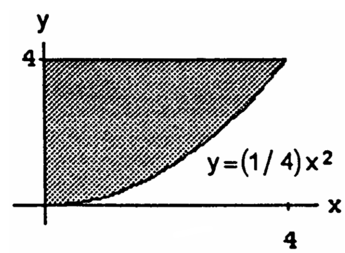
$= \frac{3}{2}\,\delta \left[-\frac{x^5}{5} - \frac{2x^3}{3} + 3x\right]_{-1}^{1} = 3\delta\left(-\frac{1}{5} - \frac{2}{3} + 3\right) = \frac{3\delta}{15}(-3 - 10 + 45) = \frac{32\delta}{5}$; $M = \int dm = 3\delta \int_{-1}^{1} (1 - x^2)\,dx$

$= 3\delta\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 6\delta\left(1 - \frac{1}{3}\right) = 4\delta \Rightarrow \overline{y} = \frac{M_x}{M} = \frac{32\delta}{5 \cdot 4\delta} = \frac{8}{5}$. Therefore, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{8}{5}\right)$.

34. Symmetry suggests that $\overline{x} = 0$. The typical *vertical* strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{x^2}{2}\right)$, length: x^2, width: dx, area: $dA = x^2\,dx$, mass: $dm = \delta \cdot dA = \delta x^2\,dx$ $\Rightarrow$ the moment about the x-axis is $\widetilde{y}\,dm = \frac{\delta}{2}\,x^2 \cdot x^2\,dx$ $= \frac{\delta}{2}\,x^4\,dx \Rightarrow M_x = \int \widetilde{y}\,dm = \frac{\delta}{2}\int_{-2}^{2} x^4\,dx = \frac{\delta}{10}\left[x^5\right]_{-2}^{2}$



35. The typical *vertical* strip has: center of mass: $(\widetilde{x}, \widetilde{y})$ $= \left(x, \frac{4 + \frac{x^2}{4}}{2}\right)$, length: $4 - \frac{x^2}{4}$, width: dx, area: $dA = \left(4 - \frac{x^2}{4}\right)dx$, mass: $dm = \delta \cdot dA$ $= \delta\left(4 - \frac{x^2}{4}\right)dx \Rightarrow$ the moment about the x-axis is $\widetilde{y}\,dm = \delta \cdot \frac{\left(4 + \frac{x^2}{4}\right)}{2}\left(4 - \frac{x^2}{4}\right)dx = \frac{\delta}{2}\left(16 - \frac{x^4}{16}\right)dx$; the



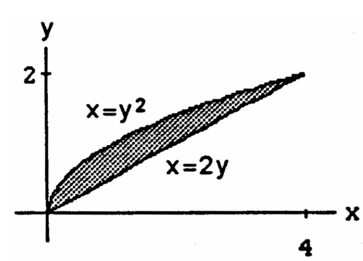
moment about the y-axis is $\widetilde{x}\,dm = \delta\left(4 - \frac{x^2}{4}\right) \cdot x\,dx = \delta\left(4x - \frac{x^3}{4}\right)dx$. Thus, $M_x = \int \widetilde{y}\,dm = \frac{\delta}{2}\int_0^4 \left(16 - \frac{x^4}{16}\right)dx$

$= \frac{\delta}{2}\left[16x - \frac{x^5}{5 \cdot 16}\right]_0^4 = \frac{\delta}{2}\left[64 - \frac{64}{5}\right] = \frac{128\delta}{5}$; $M_y = \int \widetilde{x}\,dm = \delta\int_0^4 \left(4x - \frac{x^3}{4}\right)dx = \delta\left[2x^2 - \frac{x^4}{16}\right]_0^4$

$= \delta(32 - 16) = 16\delta$; $M = \int dm = \delta\int_0^4 \left(4 - \frac{x^2}{4}\right)dx = \delta\left[4x - \frac{x^3}{12}\right]_0^4 = \delta\left(16 - \frac{64}{12}\right) = \frac{32\delta}{3}$

$\Rightarrow \overline{x} = \frac{M_y}{M} = \frac{16 \cdot \delta \cdot 3}{32 \cdot \delta} = \frac{3}{2}$ and $\overline{y} = \frac{M_x}{M} = \frac{128 \cdot \delta \cdot 3}{5 \cdot 32 \cdot \delta} = \frac{12}{5}$. Therefore, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{3}{2}, \frac{12}{5}\right)$.

36. A typical *horizontal* strip has: center of mass: $(\widetilde{x}, \widetilde{y}) = \left(\frac{y^2 + 2y}{2}, y\right)$, length: $2y - y^2$, width: dy, area: $dA = (2y - y^2)\,dy$, mass: $dm = \delta \cdot dA$ $= \delta(2y - y^2)\,dy$; the moment about the x-axis is $\widetilde{y}\,dm = \delta \cdot y \cdot (2y - y^2)\,dy = \delta(2y^2 - y^3)$; the moment about the y-axis is $\widetilde{x}\,dm = \delta \cdot \frac{(y^2+2y)}{2} \cdot (2y - y^2)\,dy$

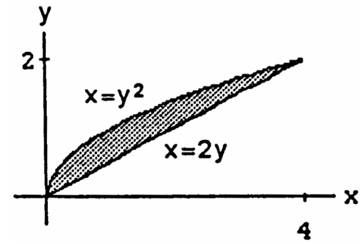


$= \frac{\delta}{2}(4y^2 - y^4)\,dy \Rightarrow M_x = \int \widetilde{y}\,dm = \delta\int_0^2 (2y^2 - y^3)\,dy$

$= \delta\left[\frac{2}{3}y^3 - \frac{y^4}{4}\right]_0^2 = \delta\left(\frac{2}{3} \cdot 8 - \frac{16}{4}\right) = \delta\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{\delta \cdot 16}{12} = \frac{4\delta}{3}$; $M_y = \int \widetilde{x}\,dm = \frac{\delta}{2}\int_0^2 (4y^2 - y^4)\,dy = \frac{\delta}{2}\left[\frac{4}{3}y^3 - \frac{y^5}{5}\right]_0^2$

$= \frac{\delta}{2}\left(\frac{4 \cdot 8}{3} - \frac{32}{5}\right) = \frac{32\delta}{15}$; $M = \int dm = \delta\int_0^2 (2y - y^2)\,dy = \delta\left[y^2 - \frac{y^3}{3}\right]_0^2 = \delta\left(4 - \frac{8}{3}\right) = \frac{4\delta}{3} \Rightarrow \overline{x} = \frac{M_y}{M} = \frac{\delta \cdot 32 \cdot 3}{15 \cdot \delta \cdot 4} = \frac{8}{5}$ and $\overline{y} = \frac{M_x}{M} = \frac{4 \cdot \delta \cdot 3}{3 \cdot 4 \cdot \delta} = 1$. Therefore, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{8}{5}, 1\right)$.

37. A typical horizontal strip has: center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{y^2+2y}{2}, y\right)$, length: $2y - y^2$, width: dy, area: $dA = (2y - y^2)\,dy$, mass: $dm = \delta \cdot dA = (1+y)(2y-y^2)\,dy \Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = y(1+y)(2y-y^2)\,dy = (2y^2 + 2y^3 - y^3 - y^4)\,dy = (2y^2 + y^3 - y^4)\,dy$; the moment about the y-axis is



$\tilde{x}\,dm = \left(\frac{y^2+2y}{2}\right)(1+y)(2y-y^2)\,dy = \frac{1}{2}(4y^2 - y^4)(1+y)\,dy = \frac{1}{2}(4y^2 + 4y^3 - y^4 - y^5)\,dy$

$\Rightarrow M_x = \int \tilde{y}\,dm = \int_0^2 (2y^2 + y^3 - y^4)\,dy = \left[\frac{2}{3}y^3 + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^2 = \left(\frac{16}{3} + \frac{16}{4} - \frac{32}{5}\right) = 16\left(\frac{1}{3} + \frac{1}{4} - \frac{2}{5}\right)$

$= \frac{16}{60}(20 + 15 - 24) = \frac{4}{15}(11) = \frac{44}{15}$; $M_y = \int \tilde{x}\,dm = \int_0^2 \frac{1}{2}(4y^2 + 4y^3 - y^4 - y^5)\,dy = \frac{1}{2}\left[\frac{4}{3}y^3 + y^4 - \frac{y^5}{5} - \frac{y^6}{6}\right]_0^2$

$= \frac{1}{2}\left(\frac{4 \cdot 2^3}{3} + 2^4 - \frac{2^5}{5} - \frac{2^6}{6}\right) = 4\left(\frac{4}{3} + 2 - \frac{4}{5} - \frac{8}{6}\right) = 4\left(2 - \frac{4}{5}\right) = \frac{24}{5}$; $M = \int dm = \int_0^2 (1+y)(2y-y^2)\,dy$

$= \int_0^2 (2y + y^2 - y^3)\,dy = \left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2 = \left(4 + \frac{8}{3} - \frac{16}{4}\right) = \frac{8}{3} \Rightarrow \bar{x} = \frac{M_y}{M} = \left(\frac{24}{5}\right)\left(\frac{3}{8}\right) = \frac{9}{5}$ and $\bar{y} = \frac{M_x}{M}$

$= \left(\frac{44}{15}\right)\left(\frac{3}{8}\right) = \frac{44}{40} = \frac{11}{10}$. Therefore, the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{9}{5}, \frac{11}{10}\right)$.

38. A typical vertical strip has: center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{3}{2x^{3/2}}\right)$, length: $\frac{3}{x^{3/2}}$, width: dx, area: $dA = \frac{3}{x^{3/2}}\,dx$, mass: $dm = \delta \cdot dA = \delta \cdot \frac{3}{x^{3/2}}\,dx \Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = \frac{3}{2x^{3/2}} \cdot \delta \frac{3}{x^{3/2}}\,dx = \frac{9\delta}{2x^3}\,dx$; the moment about the y-axis is $\tilde{x}\,dm = x \cdot \delta \frac{3}{x^{3/2}}\,dx = \frac{3\delta}{x^{1/2}}\,dx$.

(a) $M_x = \delta \int_1^9 \frac{1}{2}\left(\frac{9}{x^3}\right)dx = \frac{9\delta}{2}\left[-\frac{x^{-2}}{2}\right]_1^9 = \frac{20\delta}{9}$; $M_y = \delta \int_1^9 x\left(\frac{3}{x^{3/2}}\right)dx = 3\delta\left[2x^{1/2}\right]_1^9 = 12\delta$;

$M = \delta \int_1^9 \frac{3}{x^{3/2}}\,dx = -6\delta\left[x^{-1/2}\right]_1^9 = 4\delta \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{12\delta}{4\delta} = 3$ and $\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{20\delta}{9}\right)}{4\delta} = \frac{5}{9}$

(b) $M_x = \int_1^9 \frac{x}{2}\left(\frac{9}{x^3}\right)dx = \frac{9}{2}\left[-\frac{1}{x}\right]_1^9 = 4$; $M_y = \int_1^9 x^2\left(\frac{3}{x^{3/2}}\right)dx = \left[2x^{3/2}\right]_1^9 = 52$; $M = \int_1^9 x\left(\frac{3}{x^{3/2}}\right)dx$

$= 6\left[x^{1/2}\right]_1^9 = 12 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{13}{3}$ and $\bar{y} = \frac{M_x}{M} = \frac{1}{3}$

39. $F = \int_a^b W \cdot \binom{\text{strip}}{\text{depth}} \cdot L(y)\,dy \Rightarrow F = 2\int_0^2 (62.4)(2-y)(2y)\,dy = 249.6\int_0^2 (2y - y^2)\,dy = 249.6\left[y^2 - \frac{y^3}{3}\right]_0^2$

$= (249.6)\left(4 - \frac{8}{3}\right) = (249.6)\left(\frac{4}{3}\right) = 332.8$ lb

40. $F = \int_a^b W \cdot \binom{\text{strip}}{\text{depth}} \cdot L(y)\,dy \Rightarrow F = \int_0^{5/6} 75\left(\frac{5}{6} - y\right)(2y+4)\,dy = 75\int_0^{5/6}\left(\frac{5}{3}y + \frac{10}{3} - 2y^2 - 4y\right)dy$

$= 75\int_0^{5/6}\left(\frac{10}{3} - \frac{7}{3}y - 2y^2\right)dy = 75\left[\frac{10}{3}y - \frac{7}{6}y^2 - \frac{2}{3}y^3\right]_0^{5/6} = (75)\left[\left(\frac{50}{18}\right) - \left(\frac{7}{6}\right)\left(\frac{25}{36}\right) - \left(\frac{2}{3}\right)\left(\frac{125}{216}\right)\right]$

$= (75)\left(\frac{25}{9} - \frac{175}{216} - \frac{250}{3 \cdot 216}\right) = \left(\frac{75}{9 \cdot 216}\right)(25 \cdot 216 - 175 \cdot 9 - 250 \cdot 3) = \frac{(75)(3075)}{9 \cdot 216} \approx 118.63$ lb.

41. $F = \int_a^b W \cdot \binom{\text{strip}}{\text{depth}} \cdot L(y)\,dy \Rightarrow F = 62.4\int_0^4 (9-y)\left(2 \cdot \frac{\sqrt{y}}{2}\right)dy = 62.4\int_0^4 \left(9y^{1/2} - 3y^{3/2}\right)dy$

$= 62.4\left[6y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^4 = (62.4)\left(6 \cdot 8 - \frac{2}{5} \cdot 32\right) = \left(\frac{62.4}{5}\right)(48 \cdot 5 - 64) = \frac{(62.4)(176)}{5} = 2196.48$ lb

42. Place the origin at the bottom of the tank. Then $F = \int_0^h W \cdot \binom{\text{strip}}{\text{depth}} \cdot L(y)\,dy$, h = the height of the mercury column, strip depth $= h - y$, $L(y) = 1 \Rightarrow F = \int_0^h 849(h-y)\,1\,dy = (849)\int_0^h (h-y)\,dy = 849\left[hy - \frac{y^2}{2}\right]_0^h = 849\left(h^2 - \frac{h^2}{2}\right)$

$= \frac{849}{2}h^2$. Now solve $\frac{849}{2}h^2 = 40000$ to get $h \approx 9.707$ ft. The volume of the mercury is $s^2h = 1^2 \cdot 9.707 = 9.707\ \text{ft}^3$.

CHAPTER 6 ADDITIONAL AND ADVANCED EXERCISES

1. $V = \pi \int_a^b [f(x)]^2\, dx = b^2 - ab \Rightarrow \pi \int_a^x [f(t)]^2\, dt = x^2 - ax$ for all $x > a \Rightarrow \pi [f(x)]^2 = 2x - a \Rightarrow f(x) = \sqrt{\frac{2x-a}{\pi}}$

2. $V = \pi \int_0^a [f(x)]^2\, dx = a^2 + a \Rightarrow \pi \int_0^x [f(t)]^2\, dt = x^2 + x$ for all $x > a \Rightarrow \pi [f(x)]^2 = 2x + 1 \Rightarrow f(x) = \sqrt{\frac{2x+1}{\pi}}$

3. $s(x) = Cx \Rightarrow \int_0^x \sqrt{1 + [f'(t)]^2}\, dt = Cx \Rightarrow \sqrt{1 + [f'(x)]^2} = C \Rightarrow f'(x) = \sqrt{C^2 - 1}$ for $C \ge 1$ $\Rightarrow f(x) = \int_0^x \sqrt{C^2 - 1}\, dt + k$. Then $f(0) = a \Rightarrow a = 0 + k \Rightarrow f(x) = \int_0^x \sqrt{C^2 - 1}\, dt + a \Rightarrow f(x) = x\sqrt{C^2 - 1} + a$, where $C \ge 1$.

4. (a) The graph of $f(x) = \sin x$ traces out a path from $(0, 0)$ to $(\alpha, \sin \alpha)$ whose length is $L = \int_0^\alpha \sqrt{1 + \cos^2 \theta}\, d\theta$. The line segment from $(0, 0)$ to $(\alpha, \sin \alpha)$ has length $\sqrt{(\alpha - 0)^2 + (\sin \alpha - 0)^2} = \sqrt{\alpha^2 + \sin^2 \alpha}$. Since the shortest distance between two points is the length of the straight line segment joining them, we have immediately that $\int_0^\alpha \sqrt{1 + \cos^2 \theta}\, d\theta > \sqrt{\alpha^2 + \sin^2 \alpha}$ if $0 < \alpha \le \frac{\pi}{2}$.

   (b) In general, if $y = f(x)$ is continuously differentiable and $f(0) = 0$, then $\int_0^\alpha \sqrt{1 + [f'(t)]^2}\, dt > \sqrt{\alpha^2 + f^2(\alpha)}$ for $\alpha > 0$.

5. We can find the centroid and then use Pappus' Theorem to calculate the volume. $f(x) = x$, $g(x) = x^2$, $f(x) = g(x)$ $\Rightarrow x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x = 0, x = 1; \delta = 1; M = \int_0^1 [x - x^2]dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \left(\frac{1}{2} - \frac{1}{3}\right) - 0 = \frac{1}{6}$
$\overline{x} = \frac{1}{1/6}\int_0^1 x[x - x^2]dx = 6\int_0^1 [x^2 - x^3]dx = 6\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) - 0 = \frac{1}{2}$
$\overline{y} = \frac{1}{1/6}\int_0^1 \frac{1}{2}\left[x^2 - (x^2)^2\right]dx = 3\int_0^1 [x^2 - x^4]dx = 3\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = 3\left(\frac{1}{3} - \frac{1}{5}\right) - 0 = \frac{2}{5} \Rightarrow$ The centroid is $\left(\frac{1}{2}, \frac{2}{5}\right)$.
ρ is the distance from $\left(\frac{1}{2}, \frac{2}{5}\right)$ to the axis of rotation, $y = x$. To calculate this distance we must find the point on $y = x$ that also lies on the line perpendicular to $y = x$ that passes through $\left(\frac{1}{2}, \frac{2}{5}\right)$. The equation of this line is $y - \frac{2}{5} = -1\left(x - \frac{1}{2}\right)$ $\Rightarrow x + y = \frac{9}{10}$. The point of intersection of the lines $x + y = \frac{9}{10}$ and $y = x$ is $\left(\frac{9}{20}, \frac{9}{20}\right)$. Thus,
$\rho = \sqrt{\left(\frac{9}{20} - \frac{1}{2}\right)^2 + \left(\frac{9}{20} - \frac{2}{5}\right)^2} = \frac{1}{10\sqrt{2}}$. Thus $V = 2\pi\left(\frac{1}{10\sqrt{2}}\right)\left(\frac{1}{6}\right) = \frac{\pi}{30\sqrt{2}}$.

6. Since the slice is made at an angle of 45°, the volume of the wedge is half the volume of the cylinder of radius $\frac{1}{2}$ and height 1. Thus, $V = \frac{1}{2}\left[\pi\left(\frac{1}{2}\right)^2(1)\right] = \frac{\pi}{8}$.

7. $y = 2\sqrt{x} \Rightarrow ds = \sqrt{\frac{1}{x} + 1}\, dx \Rightarrow A = \int_0^3 2\sqrt{x}\sqrt{\frac{1}{x} + 1}\, dx = \frac{4}{3}\left[(1 + x)^{3/2}\right]_0^3 = \frac{28}{3}$

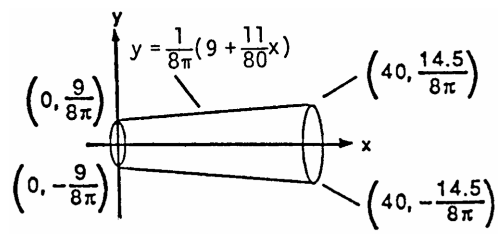
8. This surface is a triangle having a base of $2\pi a$ and a height of $2\pi ak$. Therefore the surface area is $\frac{1}{2}(2\pi a)(2\pi ak) = 2\pi^2 a^2 k$.

9. $F = ma = t^2 \Rightarrow \frac{d^2x}{dt^2} = a = \frac{t^2}{m} \Rightarrow v = \frac{dx}{dt} = \frac{t^3}{3m} + C$; $v = 0$ when $t = 0 \Rightarrow C = 0 \Rightarrow \frac{dx}{dt} = \frac{t^3}{3m} \Rightarrow x = \frac{t^4}{12m} + C_1$; $x = 0$ when $t = 0 \Rightarrow C_1 = 0 \Rightarrow x = \frac{t^4}{12m}$. Then $x = h \Rightarrow t = (12mh)^{1/4}$. The work done is
$W = \int F\, dx = \int_0^{(12mh)^{1/4}} F(t) \cdot \frac{dx}{dt}\, dt = \int_0^{(12mh)^{1/4}} t^2 \cdot \frac{t^3}{3m}\, dt = \frac{1}{3m}\left[\frac{t^6}{6}\right]_0^{(12mh)^{1/4}} = \left(\frac{1}{18m}\right)(12mh)^{6/4}$
$= \frac{(12mh)^{3/2}}{18m} = \frac{12mh \cdot \sqrt{12mh}}{18m} = \frac{2h}{3} \cdot 2\sqrt{3mh} = \frac{4h}{3}\sqrt{3mh}$

10. Converting to pounds and feet, 2 lb/in $= \frac{2\text{ lb}}{1\text{ in}} \cdot \frac{12\text{ in}}{1\text{ ft}} = 24$ lb/ft. Thus, $F = 24x \Rightarrow W = \int_0^{1/2} 24x\,dx$ $= [12x^2]_0^{1/2} = 3$ ft · lb. Since $W = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_1^2$, where W = 3 ft · lb, $m = \left(\frac{1}{10}\text{ lb}\right)\left(\frac{1}{32\text{ ft/sec}^2}\right)$ $= \frac{1}{320}$ slugs, and $v_1 = 0$ ft/sec, we have $3 = \left(\frac{1}{2}\right)\left(\frac{1}{320}v_0^2\right) \Rightarrow v_0^2 = 3 \cdot 640$. For the projectile height, $s = -16t^2 + v_0t$ (since s = 0 at t = 0) $\Rightarrow \frac{ds}{dt} = v = -32t + v_0$. At the top of the ball's path, $v = 0 \Rightarrow t = \frac{v_0}{32}$ and the height is $s = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = \frac{v_0^2}{64} = \frac{3\cdot 640}{64} = 30$ ft.

11. From the symmetry of $y = 1 - x^n$, n even, about the y-axis for $-1 \le x \le 1$, we have $\bar{x} = 0$. To find $\bar{y} = \frac{M_x}{M}$, we use the vertical strips technique. The typical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{1-x^n}{2}\right)$, length: $1 - x^n$, width: dx, area: $dA = (1 - x^n)\,dx$, mass: $dm = 1 \cdot dA = (1 - x^n)\,dx$. The moment of the strip about the x-axis is $\tilde{y}\,dm = \frac{(1-x^n)^2}{2}\,dx \Rightarrow M_x = \int_{-1}^{1} \frac{(1-x^n)^2}{2}\,dx = 2\int_0^1 \frac{1}{2}(1 - 2x^n + x^{2n})\,dx = \left[x - \frac{2x^{n+1}}{n+1} + \frac{x^{2n+1}}{2n+1}\right]_0^1$ $= 1 - \frac{2}{n+1} + \frac{1}{2n+1} = \frac{(n+1)(2n+1) - 2(2n+1) + (n+1)}{(n+1)(2n+1)} = \frac{2n^2+3n+1-4n-2+n+1}{(n+1)(2n+1)} = \frac{2n^2}{(n+1)(2n+1)}$.
Also, $M = \int_{-1}^{1} dA = \int_{-1}^{1}(1 - x^n)\,dx = 2\int_0^1 (1 - x^n)\,dx = 2\left[x - \frac{x^{n+1}}{n+1}\right]_0^1 = 2\left(1 - \frac{1}{n+1}\right) = \frac{2n}{n+1}$. Therefore, $\bar{y} = \frac{M_x}{M} = \frac{2n^2}{(n+1)(2n+1)} \cdot \frac{(n+1)}{2n} = \frac{n}{2n+1} \Rightarrow \left(0, \frac{n}{2n+1}\right)$ is the location of the centroid. As $n \to \infty$, $\bar{y} \to \frac{1}{2}$ so the limiting position of the centroid is $\left(0, \frac{1}{2}\right)$.

12. Align the telephone pole along the x-axis as shown in the accompanying figure. The slope of the top length of pole is $\frac{\left(\frac{14.5}{8\pi} - \frac{9}{8\pi}\right)}{40} = \frac{1}{8\pi} \cdot \frac{1}{40} \cdot (14.5 - 9) = \frac{5.5}{8\pi\cdot 40} = \frac{11}{8\pi\cdot 80}$. Thus, $y = \frac{9}{8\pi} + \frac{11}{8\pi\cdot 80}x = \frac{1}{8\pi}\left(9 + \frac{11}{80}x\right)$ is an equation of the line representing the top of the pole. Then,



$M_y = \int_a^b x \cdot \pi y^2\,dx = \pi \int_0^{40} x\left[\frac{1}{8\pi}\left(9 + \frac{11}{80}x\right)\right]^2 dx$ $= \frac{1}{64\pi}\int_0^{40} x\left(9 + \frac{11}{80}x\right)^2 dx$; $M = \int_a^b \pi y^2\,dx$ $= \pi\int_0^{40}\left[\frac{1}{8\pi}\left(9 + \frac{11}{80}x\right)\right]^2 dx = \frac{1}{64\pi}\int_0^{40}\left(9 + \frac{11}{80}x\right)^2 dx$. Thus, $\bar{x} = \frac{M_y}{M} \approx \frac{129{,}700}{5623.3} \approx 23.06$ (using a calculator to compute the integrals). By symmetry about the x-axis, $\bar{y} = 0$ so the center of mass is about 23 ft from the top of the pole.

13. (a) Consider a single vertical strip with center of mass $(\tilde{x}, \tilde{y})$. If the plate lies to the right of the line, then the moment of this strip about the line x = b is $(\tilde{x} - b)\,dm = (\tilde{x} - b)\,\delta\,dA \Rightarrow$ the plate's first moment about x = b is the integral $\int (x - b)\delta\,dA = \int \delta x\,dA - \int \delta b\,dA = M_y - b\delta A$.
(b) If the plate lies to the left of the line, the moment of a vertical strip about the line x = b is $(b - \tilde{x})\,dm = (b - \tilde{x})\,\delta\,dA \Rightarrow$ the plate's first moment about x = b is $\int (b - x)\delta\,dA = \int b\delta\,dA - \int \delta x\,dA$ $= b\delta A - M_y$.

14. (a) By symmetry of the plate about the x-axis, $\bar{y} = 0$. A typical vertical strip has center of mass: $(\tilde{x}, \tilde{y}) = (x, 0)$, length: $4\sqrt{ax}$, width: dx, area: $4\sqrt{ax}\,dx$, mass: $dm = \delta\,dA = kx \cdot 4\sqrt{ax}\,dx$, for some proportionality constant k. The moment of the strip about the y-axis is $M_y = \int \tilde{x}\,dm = \int_0^a 4kx^2\sqrt{ax}\,dx$ $= 4k\sqrt{a}\int_0^a x^{5/2}\,dx = 4k\sqrt{a}\left[\frac{2}{7}x^{7/2}\right]_0^a = 4ka^{1/2} \cdot \frac{2}{7}a^{7/2} = \frac{8ka^4}{7}$. Also, $M = \int dm = \int_0^a 4kx\sqrt{ax}\,dx$ $= 4k\sqrt{a}\int_0^a x^{3/2}\,dx = 4k\sqrt{a}\left[\frac{2}{5}x^{5/2}\right]_0^a = 4ka^{1/2} \cdot \frac{2}{5}a^{5/2} = \frac{8ka^3}{5}$. Thus, $\bar{x} = \frac{M_y}{M} = \frac{8ka^4}{7} \cdot \frac{5}{8ka^3} = \frac{5}{7}a$ $\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{5a}{7}, 0\right)$ is the center of mass.
(b) A typical horizontal strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{\frac{y^2}{4a} + a}{2}, y\right) = \left(\frac{y^2 + 4a^2}{8a}, y\right)$, length: $a - \frac{y^2}{4a}$, width: dy, area: $\left(a - \frac{y^2}{4a}\right)dy$, mass: $dm = \delta\,dA = |y|\left(a - \frac{y^2}{4a}\right)dy$. Thus, $M_x = \int \tilde{y}\,dm$ $= \int_{-2a}^{2a} y\,|y|\left(a - \frac{y^2}{4a}\right)dy = \int_{-2a}^{0} -y^2\left(a - \frac{y^2}{4a}\right)dy + \int_0^{2a} y^2\left(a - \frac{y^2}{4a}\right)dy$

$= \int_{-2a}^{0}\left(-ay^2 + \frac{y^4}{4a}\right) dy + \int_{0}^{2a}\left(ay^2 - \frac{y^4}{4a}\right) dy = \left[-\frac{a}{3}y^3 + \frac{y^5}{20a}\right]_{-2a}^{0} + \left[\frac{a}{3}y^3 - \frac{y^5}{20a}\right]_{0}^{2a}$

$= -\frac{8a^4}{3} + \frac{32a^5}{20a} + \frac{8a^4}{3} - \frac{32a^5}{20a} = 0$; $M_y = \int \widetilde{x}\, dm = \int_{-2a}^{2a}\left(\frac{y^2+4a^2}{8a}\right)|y|\left(a - \frac{y^2}{4a}\right) dy$

$= \frac{1}{8a}\int_{-2a}^{2a}|y|\,(y^2+4a^2)\left(\frac{4a^2-y^2}{4a}\right) dy = \frac{1}{32a^2}\int_{-2a}^{2a}|y|\,(16a^4 - y^4)\, dy$

$= \frac{1}{32a^2}\int_{-2a}^{0}(-16a^4y + y^5)\, dy + \frac{1}{32a^2}\int_{0}^{2a}(16a^4y - y^5)\, dy = \frac{1}{32a^2}\left[-8a^4y^2 + \frac{y^6}{6}\right]_{-2a}^{0} + \frac{1}{32a^2}\left[8a^4y^2 - \frac{y^6}{6}\right]_{0}^{2a}$

$= \frac{1}{32a^2}\left[8a^4 \cdot 4a^2 - \frac{64a^6}{6}\right] + \frac{1}{32a^2}\left[8a^4 \cdot 4a^2 - \frac{64a^6}{6}\right] = \frac{1}{16a^2}\left(32a^6 - \frac{32a^6}{3}\right) = \frac{1}{16a^2}\cdot\frac{2}{3}(32a^6) = \frac{4}{3}a^4$;

$M = \int dm = \int_{-2a}^{2a}|y|\left(\frac{4a^2-y^2}{4a}\right) dy = \frac{1}{4a}\int_{-2a}^{2a}|y|\,(4a^2 - y^2)\, dy$

$= \frac{1}{4a}\int_{-2a}^{0}(-4a^2y + y^3)\, dy + \frac{1}{4a}\int_{0}^{2a}(4a^2y - y^3)\, dy = \frac{1}{4a}\left[-2a^2y^2 + \frac{y^4}{4}\right]_{-2a}^{0} + \frac{1}{4a}\left[2a^2y^2 - \frac{y^4}{4}\right]_{0}^{2a}$

$= 2 \cdot \frac{1}{4a}\left(2a^2 \cdot 4a^2 - \frac{16a^4}{4}\right) = \frac{1}{2a}(8a^4 - 4a^4) = 2a^3$. Therefore, $\overline{x} = \frac{M_y}{M} = \left(\frac{4}{3}a^4\right)\left(\frac{1}{2a^3}\right) = \frac{2a}{3}$ and

$\overline{y} = \frac{M_x}{M} = 0$ is the center of mass.

15. (a) On $[0, a]$ a typical *vertical* strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{\sqrt{b^2-x^2}+\sqrt{a^2-x^2}}{2}\right)$,

length: $\sqrt{b^2-x^2} - \sqrt{a^2-x^2}$, width: dx, area: $dA = \left(\sqrt{b^2-x^2} - \sqrt{a^2-x^2}\right) dx$, mass: $dm = \delta\, dA$

$= \delta\left(\sqrt{b^2-x^2} - \sqrt{a^2-x^2}\right) dx$. On $[a, b]$ a typical *vertical* strip has center of mass:

$(\widetilde{x}, \widetilde{y}) = \left(x, \frac{\sqrt{b^2-x^2}}{2}\right)$, length: $\sqrt{b^2-x^2}$, width: dx, area: $dA = \sqrt{b^2-x^2}\, dx$,

mass: $dm = \delta\, dA = \delta\sqrt{b^2-x^2}\, dx$. Thus, $M_x = \int \widetilde{y}\, dm$

$= \int_0^a \frac{1}{2}\left(\sqrt{b^2-x^2} + \sqrt{a^2-x^2}\right)\delta\left(\sqrt{b^2-x^2} - \sqrt{a^2-x^2}\right) dx + \int_a^b \frac{1}{2}\sqrt{b^2-x^2}\,\delta\sqrt{b^2-x^2}\, dx$

$= \frac{\delta}{2}\int_0^a [(b^2-x^2) - (a^2-x^2)]\, dx + \frac{\delta}{2}\int_a^b (b^2-x^2)\, dx = \frac{\delta}{2}\int_0^a (b^2-a^2)\, dx + \frac{\delta}{2}\int_a^b (b^2-x^2)\, dx$

$= \frac{\delta}{2}\left[(b^2-a^2)\,x\right]_0^a + \frac{\delta}{2}\left[b^2x - \frac{x^3}{3}\right]_a^b = \frac{\delta}{2}\left[(b^2-a^2)\,a\right] + \frac{\delta}{2}\left[\left(b^3 - \frac{b^3}{3}\right) - \left(b^2a - \frac{a^3}{3}\right)\right]$

$= \frac{\delta}{2}(ab^2 - a^3) + \frac{\delta}{2}\left(\frac{2}{3}b^3 - ab^2 + \frac{a^3}{3}\right) = \frac{\delta b^3}{3} - \frac{\delta a^3}{3} = \delta\left(\frac{b^3-a^3}{3}\right)$; $M_y = \int \widetilde{x}\, dm$

$= \int_0^a x\delta\left(\sqrt{b^2-x^2} - \sqrt{a^2-x^2}\right) dx + \int_a^b x\delta\sqrt{b^2-x^2}\, dx$

$= \delta\int_0^a x\,(b^2-x^2)^{1/2}\, dx - \delta\int_0^a x\,(a^2-x^2)^{1/2}\, dx + \delta\int_a^b x\,(b^2-x^2)^{1/2}\, dx$

$= \frac{-\delta}{2}\left[\frac{2(b^2-x^2)^{3/2}}{3}\right]_0^a + \frac{\delta}{2}\left[\frac{2(a^2-x^2)^{3/2}}{3}\right]_0^a - \frac{\delta}{2}\left[\frac{2(b^2-x^2)^{3/2}}{3}\right]_a^b$

$= -\frac{\delta}{3}\left[(b^2-a^2)^{3/2} - (b^2)^{3/2}\right] + \frac{\delta}{3}\left[0 - (a^2)^{3/2}\right] - \frac{\delta}{3}\left[0 - (b^2-a^2)^{3/2}\right] = \frac{\delta b^3}{3} - \frac{\delta a^3}{3} = \frac{\delta(b^3-a^3)}{3} = M_x$;

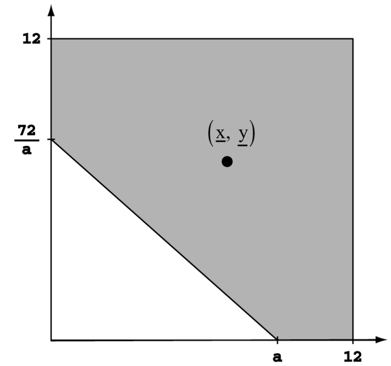
We calculate the mass geometrically: $M = \delta A = \delta\left(\frac{\pi b^2}{4}\right) - \delta\left(\frac{\pi a^2}{4}\right) = \frac{\delta\pi}{4}(b^2-a^2)$. Thus, $\overline{x} = \frac{M_y}{M}$

$= \frac{\delta(b^3-a^3)}{3}\cdot\frac{4}{\delta\pi(b^2-a^2)} = \frac{4}{3\pi}\left(\frac{b^3-a^3}{b^2-a^2}\right) = \frac{4}{3\pi}\,\frac{(b-a)(a^2+ab+b^2)}{(b-a)(b+a)} = \frac{4(a^2+ab+b^2)}{3\pi(a+b)}$; likewise

$\overline{y} = \frac{M_x}{M} = \frac{4(a^2+ab+b^2)}{3\pi(a+b)}$.

(b) $\lim_{b \to a} \frac{4}{3\pi}\left(\frac{a^2+ab+b^2}{a+b}\right) = \left(\frac{4}{3\pi}\right)\left(\frac{a^2+a^2+a^2}{a+a}\right) = \left(\frac{4}{3\pi}\right)\left(\frac{3a^2}{2a}\right) = \frac{2a}{\pi} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{2a}{\pi}, \frac{2a}{\pi}\right)$ is the limiting position of the centroid as $b \to a$. This is the centroid of a circle of radius a (and we note the two circles coincide when $b = a$).

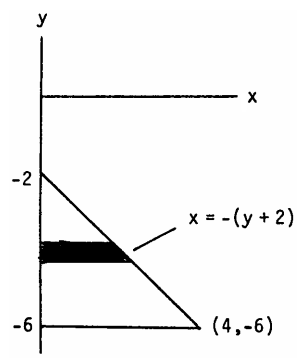
16. Since the area of the traingle is 36, the diagram may be labeled as shown at the right. The centroid of the triangle is $\left(\frac{a}{3}, \frac{24}{a}\right)$. The shaded portion is $144 - 36 = 108$. Write $(\underline{x}, \underline{y})$ for the centroid of the remaining region. The centroid of the whole square is obviously $(6, 6)$. Think of the square as a sheet of uniform density, so that the centroid of the square is the average of the centroids of the two regions, weighted by area:



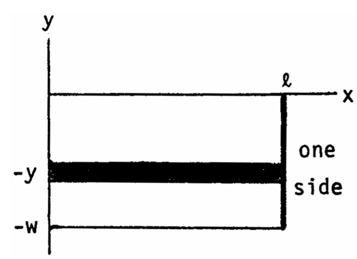
$6 = \frac{36\left(\frac{a}{3}\right) + 108(\underline{x})}{144}$ and $6 = \frac{36\left(\frac{24}{a}\right) + 108(\underline{y})}{144}$

which we solve to get $\underline{x} = 8 - \frac{a}{9}$ and $\underline{y} = \frac{8(a-1)}{a}$. Set $\underline{x} = 7$ in. (Given). It follows that $a = 9$, whence $\underline{y} = \frac{64}{9}$ $= 7\frac{1}{9}$ in. The distances of the centroid $(\underline{x}, \underline{y})$ from the other sides are easily computed. (Note that if we set $\underline{y} = 7$ in. above, we will find $\underline{x} = 7\frac{1}{9}$.)

17. The submerged triangular plate is depicted in the figure at the right. The hypotenuse of the triangle has slope -1 $\Rightarrow\ y - (-2) = -(x - 0)\ \Rightarrow\ x = -(y + 2)$ is an equation of the hypotenuse. Using a typical horizontal strip, the fluid pressure is $F = \int (62.4) \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{length}\end{smallmatrix}\right) dy$
$= \int_{-6}^{-2} (62.4)(-y)[-(y+2)]\, dy = 62.4 \int_{-6}^{-2} (y^2 + 2y)\, dy$
$= 62.4 \left[\frac{y^3}{3} + y^2\right]_{-6}^{-2} = (62.4)\left[\left(-\frac{8}{3} + 4\right) - \left(-\frac{216}{3} + 36\right)\right]$
$= (62.4)\left(\frac{208}{3} - 32\right) = \frac{(62.4)(112)}{3} \approx 2329.6$ lb



18. Consider a rectangular plate of length ℓ and width w. The length is parallel with the surface of the fluid of weight density ω. The force on one side of the plate is $F = \omega \int_{-w}^{0} (-y)(\ell)\, dy = -\omega\ell \left[\frac{y^2}{2}\right]_{-w}^{0} = \frac{\omega \ell w^2}{2}$. The average force on one side of the plate is $F_{av} = \frac{\omega}{w} \int_{-w}^{0} (-y) dy$
$= \frac{\omega}{w}\left[-\frac{y^2}{2}\right]_{-w}^{0} = \frac{\omega w}{2}$. Therefore the force $\frac{\omega \ell w^2}{2}$
$= \left(\frac{\omega w}{2}\right)(\ell w) =$ (the average pressure up and down) $\cdot$ (the area of the plate).



NOTES:

CHAPTER 7 TRANSCENDENTAL FUNCTIONS

7.1 INVERSE FUNCTIONS AND THEIR DERIVATIVES

1. Yes one-to-one, the graph passes the horizontal line test.

2. Not one-to-one, the graph fails the horizontal line test.

3. Not one-to-one since (for example) the horizontal line $y = 2$ intersects the graph twice.

4. Not one-to-one, the graph fails the horizontal line test.

5. Yes one-to-one, the graph passes the horizontal line test

6. Yes one-to-one, the graph passes the horizontal line test

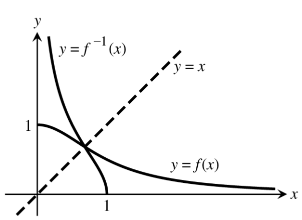
7. Not one-to-one since the horizontal line $y = 3$ intersects the graph an infinite number of times.

8. Yes one-to-one, the graph passes the horizontal line test

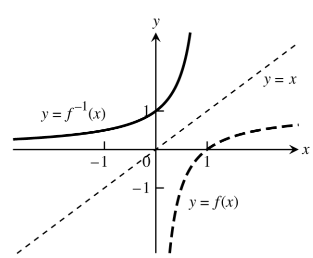
9. Yes one-to-one, the graph passes the horizontal line test

10. Not one-to-one since (for example) the horizontal line $y = 1$ intersects the graph twice.

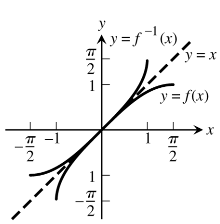
11. Domain: $0 < x \le 1$, Range: $0 \le y$



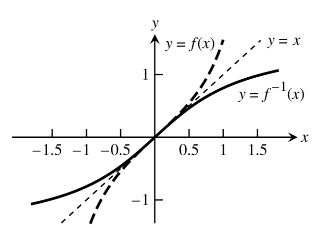
12. Domain: $x < 1$, Range: $y > 0$



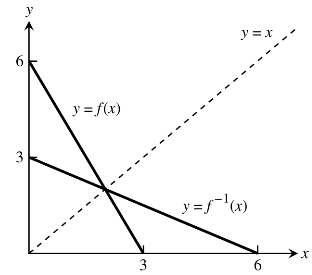
13. Domain: $-1 \le x \le 1$, Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



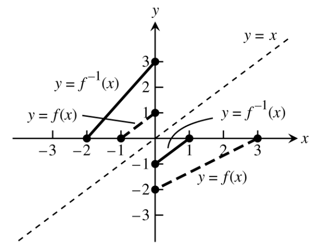
14. Domain: $-\infty < x < \infty$, Range: $-\frac{\pi}{2} < y \le \frac{\pi}{2}$



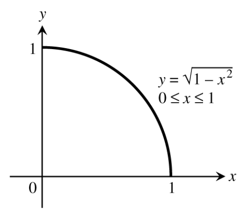
15. Domain: $0 \le x \le 6$, Range: $0 \le y \le 3$



16. Domain: $-2 \le x \le 1$, Range: $-1 \le y < 3$

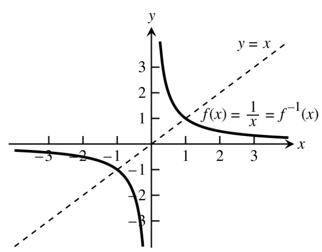


17. The graph is symmetric about $y = x$.



(b) $y = \sqrt{1 - x^2} \Rightarrow y^2 = 1 - x^2 \Rightarrow x^2 = 1 - y^2 \Rightarrow x = \sqrt{1 - y^2} \Rightarrow y = \sqrt{1 - x^2} = f^{-1}(x)$

18. The graph is symmetric about $y = x$.



$y = \frac{1}{x} \Rightarrow x = \frac{1}{y} \Rightarrow y = \frac{1}{x} = f^{-1}(x)$

19. Step 1: $y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \sqrt{y - 1}$
Step 2: $y = \sqrt{x - 1} = f^{-1}(x)$

20. Step 1: $y = x^2 \Rightarrow x = -\sqrt{y}$, since $x \le 0$.
Step 2: $y = -\sqrt{x} = f^{-1}(x)$

21. Step 1: $y = x^3 - 1 \Rightarrow x^3 = y + 1 \Rightarrow x = (y + 1)^{1/3}$
Step 2: $y = \sqrt[3]{x + 1} = f^{-1}(x)$

22. Step 1: $y = x^2 - 2x + 1 \Rightarrow y = (x - 1)^2 \Rightarrow \sqrt{y} = x - 1$, since $x \ge 1 \Rightarrow x = 1 + \sqrt{y}$
Step 2: $y = 1 + \sqrt{x} = f^{-1}(x)$

23. Step 1: $y = (x + 1)^2 \Rightarrow \sqrt{y} = x + 1$, since $x \ge -1 \Rightarrow x = \sqrt{y} - 1$
Step 2: $y = \sqrt{x} - 1 = f^{-1}(x)$

24. Step 1: $y = x^{2/3} \Rightarrow x = y^{3/2}$
Step 2: $y = x^{3/2} = f^{-1}(x)$

25. Step 1: $y = x^5 \Rightarrow x = y^{1/5}$
Step 2: $y = \sqrt[5]{x} = f^{-1}(x)$;
Domain and Range of f^{-1}: all reals;
$f(f^{-1}(x)) = \left(x^{1/5}\right)^5 = x$ and $f^{-1}(f(x)) = \left(x^5\right)^{1/5} = x$

26. Step 1: $y = x^4 \Rightarrow x = y^{1/4}$
Step 2: $y = \sqrt[4]{x} = f^{-1}(x)$;
Domain of f^{-1}: $x \ge 0$, Range of f^{-1}: $y \ge 0$;
$f(f^{-1}(x)) = \left(x^{1/4}\right)^4 = x$ and $f^{-1}(f(x)) = \left(x^4\right)^{1/4} = x$

27. Step 1: $y = x^3 + 1 \Rightarrow x^3 = y - 1 \Rightarrow x = (y-1)^{1/3}$
Step 2: $y = \sqrt[3]{x-1} = f^{-1}(x)$;
Domain and Range of f^{-1}: all reals;
$f(f^{-1}(x)) = \left((x-1)^{1/3}\right)^3 + 1 = (x-1) + 1 = x$ and $f^{-1}(f(x)) = \left((x^3+1) - 1\right)^{1/3} = \left(x^3\right)^{1/3} = x$

28. Step 1: $y = \frac{1}{2}x - \frac{7}{2} \Rightarrow \frac{1}{2}x = y + \frac{7}{2} \Rightarrow x = 2y + 7$
Step 2: $y = 2x + 7 = f^{-1}(x)$;
Domain and Range of f^{-1}: all reals;
$f(f^{-1}(x)) = \frac{1}{2}(2x+7) - \frac{7}{2} = \left(x + \frac{7}{2}\right) - \frac{7}{2} = x$ and $f^{-1}(f(x)) = 2\left(\frac{1}{2}x - \frac{7}{2}\right) + 7 = (x-7) + 7 = x$

29. Step 1: $y = \frac{1}{x^2} \Rightarrow x^2 = \frac{1}{y} \Rightarrow x = \frac{1}{\sqrt{y}}$
Step 2: $y = \frac{1}{\sqrt{x}} = f^{-1}(x)$
Domain of f^{-1}: $x > 0$, Range of f^{-1}: $y > 0$;
$f(f^{-1}(x)) = \frac{1}{\left(\frac{1}{\sqrt{x}}\right)^2} = \frac{1}{\left(\frac{1}{x}\right)} = x$ and $f^{-1}(f(x)) = \frac{1}{\sqrt{\frac{1}{x^2}}} = \frac{1}{\left(\frac{1}{x}\right)} = x$ since $x > 0$

30. Step 1: $y = \frac{1}{x^3} \Rightarrow x^3 = \frac{1}{y} \Rightarrow x = \frac{1}{y^{1/3}}$
Step 2: $y = \frac{1}{x^{1/3}} = \sqrt[3]{\frac{1}{x}} = f^{-1}(x)$;
Domain of f^{-1}: $x \ne 0$, Range of f^{-1}: $y \ne 0$;
$f(f^{-1}(x)) = \frac{1}{\left(x^{-1/3}\right)^3} = \frac{1}{x^{-1}} = x$ and $f^{-1}(f(x)) = \left(\frac{1}{x^3}\right)^{-1/3} = \left(\frac{1}{x}\right)^{-1} = x$

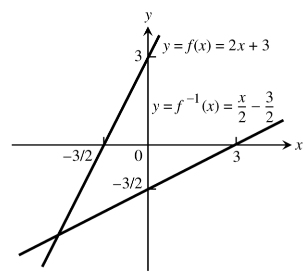
31. Step 1: $y = \frac{x+3}{x-2} \Rightarrow y(x-2) = x + 3 \Rightarrow xy - 2y = x + 3 \Rightarrow xy - x = 2y + 3 \Rightarrow x = \frac{2y+3}{y-1}$
Step 2: $y = \frac{2x+3}{x-1} = f^{-1}(x)$;
Domain of f^{-1}: $x \ne 1$, Range of f^{-1}: $y \ne 2$;
$f(f^{-1}(x)) = \frac{\left(\frac{2x+3}{x-1}\right) + 3}{\left(\frac{2x+3}{x-1}\right) - 2} = \frac{(2x+3) + 3(x-1)}{(2x+3) - 2(x-1)} = \frac{5x}{5} = x$ and $f^{-1}(f(x)) = \frac{2\left(\frac{x+3}{x-2}\right) + 3}{\left(\frac{x+3}{x-2}\right) - 1} = \frac{2(x+3) + 3(x-2)}{(x+3) - (x-2)} = \frac{5x}{5} = x$

32. Step 1: $y = \frac{\sqrt{x}}{\sqrt{x}-3} \Rightarrow y\left(\sqrt{x} - 3\right) = \sqrt{x} \Rightarrow y\sqrt{x} - 3y = \sqrt{x} \Rightarrow y\sqrt{x} - \sqrt{x} = 3y \Rightarrow x = \left(\frac{3y}{y-1}\right)^2$
Step 2: $y = \left(\frac{3x}{x-1}\right)^2 = f^{-1}(x)$;
Domain of f^{-1}: $(-\infty, 0] \cup (1, \infty)$, Range of f^{-1}: $[0, 9) \cup (9, \infty)$;
$f(f^{-1}(x)) = \frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2} - 3}$; If $x > 1$ or $x \le 0 \Rightarrow \frac{3x}{x-1} \ge 0 \Rightarrow \frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2} - 3} = \frac{\frac{3x}{x-1}}{\frac{3x}{x-1} - 3} = \frac{3x}{3x - 3(x-1)} = \frac{3x}{3} = x$ and
$f^{-1}(f(x)) = \left(\frac{3\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right)}{\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right) - 1}\right)^2 = \frac{9x}{\left(\sqrt{x} - \left(\sqrt{x} - 3\right)\right)^2} = \frac{9x}{9} = x$

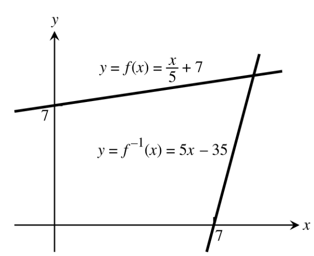
33. Step 1: $y = x^2 - 2x, x \le 1 \Rightarrow y + 1 = (x-1)^2, x \le 1 \Rightarrow -\sqrt{y+1} = x - 1, x \le 1 \Rightarrow x = 1 - \sqrt{y+1}$
Step 2: $y = 1 - \sqrt{x+1} = f^{-1}(x)$;
Domain of f^{-1}: $[-1, \infty)$, Range of f^{-1}: $(-\infty, 1]$;
$f(f^{-1}(x)) = \left(1 - \sqrt{x+1}\right)^2 - 2\left(1 - \sqrt{x+1}\right) = 1 - 2\sqrt{x+1} + x + 1 - 2 + 2\sqrt{x+1} = x$ and
$f^{-1}(f(x)) = 1 - \sqrt{(x^2 - 2x) + 1}, x \le 1 = 1 - \sqrt{(x-1)^2}, x \le 1 = 1 - |x-1| = 1 - (1-x) = x$

34. Step 1: $y = (2x^3 + 1)^{1/5} \Rightarrow y^5 = 2x^3 + 1 \Rightarrow y^5 - 1 = 2x^3 \Rightarrow \frac{y^5 - 1}{2} = x^3 \Rightarrow x = \sqrt[3]{\frac{y^5 - 1}{2}}$
Step 2: $y = \sqrt[3]{\frac{x^5 - 1}{2}} = f^{-1}(x)$;
Domain of f^{-1}: $(-\infty, \infty)$, Range of f^{-1}: $(-\infty, \infty)$;
$f(f^{-1}(x)) = \left(2\left(\sqrt[3]{\frac{x^5-1}{2}}\right)^3 + 1\right)^{1/5} = \left(2\left(\frac{x^5-1}{2}\right) + 1\right)^{1/5} = ((x^5 - 1) + 1)^{1/5} = (x^5)^{1/5} = x$ and
$f^{-1}(f(x)) = \sqrt[3]{\frac{\left[(2x^3+1)^{1/5}\right]^5 - 1}{2}} = \sqrt[3]{\frac{(2x^3+1) - 1}{2}} = \sqrt[3]{\frac{2x^3}{2}} = x$

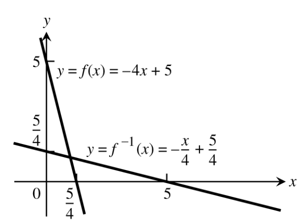
35. (a) $y = 2x + 3 \Rightarrow 2x = y - 3$
$\Rightarrow x = \frac{y}{2} - \frac{3}{2} \Rightarrow f^{-1}(x) = \frac{x}{2} - \frac{3}{2}$
(c) $\left.\frac{df}{dx}\right|_{x=-1} = 2, \left.\frac{df^{-1}}{dx}\right|_{x=1} = \frac{1}{2}$

(b)


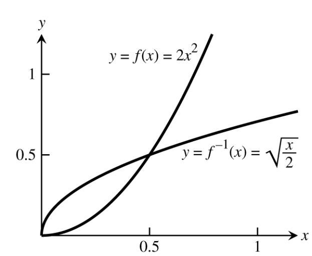
36. (a) $y = \frac{1}{5}x + 7 \Rightarrow \frac{1}{5}x = y - 7$
$\Rightarrow x = 5y - 35 \Rightarrow f^{-1}(x) = 5x - 35$
(c) $\left.\frac{df}{dx}\right|_{x=-1} = \frac{1}{5}, \left.\frac{df^{-1}}{dx}\right|_{x=34/5} = 5$

(b)


37. (a) $y = 5 - 4x \Rightarrow 4x = 5 - y$
$\Rightarrow x = \frac{5}{4} - \frac{y}{4} \Rightarrow f^{-1}(x) = \frac{5}{4} - \frac{x}{4}$
(c) $\left.\frac{df}{dx}\right|_{x=1/2} = -4, \left.\frac{df^{-1}}{dx}\right|_{x=3} = -\frac{1}{4}$

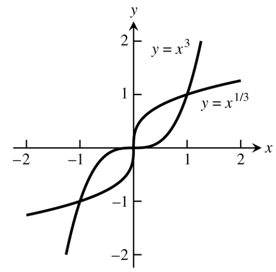
(b)


38. (a) $y = 2x^2 \Rightarrow x^2 = \frac{1}{2}y$
$\Rightarrow x = \frac{1}{\sqrt{2}}\sqrt{y} \Rightarrow f^{-1}(x) = \sqrt{\frac{x}{2}}$
(c) $\left.\frac{df}{dx}\right|_{x=5} = 4x|_{x=5} = 20$,
$\left.\frac{df^{-1}}{dx}\right|_{x=50} = \left.\frac{1}{2\sqrt{2}}x^{-1/2}\right|_{x=50} = \frac{1}{20}$

(b)


39. (a) $f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x$, $g(f(x)) = \sqrt[3]{x^3} = x$

(b)

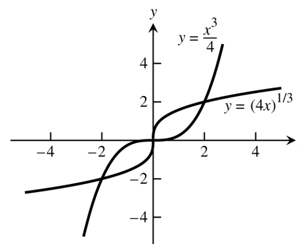


(c) $f'(x) = 3x^2 \Rightarrow f'(1) = 3, f'(-1) = 3$;
$g'(x) = \frac{1}{3}x^{-2/3} \Rightarrow g'(1) = \frac{1}{3}, g'(-1) = \frac{1}{3}$

(d) The line $y = 0$ is tangent to $f(x) = x^3$ at $(0, 0)$;
the line $x = 0$ is tangent to $g(x) = \sqrt[3]{x}$ at $(0, 0)$

40. (a) $h(k(x)) = \frac{1}{4}\left((4x)^{1/3}\right)^3 = x$,
$k(h(x)) = \left(4 \cdot \frac{x^3}{4}\right)^{1/3} = x$

(b)



(c) $h'(x) = \frac{3x^2}{4} \Rightarrow h'(2) = 3, h'(-2) = 3$;
$k'(x) = \frac{4}{3}(4x)^{-2/3} \Rightarrow k'(2) = \frac{1}{3}, k'(-2) = \frac{1}{3}$

(d) The line $y = 0$ is tangent to $h(x) = \frac{x^3}{4}$ at $(0, 0)$;
the line $x = 0$ is tangent to $k(x) = (4x)^{1/3}$ at $(0, 0)$

41. $\frac{df}{dx} = 3x^2 - 6x \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=f(3)} = \left.\frac{1}{\frac{df}{dx}}\right|_{x=3} = \frac{1}{9}$

42. $\frac{df}{dx} = 2x - 4 \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=f(5)} = \left.\frac{1}{\frac{df}{dx}}\right|_{x=5} = \frac{1}{6}$

43. $\left.\frac{df^{-1}}{dx}\right|_{x=4} = \left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \left.\frac{1}{\frac{df}{dx}}\right|_{x=2} = \frac{1}{\left(\frac{1}{3}\right)} = 3$

44. $\left.\frac{dg^{-1}}{dx}\right|_{x=0} = \left.\frac{dg^{-1}}{dx}\right|_{x=f(0)} = \left.\frac{1}{\frac{dg}{dx}}\right|_{x=0} = \frac{1}{2}$

45. (a) $y = mx \Rightarrow x = \frac{1}{m}y \Rightarrow f^{-1}(x) = \frac{1}{m}x$

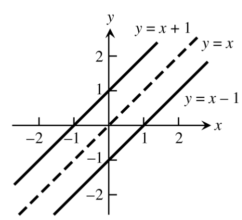
(b) The graph of $y = f^{-1}(x)$ is a line through the origin with slope $\frac{1}{m}$.

46. $y = mx + b \Rightarrow x = \frac{y}{m} - \frac{b}{m} \Rightarrow f^{-1}(x) = \frac{1}{m}x - \frac{b}{m}$; the graph of $f^{-1}(x)$ is a line with slope $\frac{1}{m}$ and y-intercept $-\frac{b}{m}$.

47. (a) $y = x + 1 \Rightarrow x = y - 1 \Rightarrow f^{-1}(x) = x - 1$

(b) $y = x + b \Rightarrow x = y - b \Rightarrow f^{-1}(x) = x - b$

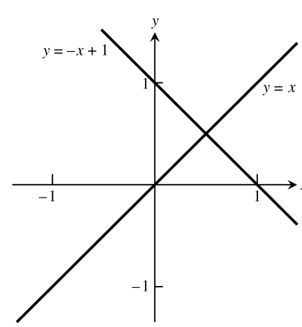
(c) Their graphs will be parallel to one another and lie on opposite sides of the line $y = x$ equidistant from that line.



48. (a) $y = -x + 1 \Rightarrow x = -y + 1 \Rightarrow f^{-1}(x) = 1 - x$;
the lines intersect at a right angle

(b) $y = -x + b \Rightarrow x = -y + b \Rightarrow f^{-1}(x) = b - x$;
the lines intersect at a right angle

(c) Such a function is its own inverse.



49. Let $x_1 \neq x_2$ be two numbers in the domain of an increasing function f. Then, either $x_1 < x_2$ or $x_1 > x_2$ which implies $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, since f(x) is increasing. In either case, $f(x_1) \neq f(x_2)$ and f is one-to-one. Similar arguments hold if f is decreasing.

50. f(x) is increasing since $x_2 > x_1 \Rightarrow \frac{1}{3}x_2 + \frac{5}{6} > \frac{1}{3}x_1 + \frac{5}{6}$; $\frac{df}{dx} = \frac{1}{3} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\left(\frac{1}{3}\right)} = 3$

51. f(x) is increasing since $x_2 > x_1 \Rightarrow 27x_2^3 > 27x_1^3$; $y = 27x^3 \Rightarrow x = \frac{1}{3}y^{1/3} \Rightarrow f^{-1}(x) = \frac{1}{3}x^{1/3}$;
$\frac{df}{dx} = 81x^2 \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{81x^2}\Big|_{\frac{1}{3}x^{1/3}} = \frac{1}{9x^{2/3}} = \frac{1}{9}x^{-2/3}$

52. f(x) is decreasing since $x_2 > x_1 \Rightarrow 1 - 8x_2^3 < 1 - 8x_1^3$; $y = 1 - 8x^3 \Rightarrow x = \frac{1}{2}(1-y)^{1/3} \Rightarrow f^{-1}(x) = \frac{1}{2}(1-x)^{1/3}$;
$\frac{df}{dx} = -24x^2 \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{-24x^2}\Big|_{\frac{1}{2}(1-x)^{1/3}} = \frac{-1}{6(1-x)^{2/3}} = -\frac{1}{6}(1-x)^{-2/3}$

53. f(x) is decreasing since $x_2 > x_1 \Rightarrow (1-x_2)^3 < (1-x_1)^3$; $y = (1-x)^3 \Rightarrow x = 1 - y^{1/3} \Rightarrow f^{-1}(x) = 1 - x^{1/3}$;
$\frac{df}{dx} = -3(1-x)^2 \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{-3(1-x)^2}\Big|_{1-x^{1/3}} = \frac{-1}{3x^{2/3}} = -\frac{1}{3}x^{-2/3}$

54. f(x) is increasing since $x_2 > x_1 \Rightarrow x_2^{5/3} > x_1^{5/3}$; $y = x^{5/3} \Rightarrow x = y^{3/5} \Rightarrow f^{-1}(x) = x^{3/5}$;
$\frac{df}{dx} = \frac{5}{3}x^{2/3} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\frac{5}{3}x^{2/3}}\Big|_{x^{3/5}} = \frac{3}{5x^{2/5}} = \frac{3}{5}x^{-2/5}$

55. The function g(x) is also one-to-one. The reasoning: f(x) is one-to-one means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, so $-f(x_1) \neq -f(x_2)$ and therefore $g(x_1) \neq g(x_2)$. Therefore g(x) is one-to-one as well.

56. The function h(x) is also one-to-one. The reasoning: f(x) is one-to-one means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, so $\frac{1}{f(x_1)} \neq \frac{1}{f(x_2)}$, and therefore $h(x_1) \neq h(x_2)$.

57. The composite is one-to-one also. The reasoning: If $x_1 \neq x_2$ then $g(x_1) \neq g(x_2)$ because g is one-to-one. Since $g(x_1) \neq g(x_2)$, we also have $f(g(x_1)) \neq f(g(x_2))$ because f is one-to-one. Thus, $f \circ g$ is one-to-one because $x_1 \neq x_2 \Rightarrow f(g(x_1)) \neq f(g(x_2))$.

58. Yes, g must be one-to-one. If g were not one-to-one, there would exist numbers $x_1 \neq x_2$ in the domain of g with $g(x_1) = g(x_2)$. For these numbers we would also have $f(g(x_1)) = f(g(x_2))$, contradicting the assumption that $f \circ g$ is one-to-one.

59. $(g \circ f)(x) = x \Rightarrow g(f(x)) = x \Rightarrow g'(f(x))f'(x) = 1$

60. $W(a) = \int_{f(a)}^{f(a)} \pi\left[\left(f^{-1}(y)\right)^2 - a^2\right] dy = 0 = \int_a^a 2\pi x[f(a) - f(x)]\, dx = S(a)$; $W'(t) = \pi\left[\left(f^{-1}(f(t))\right)^2 - a^2\right] f'(t)$
$= \pi\left(t^2 - a^2\right) f'(t)$; also $S(t) = 2\pi f(t)\int_a^t x\, dx - 2\pi\int_a^t xf(x)\, dx = \left[\pi f(t)t^2 - \pi f(t)a^2\right] - 2\pi\int_a^t xf(x)\, dx \Rightarrow S'(t)$
$= \pi t^2 f'(t) + 2\pi t f(t) - \pi a^2 f'(t) - 2\pi t f(t) = \pi\left(t^2 - a^2\right) f'(t) \Rightarrow W'(t) = S'(t)$. Therefore, $W(t) = S(t)$ for all $t \in [a, b]$.

61-68. Example CAS commands:

Maple:

```
with( plots );#63
f := x -> sqrt(3*x-2);
domain := 2/3 .. 4;
x0 := 3;
Df := D(f);                         # (a)
```

```
plot( [f(x),Df(x)], x=domain, color=[red,blue], linestyle=[1,3], legend=["y=f(x)","y=f '(x)"],
      title="#61(a) (Section 7.1)" );
q1 := solve( y=f(x), x );          # (b)
g := unapply( q1, y );
m1 := Df(x0);                      # (c)
t1 := f(x0)+m1*(x-x0);
y=t1;
m2 := 1/Df(x0);                    # (d)
t2 := g(f(x0)) + m2*(x-f(x0));
y=t2;
domaing := map(f,domain);     # (e)
p1 := plot( [f(x),x], x=domain, color=[pink,green], linestyle=[1,9], thickness=[3,0] ):
p2 := plot( g(x), x=domaing, color=cyan, linestyle=3, thickness=4 ):
p3 := plot( t1, x=x0-1..x0+1, color=red, linestyle=4, thickness=0 ):
p4 := plot( t2, x=f(x0)-1..f(x0)+1, color=blue, linestyle=7, thickness=1 ):
p5 := plot( [ [x0,f(x0)], [f(x0),x0] ], color=green ):
display( [p1,p2,p3,p4,p5], scaling=constrained, title="#63(e) (Section 7.1)" );
```

Mathematica: (assigned function and values for a, b, and x0 may vary)

If a function requires the odd root of a negative number, begin by loading the RealOnly package that allows Mathematica to do this. See section 2.5 for details.

```
<<Miscellaneous `RealOnly`
Clear[x, y]
{a,b} = {−2, 1}; x0 = 1/2 ;
f[x_] = (3x + 2) / (2x − 11)
Plot[{f[x], f'[x]}, {x, a, b}]
solx = Solve[y == f[x], x]
g[y_] = x /. solx[[1]]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x-x0)
gtan[y_] = x0 + 1/ f'[x0] (y − y0)
Plot[{f[x], ftan[x], g[x], gtan[x], Identity[x]},{x, a, b},
Epilog → Line[{{x0, y0},{y0, x0}}], PlotRange → {{a,b},{a,b}}, AspectRatio → Automatic]
```

69-70. Example CAS commands:

Maple:

```
with( plots );
eq := cos(y) = x^(1/5);
domain := 0 .. 1;
x0 := 1/2;
f := unapply( solve( eq, y ), x );   # (a)
Df := D(f);
plot( [f(x),Df(x)], x=domain, color=[red,blue], linestyle=[1,3], legend=["y=f(x)","y=f '(x)"],
      title="#70(a) (Section 7.1)" );
q1 := solve( eq, x );              # (b)
g := unapply( q1, y );
m1 := Df(x0);                      # (c)
t1 := f(x0)+m1*(x-x0);
y=t1;
m2 := 1/Df(x0);                    # (d)
```

```
t2 := g(f(x0)) + m2*(x-f(x0));
y=t2;
domaing := map(f,domain);          # (e)
p1 := plot( [f(x),x], x=domain, color=[pink,green], linestyle=[1,9], thickness=[3,0] ):
p2 := plot( g(x), x=domaing, color=cyan, linestyle=3, thickness=4 ):
p3 := plot( t1, x=x0-1..x0+1, color=red, linestyle=4, thickness=0 ):
p4 := plot( t2, x=f(x0)-1..f(x0)+1, color=blue, linestyle=7, thickness=1 ):
p5 := plot( [ [x0,f(x0)], [f(x0),x0] ], color=green ):
display( [p1,p2,p3,p4,p5], scaling=constrained, title="#70(e) (Section 7.1)" );
```

Mathematica: (assigned function and values for a, b, and x0 may vary)

For problems 69 and 70, the code is just slightly altered. At times, different "parts" of solutions need to be used, as in the definitions of f[x] and g[y]

```
Clear[x, y]
{a,b} = {0, 1}; x0 = 1/2 ;
eqn = Cos[y] == x^(1/5)
soly = Solve[eqn, y]
f[x_] = y /. soly[[2]]
Plot[{f[x], f'[x]}, {x, a, b}]
solx = Solve[eqn, x]
g[y_] = x /. solx[[1]]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x − x0)
gtan[y_] = x0 + 1/ f'[x0] (y − y0)
Plot[{f[x], ftan[x], g[x], gtan[x], Identity[x]},{x, a, b},
Epilog → Line[{{x0, y0},{y0, x0}}], PlotRange → {{a, b}, {a, b}}, AspectRatio → Automatic]
```

7.2 NATURAL LOGARITHMS

1. (a) $\ln 0.75 = \ln \frac{3}{4} = \ln 3 - \ln 4 = \ln 3 - \ln 2^2 = \ln 3 - 2\ln 2$
   (b) $\ln \frac{4}{9} = \ln 4 - \ln 9 = \ln 2^2 - \ln 3^2 = 2\ln 2 - 2\ln 3$
   (c) $\ln \frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$
   (d) $\ln \sqrt[3]{9} = \frac{1}{3}\ln 9 = \frac{1}{3}\ln 3^2 = \frac{2}{3}\ln 3$
   (e) $\ln 3\sqrt{2} = \ln 3 + \ln 2^{1/2} = \ln 3 + \frac{1}{2}\ln 2$
   (f) $\ln \sqrt{13.5} = \frac{1}{2}\ln 13.5 = \frac{1}{2}\ln \frac{27}{2} = \frac{1}{2}(\ln 3^3 - \ln 2) = \frac{1}{2}(3\ln 3 - \ln 2)$

2. (a) $\ln \frac{1}{125} = \ln 1 - 3\ln 5 = -3\ln 5$
   (b) $\ln 9.8 = \ln \frac{49}{5} = \ln 7^2 - \ln 5 = 2\ln 7 - \ln 5$
   (c) $\ln 7\sqrt{7} = \ln 7^{3/2} = \frac{3}{2}\ln 7$
   (d) $\ln 1225 = \ln 35^2 = 2\ln 35 = 2\ln 5 + 2\ln 7$
   (e) $\ln 0.056 = \ln \frac{7}{125} = \ln 7 - \ln 5^3 = \ln 7 - 3\ln 5$
   (f) $\frac{\ln 35 + \ln \frac{1}{7}}{\ln 25} = \frac{\ln 5 + \ln 7 - \ln 7}{2\ln 5} = \frac{1}{2}$

3. (a) $\ln \sin\theta - \ln\left(\frac{\sin\theta}{5}\right) = \ln\left(\frac{\sin\theta}{\left(\frac{\sin\theta}{5}\right)}\right) = \ln 5$
   (b) $\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right) = \ln\left(\frac{3x^2 - 9x}{3x}\right) = \ln(x - 3)$
   (c) $\frac{1}{2}\ln(4t^4) - \ln 2 = \ln\sqrt{4t^4} - \ln 2 = \ln 2t^2 - \ln 2 = \ln\left(\frac{2t^2}{2}\right) = \ln(t^2)$

4. (a) $\ln \sec\theta + \ln\cos\theta = \ln[(\sec\theta)(\cos\theta)] = \ln 1 = 0$
   (b) $\ln(8x + 4) - \ln 2^2 = \ln(8x + 4) - \ln 4 = \ln\left(\frac{8x+4}{4}\right) = \ln(2x + 1)$
   (c) $3\ln\sqrt[3]{t^2 - 1} - \ln(t + 1) = 3\ln(t^2 - 1)^{1/3} - \ln(t + 1) = 3\left(\frac{1}{3}\right)\ln(t^2 - 1) - \ln(t + 1) = \ln\left(\frac{(t+1)(t-1)}{(t+1)}\right)$
   $= \ln(t - 1)$

5. $y = \ln 3x \Rightarrow y' = \left(\frac{1}{3x}\right)(3) = \frac{1}{x}$

6. $y = \ln kx \Rightarrow y' = \left(\frac{1}{kx}\right)(k) = \frac{1}{x}$

7. $y = \ln(t^2) \Rightarrow \frac{dy}{dt} = \left(\frac{1}{t^2}\right)(2t) = \frac{2}{t}$

8. $y = \ln\left(t^{3/2}\right) \Rightarrow \frac{dy}{dt} = \left(\frac{1}{t^{3/2}}\right)\left(\frac{3}{2}t^{1/2}\right) = \frac{3}{2t}$

9. $y = \ln\frac{3}{x} = \ln 3x^{-1} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{3x^{-1}}\right)\left(-3x^{-2}\right) = -\frac{1}{x}$

10. $y = \ln\frac{10}{x} = \ln 10x^{-1} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{10x^{-1}}\right)\left(-10x^{-2}\right) = -\frac{1}{x}$

11. $y = \ln(\theta+1) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\theta+1}\right)(1) = \frac{1}{\theta+1}$

12. $y = \ln(2\theta+2) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{2\theta+2}\right)(2) = \frac{1}{\theta+1}$

13. $y = \ln x^3 \Rightarrow \frac{dy}{dx} = \left(\frac{1}{x^3}\right)(3x^2) = \frac{3}{x}$

14. $y = (\ln x)^3 \Rightarrow \frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{d}{dx}(\ln x) = \frac{3(\ln x)^2}{x}$

15. $y = t(\ln t)^2 \Rightarrow \frac{dy}{dt} = (\ln t)^2 + 2t(\ln t) \cdot \frac{d}{dt}(\ln t) = (\ln t)^2 + \frac{2t \ln t}{t} = (\ln t)^2 + 2\ln t$

16. $y = t\sqrt{\ln t} = t(\ln t)^{1/2} \Rightarrow \frac{dy}{dt} = (\ln t)^{1/2} + \frac{1}{2}t(\ln t)^{-1/2} \cdot \frac{d}{dt}(\ln t) = (\ln t)^{1/2} + \frac{t(\ln t)^{-1/2}}{2t}$
$= (\ln t)^{1/2} + \frac{1}{2(\ln t)^{1/2}}$

17. $y = \frac{x^4}{4}\ln x - \frac{x^4}{16} \Rightarrow \frac{dy}{dx} = x^3 \ln x + \frac{x^4}{4} \cdot \frac{1}{x} - \frac{4x^3}{16} = x^3 \ln x$

18. $y = \left(x^2 \ln x\right)^4 \Rightarrow \frac{dy}{dx} = 4\left(x^2 \ln x\right)^3\left(x^2 \cdot \frac{1}{x} + 2x\ln x\right) = 4x^6(\ln x)^3(x + 2x\ln x) = 4x^7(\ln x)^3 + 8x^7(\ln x)^4$

19. $y = \frac{\ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (\ln t)(1)}{t^2} = \frac{1 - \ln t}{t^2}$

20. $y = \frac{1 + \ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (1 + \ln t)(1)}{t^2} = \frac{1 - 1 - \ln t}{t^2} = -\frac{\ln t}{t^2}$

21. $y = \frac{\ln x}{1 + \ln x} \Rightarrow y' = \frac{(1 + \ln x)\left(\frac{1}{x}\right) - (\ln x)\left(\frac{1}{x}\right)}{(1 + \ln x)^2} = \frac{\frac{1}{x} + \frac{\ln x}{x} - \frac{\ln x}{x}}{(1 + \ln x)^2} = \frac{1}{x(1 + \ln x)^2}$

22. $y = \frac{x \ln x}{1 + \ln x} \Rightarrow y' = \frac{(1+\ln x)\left(\ln x + x \cdot \frac{1}{x}\right) - (x \ln x)\left(\frac{1}{x}\right)}{(1+\ln x)^2} = \frac{(1 + \ln x)^2 - \ln x}{(1 + \ln x)^2} = 1 - \frac{\ln x}{(1 + \ln x)^2}$

23. $y = \ln(\ln x) \Rightarrow y' = \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) = \frac{1}{x \ln x}$

24. $y = \ln(\ln(\ln x)) \Rightarrow y' = \frac{1}{\ln(\ln x)} \cdot \frac{d}{dx}(\ln(\ln x)) = \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{d}{dx}(\ln x) = \frac{1}{x(\ln x)\ln(\ln x)}$

25. $y = \theta[\sin(\ln\theta) + \cos(\ln\theta)] \Rightarrow \frac{dy}{d\theta} = [\sin(\ln\theta) + \cos(\ln\theta)] + \theta\left[\cos(\ln\theta) \cdot \frac{1}{\theta} - \sin(\ln\theta) \cdot \frac{1}{\theta}\right]$
$= \sin(\ln\theta) + \cos(\ln\theta) + \cos(\ln\theta) - \sin(\ln\theta) = 2\cos(\ln\theta)$

26. $y = \ln(\sec\theta + \tan\theta) \Rightarrow \frac{dy}{d\theta} = \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta} = \frac{\sec\theta(\tan\theta + \sec\theta)}{\tan\theta + \sec\theta} = \sec\theta$

27. $y = \ln\frac{1}{x\sqrt{x+1}} = -\ln x - \frac{1}{2}\ln(x+1) \Rightarrow y' = -\frac{1}{x} - \frac{1}{2}\left(\frac{1}{x+1}\right) = -\frac{2(x+1)+x}{2x(x+1)} = -\frac{3x+2}{2x(x+1)}$

28. $y = \frac{1}{2}\ln\frac{1+x}{1-x} = \frac{1}{2}[\ln(1+x) - \ln(1-x)] \Rightarrow y' = \frac{1}{2}\left[\frac{1}{1+x} - \left(\frac{1}{1-x}\right)(-1)\right] = \frac{1}{2}\left[\frac{1-x+1+x}{(1+x)(1-x)}\right] = \frac{1}{1-x^2}$

29. $y = \frac{1+\ln t}{1-\ln t} \Rightarrow \frac{dy}{dt} = \frac{(1-\ln t)\left(\frac{1}{t}\right) - (1+\ln t)\left(\frac{-1}{t}\right)}{(1-\ln t)^2} = \frac{\frac{1}{t} - \frac{\ln t}{t} + \frac{1}{t} + \frac{\ln t}{t}}{(1-\ln t)^2} = \frac{2}{t(1-\ln t)^2}$

30. $y = \sqrt{\ln \sqrt{t}} = \left(\ln t^{1/2}\right)^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \frac{d}{dt}\left(\ln t^{1/2}\right) = \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{d}{dt}\left(t^{1/2}\right)$
$= \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{1}{2} t^{-1/2} = \frac{1}{4t\sqrt{\ln \sqrt{t}}}$

31. $y = \ln(\sec(\ln \theta)) \Rightarrow \frac{dy}{d\theta} = \frac{1}{\sec(\ln \theta)} \cdot \frac{d}{d\theta}(\sec(\ln \theta)) = \frac{\sec(\ln \theta)\tan(\ln \theta)}{\sec(\ln \theta)} \cdot \frac{d}{d\theta}(\ln \theta) = \frac{\tan(\ln \theta)}{\theta}$

32. $y = \ln \frac{\sqrt{\sin \theta \cos \theta}}{1 + 2\ln \theta} = \frac{1}{2}(\ln \sin \theta + \ln \cos \theta) - \ln(1 + 2\ln \theta) \Rightarrow \frac{dy}{d\theta} = \frac{1}{2}\left(\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta}\right) - \frac{\frac{2}{\theta}}{1 + 2\ln \theta}$
$= \frac{1}{2}\left[\cot \theta - \tan \theta - \frac{4}{\theta(1 + 2\ln \theta)}\right]$

33. $y = \ln\left(\frac{(x^2+1)^5}{\sqrt{1-x}}\right) = 5\ln(x^2+1) - \frac{1}{2}\ln(1-x) \Rightarrow y' = \frac{5 \cdot 2x}{x^2+1} - \frac{1}{2}\left(\frac{1}{1-x}\right)(-1) = \frac{10x}{x^2+1} + \frac{1}{2(1-x)}$

34. $y = \ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}} = \frac{1}{2}[5\ln(x+1) - 20\ln(x+2)] \Rightarrow y' = \frac{1}{2}\left(\frac{5}{x+1} - \frac{20}{x+2}\right) = \frac{5}{2}\left[\frac{(x+2) - 4(x+1)}{(x+1)(x+2)}\right]$
$= -\frac{5}{2}\left[\frac{3x+2}{(x+1)(x+2)}\right]$

35. $y = \int_{x^2/2}^{x^2} \ln \sqrt{t}\, dt \Rightarrow \frac{dy}{dx} = \left(\ln \sqrt{x^2}\right) \cdot \frac{d}{dx}(x^2) - \left(\ln \sqrt{\frac{x^2}{2}}\right) \cdot \frac{d}{dx}\left(\frac{x^2}{2}\right) = 2x \ln|x| - x \ln \frac{|x|}{\sqrt{2}}$

36. $y = \int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\, dt \Rightarrow \frac{dy}{dx} = \left(\ln \sqrt[3]{x}\right) \cdot \frac{d}{dx}\left(\sqrt[3]{x}\right) - \left(\ln \sqrt{x}\right) \cdot \frac{d}{dx}\left(\sqrt{x}\right) = \left(\ln \sqrt[3]{x}\right)\left(\frac{1}{3}x^{-2/3}\right) - \left(\ln \sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right)$
$= \frac{\ln \sqrt[3]{x}}{3\sqrt[3]{x^2}} - \frac{\ln \sqrt{x}}{2\sqrt{x}}$

37. $\int_{-3}^{-2} \frac{1}{x}\, dx = [\ln|x|]_{-3}^{-2} = \ln 2 - \ln 3 = \ln \frac{2}{3}$

38. $\int_{-1}^{0} \frac{3}{3x-2}\, dx = [\ln|3x-2|]_{-1}^{0} = \ln 2 - \ln 5 = \ln \frac{2}{5}$

39. $\int \frac{2y}{y^2-25}\, dy = \ln|y^2 - 25| + C$

40. $\int \frac{8r}{4r^2-5}\, dr = \ln|4r^2 - 5| + C$

41. $\int_0^{\pi} \frac{\sin t}{2-\cos t}\, dt = [\ln|2-\cos t|]_0^{\pi} = \ln 3 - \ln 1 = \ln 3$; or let $u = 2 - \cos t \Rightarrow du = \sin t\, dt$ with $t = 0$
$\Rightarrow u = 1$ and $t = \pi \Rightarrow u = 3 \Rightarrow \int_0^{\pi} \frac{\sin t}{2-\cos t}\, dt = \int_1^3 \frac{1}{u}\, du = [\ln|u|]_1^3 = \ln 3 - \ln 1 = \ln 3$

42. $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\, d\theta = [\ln|1-4\cos\theta|]_0^{\pi/3} = \ln|1-2| = -\ln 3 = \ln \frac{1}{3}$; or let $u = 1 - 4\cos\theta \Rightarrow du = 4\sin\theta\, d\theta$
with $\theta = 0 \Rightarrow u = -3$ and $\theta = \frac{\pi}{3} \Rightarrow u = -1 \Rightarrow \int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\, d\theta = \int_{-3}^{-1} \frac{1}{u}\, du = [\ln|u|]_{-3}^{-1} = -\ln 3 = \ln \frac{1}{3}$

43. Let $u = \ln x \Rightarrow du = \frac{1}{x}\, dx$; $x = 1 \Rightarrow u = 0$ and $x = 2 \Rightarrow u = \ln 2$;
$\int_1^2 \frac{2\ln x}{x}\, dx = \int_0^{\ln 2} 2u\, du = [u^2]_0^{\ln 2} = (\ln 2)^2$

44. Let $u = \ln x \Rightarrow du = \frac{1}{x}\, dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 4 \Rightarrow u = \ln 4$;
$\int_2^4 \frac{dx}{x \ln x} = \int_{\ln 2}^{\ln 4} \frac{1}{u}\, du = [\ln u]_{\ln 2}^{\ln 4} = \ln(\ln 4) - \ln(\ln 2) = \ln\left(\frac{\ln 4}{\ln 2}\right) = \ln\left(\frac{\ln 2^2}{\ln 2}\right) = \ln\left(\frac{2\ln 2}{\ln 2}\right) = \ln 2$

45. Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 4 \Rightarrow u = \ln 4$;
$\int_2^4 \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\ln 4} u^{-2}\,du = \left[-\frac{1}{u}\right]_{\ln 2}^{\ln 4} = -\frac{1}{\ln 4} + \frac{1}{\ln 2} = -\frac{1}{\ln 2^2} + \frac{1}{\ln 2} = -\frac{1}{2\ln 2} + \frac{1}{\ln 2} = \frac{1}{2\ln 2} = \frac{1}{\ln 4}$

46. Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 16 \Rightarrow u = \ln 16$;
$\int_2^{16} \frac{dx}{2x\sqrt{\ln x}} = \frac{1}{2}\int_{\ln 2}^{\ln 16} u^{-1/2}\,du = \left[u^{1/2}\right]_{\ln 2}^{\ln 16} = \sqrt{\ln 16} - \sqrt{\ln 2} = \sqrt{4\ln 2} - \sqrt{\ln 2} = 2\sqrt{\ln 2} - \sqrt{\ln 2} = \sqrt{\ln 2}$

47. Let $u = 6 + 3\tan t \Rightarrow du = 3\sec^2 t\,dt$;
$\int \frac{3\sec^2 t}{6 + 3\tan t}\,dt = \int \frac{du}{u} = \ln|u| + C = \ln|6 + 3\tan t| + C$

48. Let $u = 2 + \sec y \Rightarrow du = \sec y \tan y\,dy$;
$\int \frac{\sec y \tan y}{2 + \sec y}\,dy = \int \frac{du}{u} = \ln|u| + C = \ln|2 + \sec y| + C$

49. Let $u = \cos\frac{x}{2} \Rightarrow du = -\frac{1}{2}\sin\frac{x}{2}\,dx \Rightarrow -2\,du = \sin\frac{x}{2}\,dx$; $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{2} \Rightarrow u = \frac{1}{\sqrt{2}}$;
$\int_0^{\pi/2} \tan\frac{x}{2}\,dx = \int_0^{\pi/2} \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\,dx = -2\int_1^{1/\sqrt{2}} \frac{du}{u} = \left[-2\ln|u|\right]_1^{1/\sqrt{2}} = -2\ln\frac{1}{\sqrt{2}} = 2\ln\sqrt{2} = \ln 2$

50. Let $u = \sin t \Rightarrow du = \cos t\,dt$; $t = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$ and $t = \frac{\pi}{2} \Rightarrow u = 1$;
$\int_{\pi/4}^{\pi/2} \cot t\,dt = \int_{\pi/4}^{\pi/2} \frac{\cos t}{\sin t}\,dt = \int_{1/\sqrt{2}}^{1} \frac{du}{u} = \left[\ln|u|\right]_{1/\sqrt{2}}^{1} = -\ln\frac{1}{\sqrt{2}} = \ln\sqrt{2}$

51. Let $u = \sin\frac{\theta}{3} \Rightarrow du = \frac{1}{3}\cos\frac{\theta}{3}\,d\theta \Rightarrow 6\,du = 2\cos\frac{\theta}{3}\,d\theta$; $\theta = \frac{\pi}{2} \Rightarrow u = \frac{1}{2}$ and $\theta = \pi \Rightarrow u = \frac{\sqrt{3}}{2}$;
$\int_{\pi/2}^{\pi} 2\cot\frac{\theta}{3}\,d\theta = \int_{\pi/2}^{\pi} \frac{2\cos\frac{\theta}{3}}{\sin\frac{\theta}{3}}\,d\theta = 6\int_{1/2}^{\sqrt{3}/2} \frac{du}{u} = 6\left[\ln|u|\right]_{1/2}^{\sqrt{3}/2} = 6\left(\ln\frac{\sqrt{3}}{2} - \ln\frac{1}{2}\right) = 6\ln\sqrt{3} = \ln 27$

52. Let $u = \cos 3x \Rightarrow du = -3\sin 3x\,dx \Rightarrow -2\,du = 6\sin 3x\,dx$; $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{12} \Rightarrow u = \frac{1}{\sqrt{2}}$;
$\int_0^{\pi/12} 6\tan 3x\,dx = \int_0^{\pi/12} \frac{6\sin 3x}{\cos 3x}\,dx = -2\int_1^{1/\sqrt{2}} \frac{du}{u} = -2\left[\ln|u|\right]_1^{1/\sqrt{2}} = -2\ln\frac{1}{\sqrt{2}} - \ln 1 = 2\ln\sqrt{2} = \ln 2$

53. $\int \frac{dx}{2\sqrt{x} + 2x} = \int \frac{dx}{2\sqrt{x}\left(1 + \sqrt{x}\right)}$; let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$; $\int \frac{dx}{2\sqrt{x}\left(1 + \sqrt{x}\right)} = \int \frac{du}{u} = \ln|u| + C$
$= \ln\left|1 + \sqrt{x}\right| + C = \ln\left(1 + \sqrt{x}\right) + C$

54. Let $u = \sec x + \tan x \Rightarrow du = \left(\sec x \tan x + \sec^2 x\right)dx = (\sec x)(\tan x + \sec x)\,dx \Rightarrow \sec x\,dx = \frac{du}{u}$;
$\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x + \tan x)}} = \int \frac{du}{u\sqrt{\ln u}} = \int (\ln u)^{-1/2} \cdot \frac{1}{u}\,du = 2(\ln u)^{1/2} + C = 2\sqrt{\ln(\sec x + \tan x)} + C$

55. $y = \sqrt{x(x+1)} = (x(x+1))^{1/2} \Rightarrow \ln y = \frac{1}{2}\ln(x(x+1)) \Rightarrow 2\ln y = \ln(x) + \ln(x+1) \Rightarrow \frac{2y'}{y} = \frac{1}{x} + \frac{1}{x+1}$
$\Rightarrow y' = \left(\frac{1}{2}\right)\sqrt{x(x+1)}\left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{\sqrt{x(x+1)}\,(2x+1)}{2x(x+1)} = \frac{2x+1}{2\sqrt{x(x+1)}}$

56. $y = \sqrt{\left(x^2+1\right)(x-1)^2} \Rightarrow \ln y = \frac{1}{2}\left[\ln\left(x^2+1\right) + 2\ln(x-1)\right] \Rightarrow \frac{y'}{y} = \frac{1}{2}\left(\frac{2x}{x^2+1} + \frac{2}{x-1}\right)$
$\Rightarrow y' = \sqrt{\left(x^2+1\right)(x-1)^2}\left(\frac{x}{x^2+1} + \frac{1}{x-1}\right) = \sqrt{\left(x^2+1\right)(x-1)^2}\left[\frac{x^2 - x + x^2 + 1}{\left(x^2+1\right)(x-1)}\right] = \frac{\left(2x^2 - x + 1\right)|x-1|}{\sqrt{x^2+1}\,(x-1)}$

57. $y = \sqrt{\frac{t}{t+1}} = \left(\frac{t}{t+1}\right)^{1/2} \Rightarrow \ln y = \frac{1}{2}\left[\ln t - \ln(t+1)\right] \Rightarrow \frac{1}{y}\frac{dy}{dt} = \frac{1}{2}\left(\frac{1}{t} - \frac{1}{t+1}\right)$
$\Rightarrow \frac{dy}{dt} = \frac{1}{2}\sqrt{\frac{t}{t+1}}\left(\frac{1}{t} - \frac{1}{t+1}\right) = \frac{1}{2}\sqrt{\frac{t}{t+1}}\left[\frac{1}{t(t+1)}\right] = \frac{1}{2\sqrt{t}\,(t+1)^{3/2}}$

58. $y = \sqrt{\frac{1}{t(t+1)}} = [t(t+1)]^{-1/2} \Rightarrow \ln y = \frac{1}{2}[\ln t + \ln(t+1)] \Rightarrow \frac{1}{y}\frac{dy}{dt} = -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right)$
$\Rightarrow \frac{dy}{dt} = -\frac{1}{2}\sqrt{\frac{1}{t(t+1)}}\left[\frac{2t+1}{t(t+1)}\right] = -\frac{2t+1}{2\left(t^2+t\right)^{3/2}}$

59. $y = \sqrt{\theta+3}\,(\sin\theta) = (\theta+3)^{1/2}\sin\theta \Rightarrow \ln y = \frac{1}{2}\ln(\theta+3) + \ln(\sin\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{1}{2(\theta+3)} + \frac{\cos\theta}{\sin\theta}$
$\Rightarrow \frac{dy}{d\theta} = \sqrt{\theta+3}\,(\sin\theta)\left[\frac{1}{2(\theta+3)} + \cot\theta\right]$

60. $y = (\tan\theta)\sqrt{2\theta+1} = (\tan\theta)(2\theta+1)^{1/2} \Rightarrow \ln y = \ln(\tan\theta) + \frac{1}{2}\ln(2\theta+1) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{\sec^2\theta}{\tan\theta} + \left(\frac{1}{2}\right)\left(\frac{2}{2\theta+1}\right)$
$\Rightarrow \frac{dy}{d\theta} = (\tan\theta)\sqrt{2\theta+1}\left(\frac{\sec^2\theta}{\tan\theta} + \frac{1}{2\theta+1}\right) = (\sec^2\theta)\sqrt{2\theta+1} + \frac{\tan\theta}{\sqrt{2\theta+1}}$

61. $y = t(t+1)(t+2) \Rightarrow \ln y = \ln t + \ln(t+1) + \ln(t+2) \Rightarrow \frac{1}{y}\frac{dy}{dt} = \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}$
$\Rightarrow \frac{dy}{dt} = t(t+1)(t+2)\left(\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}\right) = t(t+1)(t+2)\left[\frac{(t+1)(t+2)+t(t+2)+t(t+1)}{t(t+1)(t+2)}\right] = 3t^2 + 6t + 2$

62. $y = \frac{1}{t(t+1)(t+2)} \Rightarrow \ln y = \ln 1 - \ln t - \ln(t+1) - \ln(t+2) \Rightarrow \frac{1}{y}\frac{dy}{dt} = -\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2}$
$\Rightarrow \frac{dy}{dt} = \frac{1}{t(t+1)(t+2)}\left[-\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2}\right] = \frac{-1}{t(t+1)(t+2)}\left[\frac{(t+1)(t+2)+t(t+2)+t(t+1)}{t(t+1)(t+2)}\right]$
$= -\frac{3t^2+6t+2}{\left(t^3+3t^2+2t\right)^2}$

63. $y = \frac{\theta+5}{\theta\cos\theta} \Rightarrow \ln y = \ln(\theta+5) - \ln\theta - \ln(\cos\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin\theta}{\cos\theta} \Rightarrow \frac{dy}{d\theta} = \left(\frac{\theta+5}{\theta\cos\theta}\right)\left(\frac{1}{\theta+5} - \frac{1}{\theta} + \tan\theta\right)$

64. $y = \frac{\theta\sin\theta}{\sqrt{\sec\theta}} \Rightarrow \ln y = \ln\theta + \ln(\sin\theta) - \frac{1}{2}\ln(\sec\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \left[\frac{1}{\theta} + \frac{\cos\theta}{\sin\theta} - \frac{(\sec\theta)(\tan\theta)}{2\sec\theta}\right]$
$\Rightarrow \frac{dy}{d\theta} = \frac{\theta\sin\theta}{\sqrt{\sec\theta}}\left(\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right)$

65. $y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} \Rightarrow \ln y = \ln x + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1) \Rightarrow \frac{y'}{y} = \frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}$
$\Rightarrow y' = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right]$

66. $y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}} \Rightarrow \ln y = \frac{1}{2}[10\ln(x+1) - 5\ln(2x+1)] \Rightarrow \frac{y'}{y} = \frac{5}{x+1} - \frac{5}{2x+1}$
$\Rightarrow y' = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}\left(\frac{5}{x+1} - \frac{5}{2x+1}\right)$

67. $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}} \Rightarrow \ln y = \frac{1}{3}[\ln x + \ln(x-2) - \ln(x^2+1)] \Rightarrow \frac{y'}{y} = \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)$
$\Rightarrow y' = \frac{1}{3}\sqrt[3]{\frac{x(x-2)}{x^2+1}}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)$

68. $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \Rightarrow \ln y = \frac{1}{3}[\ln x + \ln(x+1) + \ln(x-2) - \ln(x^2+1) - \ln(2x+3)]$
$\Rightarrow y' = \frac{1}{3}\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}\left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right)$

69. (a) $f(x) = \ln(\cos x) \Rightarrow f'(x) = -\frac{\sin x}{\cos x} = -\tan x = 0 \Rightarrow x = 0$; $f'(x) > 0$ for $-\frac{\pi}{4} \le x < 0$ and $f'(x) < 0$ for $0 < x \le \frac{\pi}{3} \Rightarrow$ there is a relative maximum at $x = 0$ with $f(0) = \ln(\cos 0) = \ln 1 = 0$; $f\left(-\frac{\pi}{4}\right) = \ln\left(\cos\left(-\frac{\pi}{4}\right)\right)$ $= \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2$ and $f\left(\frac{\pi}{3}\right) = \ln\left(\cos\left(\frac{\pi}{3}\right)\right) = \ln\frac{1}{2} = -\ln 2$. Therefore, the absolute minimum occurs at $x = \frac{\pi}{3}$ with $f\left(\frac{\pi}{3}\right) = -\ln 2$ and the absolute maximum occurs at $x = 0$ with $f(0) = 0$.

(b) $f(x) = \cos(\ln x) \Rightarrow f'(x) = \frac{-\sin(\ln x)}{x} = 0 \Rightarrow x = 1$; $f'(x) > 0$ for $\frac{1}{2} \le x < 1$ and $f'(x) < 0$ for $1 < x \le 2$
$\Rightarrow$ there is a relative maximum at $x = 1$ with $f(1) = \cos(\ln 1) = \cos 0 = 1$; $f\left(\frac{1}{2}\right) = \cos\left(\ln\left(\frac{1}{2}\right)\right)$
$= \cos(-\ln 2) = \cos(\ln 2)$ and $f(2) = \cos(\ln 2)$. Therefore, the absolute minimum occurs at $x = \frac{1}{2}$ and $x = 2$ with $f\left(\frac{1}{2}\right) = f(2) = \cos(\ln 2)$, and the absolute maximum occurs at $x = 1$ with $f(1) = 1$.

70. (a) $f(x) = x - \ln x \Rightarrow f'(x) = 1 - \frac{1}{x}$; if $x > 1$, then $f'(x) > 0$ which means that $f(x)$ is increasing
(b) $f(1) = 1 - \ln 1 = 1 \Rightarrow f(x) = x - \ln x > 0$, if $x > 1$ by part (a) $\Rightarrow x > \ln x$ if $x > 1$

71. $\int_1^5 (\ln 2x - \ln x)\,dx = \int_1^5 (-\ln x + \ln 2 + \ln x)\,dx = (\ln 2)\int_1^5 dx = (\ln 2)(5-1) = \ln 2^4 = \ln 16$

72. $A = \int_{-\pi/4}^{0} -\tan x\,dx + \int_0^{\pi/3} \tan x\,dx = \int_{-\pi/4}^{0} \frac{-\sin x}{\cos x}\,dx - \int_0^{\pi/3} \frac{-\sin x}{\cos x}\,dx = [\ln|\cos x|]_{-\pi/4}^{0} - [\ln|\cos x|]_0^{\pi/3}$
$= \left(\ln 1 - \ln \frac{1}{\sqrt{2}}\right) - \left(\ln \frac{1}{2} - \ln 1\right) = \ln\sqrt{2} + \ln 2 = \frac{3}{2}\ln 2$

73. $V = \pi\int_0^3 \left(\frac{2}{\sqrt{y+1}}\right)^2 dy = 4\pi\int_0^3 \frac{1}{y+1}\,dy = 4\pi\,[\ln|y+1|]_0^3 = 4\pi(\ln 4 - \ln 1) = 4\pi\ln 4$

74. $V = \pi\int_{\pi/6}^{\pi/2} \cot x\,dx = \pi\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x}\,dx = \pi\,[\ln(\sin x)]_{\pi/6}^{\pi/2} = \pi\left(\ln 1 - \ln\frac{1}{2}\right) = \pi\ln 2$

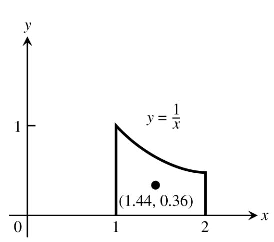
75. $V = 2\pi\int_{1/2}^{2} x\left(\frac{1}{x^2}\right) dx = 2\pi\int_{1/2}^{2} \frac{1}{x}\,dx = 2\pi\,[\ln|x|]_{1/2}^{2} = 2\pi\left(\ln 2 - \ln\frac{1}{2}\right) = 2\pi(2\ln 2) = \pi\ln 2^4 = \pi\ln 16$

76. $V = \pi\int_0^3 \left(\frac{9x}{\sqrt{x^3+9}}\right)^2 dx = 27\pi\int_0^3 dx = 27\pi\,[\ln(x^3+9)]_0^3 = 27\pi(\ln 36 - \ln 9) = 27\pi(\ln 4 + \ln 9 - \ln 9)$
$= 27\pi\ln 4 = 54\pi\ln 2$

77. (a) $y = \frac{x^2}{8} - \ln x \Rightarrow 1 + (y')^2 = 1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2 = 1 + \left(\frac{x^2-4}{4x}\right)^2 = \left(\frac{x^2+4}{4x}\right)^2 \Rightarrow L = \int_4^8 \sqrt{1+(y')^2}\,dx$
$= \int_4^8 \frac{x^2+4}{4x}\,dx = \int_4^8 \left(\frac{x}{4} + \frac{1}{x}\right) dx = \left[\frac{x^2}{8} + \ln|x|\right]_4^8 = (8 + \ln 8) - (2 + \ln 4) = 6 + \ln 2$
(b) $x = \left(\frac{y}{4}\right)^2 - 2\ln\left(\frac{y}{4}\right) \Rightarrow \frac{dx}{dy} = \frac{y}{8} - \frac{2}{y} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{y}{8} - \frac{2}{y}\right)^2 = 1 + \left(\frac{y^2-16}{8y}\right)^2 = \left(\frac{y^2+16}{8y}\right)^2$
$\Rightarrow L = \int_4^{12} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_4^{12} \frac{y^2+16}{8y}\,dy = \int_4^{12} \left(\frac{y}{8} + \frac{2}{y}\right) dy = \left[\frac{y^2}{16} + 2\ln y\right]_4^{12} = (9 + 2\ln 12) - (1 + 2\ln 4)$
$= 8 + 2\ln 3 = 8 + \ln 9$

78. $L = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\,dx \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow y = \ln|x| + C = \ln x + C$ since $x > 0 \Rightarrow 0 = \ln 1 + C \Rightarrow C = 0 \Rightarrow y = \ln x$

79. (a) $M_y = \int_1^2 x\left(\frac{1}{x}\right) dx = 1$, $M_x = \int_1^2 \left(\frac{1}{2x}\right)\left(\frac{1}{x}\right) dx = \frac{1}{2}\int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{2x}\right]_1^2 = \frac{1}{4}$, $M = \int_1^2 \frac{1}{x}\,dx = [\ln|x|]_1^2 = \ln 2$
$\Rightarrow \bar{x} = \frac{M_y}{M} = \frac{1}{\ln 2} \approx 1.44$ and $\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{1}{4}\right)}{\ln 2} \approx 0.36$
(b)



80. (a) $M_y = \int_1^{16} x\left(\frac{1}{\sqrt{x}}\right) dx = \int_1^{16} x^{1/2}\, dx = \frac{2}{3}\left[x^{3/2}\right]_1^{16} = 42$; $M_x = \int_1^{16}\left(\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{\sqrt{x}}\right) dx = \frac{1}{2}\int_1^{16} \frac{1}{x}\, dx$
$= \frac{1}{2}\left[\ln|x|\right]_1^{16} = \ln 4$, $M = \int_1^{16} \frac{1}{\sqrt{x}}\, dx = \left[2x^{1/2}\right]_1^{16} = 6 \Rightarrow \overline{x} = \frac{M_y}{M} = 7$ and $\overline{y} = \frac{M_x}{M} = \frac{\ln 4}{6}$

(b) $M_y = \int_1^{16} x\left(\frac{1}{\sqrt{x}}\right)\left(\frac{4}{\sqrt{x}}\right) dx = 4\int_1^{16} dx = 60$, $M_x = \int_1^{16}\left(\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{\sqrt{x}}\right)\left(\frac{4}{\sqrt{x}}\right) dx = 2\int_1^{16} x^{-3/2}\, dx$
$= -4\left[x^{-1/2}\right]_1^{16} = 3$, $M = \int_1^{16}\left(\frac{1}{\sqrt{x}}\right)\left(\frac{4}{\sqrt{x}}\right) dx = 4\int_1^{16} \frac{1}{x}\, dx = \left[4\ln|x|\right]_1^{16} = 4\ln 16 \Rightarrow \overline{x} = \frac{M_y}{M} = \frac{15}{\ln 16}$ and
$\overline{y} = \frac{M_x}{M} = \frac{3}{4\ln 16}$

81. $f(x) = \ln(x^3 - 1)$, domain of f: $(1, \infty) \Rightarrow f'(x) = \frac{3x^2}{x^3 - 1}$; $f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$, not in the domain;
$f'(x) =$ undefined $\Rightarrow x^3 - 1 = 0 \Rightarrow x = 1$, not in domain. On $(1, \infty)$, $f'(x) > 0 \Rightarrow$ f is increasing on $(1, \infty)$
$\Rightarrow$ f is one-to-one

82. $g(x) = \sqrt{x^2 + \ln x}$, domain of g: $x > 0.652919 \Rightarrow g'(x) = \frac{2x + \frac{1}{x}}{2\sqrt{x^2 + \ln x}} = \frac{2x^2 + 1}{2x\sqrt{x^2 + \ln x}}$; $g'(x) = 0 \Rightarrow 2x^2 + 1 = 0 \Rightarrow$ no real solutions; $g'(x) =$ undefined $\Rightarrow 2x\sqrt{x^2 + \ln x} = 0 \Rightarrow x = 0$ or $x \approx 0.652919$, neither in domain. On $x > 0.652919$, $g'(x) > 0 \Rightarrow$ g is increasing for $x > 0.652919 \Rightarrow$ g is one-to-one

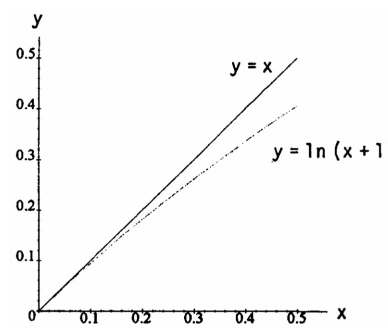
83. $\frac{dy}{dx} = 1 + \frac{1}{x}$ at $(1, 3) \Rightarrow y = x + \ln|x| + C$; $y = 3$ at $x = 1 \Rightarrow C = 2 \Rightarrow y = x + \ln|x| + 2$

84. $\frac{d^2y}{dx^2} = \sec^2 x \Rightarrow \frac{dy}{dx} = \tan x + C$ and $1 = \tan 0 + C \Rightarrow \frac{dy}{dx} = \tan x + 1 \Rightarrow y = \int(\tan x + 1)\, dx$
$= \ln|\sec x| + x + C_1$ and $0 = \ln|\sec 0| + 0 + C_1 \Rightarrow C_1 = 0 \Rightarrow y = \ln|\sec x| + x$

85. (a) $L(x) = f(0) + f'(0) \cdot x$, and $f(x) = \ln(1 + x) \Rightarrow f'(x)|_{x=0} = \frac{1}{1+x}\Big|_{x=0} = 1 \Rightarrow L(x) = \ln 1 + 1 \cdot x \Rightarrow L(x) = x$

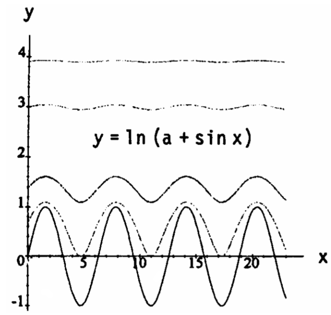
(b) Let $f(x) = \ln(x + 1)$. Since $f''(x) = -\frac{1}{(x+1)^2} < 0$ on $[0,\ 0.1]$, the graph of f is concave down on this interval and the largest error in the linear approximation will occur when $x = 0.1$. This error is $0.1 - \ln(1.1) \approx 0.00469$ to five decimal places.

(c) The approximation $y = x$ for $\ln(1 + x)$ is best for smaller positive values of x; in particular for $0 \le x \le 0.1$ in the graph. As x increases, so does the error $x - \ln(1 + x)$. From the graph an upper bound for the error is $0.5 - \ln(1 + 0.5) \approx 0.095$; i.e., $|E(x)| \le 0.095$ for $0 \le x \le 0.5$. Note from the graph that $0.1 - \ln(1 + 0.1) \approx 0.00469$ estimates the error in replacing $\ln(1 + x)$ by x over $0 \le x \le 0.1$. This is consistent with the estimate given in part (b) above.



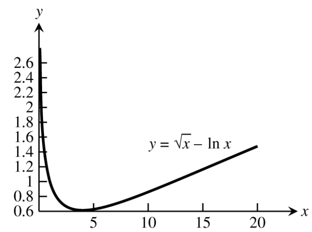
86. For all positive values of x, $\frac{d}{dx}\left[\ln \frac{a}{x}\right] = \frac{1}{\frac{a}{x}} \cdot -\frac{a}{x^2} = -\frac{1}{x}$ and $\frac{d}{dx}\left[\ln a - \ln x\right] = 0 - \frac{1}{x} = -\frac{1}{x}$. Since $\ln \frac{a}{x}$ and $\ln a - \ln x$ have the same derivative, then $\ln \frac{a}{x} = \ln a - \ln x + C$ for some constant C. Since this equation holds for all positve values of x, it must be true for $x = 1 \Rightarrow \ln \frac{1}{x} = \ln 1 - \ln x + C = 0 - \ln x + C \Rightarrow \ln \frac{1}{x} = -\ln x + C$. By part 3 we know that $\ln \frac{1}{x} = -\ln x \Rightarrow C = 0 \Rightarrow \ln \frac{a}{x} = \ln a - \ln x$.

87. (a)



(b) $y' = \frac{\cos x}{a+\sin x}$. Since $|\sin x|$ and $|\cos x|$ are less than or equal to 1, we have for $a > 1$
$\frac{-1}{a-1} \le y' \le \frac{1}{a-1}$ for all x.
Thus, $\lim_{a\to+\infty} y' = 0$ for all $x \Rightarrow$ the graph of y looks more and more horizontal as $a \to +\infty$.

88. (a) The graph of $y = \sqrt{x} - \ln x$ appears to be concave upward for all $x > 0$.



(b) $y = \sqrt{x} - \ln x \Rightarrow y' = \frac{1}{2\sqrt{x}} - \frac{1}{x} \Rightarrow y'' = -\frac{1}{4x^{3/2}} + \frac{1}{x^2} = \frac{1}{x^2}\left(-\frac{\sqrt{x}}{4} + 1\right) = 0 \Rightarrow \sqrt{x} = 4 \Rightarrow x = 16.$
Thus, $y'' > 0$ if $0 < x < 16$ and $y'' < 0$ if $x > 16$ so a point of inflection exists at $x = 16$. The graph of $y = \sqrt{x} - \ln x$ closely resembles a straight line for $x \ge 10$ and it is impossible to discuss the point of inflection visually from the graph.

7.3 EXPONENTIAL FUNCTIONS

1. (a) $e^{-0.3t} = 27 \Rightarrow \ln e^{-0.3t} = \ln 3^3 \Rightarrow (-0.3t)\ln e = 3\ln 3 \Rightarrow -0.3t = 3\ln 3 \Rightarrow t = -10\ln 3$
(b) $e^{kt} = \frac{1}{2} \Rightarrow \ln e^{kt} = \ln 2^{-1} = kt \ln e = -\ln 2 \Rightarrow t = -\frac{\ln 2}{k}$
(c) $e^{(\ln 0.2)t} = 0.4 \Rightarrow \left(e^{\ln 0.2}\right)^t = 0.4 \Rightarrow 0.2^t = 0.4 \Rightarrow \ln 0.2^t = \ln 0.4 \Rightarrow t\ln 0.2 = \ln 0.4 \Rightarrow t = \frac{\ln 0.4}{\ln 0.2}$

2. (a) $e^{-0.01t} = 1000 \Rightarrow \ln e^{-0.01t} = \ln 1000 \Rightarrow (-0.01t)\ln e = \ln 1000 \Rightarrow -0.01t = \ln 1000 \Rightarrow t = -100\ln 1000$
(b) $e^{kt} = \frac{1}{10} \Rightarrow \ln e^{kt} = \ln 10^{-1} = kt\ln e = -\ln 10 \Rightarrow kt = -\ln 10 \Rightarrow t = -\frac{\ln 10}{k}$
(c) $e^{(\ln 2)t} = \frac{1}{2} \Rightarrow \left(e^{\ln 2}\right)^t = 2^{-1} \Rightarrow 2^t = 2^{-1} \Rightarrow t = -1$

3. $e^{\sqrt{t}} = x^2 \Rightarrow \ln e^{\sqrt{t}} = \ln x^2 \Rightarrow \sqrt{t} = 2\ln x \Rightarrow t = 4(\ln x)^2$

4. $e^{x^2} e^{2x+1} = e^t \Rightarrow e^{x^2+2x+1} = e^t \Rightarrow \ln e^{x^2+2x+1} = \ln e^t \Rightarrow t = x^2 + 2x + 1$

5. $y = e^{-5x} \Rightarrow y' = e^{-5x}\frac{d}{dx}(-5x) \Rightarrow y' = -5e^{-5x}$

6. $y = e^{2x/3} \Rightarrow y' = e^{2x/3}\frac{d}{dx}\left(\frac{2x}{3}\right) \Rightarrow y' = \frac{2}{3}e^{2x/3}$

7. $y = e^{5-7x} \Rightarrow y' = e^{5-7x}\frac{d}{dx}(5-7x) \Rightarrow y' = -7e^{5-7x}$

8. $y = e^{\left(4\sqrt{x}+x^2\right)} \Rightarrow y' = e^{\left(4\sqrt{x}+x^2\right)}\frac{d}{dx}\left(4\sqrt{x}+x^2\right) \Rightarrow y' = \left(\frac{2}{\sqrt{x}} + 2x\right)e^{\left(4\sqrt{x}+x^2\right)}$

9. $y = xe^x - e^x \Rightarrow y' = (e^x + xe^x) - e^x = xe^x$

10. $y = (1+2x)\,e^{-2x} \Rightarrow y' = 2e^{-2x} + (1+2x)e^{-2x}\,\frac{d}{dx}(-2x) \Rightarrow y' = 2e^{-2x} - 2(1+2x)\,e^{-2x} = -4xe^{-2x}$

11. $y = (x^2-2x+2)\,e^x \Rightarrow y' = (2x-2)e^x + (x^2-2x+2)\,e^x = x^2e^x$

12. $y = (9x^2-6x+2)\,e^{3x} \Rightarrow y' = (18x-6)e^{3x} + (9x^2-6x+2)\,e^{3x}\,\frac{d}{dx}(3x) \Rightarrow y' = (18x-6)e^{3x} + 3\,(9x^2-6x+2)\,e^{3x}$
$= 27x^2e^{3x}$

13. $y = e^{\theta}(\sin\theta + \cos\theta) \Rightarrow y' = e^{\theta}(\sin\theta + \cos\theta) + e^{\theta}(\cos\theta - \sin\theta) = 2e^{\theta}\cos\theta$

14. $y = \ln\left(3\theta e^{-\theta}\right) = \ln 3 + \ln\theta + \ln e^{-\theta} = \ln 3 + \ln\theta - \theta \Rightarrow \frac{dy}{d\theta} = \frac{1}{\theta} - 1$

15. $y = \cos\left(e^{-\theta^2}\right) \Rightarrow \frac{dy}{d\theta} = -\sin\left(e^{-\theta^2}\right)\frac{d}{d\theta}\left(e^{-\theta^2}\right) = \left(-\sin\left(e^{-\theta^2}\right)\right)\left(e^{-\theta^2}\right)\frac{d}{d\theta}(-\theta^2) = 2\theta e^{-\theta^2}\sin\left(e^{-\theta^2}\right)$

16. $y = \theta^3e^{-2\theta}\cos 5\theta \Rightarrow \frac{dy}{d\theta} = (3\theta^2)\left(e^{-2\theta}\cos 5\theta\right) + (\theta^3\cos 5\theta)\,e^{-2\theta}\,\frac{d}{d\theta}(-2\theta) - 5(\sin 5\theta)\left(\theta^3e^{-2\theta}\right)$
$= \theta^2e^{-2\theta}\,(3\cos 5\theta - 2\theta\cos 5\theta - 5\theta\sin 5\theta)$

17. $y = \ln(3te^{-t}) = \ln 3 + \ln t + \ln e^{-t} = \ln 3 + \ln t - t \Rightarrow \frac{dy}{dt} = \frac{1}{t} - 1 = \frac{1-t}{t}$

18. $y = \ln(2e^{-t}\sin t) = \ln 2 + \ln e^{-t} + \ln\sin t = \ln 2 - t + \ln\sin t \Rightarrow \frac{dy}{dt} = -1 + \left(\frac{1}{\sin t}\right)\frac{d}{dt}(\sin t) = -1 + \frac{\cos t}{\sin t}$
$= \frac{\cos t - \sin t}{\sin t}$

19. $y = \ln\frac{e^{\theta}}{1+e^{\theta}} = \ln e^{\theta} - \ln\left(1+e^{\theta}\right) = \theta - \ln\left(1+e^{\theta}\right) \Rightarrow \frac{dy}{d\theta} = 1 - \left(\frac{1}{1+e^{\theta}}\right)\frac{d}{d\theta}\left(1+e^{\theta}\right) = 1 - \frac{e^{\theta}}{1+e^{\theta}} = \frac{1}{1+e^{\theta}}$

20. $y = \ln\frac{\sqrt{\theta}}{1+\sqrt{\theta}} = \ln\sqrt{\theta} - \ln\left(1+\sqrt{\theta}\right) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\sqrt{\theta}}\right)\frac{d}{d\theta}\left(\sqrt{\theta}\right) - \left(\frac{1}{1+\sqrt{\theta}}\right)\frac{d}{d\theta}\left(1+\sqrt{\theta}\right)$
$= \left(\frac{1}{\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right) - \left(\frac{1}{1+\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right) = \frac{\left(1+\sqrt{\theta}\right)-\sqrt{\theta}}{2\theta\left(1+\sqrt{\theta}\right)} = \frac{1}{2\theta\left(1+\sqrt{\theta}\right)} = \frac{1}{2\theta\,(1+\theta^{1/2})}$

21. $y = e^{(\cos t + \ln t)} = e^{\cos t}\,e^{\ln t} = te^{\cos t} \Rightarrow \frac{dy}{dt} = e^{\cos t} + te^{\cos t}\,\frac{d}{dt}(\cos t) = (1 - t\sin t)\,e^{\cos t}$

22. $y = e^{\sin t}\,(\ln t^2 + 1) \Rightarrow \frac{dy}{dt} = e^{\sin t}(\cos t)\,(\ln t^2 + 1) + \frac{2}{t}\,e^{\sin t} = e^{\sin t}\left[(\ln t^2 + 1)\,(\cos t) + \frac{2}{t}\right]$

23. $\int_0^{\ln x}\sin e^t\,dt \Rightarrow y' = \left(\sin e^{\ln x}\right)\cdot\frac{d}{dx}(\ln x) = \frac{\sin x}{x}$

24. $y = \int_{e^{4\sqrt{x}}}^{e^{2x}}\ln t\,dt \Rightarrow y' = (\ln e^{2x})\cdot\frac{d}{dx}(e^{2x}) - \left(\ln e^{4\sqrt{x}}\right)\cdot\frac{d}{dx}\left(e^{4\sqrt{x}}\right) = (2x)\,(2e^{2x}) - \left(4\sqrt{x}\right)\left(e^{4\sqrt{x}}\right)\cdot\frac{d}{dx}\left(4\sqrt{x}\right)$
$= 4xe^{2x} - 4\sqrt{x}\,e^{4\sqrt{x}}\left(\frac{2}{\sqrt{x}}\right) = 4xe^{2x} - 8e^{4\sqrt{x}}$

25. $\ln y = e^y\sin x \Rightarrow \left(\frac{1}{y}\right)y' = (y'e^y)\,(\sin x) + e^y\cos x \Rightarrow y'\left(\frac{1}{y} - e^y\sin x\right) = e^y\cos x$
$\Rightarrow y'\left(\frac{1-ye^y\sin x}{y}\right) = e^y\cos x \Rightarrow y' = \frac{ye^y\cos x}{1-ye^y\sin x}$

26. $\ln xy = e^{x+y} \Rightarrow \ln x + \ln y = e^{x+y} \Rightarrow \frac{1}{x} + \left(\frac{1}{y}\right)y' = (1+y')\,e^{x+y} \Rightarrow y'\left(\frac{1}{y} - e^{x+y}\right) = e^{x+y} - \frac{1}{x}$
$\Rightarrow y'\left(\frac{1-ye^{x+y}}{y}\right) = \frac{xe^{x+y}-1}{x} \Rightarrow y' = \frac{y\,(xe^{x+y}-1)}{x\,(1-ye^{x+y})}$

27. $e^{2x} = \sin(x+3y) \Rightarrow 2e^{2x} = (1+3y')\cos(x+3y) \Rightarrow 1+3y' = \frac{2e^{2x}}{\cos(x+3y)} \Rightarrow 3y' = \frac{2e^{2x}}{\cos(x+3y)} - 1 \Rightarrow y' = \frac{2e^{2x} - \cos(x+3y)}{3\cos(x+3y)}$

28. $\tan y = e^x + \ln x \;\Rightarrow\; (\sec^2 y)\, y' = e^x + \frac{1}{x} \;\Rightarrow\; y' = \frac{(xe^x+1)\cos^2 y}{x}$

29. $\int (e^{3x} + 5e^{-x})\, dx = \frac{e^{3x}}{3} - 5e^{-x} + C$

30. $\int (2e^x - 3e^{-2x})\, dx = 2e^x + \frac{3}{2}\, e^{-2x} + C$

31. $\int_{\ln 2}^{\ln 3} e^x\, dx = [e^x]_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2} = 3 - 2 = 1$

32. $\int_{-\ln 2}^{0} e^{-x}\, dx = [-e^{-x}]_{-\ln 2}^{0} = -e^0 + e^{\ln 2} = -1 + 2 = 1$

33. $\int 8e^{(x+1)}\, dx = 8e^{(x+1)} + C$

34. $\int 2e^{(2x-1)}\, dx = e^{(2x-1)} + C$

35. $\int_{\ln 4}^{\ln 9} e^{x/2}\, dx = \left[2e^{x/2}\right]_{\ln 4}^{\ln 9} = 2\left[e^{(\ln 9)/2} - e^{(\ln 4)/2}\right] = 2\left(e^{\ln 3} - e^{\ln 2}\right) = 2(3-2) = 2$

36. $\int_0^{\ln 16} e^{x/4}\, dx = \left[4e^{x/4}\right]_0^{\ln 16} = 4\left(e^{(\ln 16)/4} - e^0\right) = 4\left(e^{\ln 2} - 1\right) = 4(2-1) = 4$

37. Let $u = r^{1/2} \;\Rightarrow\; du = \frac{1}{2}\, r^{-1/2}\, dr \;\Rightarrow\; 2\, du = r^{-1/2}\, dr$;
$\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\, dr = \int e^{r^{1/2}} \cdot r^{-1/2}\, dr = 2\int e^u\, du = 2e^u + C = 2e^{r^{1/2}} + C = 2e^{\sqrt{r}} + C$

38. Let $u = -r^{1/2} \;\Rightarrow\; du = -\frac{1}{2}\, r^{-1/2}\, dr \;\Rightarrow\; -2\, du = r^{-1/2}\, dr$;
$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}\, dr = \int e^{-r^{1/2}} \cdot r^{-1/2}\, dr = -2\int e^u\, du = -2e^{-r^{1/2}} + C = -2e^{-\sqrt{r}} + C$

39. Let $u = -t^2 \;\Rightarrow\; du = -2t\, dt \;\Rightarrow\; -du = 2t\, dt$;
$\int 2te^{-t^2}\, dt = -\int e^u\, du = -e^u + C = -e^{-t^2} + C$

40. Let $u = t^4 \;\Rightarrow\; du = 4t^3\, dt \;\Rightarrow\; \frac{1}{4}\, du = t^3\, dt$;
$\int t^3 e^{t^4}\, dt = \frac{1}{4}\int e^u\, du = \frac{1}{4}\, e^{t^4} + C$

41. Let $u = \frac{1}{x} \;\Rightarrow\; du = -\frac{1}{x^2}\, dx \;\Rightarrow\; -du = \frac{1}{x^2}\, dx$;
$\int \frac{e^{1/x}}{x^2}\, dx = \int -e^u\, du = -e^u + C = -e^{1/x} + C$

42. Let $u = -x^{-2} \;\Rightarrow\; du = 2x^{-3}\, dx \;\Rightarrow\; \frac{1}{2}\, du = x^{-3}\, dx$;
$\int \frac{e^{-1/x^2}}{x^3}\, dx = \int e^{-x^{-2}} \cdot x^{-3}\, dx = \frac{1}{2}\int e^u\, du = \frac{1}{2}\, e^u + C = \frac{1}{2}\, e^{-x^{-2}} + C = \frac{1}{2}\, e^{-1/x^2} + C$

43. Let $u = \tan\theta \;\Rightarrow\; du = \sec^2\theta\, d\theta;\ \theta = 0 \;\Rightarrow\; u = 0,\ \theta = \frac{\pi}{4} \;\Rightarrow\; u = 1$;
$\int_0^{\pi/4} \left(1 + e^{\tan\theta}\right)\sec^2\theta\, d\theta = \int_0^{\pi/4} \sec^2\theta\, d\theta + \int_0^1 e^u\, du = [\tan\theta]_0^{\pi/4} + [e^u]_0^1 = \left[\tan\left(\frac{\pi}{4}\right) - \tan(0)\right] + (e^1 - e^0)$
$= (1-0) + (e-1) = e$

44. Let $u = \cot\theta \;\Rightarrow\; du = -\csc^2\theta\, d\theta;\ \theta = \frac{\pi}{4} \;\Rightarrow\; u = 1,\ \theta = \frac{\pi}{2} \;\Rightarrow\; u = 0$;
$\int_{\pi/4}^{\pi/2} \left(1 + e^{\cot\theta}\right)\csc^2\theta\, d\theta = \int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta - \int_1^0 e^u\, du = [-\cot\theta]_{\pi/4}^{\pi/2} - [e^u]_1^0 = \left[-\cot\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{4}\right)\right] - (e^0 - e^1)$
$= (0+1) - (1-e) = e$

45. Let $u = \sec \pi t \Rightarrow du = \pi \sec \pi t \tan \pi t\, dt \Rightarrow \frac{du}{\pi} = \sec \pi t \tan \pi t\, dt$;
$\int e^{\sec(\pi t)} \sec(\pi t) \tan(\pi t)\, dt = \frac{1}{\pi} \int e^u\, du = \frac{e^u}{\pi} + C = \frac{e^{\sec(\pi t)}}{\pi} + C$

46. Let $u = \csc(\pi + t) \Rightarrow du = -\csc(\pi + t)\cot(\pi + t)\, dt$;
$\int e^{\csc(\pi+t)} \csc(\pi + t)\cot(\pi + t)\, dt = -\int e^u\, du = -e^u + C = -e^{\csc(\pi+t)} + C$

47. Let $u = e^v \Rightarrow du = e^v\, dv \Rightarrow 2\, du = 2e^v\, dv;\ v = \ln \frac{\pi}{6} \Rightarrow u = \frac{\pi}{6},\ v = \ln \frac{\pi}{2} \Rightarrow u = \frac{\pi}{2}$;
$\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v\, dv = 2\int_{\pi/6}^{\pi/2} \cos u\, du = [2 \sin u]_{\pi/6}^{\pi/2} = 2\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)\right] = 2\left(1 - \frac{1}{2}\right) = 1$

48. Let $u = e^{x^2} \Rightarrow du = 2xe^{x^2}\, dx;\ x = 0 \Rightarrow u = 1,\ x = \sqrt{\ln \pi} \Rightarrow u = e^{\ln \pi} = \pi$;
$\int_0^{\sqrt{\ln \pi}} 2xe^{x^2} \cos\left(e^{x^2}\right) dx = \int_1^{\pi} \cos u\, du = [\sin u]_1^{\pi} = \sin(\pi) - \sin(1) = -\sin(1) \approx -0.84147$

49. Let $u = 1 + e^r \Rightarrow du = e^r\, dr$;
$\int \frac{e^r}{1 + e^r}\, dr = \int \frac{1}{u}\, du = \ln|u| + C = \ln(1 + e^r) + C$

50. $\int \frac{1}{1 + e^x}\, dx = \int \frac{e^{-x}}{e^{-x} + 1}\, dx$;
let $u = e^{-x} + 1 \Rightarrow du = -e^{-x}\, dx \Rightarrow -du = e^{-x}\, dx$;
$\int \frac{e^{-x}}{e^{-x} + 1}\, dx = -\int \frac{1}{u}\, du = -\ln|u| + C = -\ln(e^{-x} + 1) + C$

51. $\frac{dy}{dt} = e^t \sin(e^t - 2) \Rightarrow y = \int e^t \sin(e^t - 2)\, dt$;
let $u = e^t - 2 \Rightarrow du = e^t\, dt \Rightarrow y = \int \sin u\, du = -\cos u + C = -\cos(e^t - 2) + C;\ y(\ln 2) = 0$
$\Rightarrow -\cos\left(e^{\ln 2} - 2\right) + C = 0 \Rightarrow -\cos(2 - 2) + C = 0 \Rightarrow C = \cos 0 = 1$; thus, $y = 1 - \cos(e^t - 2)$

52. $\frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t}) \Rightarrow y = \int e^{-t} \sec^2(\pi e^{-t})\, dt$;
let $u = \pi e^{-t} \Rightarrow du = -\pi e^{-t}\, dt \Rightarrow -\frac{1}{\pi}\, du = e^{-t}\, dt \Rightarrow y = -\frac{1}{\pi}\int \sec^2 u\, du = -\frac{1}{\pi} \tan u + C$
$= -\frac{1}{\pi} \tan(\pi e^{-t}) + C;\ y(\ln 4) = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi} \tan\left(\pi e^{-\ln 4}\right) + C = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi} \tan\left(\pi \cdot \frac{1}{4}\right) + C = \frac{2}{\pi}$
$\Rightarrow -\frac{1}{\pi}(1) + C = \frac{2}{\pi} \Rightarrow C = \frac{3}{\pi}$; thus, $y = \frac{3}{\pi} - \frac{1}{\pi} \tan(\pi e^{-t})$

53. $\frac{d^2y}{dx^2} = 2e^{-x} \Rightarrow \frac{dy}{dx} = -2e^{-x} + C;\ x = 0$ and $\frac{dy}{dx} = 0 \Rightarrow 0 = -2e^0 + C \Rightarrow C = 2$; thus $\frac{dy}{dx} = -2e^{-x} + 2$
$\Rightarrow y = 2e^{-x} + 2x + C_1;\ x = 0$ and $y = 1 \Rightarrow 1 = 2e^0 + C_1 \Rightarrow C_1 = -1 \Rightarrow y = 2e^{-x} + 2x - 1 = 2(e^{-x} + x) - 1$

54. $\frac{d^2y}{dt^2} = 1 - e^{2t} \Rightarrow \frac{dy}{dt} = t - \frac{1}{2}e^{2t} + C;\ t = 1$ and $\frac{dy}{dt} = 0 \Rightarrow 0 = 1 - \frac{1}{2}e^2 + C \Rightarrow C = \frac{1}{2}e^2 - 1$; thus
$\frac{dy}{dt} = t - \frac{1}{2}e^{2t} + \frac{1}{2}e^2 - 1 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t + C_1;\ t = 1$ and $y = -1 \Rightarrow -1 = \frac{1}{2} - \frac{1}{4}e^2 + \frac{1}{2}e^2 - 1 + C_1$
$\Rightarrow C_1 = -\frac{1}{2} - \frac{1}{4}e^2 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t - \left(\frac{1}{2} + \frac{1}{4}e^2\right)$

55. $y = 2^x \Rightarrow y' = 2^x \ln 2$

56. $y = 3^{-x} \Rightarrow y' = 3^{-x}(\ln 3)(-1) = -3^{-x} \ln 3$

57. $y = 5^{\sqrt{s}} \Rightarrow \frac{dy}{ds} = 5^{\sqrt{s}}(\ln 5)\left(\frac{1}{2}s^{-1/2}\right) = \left(\frac{\ln 5}{2\sqrt{s}}\right)5^{\sqrt{s}}$

58. $y = 2^{s^2} \Rightarrow \frac{dy}{ds} = 2^{s^2}(\ln 2)2s = (\ln 2^2)\left(s2^{s^2}\right) = (\ln 4)s2^{s^2}$

59. $y = x^{\pi} \Rightarrow y' = \pi x^{(\pi - 1)}$

60. $y = t^{1-e} \Rightarrow \frac{dy}{dt} = (1 - e)\, t^{-e}$

61. $y = (\cos\theta)^{\sqrt{2}} \Rightarrow \frac{dy}{d\theta} = -\sqrt{2}\,(\cos\theta)^{\left(\sqrt{2}-1\right)}(\sin\theta)$

62. $y = (\ln\theta)^{\pi} \Rightarrow \frac{dy}{d\theta} = \pi(\ln\theta)^{(\pi-1)}\left(\frac{1}{\theta}\right) = \frac{\pi(\ln\theta)^{(\pi-1)}}{\theta}$

63. $y = 7^{\sec\theta}\ln 7 \Rightarrow \frac{dy}{d\theta} = (7^{\sec\theta}\ln 7)(\ln 7)(\sec\theta\tan\theta) = 7^{\sec\theta}(\ln 7)^2(\sec\theta\tan\theta)$

64. $y = 3^{\tan\theta}\ln 3 \Rightarrow \frac{dy}{d\theta} = (3^{\tan\theta}\ln 3)(\ln 3)\sec^2\theta = 3^{\tan\theta}(\ln 3)^2\sec^2\theta$

65. $y = 2^{\sin 3t} \Rightarrow \frac{dy}{dt} = (2^{\sin 3t}\ln 2)(\cos 3t)(3) = (3\cos 3t)\left(2^{\sin 3t}\right)(\ln 2)$

66. $y = 5^{-\cos 2t} \Rightarrow \frac{dy}{dt} = (5^{-\cos 2t}\ln 5)(\sin 2t)(2) = (2\sin 2t)\left(5^{-\cos 2t}\right)(\ln 5)$

67. $y = \log_2 5\theta = \frac{\ln 5\theta}{\ln 2} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 2}\right)\left(\frac{1}{5\theta}\right)(5) = \frac{1}{\theta\ln 2}$

68. $y = \log_3(1 + \theta\ln 3) = \frac{\ln(1+\theta\ln 3)}{\ln 3} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 3}\right)\left(\frac{1}{1+\theta\ln 3}\right)(\ln 3) = \frac{1}{1+\theta\ln 3}$

69. $y = \frac{\ln x}{\ln 4} + \frac{\ln x^2}{\ln 4} = \frac{\ln x}{\ln 4} + 2\,\frac{\ln x}{\ln 4} = 3\,\frac{\ln x}{\ln 4} \Rightarrow y' = \frac{3}{x\ln 4}$

70. $y = \frac{x\ln e}{\ln 25} - \frac{\ln x}{2\ln 5} = \frac{x}{2\ln 5} - \frac{\ln x}{2\ln 5} = \left(\frac{1}{2\ln 5}\right)(x - \ln x) \Rightarrow y' = \left(\frac{1}{2\ln 5}\right)\left(1 - \frac{1}{x}\right) = \frac{x-1}{2x\ln 5}$

71. $y = x^3\log_{10} x = x^3\left(\frac{\ln x}{\ln 10}\right) = \frac{1}{\ln 10}x^3\ln x \Rightarrow y' = \frac{1}{\ln 10}\left(x^3 \cdot \frac{1}{x} + 3x^2\ln x\right) = \frac{1}{\ln 10}x^2 + 3x^2\frac{\ln x}{\ln 10} = \frac{1}{\ln 10}x^2 + 3x^2\log_{10} x$

72. $y = \log_3 r \cdot \log_9 r = \left(\frac{\ln r}{\ln 3}\right)\left(\frac{\ln r}{\ln 9}\right) = \frac{\ln^2 r}{(\ln 3)(\ln 9)} \Rightarrow \frac{dy}{dr} = \left[\frac{1}{(\ln 3)(\ln 9)}\right](2\ln r)\left(\frac{1}{r}\right) = \frac{2\ln r}{r(\ln 3)(\ln 9)}$

73. $y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right) = \frac{\ln\left(\frac{x+1}{x-1}\right)^{\ln 3}}{\ln 3} = \frac{(\ln 3)\ln\left(\frac{x+1}{x-1}\right)}{\ln 3} = \ln\left(\frac{x+1}{x-1}\right) = \ln(x+1) - \ln(x-1)$
$\Rightarrow \frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$

74. $y = \log_5\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}} = \log_5\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2} = \frac{\ln\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2}}{\ln 5} = \left(\frac{\ln 5}{2}\right)\left[\frac{\ln\left(\frac{7x}{3x+2}\right)}{\ln 5}\right] = \frac{1}{2}\ln\left(\frac{7x}{3x+2}\right)$
$= \frac{1}{2}\ln 7x - \frac{1}{2}\ln(3x+2) \Rightarrow \frac{dy}{dx} = \frac{7}{2\cdot 7x} - \frac{3}{2\cdot(3x+2)} = \frac{(3x+2)-3x}{2x(3x+2)} = \frac{1}{x(3x+2)}$

75. $y = \theta\sin(\log_7\theta) = \theta\sin\left(\frac{\ln\theta}{\ln 7}\right) \Rightarrow \frac{dy}{d\theta} = \sin\left(\frac{\ln\theta}{\ln 7}\right) + \theta\left[\cos\left(\frac{\ln\theta}{\ln 7}\right)\right]\left(\frac{1}{\theta\ln 7}\right) = \sin(\log_7\theta) + \frac{1}{\ln 7}\cos(\log_7\theta)$

76. $y = \log_7\left(\frac{\sin\theta\cos\theta}{e^{\theta}2^{\theta}}\right) = \frac{\ln(\sin\theta) + \ln(\cos\theta) - \ln e^{\theta} - \ln 2^{\theta}}{\ln 7} = \frac{\ln(\sin\theta) + \ln(\cos\theta) - \theta - \theta\ln 2}{\ln 7}$
$\Rightarrow \frac{dy}{d\theta} = \frac{\cos\theta}{(\sin\theta)(\ln 7)} - \frac{\sin\theta}{(\cos\theta)(\ln 7)} - \frac{1}{\ln 7} - \frac{\ln 2}{\ln 7} = \left(\frac{1}{\ln 7}\right)(\cot\theta - \tan\theta - 1 - \ln 2)$

77. $y = \log_{10} e^x = \frac{\ln e^x}{\ln 10} = \frac{x}{\ln 10} \Rightarrow y' = \frac{1}{\ln 10}$

78. $y = \frac{\theta\cdot 5^{\theta}}{2 - \log_5\theta} = \frac{\theta\cdot 5^{\theta}}{2 - \frac{\ln\theta}{\ln 5}} \Rightarrow y' = \frac{\left(2 - \frac{\ln\theta}{\ln 5}\right)\left(\theta\cdot 5^{\theta}\ln 5 + 5^{\theta}(1)\right) - \left(\theta\cdot 5^{\theta}\right)\left(-\frac{1}{\theta\ln 5}\right)}{\left(2 - \frac{\ln\theta}{\ln 5}\right)^2} = \frac{5^{\theta}\ln 5(2 - \log_5\theta)(\theta\ln 5 + 1) + 5^{\theta}}{\ln 5(2 - \log_5\theta)^2}$

79. $y = 3^{\log_2 t} = 3^{(\ln t)/(\ln 2)} \Rightarrow \frac{dy}{dt} = \left[3^{(\ln t)/(\ln 2)}(\ln 3)\right]\left(\frac{1}{t \ln 2}\right) = \frac{1}{t}\,(\log_2 3)\, 3^{\log_2 t}$

80. $y = 3\log_8(\log_2 t) = \frac{3\ln(\log_2 t)}{\ln 8} = \frac{3\ln\left(\frac{\ln t}{\ln 2}\right)}{\ln 8} \Rightarrow \frac{dy}{dt} = \left(\frac{3}{\ln 8}\right)\left[\frac{1}{(\ln t)/(\ln 2)}\right]\left(\frac{1}{t\ln 2}\right) = \frac{3}{t(\ln t)(\ln 8)} = \frac{1}{t(\ln t)(\ln 2)}$

81. $y = \log_2\left(8t^{\ln 2}\right) = \frac{\ln 8 + \ln\left(t^{\ln 2}\right)}{\ln 2} = \frac{3\ln 2 + (\ln 2)(\ln t)}{\ln 2} = 3 + \ln t \Rightarrow \frac{dy}{dt} = \frac{1}{t}$

82. $y = \frac{t\ln\left(\left(e^{\ln 3}\right)^{\sin t}\right)}{\ln 3} = \frac{t\ln\left(3^{\sin t}\right)}{\ln 3} = \frac{t(\sin t)(\ln 3)}{\ln 3} = t\sin t \Rightarrow \frac{dy}{dt} = \sin t + t\cos t$

83. $\int 5^x\,dx = \frac{5^x}{\ln 5} + C$

84. Let $u = 3 - 3^x \Rightarrow du = -3^x \ln 3\,dx \Rightarrow -\frac{1}{\ln 3}du = 3^x dx;$
$\int \frac{3^x}{3-3^x}dx = -\frac{1}{\ln 3}\int \frac{1}{u}du = -\frac{1}{\ln 3}\ln|u| + C = -\frac{\ln|3-3^x|}{\ln 3} + C$

85. $\int_0^1 2^{-\theta}\,d\theta = \int_0^1 \left(\frac{1}{2}\right)^\theta d\theta = \left[\frac{\left(\frac{1}{2}\right)^\theta}{\ln\left(\frac{1}{2}\right)}\right]_0^1 = \frac{\frac{1}{2}}{\ln\left(\frac{1}{2}\right)} - \frac{1}{\ln\left(\frac{1}{2}\right)} = -\frac{\frac{1}{2}}{\ln\left(\frac{1}{2}\right)} = \frac{-1}{2(\ln 1 - \ln 2)} = \frac{1}{2\ln 2}$

86. $\int_{-2}^0 5^{-\theta}\,d\theta = \int_{-2}^0 \left(\frac{1}{5}\right)^\theta d\theta = \left[\frac{\left(\frac{1}{5}\right)^\theta}{\ln\left(\frac{1}{5}\right)}\right]_{-2}^0 = \frac{1}{\ln\left(\frac{1}{5}\right)} - \frac{\left(\frac{1}{5}\right)^{-2}}{\ln\left(\frac{1}{5}\right)} = \frac{1}{\ln\left(\frac{1}{5}\right)}(1-25) = \frac{-24}{\ln 1 - \ln 5} = \frac{24}{\ln 5}$

87. Let $u = x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx;\ x = 1 \Rightarrow u = 1,\ x = \sqrt{2} \Rightarrow u = 2;$
$\int_1^{\sqrt{2}} x2^{(x^2)}\,dx = \int_1^2 \left(\frac{1}{2}\right)2^u\,du = \frac{1}{2}\left[\frac{2^u}{\ln 2}\right]_1^2 = \left(\frac{1}{2\ln 2}\right)\left(2^2 - 2^1\right) = \frac{1}{\ln 2}$

88. Let $u = x^{1/2} \Rightarrow du = \frac{1}{2}x^{-1/2}\,dx \Rightarrow 2\,du = \frac{dx}{\sqrt{x}};\ x = 1 \Rightarrow u = 1,\ x = 4 \Rightarrow u = 2;$
$\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^4 2^{x^{1/2}} \cdot x^{-1/2}\,dx = 2\int_1^2 2^u\,du = \left[\frac{2^{(u+1)}}{\ln 2}\right]_1^2 = \left(\frac{1}{\ln 2}\right)\left(2^3 - 2^2\right) = \frac{4}{\ln 2}$

89. Let $u = \cos t \Rightarrow du = -\sin t\,dt \Rightarrow -du = \sin t\,dt;\ t = 0 \Rightarrow u = 1,\ t = \frac{\pi}{2} \Rightarrow u = 0;$
$\int_0^{\pi/2} 7^{\cos t}\sin t\,dt = -\int_1^0 7^u\,du = \left[-\frac{7^u}{\ln 7}\right]_1^0 = \left(\frac{-1}{\ln 7}\right)\left(7^0 - 7\right) = \frac{6}{\ln 7}$

90. Let $u = \tan t \Rightarrow du = \sec^2 t\,dt;\ t = 0 \Rightarrow u = 0,\ t = \frac{\pi}{4} \Rightarrow u = 1;$
$\int_0^{\pi/4}\left(\frac{1}{3}\right)^{\tan t}\sec^2 t\,dt = \int_0^1 \left(\frac{1}{3}\right)^u du = \left[\frac{\left(\frac{1}{3}\right)^u}{\ln\left(\frac{1}{3}\right)}\right]_0^1 = \left(-\frac{1}{\ln 3}\right)\left[\left(\frac{1}{3}\right)^1 - \left(\frac{1}{3}\right)^0\right] = \frac{2}{3\ln 3}$

91. Let $u = x^{2x} \Rightarrow \ln u = 2x\ln x \Rightarrow \frac{1}{u}\frac{du}{dx} = 2\ln x + (2x)\left(\frac{1}{x}\right) \Rightarrow \frac{du}{dx} = 2u(\ln x + 1) \Rightarrow \frac{1}{2}du = x^{2x}(1 + \ln x)\,dx;$
$x = 2 \Rightarrow u = 2^4 = 16,\ x = 4 \Rightarrow u = 4^8 = 65{,}536;$
$\int_2^4 x^{2x}(1 + \ln x)\,dx = \frac{1}{2}\int_{16}^{65{,}536} du = \frac{1}{2}[u]_{16}^{65{,}536} = \frac{1}{2}(65{,}536 - 16) = \frac{65{,}520}{2} = 32{,}760$

92. Let $u = 1 + 2^{x^2} \Rightarrow du = 2^{x^2}(2x)\ln 2\,dx \Rightarrow \frac{1}{2\ln 2}du = 2^{x^2}x\,dx$
$\int \frac{x\,2^{x^2}}{1+2^{x^2}}dx = \frac{1}{2\ln 2}\int\frac{1}{u}du = \frac{1}{2\ln 2}\ln|u| + C = \frac{\ln\left(1+2^{x^2}\right)}{2\ln 2} + C$

93. $\int 3x^{\sqrt{3}}\,dx = \frac{3x^{\left(\sqrt{3}+1\right)}}{\sqrt{3}+1} + C$

94. $\int x^{\left(\sqrt{2}-1\right)}\,dx = \frac{x^{\sqrt{2}}}{\sqrt{2}} + C$

95. $\int_0^3 \left(\sqrt{2}+1\right) x^{\sqrt{2}}\, dx = \left[x^{\left(\sqrt{2}+1\right)}\right]_0^3 = 3^{\left(\sqrt{2}+1\right)}$

96. $\int_1^e x^{(\ln 2)-1}\, dx = \left[\frac{x^{\ln 2}}{\ln 2}\right]_1^e = \frac{e^{\ln 2} - 1^{\ln 2}}{\ln 2} = \frac{2-1}{\ln 2} = \frac{1}{\ln 2}$

97. $\int \frac{\log_{10} x}{x}\, dx = \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right) dx$; $\left[u = \ln x \Rightarrow du = \frac{1}{x}\, dx\right]$
$\rightarrow \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right) dx = \frac{1}{\ln 10} \int u\, du = \left(\frac{1}{\ln 10}\right)\left(\frac{1}{2} u^2\right) + C = \frac{(\ln x)^2}{2 \ln 10} + C$

98. $\int_1^4 \frac{\log_2 x}{x}\, dx = \int_1^4 \left(\frac{\ln x}{\ln 2}\right)\left(\frac{1}{x}\right) dx$; $\left[u = \ln x \Rightarrow du = \frac{1}{x}\, dx; x = 1 \Rightarrow u = 0, x = 4 \Rightarrow u = \ln 4\right]$
$\rightarrow \int_1^4 \left(\frac{\ln x}{\ln 2}\right)\left(\frac{1}{x}\right) dx = \int_0^{\ln 4} \left(\frac{1}{\ln 2}\right) u\, du = \left(\frac{1}{\ln 2}\right)\left[\frac{1}{2} u^2\right]_0^{\ln 4} = \left(\frac{1}{\ln 2}\right)\left[\frac{1}{2}(\ln 4)^2\right] = \frac{(\ln 4)^2}{2 \ln 2} = \frac{(\ln 4)^2}{\ln 4} = \ln 4$

99. $\int_1^4 \frac{\ln 2 \log_2 x}{x}\, dx = \int_1^4 \left(\frac{\ln 2}{x}\right)\left(\frac{\ln x}{\ln 2}\right) dx = \int_1^4 \frac{\ln x}{x}\, dx = \left[\frac{1}{2}(\ln x)^2\right]_1^4 = \frac{1}{2}\left[(\ln 4)^2 - (\ln 1)^2\right] = \frac{1}{2}(\ln 4)^2 = \frac{1}{2}(2 \ln 2)^2 = 2(\ln 2)^2$

100. $\int_1^e \frac{2 \ln 10 (\log_{10} x)}{x}\, dx = \int_1^e \frac{(\ln 10)(2 \ln x)}{(\ln 10)}\left(\frac{1}{x}\right) dx = \left[(\ln x)^2\right]_1^e = (\ln e)^2 - (\ln 1)^2 = 1$

101. $\int_0^2 \frac{\log_2 (x+2)}{x+2}\, dx = \frac{1}{\ln 2}\int_0^2 [\ln (x+2)]\left(\frac{1}{x+2}\right) dx = \left(\frac{1}{\ln 2}\right)\left[\frac{(\ln (x+2))^2}{2}\right]_0^2 = \left(\frac{1}{\ln 2}\right)\left[\frac{(\ln 4)^2}{2} - \frac{(\ln 2)^2}{2}\right]$
$= \left(\frac{1}{\ln 2}\right)\left[\frac{4(\ln 2)^2}{2} - \frac{(\ln 2)^2}{2}\right] = \frac{3}{2} \ln 2$

102. $\int_{1/10}^{10} \frac{\log_{10} (10x)}{x}\, dx = \frac{10}{\ln 10}\int_{1/10}^{10} [\ln (10x)]\left(\frac{1}{10x}\right) dx = \left(\frac{10}{\ln 10}\right)\left[\frac{(\ln (10x))^2}{20}\right]_{1/10}^{10} = \left(\frac{10}{\ln 10}\right)\left[\frac{(\ln 100)^2}{20} - \frac{(\ln 1)^2}{2}\right]$
$= \left(\frac{10}{\ln 10}\right)\left[\frac{4(\ln 10)^2}{20}\right] = 2 \ln 10$

103. $\int_0^9 \frac{2 \log_{10} (x+1)}{x+1}\, dx = \frac{2}{\ln 10}\int_0^9 \ln (x+1)\left(\frac{1}{x+1}\right) dx = \left(\frac{2}{\ln 10}\right)\left[\frac{(\ln (x+1))^2}{2}\right]_0^9 = \left(\frac{2}{\ln 10}\right)\left[\frac{(\ln 10)^2}{2} - \frac{(\ln 1)^2}{2}\right] = \ln 10$

104. $\int_2^3 \frac{2 \log_2 (x-1)}{x-1}\, dx = \frac{2}{\ln 2}\int_2^3 \ln (x-1)\left(\frac{1}{x-1}\right) dx = \left(\frac{2}{\ln 2}\right)\left[\frac{(\ln (x-1))^2}{2}\right]_2^3 = \left(\frac{2}{\ln 2}\right)\left[\frac{(\ln 2)^2}{2} - \frac{(\ln 1)^2}{2}\right] = \ln 2$

105. $\int \frac{dx}{x \log_{10} x} = \int \left(\frac{\ln 10}{\ln x}\right)\left(\frac{1}{x}\right) dx = (\ln 10)\int \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) dx$; $\left[u = \ln x \Rightarrow du = \frac{1}{x}\, dx\right]$
$\rightarrow (\ln 10)\int \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) dx = (\ln 10)\int \frac{1}{u}\, du = (\ln 10) \ln |u| + C = (\ln 10) \ln |\ln x| + C$

106. $\int \frac{dx}{x (\log_8 x)^2} = \int \frac{dx}{x\left(\frac{\ln x}{\ln 8}\right)^2} = (\ln 8)^2 \int \frac{(\ln x)^{-2}}{x}\, dx = (\ln 8)^2\, \frac{(\ln x)^{-1}}{-1} + C = -\frac{(\ln 8)^2}{\ln x} + C$

107. $\int_1^{\ln x} \frac{1}{t}\, dt = \left[\ln |t|\right]_1^{\ln x} = \ln |\ln x| - \ln 1 = \ln (\ln x),\ x > 1$

108. $\int_1^{e^x} \frac{1}{t}\, dt = \left[\ln |t|\right]_1^{e^x} = \ln e^x - \ln 1 = x \ln e = x$

109. $\int_1^{1/x} \frac{1}{t}\, dt = \left[\ln |t|\right]_1^{1/x} = \ln \left|\frac{1}{x}\right| - \ln 1 = (\ln 1 - \ln |x|) - \ln 1 = -\ln x,\ x > 0$

110. $\frac{1}{\ln a}\int_1^x \frac{1}{t}\, dt = \left[\frac{1}{\ln a} \ln |t|\right]_1^x = \frac{\ln x}{\ln a} - \frac{\ln 1}{\ln a} = \log_a x,\ x > 0$

111. $y = (x+1)^x \Rightarrow \ln y = \ln (x+1)^x = x \ln (x+1) \Rightarrow \frac{y'}{y} = \ln (x+1) + x \cdot \frac{1}{(x+1)} \Rightarrow y' = (x+1)^x \left[\frac{x}{x+1} + \ln (x+1)\right]$

112. $y = x^2 + x^{2x} \Rightarrow y - x^2 = x^{2x} \Rightarrow \ln(y - x^2) = \ln x^{2x} = 2x \ln x \Rightarrow \frac{1}{y - x^2}(y' - 2x) = 2x \cdot \frac{1}{x} + 2 \cdot \ln x = 2 + 2\ln x$
$\Rightarrow y' - 2x = (y - x^2)(2 + 2\ln x) \Rightarrow y' = ((x^2 + x^{2x}) - x^2)(2 + 2\ln x) + 2x = 2(x + x^{2x} + x^{2x} \ln x)$

113. $y = \left(\sqrt{t}\right)^t = \left(t^{1/2}\right)^t = t^{t/2} \Rightarrow \ln y = \ln t^{t/2} = \left(\frac{t}{2}\right) \ln t \Rightarrow \frac{1}{y} \frac{dy}{dt} = \left(\frac{1}{2}\right)(\ln t) + \left(\frac{t}{2}\right)\left(\frac{1}{t}\right) = \frac{\ln t}{2} + \frac{1}{2}$
$\Rightarrow \frac{dy}{dt} = \left(\sqrt{t}\right)^t \left(\frac{\ln t}{2} + \frac{1}{2}\right)$

114. $y = t^{\sqrt{t}} = t^{\left(t^{1/2}\right)} \Rightarrow \ln y = \ln t^{\left(t^{1/2}\right)} = \left(t^{1/2}\right)(\ln t) \Rightarrow \frac{1}{y} \frac{dy}{dt} = \left(\frac{1}{2} t^{-1/2}\right)(\ln t) + t^{1/2}\left(\frac{1}{t}\right) = \frac{\ln t + 2}{2\sqrt{t}} \Rightarrow \frac{dy}{dt} = \left(\frac{\ln t + 2}{2\sqrt{t}}\right) t^{\sqrt{t}}$

115. $y = (\sin x)^x \Rightarrow \ln y = \ln(\sin x)^x = x \ln(\sin x) \Rightarrow \frac{y'}{y} = \ln(\sin x) + x\left(\frac{\cos x}{\sin x}\right) \Rightarrow y' = (\sin x)^x [\ln(\sin x) + x \cot x]$

116. $y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} = (\sin x)(\ln x) \Rightarrow \frac{y'}{y} = (\cos x)(\ln x) + (\sin x)\left(\frac{1}{x}\right) = \frac{\sin x + x(\ln x)(\cos x)}{x}$
$\Rightarrow y' = x^{\sin x}\left[\frac{\sin x + x(\ln x)(\cos x)}{x}\right]$

117. $y = \sin x^x \Rightarrow y' = \cos x^x \frac{d}{dx}(x^x)$; if $u = x^x \Rightarrow \ln u = \ln x^x = x \ln x \Rightarrow \frac{u'}{u} = x \cdot \frac{1}{x} + 1 \cdot \ln x = 1 + \ln x$
$\Rightarrow u' = x^x(1 + \ln x) \Rightarrow y' = \cos x^x \cdot x^x(1 + \ln x) = x^x \cos x^x(1 + \ln x)$

118. $y = (\ln x)^{\ln x} \Rightarrow \ln y = (\ln x)\ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{x}\right)\ln(\ln x) + (\ln x)\left(\frac{1}{\ln x}\right)\frac{d}{dx}(\ln x) = \frac{\ln(\ln x)}{x} + \frac{1}{x}$
$\Rightarrow y' = \left(\frac{\ln(\ln x) + 1}{x}\right)(\ln x)^{\ln x}$

119. $f(x) = e^x - 2x \Rightarrow f'(x) = e^x - 2$; $f'(x) = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2$; $f(0) = 1$, the absolute maximum; $f(\ln 2) = 2 - 2\ln 2 \approx 0.613706$, the absolute minimum; $f(1) = e - 2 \approx 0.71828$, a relative or local maximum since $f''(x) = e^x$ is always positive.

120. The function $f(x) = 2e^{\sin(x/2)}$ has a maximum whenever $\sin\frac{x}{2} = 1$ and a minimum whenever $\sin\frac{x}{2} = -1$. Therefore the maximums occur at $x = \pi + 2k(2\pi)$ and the minimums occur at $x = 3\pi + 2k(2\pi)$, where k is any integer. The maximum is $2e \approx 5.43656$ and the minimum is $\frac{2}{e} \approx 0.73576$.

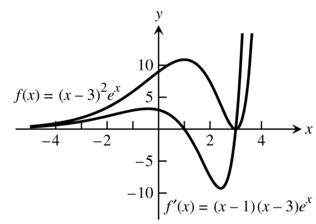
121. $f(x) = x e^{-x} \Rightarrow f'(x) = x e^{-x}(-1) + e^{-x} = e^{-x} - x e^{-x} \Rightarrow f''(x) = -e^{-x} - (x e^{-x}(-1) + e^{-x}) = x e^{-x} - 2e^{-x}$
(a) $f'(x) = 0 \Rightarrow e^{-x} - x e^{-x} = e^{-x}(1 - x) = 0 \Rightarrow e^{-x} = 0$ or $1 - x = 0 \Rightarrow x = 1$, $f(1) = (1)e^{-1} = \frac{1}{e}$; using second derivative test, $f''(1) = (1)e^{-1} - 2e^{-1} = -\frac{1}{e} < 0 \Rightarrow$ absolute maximum at $\left(1, \frac{1}{e}\right)$
(b) $f''(x) = 0 \Rightarrow x e^{-x} - 2e^{-x} = e^{-x}(x - 2) = 0 \Rightarrow e^{-x} = 0$ or $x - 2 = 0 \Rightarrow x = 2$, $f(2) = (2)e^{-2} = \frac{2}{e^2}$; since $f''(1) < 0$ and $f''(3) = e^{-3}(3 - 2) = \frac{1}{e^3} > 0 \Rightarrow$ point of inflection at $\left(2, \frac{2}{e^2}\right)$

122. $f(x) = \frac{e^x}{1 + e^{2x}} \Rightarrow f'(x) = \frac{(1 + e^{2x})e^x - e^x(2e^{2x})}{(1 + e^{2x})^2} = \frac{e^x - e^{3x}}{(1 + e^{2x})^2} \Rightarrow f''(x) = \frac{(1 + e^{2x})^2(e^x - 3e^{3x}) - (e^x - e^{3x})2(1 + e^{2x})(2e^{2x})}{\left[(1 + e^{2x})^2\right]^2}$
$= \frac{e^x(1 - 6e^{2x} + e^{4x})}{(1 + e^{2x})^3}$
(a) $f'(x) = 0 \Rightarrow e^x - e^{3x} = 0 \Rightarrow e^x(1 - e^{2x}) = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0$; $f(0) = \frac{e^0}{1 + e^{2(0)}} = \frac{1}{2}$;
$f'(x) =$ undefined $\Rightarrow (1 + e^{2x})^2 = 0 \Rightarrow e^{2x} = -1 \Rightarrow$ no real solutions. Using the second derivative test,
$f''(0) = \frac{e^0(1 - 6e^{2(0)} + e^{4(0)})}{(1 + e^{2(0)})^3} = \frac{-4}{8} < 0 \Rightarrow$ absolute maximum at $\left(0, \frac{1}{2}\right)$
(b) $f''(x) = 0 \Rightarrow e^x(1 - 6e^{2x} + e^{4x}) \Rightarrow e^x = 0$ or $1 - 6e^{2x} + e^{4x} = 0 \Rightarrow e^{2x} = \frac{-(-6) \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}$,
$\Rightarrow x = \frac{\ln\left(3 + 2\sqrt{2}\right)}{2}$ or $x = \frac{\ln\left(3 - 2\sqrt{2}\right)}{2}$. $f\left(\frac{\ln\left(3 + 2\sqrt{2}\right)}{2}\right) = \frac{\sqrt{3 + 2\sqrt{2}}}{4 + 2\sqrt{2}}$ and $f\left(\frac{\ln\left(3 - 2\sqrt{2}\right)}{2}\right) = \frac{\sqrt{3 - 2\sqrt{2}}}{4 - 2\sqrt{2}}$;

since $f''(-1) > 0$, $f''(0) < 0$, and $f''(1) > 0 \Rightarrow$ points of inflection at $\left(\frac{\ln(3+2\sqrt{2})}{2}, \frac{\sqrt{3+2\sqrt{2}}}{4+2\sqrt{2}}\right)$ and $\left(\frac{\ln(3-2\sqrt{2})}{2}, \frac{\sqrt{3-2\sqrt{2}}}{4-2\sqrt{2}}\right)$.

123. $f(x) = x^2 \ln \frac{1}{x} \Rightarrow f'(x) = 2x \ln \frac{1}{x} + x^2\left(\frac{1}{\frac{1}{x}}\right)(-x^{-2}) = 2x \ln \frac{1}{x} - x = -x(2 \ln x + 1)$; $f'(x) = 0 \Rightarrow x = 0$ or $\ln x = -\frac{1}{2}$. Since $x = 0$ is not in the domain of f, $x = e^{-1/2} = \frac{1}{\sqrt{e}}$. Also, $f'(x) > 0$ for $0 < x < \frac{1}{\sqrt{e}}$ and $f'(x) < 0$ for $x > \frac{1}{\sqrt{e}}$. Therefore, $f\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{e} \ln \sqrt{e} = \frac{1}{e} \ln e^{1/2} = \frac{1}{2e} \ln e = \frac{1}{2e}$ is the absolute maximum value of f assumed at $x = \frac{1}{\sqrt{e}}$.

124. $f(x) = (x-3)^2 e^x \Rightarrow f'(x) = 2(x-3)e^x + (x-3)^2 e^x = (x-3)e^x(2 + x - 3) = (x-1)(x-3)e^x$; thus $f'(x) > 0$ for $x < 1$ or $x > 3$, and $f'(x) < 0$ for $1 < x < 3 \Rightarrow f(1) = 4e \approx 10.87$ is a local maximum and $f(3) = 0$ is a local minimum. Since $f(x) \geq 0$ for all x, $f(3) = 0$ is also an absolute minimum.



125. $\int_0^{\ln 3} (e^{2x} - e^x)\,dx = \left[\frac{e^{2x}}{2} - e^x\right]_0^{\ln 3} = \left(\frac{e^{2\ln 3}}{2} - e^{\ln 3}\right) - \left(\frac{e^0}{2} - e^0\right) = \left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right) = \frac{8}{2} - 2 = 2$

126. $\int_0^{2\ln 2} \left(e^{x/2} - e^{-x/2}\right) dx = \left[2e^{x/2} + 2e^{-x/2}\right]_0^{2\ln 2} = \left(2e^{\ln 2} + 2e^{-\ln 2}\right) - \left(2e^0 + 2e^0\right) = (4+1) - (2+2) = 5 - 4 = 1$

127. $L = \int_0^1 \sqrt{1 + \frac{e^x}{4}}\,dx \Rightarrow \frac{dy}{dx} = \frac{e^{x/2}}{2} \Rightarrow y = e^{x/2} + C$; $y(0) = 0 \Rightarrow 0 = e^0 + C \Rightarrow C = -1 \Rightarrow y = e^{x/2} - 1$

128. $S = 2\pi \int_0^{\ln 2} \left(\frac{e^y + e^{-y}}{2}\right) \sqrt{1 + \left(\frac{e^y - e^{-y}}{2}\right)^2}\,dy = 2\pi \int_0^{\ln 2} \left(\frac{e^y + e^{-y}}{2}\right) \sqrt{1 + \frac{1}{4}\left(e^{2y} - 2 + e^{-2y}\right)}\,dy$
$= 2\pi \int_0^{\ln 2} \left(\frac{e^y + e^{-y}}{2}\right) \sqrt{\left(\frac{e^y + e^{-y}}{2}\right)^2}\,dy = 2\pi \int_0^{\ln 2} \left(\frac{e^y + e^{-y}}{2}\right)^2 dy = \frac{\pi}{2} \int_0^{\ln 2} \left(e^{2y} + 2 + e^{-2y}\right) dy$
$= \frac{\pi}{2}\left[\frac{1}{2}e^{2y} + 2y - \frac{1}{2}e^{-2y}\right]_0^{\ln 2} = \frac{\pi}{2}\left[\left(\frac{1}{2}e^{2\ln 2} + 2\ln 2 - \frac{1}{2}e^{-2\ln 2}\right) - \left(\frac{1}{2} + 0 - \frac{1}{2}\right)\right]$
$= \frac{\pi}{2}\left(\frac{1}{2} \cdot 4 + 2\ln 2 - \frac{1}{2} \cdot \frac{1}{4}\right) = \frac{\pi}{2}\left(2 - \frac{1}{8} + 2\ln 2\right) = \pi\left(\frac{15}{16} + \ln 2\right)$

129. $y = \frac{1}{2}(e^x + e^{-x}) \Rightarrow \frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$; $L = \int_0^1 \sqrt{1 + \left(\frac{1}{2}(e^x - e^{-x})\right)^2}\,dx = \int_0^1 \sqrt{1 + \frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}}\,dx$
$= \int_0^1 \sqrt{\frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4}}\,dx = \int_0^1 \sqrt{\left(\frac{1}{2}(e^x + e^{-x})\right)^2}\,dx = \int_0^1 \frac{1}{2}(e^x + e^{-x})\,dx = \frac{1}{2}\left[e^x - e^{-x}\right]_0^1 = \frac{1}{2}\left(e - \frac{1}{e}\right) - 0 = \frac{e^2 - 1}{2e}$

130. $y = \ln(e^x - 1) - \ln(e^x + 1) \Rightarrow \frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1} = \frac{2e^x}{e^{2x} - 1}$; $L = \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{2e^x}{e^{2x} - 1}\right)^2}\,dx = \int_{\ln 2}^{\ln 3} \sqrt{1 + \frac{4e^{2x}}{(e^{2x} - 1)^2}}\,dx$
$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{(e^{2x} - 1)^2}}\,dx = \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{(e^{2x} - 1)^2}}\,dx = \int_{\ln 2}^{\ln 3} \sqrt{\frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}}\,dx = \int_{\ln 2}^{\ln 3} \frac{e^{2x} + 1}{e^{2x} - 1}\,dx = \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x} + 1}{e^x}}{\frac{e^{2x} - 1}{e^x}}\,dx$
$= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$; $\Big[$let $u = e^x - e^{-x} \Rightarrow du = (e^x + e^{-x})dx$, $x = \ln 2 \Rightarrow u = e^{\ln 2} - e^{-\ln 2} = 2 - \frac{1}{2} = \frac{3}{2}$, $x = \ln 3$
$\Rightarrow u = e^{\ln 3} - e^{-\ln 3} = 3 - \frac{1}{3} = \frac{8}{3}\Big] \to \int_{3/2}^{8/3} \frac{1}{u}\,du = \left[\ln|u|\right]_{3/2}^{8/3} = \ln\left(\frac{8}{3}\right) - \ln\left(\frac{3}{2}\right) = \ln\left(\frac{16}{9}\right)$

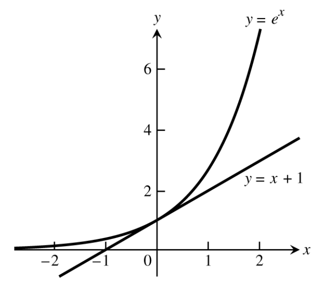
131. $y = \ln \cos x \Rightarrow \frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$; $L = \int_0^{\pi/4} \sqrt{1 + (-\tan x)^2}\,dx = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx$
$= \int_0^{\pi/4} \sec x\,dx = \left[\ln|\sec x + \tan x|\right]_0^{\pi/4} = \left(\ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right|\right) - (0) = \ln\left(\sqrt{2} + 1\right)$

132. $y = \ln \csc x \Rightarrow \frac{dy}{dx} = \frac{-\csc x \cot x}{\csc x} = -\cot x$; $L = \int_{\pi/6}^{\pi/4} \sqrt{1 + (-\cot x)^2}\, dx = \int_{\pi/6}^{\pi/4} \sqrt{1 + \cot^2 x}\, dx = \int_{\pi/6}^{\pi/4} \sqrt{\csc^2 x}\, dx$
$= \int_{\pi/6}^{\pi/4} \csc x\, dx = [-\ln|\csc x + \cot x|]_{\pi/6}^{\pi/4} = \left(-\ln\left|\csc\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right)\right|\right) + \left(\ln\left|\csc\left(\frac{\pi}{6}\right) + \cot\left(\frac{\pi}{6}\right)\right|\right)$
$= -\ln\left(\sqrt{2} + 1\right) + \ln\left(2 + \sqrt{3}\right) = \ln\left(\frac{2+\sqrt{3}}{\sqrt{2}+1}\right)$

133. (a) $\frac{d}{dx}(x \ln x - x + C) = x \cdot \frac{1}{x} + \ln x - 1 + 0 = \ln x$
(b) average value $= \frac{1}{e-1} \int_1^e \ln x\, dx = \frac{1}{e-1}[x \ln x - x]_1^e = \frac{1}{e-1}[(e \ln e - e) - (1 \ln 1 - 1)] = \frac{1}{e-1}(e - e + 1) = \frac{1}{e-1}$

134. average value $= \frac{1}{2-1} \int_1^2 \frac{1}{x}\, dx = [\ln|x|]_1^2 = \ln 2 - \ln 1 = \ln 2$

135. (a) $f(x) = e^x \Rightarrow f'(x) = e^x$; $L(x) = f(0) + f'(0)(x - 0) \Rightarrow L(x) = 1 + x$
(b) $f(0) = 1$ and $L(0) = 1 \Rightarrow$ error $= 0$; $f(0.2) = e^{0.2} \approx 1.22140$ and $L(0.2) = 1.2 \Rightarrow$ error ≈ 0.02140
(c) Since $y'' = e^x > 0$, the tangent line approximation always lies below the curve $y = e^x$. Thus $L(x) = x + 1$ never overestimates e^x.

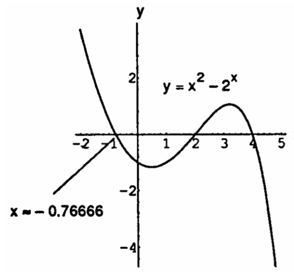


136. (a) $y = e^x \Rightarrow y'' = e^x > 0$ for all $x \Rightarrow$ the graph of $y = e^x$ is always concave upward
(b) area of the trapezoid ABCD $< \int_{\ln a}^{\ln b} e^x\, dx <$ area of the trapezoid AEFD $\Rightarrow \frac{1}{2}(AB + CD)(\ln b - \ln a)$
$< \int_{\ln a}^{\ln b} e^x\, dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$. Now $\frac{1}{2}(AB + CD)$ is the height of the midpoint
$M = e^{(\ln a + \ln b)/2}$ since the curve containing the points B and C is linear $\Rightarrow e^{(\ln a + \ln b)/2}(\ln b - \ln a)$
$< \int_{\ln a}^{\ln b} e^x\, dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$
(c) $\int_{\ln a}^{\ln b} e^x\, dx = [e^x]_{\ln a}^{\ln b} = e^{\ln b} - e^{\ln a} = b - a$, so part (b) implies that
$e^{(\ln a + \ln b)/2}(\ln b - \ln a) < b - a < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a) \Rightarrow e^{(\ln a + \ln b)/2} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$
$\Rightarrow e^{\ln a/2} \cdot e^{\ln b/2} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2} \Rightarrow \sqrt{e^{\ln a}}\sqrt{e^{\ln b}} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2} \Rightarrow \sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$

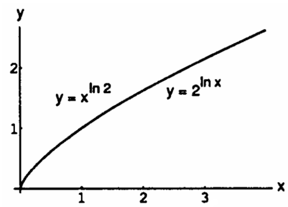
137. $A = \int_{-2}^{2} \frac{2x}{1+x^2}\, dx = 2\int_0^2 \frac{2x}{1+x^2}\, dx$; $[u = 1 + x^2 \Rightarrow du = 2x\, dx;\ x = 0 \Rightarrow u = 1,\ x = 2 \Rightarrow u = 5]$
$\to A = 2\int_1^5 \frac{1}{u}\, du = 2[\ln|u|]_1^5 = 2(\ln 5 - \ln 1) = 2 \ln 5$

138. $A = \int_{-1}^{1} 2^{(1-x)}\, dx = 2\int_{-1}^{1} \left(\frac{1}{2}\right)^x dx = 2\left[\frac{\left(\frac{1}{2}\right)^x}{\ln\left(\frac{1}{2}\right)}\right]_{-1}^{1} = -\frac{2}{\ln 2}\left(\frac{1}{2} - 2\right) = \left(-\frac{2}{\ln 2}\right)\left(-\frac{3}{2}\right) = \frac{3}{\ln 2}$

139. From zooming in on the graph at the right, we estimate the third root to be $x \approx -0.76666$

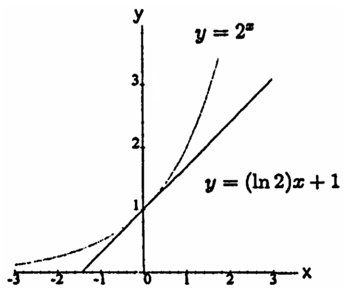


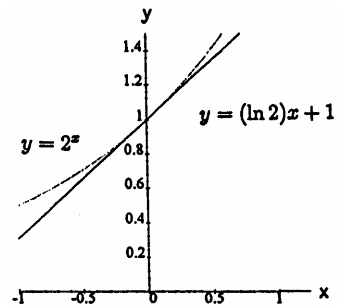
140. The functions $f(x) = x^{\ln 2}$ and $g(x) = 2^{\ln x}$ appear to have identical graphs for $x > 0$. This is no accident, because $x^{\ln 2} = e^{\ln 2 \cdot \ln x} = \left(e^{\ln 2}\right)^{\ln x} = 2^{\ln x}$.



141. (a) $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2$; $L(x) = \left(2^0 \ln 2\right) x + 2^0 = x \ln 2 + 1 \approx 0.69x + 1$

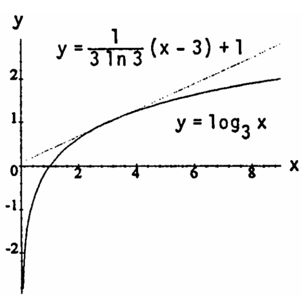
(b)

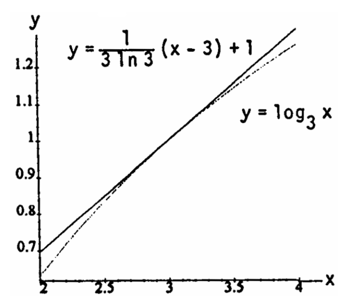




142. (a) $f(x) = \log_3 x \Rightarrow f'(x) = \frac{1}{x \ln 3}$, and $f(3) = \frac{\ln 3}{\ln 3} \Rightarrow L(x) = \frac{1}{3 \ln 3}(x-3) + \frac{\ln 3}{\ln 3} = \frac{x}{3 \ln 3} - \frac{1}{\ln 3} + 1 \approx 0.30x + 0.09$

(b)





143. (a) The point of tangency is $(p, \ln p)$ and $m_{\text{tangent}} = \frac{1}{p}$ since $\frac{dy}{dx} = \frac{1}{x}$. The tangent line passes through $(0, 0) \Rightarrow$ the equation of the tangent line is $y = \frac{1}{p}x$. The tangent line also passes through $(p, \ln p) \Rightarrow \ln p = \frac{1}{p}p = 1 \Rightarrow p = e$, and the tangent line equation is $y = \frac{1}{e}x$.

(b) $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$ for $x \neq 0 \Rightarrow y = \ln x$ is concave downward over its domain. Therefore, $y = \ln x$ lies below the graph of $y = \frac{1}{e}x$ for all $x > 0$, $x \neq e$, and $\ln x < \frac{x}{e}$ for $x > 0$, $x \neq e$.

(c) Multiplying by e, $e \ln x < x$ or $\ln x^e < x$.

(d) Exponentiating both sides of $\ln x^e < x$, we have $e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

(e) Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, e^π is bigger.

144. Using Newton's Method: $f(x) = \ln(x) - 1 \Rightarrow f'(x) = \frac{1}{x} \Rightarrow x_{n+1} = x_n - \frac{\ln(x_n) - 1}{\frac{1}{xn}} \Rightarrow x_{n+1} = x_n\left[2 - \ln(x_n)\right]$.
Then, $x_1 = 2$, $x_2 = 2.61370564$, $x_3 = 2.71624393$, and $x_5 = 2.71828183$. Many other methods may be used. For example, graph $y = \ln x - 1$ and determine the zero of y.

7.4 EXPONENTIAL CHANGE AND SEPARABLE DIFFERENTIAL EQUATIONS

1. (a) $y = e^{-x} \Rightarrow y' = -e^{-x} \Rightarrow 2y' + 3y = 2(-e^{-x}) + 3e^{-x} = e^{-x}$
(b) $y = e^{-x} + e^{-3x/2} \Rightarrow y' = -e^{-x} - \frac{3}{2}e^{-3x/2} \Rightarrow 2y' + 3y = 2\left(-e^{-x} - \frac{3}{2}e^{-3x/2}\right) + 3\left(e^{-x} + e^{-3x/2}\right) = e^{-x}$
(c) $y = e^{-x} + Ce^{-3x/2} \Rightarrow y' = -e^{-x} - \frac{3}{2}Ce^{-3x/2} \Rightarrow 2y' + 3y = 2\left(-e^{-x} - \frac{3}{2}Ce^{-3x/2}\right) + 3\left(e^{-x} + Ce^{-3x/2}\right) = e^{-x}$

2. (a) $y = -\frac{1}{x} \Rightarrow y' = \frac{1}{x^2} = \left(-\frac{1}{x}\right)^2 = y^2$
(b) $y = -\frac{1}{x+3} \Rightarrow y' = \frac{1}{(x+3)^2} = \left[-\frac{1}{(x+3)}\right]^2 = y^2$
(c) $y = \frac{1}{x+C} \Rightarrow y' = \frac{1}{(x+C)^2} = \left[-\frac{1}{x+C}\right]^2 = y^2$

3. $y = \frac{1}{x}\int_1^x \frac{e^t}{t}\,dt \Rightarrow y' = -\frac{1}{x^2}\int_1^x \frac{e^t}{t}\,dt + \left(\frac{1}{x}\right)\left(\frac{e^x}{x}\right) \Rightarrow x^2y' = -\int_1^x \frac{e^t}{t}\,dt + e^x = -x\left(\frac{1}{x}\int_1^x \frac{e^t}{t}\,dt\right) + e^x = -xy + e^x$
$\Rightarrow x^2y' + xy = e^x$

4. $y = \frac{1}{\sqrt{1+x^4}}\int_1^x \sqrt{1+t^4}\,dt \Rightarrow y' = -\frac{1}{2}\left[\frac{4x^3}{\left(\sqrt{1+x^4}\right)^3}\right]\int_1^x \sqrt{1+t^4}\,dt + \frac{1}{\sqrt{1+x^4}}\left(\sqrt{1+x^4}\right)$
$\Rightarrow y' = \left(\frac{-2x^3}{1+x^4}\right)\left(\frac{1}{\sqrt{1+x^4}}\int_1^x \sqrt{1+t^4}\,dt\right) + 1 \Rightarrow y' = \left(\frac{-2x^3}{1+x^4}\right)y + 1 \Rightarrow y' + \frac{2x^3}{1+x^4}\cdot y = 1$

5. $y = e^{-x}\tan^{-1}(2e^x) \Rightarrow y' = -e^{-x}\tan^{-1}(2e^x) + e^{-x}\left[\frac{1}{1+(2e^x)^2}\right](2e^x) = -e^{-x}\tan^{-1}(2e^x) + \frac{2}{1+4e^{2x}}$
$\Rightarrow y' = -y + \frac{2}{1+4e^{2x}} \Rightarrow y' + y = \frac{2}{1+4e^{2x}}$; $y(-\ln 2) = e^{-(-\ln 2)}\tan^{-1}(2e^{-\ln 2}) = 2\tan^{-1}1 = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

6. $y = (x-2)e^{-x^2} \Rightarrow y' = e^{-x^2} + \left(-2xe^{-x^2}\right)(x-2) \Rightarrow y' = e^{-x^2} - 2xy$; $y(2) = (2-2)e^{-2^2} = 0$

7. $y = \frac{\cos x}{x} \Rightarrow y' = \frac{-x\sin x - \cos x}{x^2} \Rightarrow y' = -\frac{\sin x}{x} - \frac{1}{x}\left(\frac{\cos x}{x}\right) \Rightarrow y' = -\frac{\sin x}{x} - \frac{y}{x} \Rightarrow xy' = -\sin x - y \Rightarrow xy' + y = -\sin x$;
$y\left(\frac{\pi}{2}\right) = \frac{\cos(\pi/2)}{(\pi/2)} = 0$

8. $y = \frac{x}{\ln x} \Rightarrow y' = \frac{\ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} \Rightarrow y' = \frac{1}{\ln x} - \frac{1}{(\ln x)^2} \Rightarrow x^2y' = \frac{x^2}{\ln x} - \frac{x^2}{(\ln x)^2} \Rightarrow x^2y' = xy - y^2$; $y(e) = \frac{e}{\ln e} = e$.

9. $2\sqrt{xy}\,\frac{dy}{dx} = 1 \Rightarrow 2x^{1/2}y^{1/2}\,dy = dx \Rightarrow 2y^{1/2}\,dy = x^{-1/2}\,dx \Rightarrow \int 2y^{1/2}\,dy = \int x^{-1/2}\,dx \Rightarrow 2\left(\frac{2}{3}y^{3/2}\right) = 2x^{1/2} + C_1$
$\Rightarrow \frac{2}{3}y^{3/2} - x^{1/2} = C$, where $C = \frac{1}{2}C_1$

10. $\frac{dy}{dx} = x^2\sqrt{y} \Rightarrow dy = x^2y^{1/2}\,dx \Rightarrow y^{-1/2}\,dy = x^2\,dx \Rightarrow \int y^{-1/2}\,dy = \int x^2\,dx \Rightarrow 2y^{1/2} = \frac{x^3}{3} + C \Rightarrow 2y^{1/2} - \frac{1}{3}x^3 = C$

11. $\frac{dy}{dx} = e^{x-y} \Rightarrow dy = e^xe^{-y}\,dx \Rightarrow e^y\,dy = e^x\,dx \Rightarrow \int e^y\,dy = \int e^x\,dx \Rightarrow e^y = e^x + C \Rightarrow e^y - e^x = C$

12. $\frac{dy}{dx} = 3x^2e^{-y} \Rightarrow dy = 3x^2e^{-y}dx \Rightarrow e^y\,dy = 3x^2dx \Rightarrow \int e^y\,dy = \int 3x^2dx \Rightarrow e^y = x^3 + C \Rightarrow e^y - x^3 = C$

13. $\frac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y} \Rightarrow dy = \left(\sqrt{y}\cos^2\sqrt{y}\right)dx \Rightarrow \frac{\sec^2\sqrt{y}}{\sqrt{y}}dy = dx \Rightarrow \int \frac{\sec^2\sqrt{y}}{\sqrt{y}}dy = \int dx$. In the integral on the left-hand side, substitute $u = \sqrt{y} \Rightarrow du = \frac{1}{2\sqrt{y}}dy \Rightarrow 2\,du = \frac{1}{\sqrt{y}}dy$, and we have $\int \sec^2 u\,du = \int dx \Rightarrow 2\tan u = x + C$
$\Rightarrow -x + 2\tan\sqrt{y} = C$

14. $\sqrt{2xy}\,\frac{dy}{dx} = 1 \Rightarrow dy = \frac{1}{\sqrt{2xy}}\,dx \Rightarrow \sqrt{2}\sqrt{y}dy = \frac{1}{\sqrt{x}}\,dx \Rightarrow \sqrt{2}\,y^{1/2}\,dy = x^{-1/2}\,dx \Rightarrow \sqrt{2}\int y^{1/2}\,dy = \int x^{-1/2}\,dx$
$\Rightarrow \sqrt{2}\,\frac{y^{3/2}}{\frac{3}{2}}\,dy = \frac{x^{1/2}}{\frac{1}{2}} + C_1 \Rightarrow \sqrt{2}\,y^{3/2} = 3\sqrt{x} + \frac{3}{2}C_1 \Rightarrow \sqrt{2}\left(\sqrt{y}\right)^3 - 3\sqrt{x} = C$, where $C = \frac{3}{2}C_1$

15. $\sqrt{x}\,\frac{dy}{dx} = e^{y+\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{e^y e^{\sqrt{x}}}{\sqrt{x}} \Rightarrow dy = \frac{e^y e^{\sqrt{x}}}{\sqrt{x}}dx \Rightarrow e^{-y}\,dy = \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx \Rightarrow \int e^{-y}\,dy = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. In the integral on the right-hand side, substitute $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}\,dx$, and we have $\int e^{-y}\,dy = 2\int e^u\,du \Rightarrow -e^{-y} = 2e^u + C_1$
$\Rightarrow -e^{-y} = 2e^{\sqrt{x}} + C$, where $C = -C_1$

16. $(\sec x)\,\frac{dy}{dx} = e^{y+\sin x} \Rightarrow \frac{dy}{dx} = e^{y+\sin x}\cos x \Rightarrow dy = (e^y e^{\sin x}\cos x)dx \Rightarrow e^{-y}\,dy = e^{\sin x}\cos x\,dx$
$\Rightarrow \int e^{-y}\,dy = \int e^{\sin x}\cos x\,dx \Rightarrow -e^{-y} = e^{\sin x} + C_1 \Rightarrow e^{-y} + e^{\sin x} = C$, where $C = -C_1$

17. $\frac{dy}{dx} = 2x\sqrt{1-y^2} \Rightarrow dy = 2x\sqrt{1-y^2}dx \Rightarrow \frac{dy}{\sqrt{1-y^2}} = 2x\,dx \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int 2x\,dx \Rightarrow \sin^{-1}y = x^2 + C$ since $|y| < 1$
$\Rightarrow y = \sin(x^2 + C)$

18. $\frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}} \Rightarrow dy = \frac{e^{2x-y}}{e^{x+y}}dx \Rightarrow dy = \frac{e^{2x}e^{-y}}{e^x e^y}dx = \frac{e^x}{e^{2y}}dx \Rightarrow e^{2y}\,dy = e^x\,dx \Rightarrow \int e^{2y}\,dy = \int e^x\,dx \Rightarrow \frac{e^{2y}}{2} = e^x + C_1$
$\Rightarrow e^{2y} - 2e^x = C$ where $C = 2C_1$

19. $y^2\frac{dy}{dx} = 3x^2y^3 - 6x^2 \Rightarrow y^2\,dy = 3x^2(y^3-2)dx \Rightarrow \frac{y^2}{y^3-2}\,dy = 3x^2dx \Rightarrow \int \frac{y^2}{y^3-2}\,dy = \int 3x^2dx \Rightarrow \frac{1}{3}\ln|y^3-2| = x^3 + C$

20. $\frac{dy}{dx} = x\,y + 3x - 2y - 6 = (y+3)(x-2) \Rightarrow \frac{1}{y+3}dy = (x-2)dx \Rightarrow \int \frac{1}{y+3}dy = \int (x-2)dx$
$\Rightarrow \ln|y+3| = \frac{1}{2}x^2 - 2x + C$

21. $\frac{1}{x}\frac{dy}{dx} = ye^{x^2} + 2\sqrt{y}\,e^{x^2} = e^{x^2}\left(y + 2\sqrt{y}\right) \Rightarrow \frac{1}{y+2\sqrt{y}}dy = x\,e^{x^2}dx \Rightarrow \int \frac{1}{y+2\sqrt{y}}dy = \int x\,e^{x^2}dx$
$\Rightarrow \int \frac{1}{\sqrt{y}(\sqrt{y}+2)}dy = \int x\,e^{x^2}dx \Rightarrow 2\ln|\sqrt{y}+2| = \frac{1}{2}e^{x^2} + C \Rightarrow 4\ln|\sqrt{y}+2| = e^{x^2} + C \Rightarrow 4\ln\left(\sqrt{y}+2\right) = e^{x^2} + C$

22. $\frac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1 = (e^{-y}+1)(e^x+1) \Rightarrow \frac{1}{e^{-y}+1}dy = (e^x+1)dx \Rightarrow \int \frac{1}{e^{-y}+1}dy = \int (e^x+1)dx$
$\Rightarrow \int \frac{e^y}{1+e^y}dy = \int (e^x+1)dx \Rightarrow \ln|1+e^y| = e^x + x + C \Rightarrow \ln(1+e^y) = e^x + x + C$

23. (a) $y = y_0e^{kt} \Rightarrow 0.99y_0 = y_0e^{1000k} \Rightarrow k = \frac{\ln 0.99}{1000} \approx -0.00001$
(b) $0.9 = e^{(-0.00001)t} \Rightarrow (-0.00001)t = \ln(0.9) \Rightarrow t = \frac{\ln(0.9)}{-0.00001} \approx 10{,}536$ years
(c) $y = y_0e^{(20{,}000)k} \approx y_0e^{-0.2} = y_0(0.82) \Rightarrow 82\%$

24. (a) $\frac{dp}{dh} = kp \Rightarrow p = p_0e^{kh}$ where $p_0 = 1013$; $90 = 1013e^{20k} \Rightarrow k = \frac{\ln(90) - \ln(1013)}{20} \approx -0.121$
(b) $p = 1013e^{-6.05} \approx 2.389$ millibars
(c) $900 = 1013e^{(-0.121)h} \Rightarrow -0.121h = \ln\left(\frac{900}{1013}\right) \Rightarrow h = \frac{\ln(1013) - \ln(900)}{0.121} \approx 0.977$ km

25. $\frac{dy}{dt} = -0.6y \Rightarrow y = y_0e^{-0.6t}$; $y_0 = 100 \Rightarrow y = 100e^{-0.6t} \Rightarrow y = 100e^{-0.6} \approx 54.88$ grams when $t = 1$ hr

26. $A = A_0e^{kt} \Rightarrow 800 = 1000e^{10k} \Rightarrow k = \frac{\ln(0.8)}{10} \Rightarrow A = 1000e^{(\ln(0.8)/10)t}$, where A represents the amount of sugar that remains after time t. Thus after another 14 hrs, $A = 1000e^{(\ln(0.8)/10)24} \approx 585.35$ kg

27. $L(x) = L_0e^{-kx} \Rightarrow \frac{L_0}{2} = L_0e^{-18k} \Rightarrow \ln\frac{1}{2} = -18k \Rightarrow k = \frac{\ln 2}{18} \approx 0.0385 \Rightarrow L(x) = L_0e^{-0.0385x}$; when the intensity is one-tenth of the surface value, $\frac{L_0}{10} = L_0e^{-0.0385x} \Rightarrow \ln 10 = 0.0385x \Rightarrow x \approx 59.8$ ft

28. $V(t) = V_0e^{-t/40} \Rightarrow 0.1V_0 = V_0e^{-t/40}$ when the voltage is 10% of its original value $\Rightarrow t = -40\ln(0.1) \approx 92.1$ sec

29. $y = y_0e^{kt}$ and $y_0 = 1 \Rightarrow y = e^{kt} \Rightarrow$ at $y = 2$ and $t = 0.5$ we have $2 = e^{0.5k} \Rightarrow \ln 2 = 0.5k \Rightarrow k = \frac{\ln 2}{0.5} = \ln 4$. Therefore, $y = e^{(\ln 4)t} \Rightarrow y = e^{24\ln 4} = 4^{24} = 2.81474978 \times 10^{14}$ at the end of 24 hrs

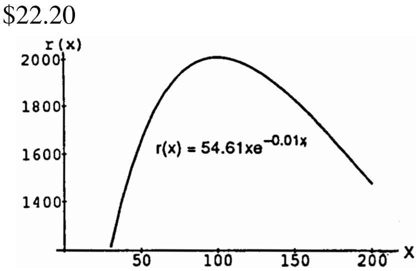
30. $y = y_0e^{kt}$ and $y(3) = 10{,}000 \Rightarrow 10{,}000 = y_0e^{3k}$; also $y(5) = 40{,}000 = y_0e^{5k}$. Therefore $y_0e^{5k} = 4y_0e^{3k}$
$\Rightarrow e^{5k} = 4e^{3k} \Rightarrow e^{2k} = 4 \Rightarrow k = \ln 2$. Thus, $y = y_0e^{(\ln 2)t} \Rightarrow 10{,}000 = y_0e^{3\ln 2} = y_0e^{\ln 8} \Rightarrow 10{,}000 = 8y_0$
$\Rightarrow y_0 = \frac{10{,}000}{8} = 1250$

31. (a) $10{,}000e^{k(1)} = 7500 \Rightarrow e^k = 0.75 \Rightarrow k = \ln 0.75$ and $y = 10{,}000e^{(\ln 0.75)t}$. Now $1000 = 10{,}000e^{(\ln 0.75)t}$
$\Rightarrow \ln 0.1 = (\ln 0.75)t \Rightarrow t = \frac{\ln 0.1}{\ln 0.75} \approx 8.00$ years (to the nearest hundredth of a year)
(b) $1 = 10{,}000e^{(\ln 0.75)t} \Rightarrow \ln 0.0001 = (\ln 0.75)t \Rightarrow t = \frac{\ln 0.0001}{\ln 0.75} \approx 32.02$ years (to the nearest hundredth of a year)

32. (a) There are $(60)(60)(24)(365) = 31{,}536{,}000$ seconds in a year. Thus, assuming exponential growth,
$P = 257{,}313{,}431e^{kt}$ and $257{,}313{,}432 = 257{,}313{,}431e^{(14k/31{,}536{,}000)} \Rightarrow \ln\left(\frac{257{,}313{,}432}{257{,}313{,}431}\right) = \frac{14k}{31{,}536{,}000} \Rightarrow k \approx 0.0087542$
(b) $P = 257{,}313{,}431e^{(0.0087542)(15)} \approx 293{,}420{,}847$ (to the nearest integer). Answers will vary considerably with the number of decimal places retained.

33. $0.9P_0 = P_0e^{k} \Rightarrow k = \ln 0.9$; when the well's output falls to one-fifth of its present value $P = 0.2P_0$
$\Rightarrow 0.2P_0 = P_0e^{(\ln 0.9)t} \Rightarrow 0.2 = e^{(\ln 0.9)t} \Rightarrow \ln(0.2) = (\ln 0.9)t \Rightarrow t = \frac{\ln 0.2}{\ln 0.9} \approx 15.28$ yr

34. (a) $\frac{dp}{dx} = -\frac{1}{100}p \Rightarrow \frac{dp}{p} = -\frac{1}{100}dx \Rightarrow \ln p = -\frac{1}{100}x + C \Rightarrow p = e^{(-0.01x+C)} = e^Ce^{-0.01x} = C_1e^{-0.01x}$;
$p(100) = 20.09 \Rightarrow 20.09 = C_1e^{(-0.01)(100)} \Rightarrow C_1 = 20.09e \approx 54.61 \Rightarrow p(x) = 54.61e^{-0.01x}$ (in dollars)
(b) $p(10) = 54.61e^{(-0.01)(10)} = \$49.41$, and $p(90) = 54.61e^{(-0.01)(90)} = \$22.20$
(c) $r(x) = xp(x) \Rightarrow r'(x) = p(x) + xp'(x)$;
$p'(x) = -.5461e^{-0.01x} \Rightarrow r'(x)$
$= (54.61 - .5461x)e^{-0.01x}$. Thus, $r'(x) = 0$
$\Rightarrow 54.61 = .5461x \Rightarrow x = 100$. Since $r' > 0$ for any $x < 100$ and $r' < 0$ for $x > 100$, then $r(x)$ must be a maximum at $x = 100$.



35. $A = A_0e^{kt}$ and $A_0 = 10 \Rightarrow A = 10\,e^{kt}$, $5 = 10\,e^{k(24360)} \Rightarrow k = \frac{\ln(0.5)}{24360} \approx -0.000028454 \Rightarrow A = 10\,e^{-0.000028454t}$, then $0.2(10) = 10\,e^{-0.000028254t} \Rightarrow t = \frac{\ln 0.2}{-0.000028454} \approx 56563$ years

36. $A = A_0e^{kt}$ and $\frac{1}{2}A_0 = A_0e^{139k} \Rightarrow \frac{1}{2} = e^{139k} \Rightarrow k = \frac{\ln(0.5)}{139} \approx -0.00499$; then $0.05A_0 = A_0e^{-0.00499t}$
$\Rightarrow t = \frac{\ln 0.05}{-0.00499} \approx 600$ days

37. $y = y_0e^{-kt} = y_0e^{-(k)(3/k)} = y_0e^{-3} = \frac{y_0}{e^3} < \frac{y_0}{20} = (0.05)(y_0) \Rightarrow$ after three mean lifetimes less than 5% remains

38. (a) $A = A_0 e^{-kt} \Rightarrow \frac{1}{2} = e^{-2.645k} \Rightarrow k = \frac{\ln 2}{2.645} \approx 0.262$

(b) $\frac{1}{k} \approx 3.816$ years

(c) $(0.05)A = A \exp\left(-\frac{\ln 2}{2.645} t\right) \Rightarrow -\ln 20 = \left(-\frac{\ln 2}{2.645}\right) t \Rightarrow t = \frac{2.645 \ln 20}{\ln 2} \approx 11.431$ years

39. $T - T_s = (T_0 - T_s)\, e^{-kt}$, $T_0 = 90°C$, $T_s = 20°C$, $T = 60°C \Rightarrow 60 - 20 = 70e^{-10k} \Rightarrow \frac{4}{7} = e^{-10k} \Rightarrow k = \frac{\ln\left(\frac{7}{4}\right)}{10} \approx 0.05596$

(a) $35 - 20 = 70e^{-0.05596t} \Rightarrow t \approx 27.5$ min is the total time $\Rightarrow$ it will take $27.5 - 10 = 17.5$ minutes longer to reach 35°C

(b) $T - T_s = (T_0 - T_s)\, e^{-kt}$, $T_0 = 90°C$, $T_s = -15°C \Rightarrow 35 + 15 = 105e^{-0.05596t} \Rightarrow t \approx 13.26$ min

40. $T - 65° = (T_0 - 65°)\, e^{-kt} \Rightarrow 35° - 65° = (T_0 - 65°)\, e^{-10k}$ and $50° - 65° = (T_0 - 65°)\, e^{-20k}$. Solving $-30° = (T_0 - 65°)\, e^{-10k}$ and $-15° = (T_0 - 65°)\, e^{-20k}$ simultaneously $\Rightarrow (T_0 - 65°)\, e^{-10k} = 2(T_0 - 65°)\, e^{-20k}$ $\Rightarrow e^{10k} = 2 \Rightarrow k = \frac{\ln 2}{10}$ and $-30° = \frac{T_0 - 65°}{e^{10k}} \Rightarrow -30°\left[e^{10\left(\frac{\ln 2}{10}\right)}\right] = T_0 - 65° \Rightarrow T_0 = 65° - 30°\left(e^{\ln 2}\right) = 65° - 60° = 5°$

41. $T - T_s = (T_0 - T_s)\, e^{-kt} \Rightarrow 39 - T_s = (46 - T_s)\, e^{-10k}$ and $33 - T_s = (46 - T_s)\, e^{-20k} \Rightarrow \frac{39 - T_s}{46 - T_s} = e^{-10k}$ and $\frac{33 - T_s}{46 - T_s} = e^{-20k} = \left(e^{-10k}\right)^2 \Rightarrow \frac{33 - T_s}{46 - T_s} = \left(\frac{39 - T_s}{46 - T_s}\right)^2 \Rightarrow (33 - T_s)(46 - T_s) = (39 - T_s)^2 \Rightarrow 1518 - 79T_s + T_s^2$ $= 1521 - 78T_s + T_s^2 \Rightarrow -T_s = 3 \Rightarrow T_s = -3°C$

42. Let x represent how far above room temperature the silver will be 15 min from now, y how far above room temperature the silver will be 120 min from now, and t_0 the time the silver will be 10°C above room temperature. We then have the following time-temperature table:

time in min.	0	20 (Now)	35	140	t_0
temperature	$T_s + 70°$	$T_s + 60°$	$T_s + x$	$T_s + y$	$T_s + 10°$

$T - T_s = (T_0 - T_s)\, e^{-kt} \Rightarrow (60 + T_s) - T_s = [(70 + T_s) - T_s]\, e^{-20k} \Rightarrow 60 = 70e^{-20k} \Rightarrow k = \left(-\frac{1}{20}\right) \ln\left(\frac{6}{7}\right) \approx 0.00771$

(a) $T - T_s = (T_0 - T_s)\, e^{-0.00771t} \Rightarrow (T_s + x) - T_s = [(70 + T_s) - T_s]\, e^{-(0.00771)(35)} \Rightarrow x = 70e^{-0.26985} \approx 53.44°C$

(b) $T - T_s = (T_0 - T_s)\, e^{-0.00771t} \Rightarrow (T_s + y) - T_s = [(70 + T_s) - T_s]\, e^{-(0.00771)(140)} \Rightarrow y = 70e^{-1.0794} \approx 23.79°C$

(c) $T - T_s = (T_0 - T_s)\, e^{-0.00771t} \Rightarrow (T_s + 10) - T_s = [(70 + T_s) - T_s]\, e^{-(0.00771)\, t_0} \Rightarrow 10 = 70e^{-0.00771 t_0}$ $\Rightarrow \ln\left(\frac{1}{7}\right) = -0.00771 t_0 \Rightarrow t_0 = \left(-\frac{1}{0.00771}\right) \ln\left(\frac{1}{7}\right) = 252.39 \Rightarrow 252.39 - 20 \approx 232$ minutes from now the silver will be 10°C above room temperature

43. From Example 4, the half-life of carbon-14 is 5700 yr $\Rightarrow \frac{1}{2} c_0 = c_0 e^{-k(5700)} \Rightarrow k = \frac{\ln 2}{5700} \approx 0.0001216 \Rightarrow c = c_0 e^{-0.0001216t}$ $\Rightarrow (0.445)c_0 = c_0 e^{-0.0001216t} \Rightarrow t = \frac{\ln(0.445)}{-0.0001216} \approx 6659$ years

44. From Exercise 43, $k \approx 0.0001216$ for carbon-14.

(a) $c = c_0 e^{-0.0001216t} \Rightarrow (0.17)c_0 = c_0 e^{-0.0001216t} \Rightarrow t \approx 14{,}571.44$ years $\Rightarrow$ 12,571 BC

(b) $(0.18)c_0 = c_0 e^{-0.0001216t} \Rightarrow t \approx 14{,}101.41$ years $\Rightarrow$ 12,101 BC

(c) $(0.16)c_0 = c_0 e^{-0.0001216t} \Rightarrow t \approx 15{,}069.98$ years $\Rightarrow$ 13,070 BC

45. From Exercise 43, $k \approx 0.0001216$ for carbon-14 $\Rightarrow y = y_0 e^{-0.0001216t}$. When $t = 5000$ $\Rightarrow y = y_0 e^{-0.0001216(5000)} \approx 0.5444 y_0 \Rightarrow \frac{y}{y_0} \approx 0.5444 \Rightarrow$ approximately 54.44% remains

46. From Exercise 43, $k \approx 0.0001216$ for carbon-14. Thus, $c = c_0 e^{-0.0001216t} \Rightarrow (0.995)c_0 = c_0 e^{-0.0001216t}$ $\Rightarrow t = \frac{\ln(0.995)}{-0.0001216} \approx 41$ years old

7.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

1. l'Hôpital: $\lim_{x \to 2} \frac{x-2}{x^2-4} = \frac{1}{2x}\Big|_{x=2} = \frac{1}{4}$ or $\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$

2. l'Hôpital: $\lim_{x \to 0} \frac{\sin 5x}{x} = \frac{5\cos 5x}{1}\Big|_{x=0} = 5$ or $\lim_{x \to 0} \frac{\sin 5x}{x} = 5 \lim_{5x \to 0} \frac{\sin 5x}{5x} = 5 \cdot 1 = 5$

3. l'Hôpital: $\lim_{x \to \infty} \frac{5x^2-3x}{7x^2+1} = \lim_{x \to \infty} \frac{10x-3}{14x} = \lim_{x \to \infty} \frac{10}{14} = \frac{5}{7}$ or $\lim_{x \to \infty} \frac{5x^2-3x}{7x^2+1} = \lim_{x \to \infty} \frac{5-\frac{3}{x}}{7+\frac{1}{x^2}} = \frac{5}{7}$

4. l'Hôpital: $\lim_{x \to 1} \frac{x^3-1}{4x^3-x-3} = \lim_{x \to 1} \frac{3x^2}{12x^2-1} = \frac{3}{11}$ or $\lim_{x \to 1} \frac{x^3-1}{4x^3-x-3} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(4x^2+4x+3)}$
$= \lim_{x \to 1} \frac{(x^2+x+1)}{(4x^2+4x+3)} = \frac{3}{11}$

5. l'Hôpital: $\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$ or $\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \left[\frac{(1-\cos x)}{x^2}\left(\frac{1+\cos x}{1+\cos x}\right)\right]$
$= \lim_{x \to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim_{x \to 0} \left[\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{x}\right)\left(\frac{1}{1+\cos x}\right)\right] = \frac{1}{2}$

6. l'Hôpital: $\lim_{x \to \infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x \to \infty} \frac{4x+3}{3x^2+1} = \lim_{x \to \infty} \frac{4}{6x} = 0$ or $\lim_{x \to \infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x \to \infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}+\frac{1}{x^3}} = \frac{0}{1} = 0$

7. $\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{1}{2x} = \frac{1}{4}$

8. $\lim_{x \to -5} \frac{x^2-25}{x+5} = \lim_{x \to -5} \frac{2x}{1} = -10$

9. $\lim_{t \to -3} \frac{t^3-4t+15}{t^2-t-12} = \lim_{t \to -3} \frac{3t^2-4}{2t-1} = \frac{3(-3)^2-4}{2(-3)-1} = -\frac{23}{7}$

10. $\lim_{t \to 1} \frac{t^3-1}{4t^3-t-3} = \lim_{t \to 1} \frac{3t^2}{12t^2-1} = \frac{3}{11}$

11. $\lim_{x \to \infty} \frac{5x^3-2x}{7x^3+3} = \lim_{x \to \infty} \frac{15x^2-2}{21x^2} = \lim_{x \to \infty} \frac{30x}{42x} = \lim_{x \to \infty} \frac{30}{42} = \frac{5}{7}$

12. $\lim_{x \to \infty} \frac{x-8x^2}{12x^2+5x} = \lim_{x \to \infty} \frac{1-16x}{24x+5} = \lim_{x \to \infty} \frac{-16}{24} = -\frac{2}{3}$

13. $\lim_{t \to 0} \frac{\sin t^2}{t} = \lim_{t \to 0} \frac{(\cos t^2)(2t)}{1} = 0$

14. $\lim_{t \to 0} \frac{\sin 5t}{2t} = \lim_{t \to 0} \frac{5\cos 5t}{2} = \frac{5}{2}$

15. $\lim_{x \to 0} \frac{8x^2}{\cos x - 1} = \lim_{x \to 0} \frac{16x}{-\sin x} = \lim_{x \to 0} \frac{16}{-\cos x} = \frac{16}{-1} = -16$

16. $\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \lim_{x \to 0} \frac{-\sin x}{6x} = \lim_{x \to 0} \frac{-\cos x}{6} = -\frac{1}{6}$

17. $\lim_{\theta \to \pi/2} \frac{2\theta-\pi}{\cos(2\pi-\theta)} = \lim_{\theta \to \pi/2} \frac{2}{\sin(2\pi-\theta)} = \frac{2}{\sin\left(\frac{3\pi}{2}\right)} = -2$

18. $\lim_{\theta \to -\pi/3} \frac{3\theta+\pi}{\sin\left(\theta+\frac{\pi}{3}\right)} = \lim_{\theta \to -\pi/3} \frac{3}{\cos\left(\theta+\frac{\pi}{3}\right)} = 3$

19. $\lim_{\theta \to \pi/2} \frac{1-\sin\theta}{1+\cos 2\theta} = \lim_{\theta \to \pi/2} \frac{-\cos\theta}{-2\sin 2\theta} = \lim_{\theta \to \pi/2} \frac{\sin\theta}{-4\cos 2\theta} = \frac{1}{(-4)(-1)} = \frac{1}{4}$

20. $\lim_{x \to 1} \frac{x-1}{\ln x - \sin(\pi x)} = \lim_{x \to 1} \frac{1}{\frac{1}{x} - \pi \cos(\pi x)} = \frac{1}{1+\pi}$

21. $\lim_{x \to 0} \frac{x^2}{\ln(\sec x)} = \lim_{x \to 0} \frac{2x}{\left(\frac{\sec x \tan x}{\sec x}\right)} = \lim_{x \to 0} \frac{2x}{\tan x} = \lim_{x \to 0} \frac{2}{\sec^2 x} = \frac{2}{1^2} = 2$

22. $\lim_{x \to \pi/2} \frac{\ln(\csc x)}{\left(x - \left(\frac{\pi}{2}\right)\right)^2} = \lim_{x \to \pi/2} \frac{-\left(\frac{\csc x \cot x}{\csc x}\right)}{2\left(x - \left(\frac{\pi}{2}\right)\right)} = \lim_{x \to \pi/2} \frac{-\cot x}{2\left(x - \left(\frac{\pi}{2}\right)\right)} = \lim_{x \to \pi/2} \frac{\csc^2 x}{2} = \frac{1^2}{2} = \frac{1}{2}$

23. $\lim_{t \to 0} \frac{t(1-\cos t)}{t - \sin t} = \lim_{t \to 0} \frac{(1-\cos t) + t(\sin t)}{1 - \cos t} = \lim_{t \to 0} \frac{\sin t + (\sin t + t \cos t)}{\sin t} = \lim_{t \to 0} \frac{\cos t + \cos t + \cos t - t \sin t}{\cos t} = \frac{1+1+1-0}{1} = 3$

24. $\lim_{t \to 0} \frac{t \sin t}{1 - \cos t} = \lim_{t \to 0} \frac{\sin t + t \cos t}{\sin t} = \lim_{t \to 0} \frac{\cos t + (\cos t - t \sin t)}{\cos t} = \frac{1 + (1-0)}{1} = 2$

25. $\lim_{x \to (\pi/2)^-} \left(x - \frac{\pi}{2}\right) \sec x = \lim_{x \to (\pi/2)^-} \frac{\left(x - \frac{\pi}{2}\right)}{\cos x} = \lim_{x \to (\pi/2)^-} \left(\frac{1}{-\sin x}\right) = \frac{1}{-1} = -1$

26. $\lim_{x \to (\pi/2)^-} \left(\frac{\pi}{2} - x\right) \tan x = \lim_{x \to (\pi/2)^-} \frac{\left(\frac{\pi}{2} - x\right)}{\cot x} = \lim_{x \to (\pi/2)^-} \left(\frac{-1}{-\csc^2 x}\right) = \lim_{x \to (\pi/2)^-} \sin^2 x = 1$

27. $\lim_{\theta \to 0} \frac{3^{\sin \theta} - 1}{\theta} = \lim_{\theta \to 0} \frac{3^{\sin \theta}(\ln 3)(\cos \theta)}{1} = \frac{(3^0)(\ln 3)(1)}{1} = \ln 3$

28. $\lim_{\theta \to 0} \frac{\left(\frac{1}{2}\right)^{\theta} - 1}{\theta} = \lim_{\theta \to 0} \frac{\left(\ln\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)^{\theta}}{1} = \ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2$

29. $\lim_{x \to 0} \frac{x\, 2^x}{2^x - 1} = \lim_{x \to 0} \frac{(1)(2^x) + (x)(\ln 2)(2^x)}{(\ln 2)(2^x)} = \frac{1 \cdot 2^0 + 0}{(\ln 2) \cdot 2^0} = \frac{1}{\ln 2}$

30. $\lim_{x \to 0} \frac{3^x - 1}{2^x - 1} = \lim_{x \to 0} \frac{3^x \ln 3}{2^x \ln 2} = \frac{3^0 \cdot \ln 3}{2^0 \cdot \ln 2} = \frac{\ln 3}{\ln 2}$

31. $\lim_{x \to \infty} \frac{\ln(x+1)}{\log_2 x} = \lim_{x \to \infty} \frac{\ln(x+1)}{\left(\frac{\ln x}{\ln 2}\right)} = (\ln 2) \lim_{x \to \infty} \frac{\left(\frac{1}{x+1}\right)}{\left(\frac{1}{x}\right)} = (\ln 2) \lim_{x \to \infty} \frac{x}{x+1} = (\ln 2) \lim_{x \to \infty} \frac{1}{1} = \ln 2$

32. $\lim_{x \to \infty} \frac{\log_2 x}{\log_3 (x+3)} = \lim_{x \to \infty} \frac{\left(\frac{\ln x}{\ln 2}\right)}{\left(\frac{\ln(x+3)}{\ln 3}\right)} = \left(\frac{\ln 3}{\ln 2}\right) \lim_{x \to \infty} \frac{\ln x}{\ln(x+3)} = \left(\frac{\ln 3}{\ln 2}\right) \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x+3}\right)} = \left(\frac{\ln 3}{\ln 2}\right) \lim_{x \to \infty} \frac{x+3}{x}$
$= \left(\frac{\ln 3}{\ln 2}\right) \lim_{x \to \infty} \frac{1}{1} = \frac{\ln 3}{\ln 2}$

33. $\lim_{x \to 0^+} \frac{\ln(x^2+2x)}{\ln x} = \lim_{x \to 0^+} \frac{\left(\frac{2x+2}{x^2+2x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to 0^+} \frac{2x^2+2x}{x^2+2x} = \lim_{x \to 0^+} \frac{4x+2}{2x+2} = \lim_{x \to 0^+} \frac{2}{2} = 1$

34. $\lim_{x \to 0^+} \frac{\ln(e^x - 1)}{\ln x} = \lim_{x \to 0^+} \frac{\left(\frac{e^x}{e^x - 1}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to 0^+} \frac{xe^x}{e^x - 1} = \lim_{x \to 0^+} \frac{e^x + xe^x}{e^x} = \frac{1+0}{1} = 1$

35. $\lim_{y \to 0} \frac{\sqrt{5y+25} - 5}{y} = \lim_{y \to 0} \frac{(5y+25)^{1/2} - 5}{y} = \lim_{y \to 0} \frac{\left(\frac{1}{2}\right)(5y+25)^{-1/2}(5)}{1} = \lim_{y \to 0} \frac{5}{2\sqrt{5y+25}} = \frac{1}{2}$

36. $\lim_{y \to 0} \frac{\sqrt{ay+a^2} - a}{y} = \lim_{y \to 0} \frac{(ay+a^2)^{1/2} - a}{y} = \lim_{y \to 0} \frac{\left(\frac{1}{2}\right)(ay+a^2)^{-1/2}(a)}{1} = \lim_{y \to 0} \frac{a}{2\sqrt{ay+a^2}} = \frac{1}{2}, a > 0$

37. $\lim_{x \to \infty} [\ln 2x - \ln(x+1)] = \lim_{x \to \infty} \ln\left(\frac{2x}{x+1}\right) = \ln\left(\lim_{x \to \infty} \frac{2x}{x+1}\right) = \ln\left(\lim_{x \to \infty} \frac{2}{1}\right) = \ln 2$

38. $\lim\limits_{x \to 0^+} (\ln x - \ln \sin x) = \lim\limits_{x \to 0^+} \ln\left(\frac{x}{\sin x}\right) = \ln\left(\lim\limits_{x \to 0^+} \frac{x}{\sin x}\right) = \ln\left(\lim\limits_{x \to 0^+} \frac{1}{\cos x}\right) = \ln 1 = 0$

39. $\lim\limits_{x\to0^+} \frac{(\ln x)^2}{\ln(\sin x)} = \lim\limits_{x\to0^+} \frac{2(\ln x)\left(\frac{1}{x}\right)}{\frac{\cos x}{\sin x}} = \lim\limits_{x\to0^+} \frac{2(\ln x)(\sin x)}{x\cos x} = \lim\limits_{x\to0^+} \left[\frac{2(\ln x)}{\cos x} \cdot \frac{\sin x}{x}\right] = -\infty \cdot 1 = -\infty$

40. $\lim\limits_{x \to 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x}\right) = \lim\limits_{x \to 0^+} \left(\frac{(3x+1)(\sin x) - x}{x \sin x}\right) = \lim\limits_{x \to 0^+} \frac{3\sin x + (3x+1)(\cos x) - 1}{\sin x + x\cos x}$
$= \lim\limits_{x \to 0^+} \left(\frac{3\cos x + 3\cos x + (3x+1)(-\sin x)}{\cos x + \cos x - x\sin x}\right) = \frac{3+3+(1)(0)}{1+1-0} = \frac{6}{2} = 3$

41. $\lim\limits_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \lim\limits_{x \to 1^+} \left(\frac{\ln x - (x-1)}{(x-1)(\ln x)}\right) = \lim\limits_{x \to 1^+} \left(\frac{\frac{1}{x} - 1}{(\ln x) + (x-1)\left(\frac{1}{x}\right)}\right) = \lim\limits_{x \to 1^+} \left(\frac{1-x}{(x\ln x) + x - 1}\right)$
$= \lim\limits_{x \to 1^+} \left(\frac{-1}{(\ln x + 1) + 1}\right) = \frac{-1}{(0+1)+1} = -\frac{1}{2}$

42. $\lim\limits_{x \to 0^+} (\csc x - \cot x + \cos x) = \lim\limits_{x \to 0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} + \cos x\right) = \lim\limits_{x \to 0^+} \left(\frac{(1-\cos x) + (\sin x)(\cos x)}{\sin x}\right)$
$= \lim\limits_{x \to 0^+} \left(\frac{\sin x + \cos^2 x - \sin^2 x}{\cos x}\right) = \frac{0+1-0}{1} = 1$

43. $\lim\limits_{\theta \to 0} \frac{\cos\theta - 1}{e^\theta - \theta - 1} = \lim\limits_{\theta \to 0} \frac{-\sin\theta}{e^\theta - 1} = \lim\limits_{\theta \to 0} \frac{-\cos\theta}{e^\theta} = -1$

44. $\lim\limits_{h \to 0} \frac{e^h - (1+h)}{h^2} = \lim\limits_{h \to 0} \frac{e^h - 1}{2h} = \lim\limits_{h \to 0} \frac{e^h}{2} = \frac{1}{2}$

45. $\lim\limits_{t \to \infty} \frac{e^t + t^2}{e^t - 1} = \lim\limits_{t \to \infty} \frac{e^t + 2t}{e^t} = \lim\limits_{t \to \infty} \frac{e^t + 2}{e^t} = \lim\limits_{t \to \infty} \frac{e^t}{e^t} = 1$

46. $\lim\limits_{x \to \infty} x^2 e^{-x} = \lim\limits_{x \to \infty} \frac{x^2}{e^x} = \lim\limits_{x \to \infty} \frac{2x}{e^x} = \lim\limits_{x \to \infty} \frac{2}{e^x} = 0$

47. $\lim\limits_{x\to0} \frac{x - \sin x}{x\tan x} = \lim\limits_{x\to0} \frac{1 - \cos x}{x\sec^2 x + \tan x} = \lim\limits_{x\to0} \frac{\sin x}{2x\sec^2 x\tan x + 2\sec^2 x} = \frac{0}{2} = 0$

48. $\lim\limits_{x\to0} \frac{(e^x - 1)^2}{x\sin x} = \lim\limits_{x\to0} \frac{2(e^x - 1)e^x}{x\cos x + \sin x} = \lim\limits_{x\to0} \frac{2e^{2x} - 2e^x}{x\cos x + \sin x} = \lim\limits_{x\to0} \frac{4e^{2x} - 2e^x}{-x\sin x + 2\cos x} = \frac{2}{2} = 1$

49. $\lim\limits_{\theta\to0} \frac{\theta - \sin\theta\cos\theta}{\tan\theta - \theta} = \lim\limits_{\theta\to0} \frac{1 + \sin^2\theta - \cos^2\theta}{\sec^2\theta - 1} = \lim\limits_{\theta\to0} \frac{2\sin^2\theta}{\tan^2\theta} = \lim\limits_{\theta\to0} 2\cos^2\theta = 2$

50. $\lim\limits_{x\to0} \frac{\sin 3x - 3x + x^2}{\sin x \sin 2x} = \lim\limits_{x\to0} \frac{3\cos 3x - 3 + 2x}{2\sin x\cos 2x + \cos x\sin 2x} = \lim\limits_{x\to0} \frac{3\cos 3x - 3 + 2x}{\sin x\cos 2x + \sin 3x} = \lim\limits_{x\to0} \frac{-9\sin 3x + 2}{-2\sin x\sin 2x + \cos x\cos 2x + 3\cos 3x} = \frac{2}{4} = \frac{1}{2}$

51. The limit leads to the indeterminate form 1^∞. Let $f(x) = x^{1/(1-x)} \Rightarrow \ln f(x) = \ln\left(x^{1/(1-x)}\right) = \frac{\ln x}{1-x}$. Now $\lim\limits_{x \to 1^+} \ln f(x) = \lim\limits_{x \to 1^+} \frac{\ln x}{1-x} = \lim\limits_{x \to 1^+} \frac{\left(\frac{1}{x}\right)}{-1} = -1$. Therefore $\lim\limits_{x \to 1^+} x^{1/(1-x)} = \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$

52. The limit leads to the indeterminate form 1^∞. Let $f(x) = x^{1/(x-1)} \Rightarrow \ln f(x) = \ln\left(x^{1/(x-1)}\right) = \frac{\ln x}{x-1}$. Now $\lim\limits_{x \to 1^+} \ln f(x) = \lim\limits_{x \to 1^+} \frac{\ln x}{x-1} = \lim\limits_{x \to 1^+} \frac{\left(\frac{1}{x}\right)}{1} = 1$. Therefore $\lim\limits_{x \to 1^+} x^{1/(x-1)} = \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} e^{\ln f(x)} = e^1 = e$

53. The limit leads to the indeterminate form ∞^0. Let $f(x) = (\ln x)^{1/x} \Rightarrow \ln f(x) = \ln(\ln x)^{1/x} = \frac{\ln(\ln x)}{x}$. Now $\lim\limits_{x \to \infty} \ln f(x) = \lim\limits_{x \to \infty} \frac{\ln(\ln x)}{x} = \lim\limits_{x \to \infty} \frac{\left(\frac{1}{x\ln x}\right)}{1} = 0$. Therefore $\lim\limits_{x \to \infty} (\ln x)^{1/x} = \lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} e^{\ln f(x)} = e^0 = 1$

54. The limit leads to the indeterminate form 1^∞. Let $f(x) = (\ln x)^{1/(x-e)} \Rightarrow \ln f(x) = \frac{\ln(\ln x)}{x-e} = \lim\limits_{x \to e^+} \ln f(x)$
$= \lim\limits_{x \to e^+} \frac{\ln(\ln x)}{x-e} = \lim\limits_{x \to e^+} \frac{\left(\frac{1}{x \ln x}\right)}{1} = \frac{1}{e}$. Therefore $(\ln x)^{1/(x-e)} = \lim\limits_{x \to e^+} f(x) = \lim\limits_{x \to e^+} e^{\ln f(x)} = e^{1/e}$

55. The limit leads to the indeterminate form 0^0. Let $f(x) = x^{-1/\ln x} \Rightarrow \ln f(x) = -\frac{\ln x}{\ln x} = -1$. Therefore
$\lim\limits_{x \to 0^+} x^{-1/\ln x} = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$

56. The limit leads to the indeterminate form ∞^0. Let $f(x) = x^{1/\ln x} \Rightarrow \ln f(x) = \frac{\ln x}{\ln x} = 1$. Therefore $\lim\limits_{x \to \infty} x^{1/\ln x}$
$= \lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} e^{\ln f(x)} = e^1 = e$

57. The limit leads to the indeterminate form ∞^0. Let $f(x) = (1+2x)^{1/(2 \ln x)} \Rightarrow \ln f(x) = \frac{\ln(1+2x)}{2 \ln x}$
$\Rightarrow \lim\limits_{x \to \infty} \ln f(x) = \lim\limits_{x \to \infty} \frac{\ln(1+2x)}{2 \ln x} = \lim\limits_{x \to \infty} \frac{x}{1+2x} = \lim\limits_{x \to \infty} \frac{1}{2} = \frac{1}{2}$. Therefore $\lim\limits_{x \to \infty} (1+2x)^{1/(2 \ln x)}$
$= \lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} e^{\ln f(x)} = e^{1/2}$

58. The limit leads to the indeterminate form 1^∞. Let $f(x) = (e^x + x)^{1/x} \Rightarrow \ln f(x) = \frac{\ln(e^x + x)}{x}$
$\Rightarrow \lim\limits_{x \to 0} \ln f(x) = \lim\limits_{x \to 0} \frac{\ln(e^x+x)}{x} = \lim\limits_{x \to 0} \frac{e^x+1}{e^x+x} = 2$. Therefore $\lim\limits_{x \to 0} (e^x + x)^{1/x} = \lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} e^{\ln f(x)} = e^2$

59. The limit leads to the indeterminate form 0^0. Let $f(x) = x^x \Rightarrow \ln f(x) = x \ln x \Rightarrow \ln f(x) = \frac{\ln x}{\left(\frac{1}{x}\right)}$
$= \lim\limits_{x \to 0^+} \ln f(x) = \lim\limits_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} = \lim\limits_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim\limits_{x \to 0^+} (-x) = 0$. Therefore $\lim\limits_{x \to 0^+} x^x = \lim\limits_{x \to 0^+} f(x)$
$= \lim\limits_{x \to 0^+} e^{\ln f(x)} = e^0 = 1$

60. The limit leads to the indeterminate form ∞^0. Let $f(x) = \left(1 + \frac{1}{x}\right)^x \Rightarrow \ln f(x) = \frac{\ln(1+x^{-1})}{x^{-1}} \Rightarrow \lim\limits_{x \to 0^+} \ln f(x)$
$= \lim\limits_{x \to 0^+} \frac{\left(\frac{-x^{-2}}{1+x^{-1}}\right)}{-x^{-2}} = \lim\limits_{x \to 0^+} \frac{1}{1+x^{-1}} = \lim\limits_{x \to 0^+} \frac{x}{x+1} = 0$. Therefore $\lim\limits_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x = \lim\limits_{x \to 0^+} f(x)$
$= \lim\limits_{x \to 0^+} e^{\ln f(x)} = e^0 = 1$

61. The limit leads to the indeterminate form 1^∞. Let $f(x) = \left(\frac{x+2}{x-1}\right)^x \Rightarrow \ln f(x) = \ln\left(\frac{x+2}{x-1}\right)^x = x \ln\left(\frac{x+2}{x-1}\right) \Rightarrow \lim\limits_{x\to\infty} \ln f(x)$
$= \lim\limits_{x\to\infty} x \ln\left(\frac{x+2}{x-1}\right) = \lim\limits_{x\to\infty} \left(\frac{\ln\left(\frac{x+2}{x-1}\right)}{\frac{1}{x}}\right) = \lim\limits_{x\to\infty} \left(\frac{\ln(x+2) - \ln(x-1)}{\frac{1}{x}}\right) = \lim\limits_{x\to\infty} \left(\frac{\frac{1}{x+2} - \frac{1}{x-1}}{-\frac{1}{x^2}}\right) = \lim\limits_{x\to\infty} \left(\frac{\frac{-3}{(x+2)(x-1)}}{-\frac{1}{x^2}}\right)$
$= \lim\limits_{x\to\infty} \left(\frac{3x^2}{(x+2)(x-1)}\right) = \lim\limits_{x\to\infty} \left(\frac{6x}{2x+1}\right) = \lim\limits_{x\to\infty} \left(\frac{6}{2}\right) = 3$. Therefore, $\lim\limits_{x\to\infty} \left(\frac{x+2}{x-1}\right)^x = \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^3$

62. The limit leads to the indeterminate form ∞^0. Let $f(x) = \left(\frac{x^2+1}{x+2}\right)^{1/x} \Rightarrow \ln f(x) = \ln\left(\frac{x^2+1}{x+2}\right)^{1/x} = \frac{1}{x} \ln\left(\frac{x^2+1}{x+2}\right)$
$\Rightarrow \lim\limits_{x\to\infty} \ln f(x) = \lim\limits_{x\to\infty} \frac{1}{x} \ln\left(\frac{x^2+1}{x+2}\right) = \lim\limits_{x\to\infty} \frac{\ln\left(\frac{x^2+1}{x+2}\right)}{x} = \lim\limits_{x\to\infty} \frac{\ln(x^2+1) - \ln(x+2)}{x} = \lim\limits_{x\to\infty} \frac{\frac{2x}{x^2+1} - \frac{1}{x+2}}{1} = \lim\limits_{x\to\infty} \frac{x^2+4x-1}{(x^2+1)(x+2)}$
$= \lim\limits_{x\to\infty} \frac{x^2+4x-1}{x^3+2x^2+x+2} = \lim\limits_{x\to\infty} \frac{2x+4}{3x^2+4x+1} = \lim\limits_{x\to\infty} \frac{2}{6x+4} = 0$. Therefore, $\lim\limits_{x\to\infty} \left(\frac{x^2+1}{x+2}\right)^{1/x} = \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^0 = 1$

63. $\lim\limits_{x\to 0^+} x^2 \ln x = \lim\limits_{x\to 0^+} \left(\frac{\ln x}{\frac{1}{x^2}}\right) = \lim\limits_{x\to 0^+} \left(\frac{\frac{1}{x}}{-\frac{2}{x^3}}\right) = \lim\limits_{x\to 0^+} \left(-\frac{x^3}{2x}\right) = \lim\limits_{x\to 0^+} \left(-\frac{3x^2}{2}\right) = 0$

64. $\lim\limits_{x\to 0^+} x(\ln x)^2 = \lim\limits_{x\to 0^+} \left(\frac{(\ln x)^2}{\frac{1}{x}}\right) = \lim\limits_{x\to 0^+} \left(\frac{2(\ln x)\frac{1}{x}}{-\frac{1}{x^2}}\right) = \lim\limits_{x\to 0^+} \left(\frac{2 \ln x}{-\frac{1}{x}}\right) = \lim\limits_{x\to 0^+} \left(\frac{\frac{2}{x}}{\frac{1}{x^2}}\right) = \lim\limits_{x\to 0^+} \left(\frac{2x^2}{x}\right) = \lim\limits_{x\to 0^+} (2x) = 0$

65. $\lim_{x\to 0^+} x\tan\left(\frac{\pi}{2} - x\right) = \lim_{x\to 0^+}\left(\frac{x}{\cot\left(\frac{\pi}{2}-x\right)}\right) = \lim_{x\to 0^+}\left(\frac{1}{\csc^2\left(\frac{\pi}{2}-x\right)}\right) = \frac{1}{1} = 1$

66. $\lim_{x\to 0^+} \sin x \cdot \ln x = \lim_{x\to 0^+}\left(\frac{\ln x}{\csc x}\right) = \lim_{x\to 0^+}\left(\frac{\frac{1}{x}}{-\csc x \cot x}\right) = \lim_{x\to 0^+}\left(-\frac{\sin x \tan x}{x}\right) = \lim_{x\to 0^+}\left(-\frac{\sin x \sec^2 x + \cos x \tan x}{1}\right) = \frac{0}{1} = 0$

67. $\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}} = \sqrt{\lim_{x\to\infty} \frac{9x+1}{x+1}} = \sqrt{\lim_{x\to\infty} \frac{9}{1}} = \sqrt{9} = 3$

68. $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}} = \sqrt{\frac{1}{\lim_{x\to 0^+} \frac{\sin x}{x}}} = \sqrt{\frac{1}{1}} = 1$

69. $\lim_{x\to \pi/2^-} \frac{\sec x}{\tan x} = \lim_{x\to \pi/2^-} \left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right) = \lim_{x\to \pi/2^-} \frac{1}{\sin x} = 1$

70. $\lim_{x\to 0^+} \frac{\cot x}{\csc x} = \lim_{x\to 0^+} \frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)} = \lim_{x\to 0^+} \cos x = 1$

71. $\lim_{x\to\infty} \frac{2^x - 3^x}{3^x + 4^x} = \lim_{x\to\infty} \frac{\left(\frac{2}{3}\right)^x - 1}{1 + \left(\frac{4}{3}\right)^x} = 0$

72. $\lim_{x\to -\infty} \frac{2^x + 4^x}{5^x - 2^x} = \lim_{x\to -\infty} \frac{1 + \left(\frac{4}{2}\right)^x}{\left(\frac{5}{2}\right)^x - 1} = \lim_{x\to -\infty} \frac{1 + 2^x}{\left(\frac{5}{2}\right)^x - 1} = \frac{1+0}{0-1} = -1$

73. $\lim_{x\to\infty} \frac{e^{x^2}}{x e^x} = \lim_{x\to\infty} \frac{e^{x^2 - x}}{x} = \lim_{x\to\infty} \frac{e^{x(x-1)}}{x} = \lim_{x\to\infty} \frac{e^{x(x-1)}(2x-1)}{1} = \infty$

74. $\lim_{x\to 0^+} \frac{x}{e^{-1/x}} = \lim_{x\to 0^+} \frac{e^{1/x}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{e^{1/x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to 0^+} e^{1/x} = \infty$

75. Part (b) is correct because part (a) is neither in the $\frac{0}{0}$ nor $\frac{\infty}{\infty}$ form and so l'Hôpital's rule may not be used.

76. Part (b) is correct; the step $\lim_{x\to 0} \frac{2x-2}{2x-\cos x} = \lim_{x\to 0} \frac{2}{2+\sin x}$ in part (a) is false because $\lim_{x\to 0} \frac{2x-2}{2x-\cos x}$ is not an indeterminate quotient form.

77. Part (d) is correct, the other parts are indeterminate forms and cannot be calculated by the incorrect arithmetic

78. (a) We seek c in $(-2, 0)$ so that $\frac{f'(c)}{g'(c)} = \frac{f(0)-f(-2)}{g(0)-g(-2)} = \frac{0+2}{0-4} = -\frac{1}{2}$. Since $f'(c) = 1$ and $g'(c) = 2c$ we have that $\frac{1}{2c} = -\frac{1}{2}$ $\Rightarrow c = -1$.

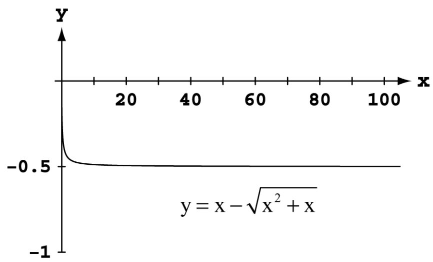
(b) We seek c in (a, b) so that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{b-a}{b^2-a^2} = \frac{1}{b+a}$. Since $f'(c) = 1$ and $g'(c) = 2c$ we have that $\frac{1}{2c} = \frac{1}{b+a}$ $\Rightarrow c = \frac{b+a}{2}$.

(c) We seek c in $(0, 3)$ so that $\frac{f'(c)}{g'(c)} = \frac{f(3)-f(0)}{g(3)-g(0)} = \frac{-3-0}{9-0} = -\frac{1}{3}$. Since $f'(c) = c^2 - 4$ and $g'(c) = 2c$ we have that $\frac{c^2-4}{2c} = -\frac{1}{3} \Rightarrow c = \frac{-1 \pm \sqrt{37}}{3} \Rightarrow c = \frac{-1+\sqrt{37}}{3}$.

79. If f(x) is to be continuous at $x = 0$, then $\lim_{x\to 0} f(x) = f(0) \Rightarrow c = f(0) = \lim_{x\to 0} \frac{9x - 3\sin 3x}{5x^3} = \lim_{x\to 0} \frac{9 - 9\cos 3x}{15x^2}$ $= \lim_{x\to 0} \frac{27\sin 3x}{30x} = \lim_{x\to 0} \frac{81\cos 3x}{30} = \frac{27}{10}$.

80. $\lim_{x\to 0}\left(\frac{\tan 2x}{x^3} + \frac{a}{x^2} + \frac{\sin bx}{x}\right) = \lim_{x\to 0}\left(\frac{\tan 2x + ax + x^2 \sin bx}{x^3}\right) = \lim_{x\to 0}\left(\frac{2\sec^2 2x + a + bx^2 \cos bx + 2x \sin bx}{3x^2}\right)$ will be in $\frac{0}{0}$ form if $\lim_{x\to 0}(2\sec^2 2x + a + bx^2 \cos bx + 2x \sin bx) = a + 2 = 0 \Rightarrow a = -2;$ $\lim_{x\to 0}\left(\frac{2\sec^2 2x - 2 + bx^2 \cos bx + 2x \sin bx}{3x^2}\right)$
$= \lim_{x\to 0}\left(\frac{8\sec^2 2x \tan 2x - b^2x^2 \sin bx + 4bx \cos bx + 2\sin bx}{6x}\right) = \lim_{x\to 0}\left(\frac{32\sec^2 2x \tan^2 2x + 16\sec^4 2x - b^3x^2 \cos bx - 6b^2x \sin bx + 6b \cos bx}{6}\right)$
$= \frac{16+6b}{6} = 0 \Rightarrow 16 + 6b = 0 \Rightarrow b = -\frac{8}{3}$

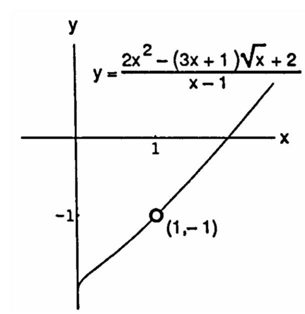
81. (a)



(b) The limit leads to the indeterminate form $\infty - \infty$:
$\lim_{x\to\infty}\left(x - \sqrt{x^2+x}\right) = \lim_{x\to\infty}\left(x - \sqrt{x^2+x}\right)\left(\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}\right) = \lim_{x\to\infty}\left(\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}}\right) = \lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}}$
$= \lim_{x\to\infty}\frac{-1}{1+\sqrt{1+\frac{1}{x}}} = \frac{-1}{1+\sqrt{1+0}} = -\frac{1}{2}$

82. $\lim_{x\to\infty}\left(\sqrt{x^2+1} - \sqrt{x}\right) = \lim_{x\to\infty} x\left(\frac{\sqrt{x^2+1}}{x} - \frac{\sqrt{x}}{x}\right) = \lim_{x\to\infty} x\left(\sqrt{\frac{x^2+1}{x^2}} - \sqrt{\frac{x}{x^2}}\right) = \lim_{x\to\infty} x\left(\sqrt{1+\frac{1}{x^2}} - \sqrt{\frac{1}{x}}\right) = \infty$

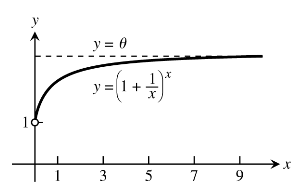
83. The graph indicates a limit near -1. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 1}\frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$
$= \lim_{x\to 1}\frac{2x^2-3x^{3/2}-x^{1/2}+2}{x-1} = \lim_{x\to 1}\frac{4x-\frac{9}{2}x^{1/2}-\frac{1}{2}x^{-1/2}}{1}$
$= \frac{4-\frac{9}{2}-\frac{1}{2}}{1} = \frac{4-5}{1} = -1$



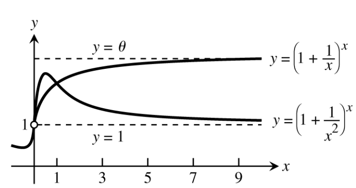
84. (a) The limit leads to the indeterminate form 1^∞. Let $f(x) = \left(1+\frac{1}{x}\right)^x \Rightarrow \ln f(x) = x \ln\left(1+\frac{1}{x}\right) \Rightarrow \lim_{x\to\infty} \ln f(x)$
$= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty}\frac{\ln(1+x^{-1})}{x^{-1}} = \lim_{x\to\infty}\frac{\left(\frac{-x^{-2}}{1+x^{-1}}\right)}{-x^{-2}} = \lim_{x\to\infty}\frac{1}{1+\left(\frac{1}{x}\right)} = \frac{1}{1+0} = 1$
$\Rightarrow \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^1 = e$

(b)

x	$\left(1+\frac{1}{x}\right)^x$
10	2.5937424601
100	2.70481382942
1000	2.71692393224
10,000	2.71814592683
100,000	2.71826823717



Both functions have limits as x approaches infinity. The function f has a maximum but no minimum while g has no extrema. The limit of f(x) leads to the indeterminate form 1^∞.



(c) Let $f(x) = \left(1 + \frac{1}{x^2}\right)^x \Rightarrow \ln f(x) = x \ln\left(1 + x^{-2}\right)$

$\Rightarrow \lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln(1 + x^{-2})}{x^{-1}} = \lim_{x \to \infty} \frac{\left(\frac{-2x^{-3}}{1+x^{-2}}\right)}{-x^{-2}} = \lim_{x \to \infty} \frac{2x^2}{(x^3 + x)} = \lim_{x \to \infty} \frac{4x}{(3x^2 + 1)} = \lim_{x \to \infty} \frac{4}{6x} = 0.$

Therefore $\lim_{x \to \infty} \left(1 + \frac{1}{x^2}\right)^x = \lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{\ln f(x)} = e^0 = 1$

85. Let $f(k) = \left(1 + \frac{r}{k}\right)^k \Rightarrow \ln f(k) = \frac{\ln(1 + rk^{-1})}{k^{-1}} \Rightarrow \lim_{k \to \infty} \frac{\ln(1 + rk^{-1})}{k^{-1}} = \lim_{k \to \infty} \frac{\left(\frac{-rk^{-2}}{1 + rk^{-1}}\right)}{-k^{-2}} = \lim_{k \to \infty} \frac{r}{1 + rk^{-1}}$

$= \lim_{k \to \infty} \frac{rk}{k + r} = \lim_{k \to \infty} \frac{r}{1} = r$. Therefore $\lim_{k \to \infty} \left(1 + \frac{r}{k}\right)^k = \lim_{k \to \infty} f(k) = \lim_{k \to \infty} e^{\ln f(k)} = e^r$.

86. (a) $y = x^{1/x} \Rightarrow \ln y = \frac{\ln x}{x} \Rightarrow \frac{y'}{y} = \frac{\left(\frac{1}{x}\right)(x) - \ln x}{x^2} \Rightarrow y' = \left(\frac{1 - \ln x}{x^2}\right)\left(x^{1/x}\right)$. The sign pattern is

$y' = \underset{0}{|} + + + + + \underset{e}{|} - - - -$ which indicates a maximum value of $y = e^{1/e}$ when $x = e$

(b) $y = x^{1/x^2} \Rightarrow \ln y = \frac{\ln x}{x^2} \Rightarrow \frac{y'}{y} = \frac{\left(\frac{1}{x}\right)(x^2) - 2x \ln x}{x^4} \Rightarrow y' = \left(\frac{1 - 2\ln x}{x^3}\right)\left(x^{1/x^2}\right)$. The sign pattern is

$y' = \underset{0}{|} + + + \underset{\sqrt{e}}{|} - - - -$ which indicates a maximum of $y = e^{1/2e}$ when $x = \sqrt{e}$

(c) $y = x^{1/x^n} \Rightarrow \ln y = \frac{\ln x}{x^n} = \frac{\left(\frac{1}{x}\right)(x^n) - (\ln x)(nx^{n-1})}{x^{2n}} \Rightarrow y' = \frac{x^{n-1}(1 - n \ln x)}{x^{2n}} \cdot x^{1/x^n}$. The sign pattern is

$y' = \underset{0}{|} + + + \underset{\sqrt[n]{e}}{|} - - - -$ which indicates a maximum of $y = e^{1/ne}$ when $x = \sqrt[n]{e}$

(d) $\lim_{x \to \infty} x^{1/x^n} = \lim_{x \to \infty} \left(e^{\ln x}\right)^{1/x^n} = \lim_{x \to \infty} e^{(\ln x)/x^n} = \exp\left(\lim_{x \to \infty} \frac{\ln x}{x^n}\right) = \exp\left(\lim_{x \to \infty} \left(\frac{1}{nx^n}\right)\right) = e^0 = 1$

87. (a) $y = x\tan\left(\frac{1}{x}\right)$, $\lim_{x\to\infty}\left(x\tan\left(\frac{1}{x}\right)\right) = \lim_{x\to\infty}\left(\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}\right) = \lim_{x\to\infty}\left(\frac{\sec^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right) = \lim_{x\to\infty}\sec^2\left(\frac{1}{x}\right) = 1$; $\lim_{x\to-\infty}\left(x\tan\left(\frac{1}{x}\right)\right)$

$= \lim_{x\to-\infty}\left(\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}\right) = \lim_{x\to-\infty}\left(\frac{\sec^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right) = \lim_{x\to-\infty}\sec^2\left(\frac{1}{x}\right) = 1 \Rightarrow$ the horizontal asymptote is $y = 1$ as $x \to \infty$ and as $x \to -\infty$.

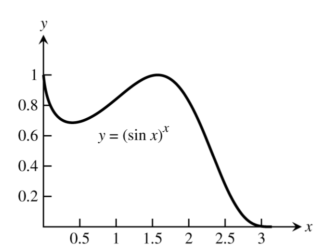
(b) $y = \frac{3x + e^{2x}}{2x + e^{3x}}$, $\lim_{x\to\infty}\left(\frac{3x + e^{2x}}{2x + e^{3x}}\right) = \lim_{x\to\infty}\left(\frac{3 + 2e^{2x}}{2 + 3e^{3x}}\right) = \lim_{x\to\infty}\left(\frac{4e^{2x}}{9e^{3x}}\right) = \lim_{x\to\infty}\left(\frac{4}{9e^x}\right) = 0$; $\lim_{x\to-\infty}\left(\frac{3x + e^{2x}}{2x + e^{3x}}\right)$

$= \lim_{x\to-\infty}\left(\frac{3 + 2e^{2x}}{2 + 3e^{3x}}\right) = \frac{3}{2} \Rightarrow$ the horizontal asymptotes are $y = 0$ as $x \to \infty$ and $y = \frac{3}{2}$ as $x \to -\infty$.

88. $f'(0) = \lim_{h\to0}\frac{f(0 + h) - f(0)}{h} = \lim_{h\to0}\frac{e^{-1/h^2} - 0}{h} = \lim_{h\to0}\frac{e^{-1/h^2}}{h} = \lim_{h\to0}\left(\frac{\frac{1}{h}}{e^{1/h^2}}\right) = \lim_{h\to0}\left(\frac{-\frac{1}{h^2}}{e^{1/h^2}\left(-\frac{2}{h^3}\right)}\right) = \lim_{h\to0}\left(\frac{h}{2e^{1/h^2}}\right)$

$= \lim_{h\to0}\left(\frac{h}{2}e^{-1/h^2}\right) = 0$

89. (a) We should assign the value 1 to $f(x) = (\sin x)^x$ to make it continuous at $x = 0$.

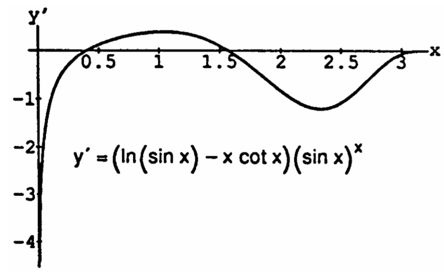


(b) $\ln f(x) = x \ln(\sin x) = \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} \Rightarrow \lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} = \lim_{x \to 0^+} \frac{\left(\frac{1}{\sin x}\right)(\cos x)}{\left(-\frac{1}{x^2}\right)}$

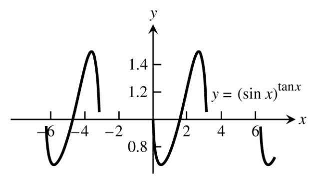
$= \lim_{x \to 0} \frac{-x^2}{\tan x} = \lim_{x \to 0} \frac{-2x}{\sec^2 x} = 0 \Rightarrow \lim_{x \to 0} f(x) = e^0 = 1$

(c) The maximum value of f(x) is close to 1 near the point $x \approx 1.55$ (see the graph in part (a)).

(d) The root in question is near 1.57.



90. (a) When $\sin x < 0$ there are gaps in the sketch. The width of each gap is π.



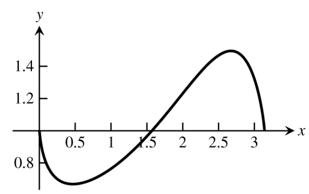
(b) Let $f(x) = (\sin x)^{\tan x} \Rightarrow \ln f(x) = (\tan x) \ln(\sin x)$

$\Rightarrow \lim_{x \to \pi/2^-} \ln f(x) = \lim_{x \to \pi/2^-} \frac{\ln(\sin x)}{\cot x}$

$= \lim_{x \to \pi/2^-} \frac{\left(\frac{1}{\sin x}\right)(\cos x)}{-\csc^2 x} = \lim_{x \to \pi/2^-} \frac{\cos x}{(-\csc x)} = 0$

$\Rightarrow \lim_{x \to \pi/2^-} f(x) = e^0 = 1$. Similarly,

$\lim_{x \to \pi/2^+} f(x) = e^0 = 1$. Therefore, $\lim_{x \to \pi/2} f(x) = 1$.



(c) From the graph in part (b) we have a minimum of about 0.665 at $x \approx 0.47$ and the maximum is about 1.491 at $x \approx 2.66$.

7.6 INVERSE TRIGONOMETRIC FUNCTIONS

1. (a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$

2. (a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $-\frac{\pi}{6}$

3. (a) $-\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $-\frac{\pi}{3}$

4. (a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{4}$ (c) $\frac{\pi}{3}$

5. (a) $\frac{\pi}{3}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{6}$

6. (a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$

7. (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$

8. (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$

9. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

10. $\sec\left(\cos^{-1}\frac{1}{2}\right) = \sec\left(\frac{\pi}{3}\right) = 2$

11. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

12. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cot\left(-\frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}}$

13. $\lim_{x \to 1^-} \sin^{-1} x = \frac{\pi}{2}$

14. $\lim_{x \to -1^+} \cos^{-1} x = \pi$

15. $\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$

16. $\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

17. $\lim_{x \to \infty} \sec^{-1} x = \frac{\pi}{2}$

18. $\lim_{x \to -\infty} \sec^{-1} x = \lim_{x \to -\infty} \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$

19. $\lim_{x \to \infty} \csc^{-1} x = \lim_{x \to \infty} \sin^{-1}\left(\frac{1}{x}\right) = 0$

20. $\lim_{x \to -\infty} \csc^{-1} x = \lim_{x \to -\infty} \sin^{-1}\left(\frac{1}{x}\right) = 0$

21. $y = \cos^{-1}(x^2) \Rightarrow \frac{dy}{dx} = -\frac{2x}{\sqrt{1-(x^2)^2}} = \frac{-2x}{\sqrt{1-x^4}}$

22. $y = \cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

23. $y = \sin^{-1}\sqrt{2}t \Rightarrow \frac{dy}{dt} = \frac{\sqrt{2}}{\sqrt{1-\left(\sqrt{2}t\right)^2}} = \frac{\sqrt{2}}{\sqrt{1-2t^2}}$

24. $y = \sin^{-1}(1-t) \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-(1-t)^2}} = \frac{-1}{\sqrt{2t-t^2}}$

25. $y = \sec^{-1}(2s+1) \Rightarrow \frac{dy}{ds} = \frac{2}{|2s+1|\sqrt{(2s+1)^2-1}} = \frac{2}{|2s+1|\sqrt{4s^2+4s}} = \frac{1}{|2s+1|\sqrt{s^2+s}}$

26. $y = \sec^{-1} 5s \Rightarrow \frac{dy}{ds} = \frac{5}{|5s|\sqrt{(5s)^2-1}} = \frac{1}{|s|\sqrt{25s^2-1}}$

27. $y = \csc^{-1}(x^2+1) \Rightarrow \frac{dy}{dx} = -\frac{2x}{|x^2+1|\sqrt{(x^2+1)^2-1}} = \frac{-2x}{(x^2+1)\sqrt{x^4+2x^2}}$

28. $y = \csc^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = -\frac{\left(\frac{1}{2}\right)}{\left|\frac{x}{2}\right|\sqrt{\left(\frac{x}{2}\right)^2-1}} = \frac{-1}{|x|\sqrt{\frac{x^2-4}{4}}} = \frac{-2}{|x|\sqrt{x^2-4}}$

29. $y = \sec^{-1}\left(\frac{1}{t}\right) = \cos^{-1} t \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$

30. $y = \sin^{-1}\left(\frac{3}{t^2}\right) = \csc^{-1}\left(\frac{t^2}{3}\right) \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{2t}{3}\right)}{\left|\frac{t^2}{3}\right|\sqrt{\left(\frac{t^2}{3}\right)^2-1}} = \frac{-2t}{t^2\sqrt{\frac{t^4-9}{9}}} = \frac{-6}{t\sqrt{t^4-9}}$

31. $y = \cot^{-1}\sqrt{t} = \cot^{-1} t^{1/2} \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1+(t^{1/2})^2} = \frac{-1}{2\sqrt{t}(1+t)}$

32. $y = \cot^{-1}\sqrt{t-1} = \cot^{-1}(t-1)^{1/2} \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)(t-1)^{-1/2}}{1+[(t-1)^{1/2}]^2} = \frac{-1}{2\sqrt{t-1}\,(1+t-1)} = \frac{-1}{2t\sqrt{t-1}}$

33. $y = \ln(\tan^{-1} x) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{1+x^2}\right)}{\tan^{-1} x} = \frac{1}{(\tan^{-1} x)(1+x^2)}$

34. $y = \tan^{-1}(\ln x) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)}{1+(\ln x)^2} = \frac{1}{x[1+(\ln x)^2]}$

35. $y = \csc^{-1}(e^t) \Rightarrow \frac{dy}{dt} = -\frac{e^t}{|e^t|\sqrt{(e^t)^2-1}} = \frac{-1}{\sqrt{e^{2t}-1}}$

36. $y = \cos^{-1}(e^{-t}) \Rightarrow \frac{dy}{dt} = -\frac{-e^{-t}}{\sqrt{1-(e^{-t})^2}} = \frac{e^{-t}}{\sqrt{1-e^{-2t}}}$

37. $y = s\sqrt{1-s^2} + \cos^{-1} s = s(1-s^2)^{1/2} + \cos^{-1} s \Rightarrow \frac{dy}{ds} = (1-s^2)^{1/2} + s\left(\frac{1}{2}\right)(1-s^2)^{-1/2}(-2s) - \frac{1}{\sqrt{1-s^2}}$
$= \sqrt{1-s^2} - \frac{s^2}{\sqrt{1-s^2}} - \frac{1}{\sqrt{1-s^2}} = \sqrt{1-s^2} - \frac{s^2+1}{\sqrt{1-s^2}} = \frac{1-s^2-s^2-1}{\sqrt{1-s^2}} = \frac{-2s^2}{\sqrt{1-s^2}}$

38. $y = \sqrt{s^2-1} - \sec^{-1} s = (s^2-1)^{1/2} - \sec^{-1} s \Rightarrow \frac{dy}{dx} = \left(\frac{1}{2}\right)(s^2-1)^{-1/2}(2s) - \frac{1}{|s|\sqrt{s^2-1}} = \frac{s}{\sqrt{s^2-1}} - \frac{1}{|s|\sqrt{s^2-1}}$
$= \frac{s|s|-1}{|s|\sqrt{s^2-1}}$

39. $y = \tan^{-1}\sqrt{x^2-1} + \csc^{-1} x = \tan^{-1}(x^2-1)^{1/2} + \csc^{-1} x \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)(x^2-1)^{-1/2}(2x)}{1+\left[(x^2-1)^{1/2}\right]^2} - \frac{1}{|x|\sqrt{x^2-1}}$
$= \frac{1}{x\sqrt{x^2-1}} - \frac{1}{|x|\sqrt{x^2-1}} = 0$, for $x > 1$

40. $y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1} x = \frac{\pi}{2} - \tan^{-1}(x^{-1}) - \tan^{-1} x \Rightarrow \frac{dy}{dx} = 0 - \frac{-x^{-2}}{1+(x^{-1})^2} - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0$

41. $y = x\sin^{-1} x + \sqrt{1-x^2} = x\sin^{-1} x + (1-x^2)^{1/2} \Rightarrow \frac{dy}{dx} = \sin^{-1} x + x\left(\frac{1}{\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)$
$= \sin^{-1} x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \sin^{-1} x$

42. $y = \ln(x^2+4) - x\tan^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - x\left[\frac{\left(\frac{1}{2}\right)}{1+\left(\frac{x}{2}\right)^2}\right] = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{2x}{4+x^2} = -\tan^{-1}\left(\frac{x}{2}\right)$

43. $\int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right) + C$

44. $\int \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int \frac{2}{\sqrt{1-(2x)^2}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = 2x$ and $du = 2\,dx$
$= \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1}(2x) + C$

45. $\int \frac{1}{17+x^2}\,dx = \int \frac{1}{\left(\sqrt{17}\right)^2+x^2}\,dx = \frac{1}{\sqrt{17}}\tan^{-1}\frac{x}{\sqrt{17}} + C$

46. $\int \frac{1}{9+3x^2}\,dx = \frac{1}{3}\int \frac{1}{\left(\sqrt{3}\right)^2+x^2}\,dx = \frac{1}{3\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C = \frac{\sqrt{3}}{9}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

47. $\int \frac{dx}{x\sqrt{25x^2-2}} = \int \frac{du}{u\sqrt{u^2-2}}$, where $u = 5x$ and $du = 5\,dx$
$= \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{u}{\sqrt{2}}\right| + C = \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{5x}{\sqrt{2}}\right| + C$

48. $\int \frac{dx}{x\sqrt{5x^2-4}} = \int \frac{du}{u\sqrt{u^2-4}}$, where $u = \sqrt{5}x$ and $du = \sqrt{5}\,dx$
$= \frac{1}{2}\sec^{-1}\left|\frac{u}{2}\right| + C = \frac{1}{2}\sec^{-1}\left|\frac{\sqrt{5}x}{2}\right| + C$

49. $\int_0^1 \frac{4\,ds}{\sqrt{4-s^2}} = \left[4\sin^{-1}\frac{s}{2}\right]_0^1 = 4\left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right) = 4\left(\frac{\pi}{6} - 0\right) = \frac{2\pi}{3}$

50. $\int_0^{3\sqrt{2}/4} \frac{ds}{\sqrt{9-4s^2}} = \frac{1}{2}\int_0^{3\sqrt{2}/4} \frac{du}{\sqrt{9-u^2}}$, where $u = 2s$ and $du = 2\,ds$; $s = 0 \Rightarrow u = 0$, $s = \frac{3\sqrt{2}}{4} \Rightarrow u = \frac{3\sqrt{2}}{2}$
$= \left[\frac{1}{2}\sin^{-1}\frac{u}{3}\right]_0^{3\sqrt{2}/2} = \frac{1}{2}\left(\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1} 0\right) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$

51. $\int_0^2 \frac{dt}{8+2t^2} = \frac{1}{\sqrt{2}}\int_0^{2\sqrt{2}} \frac{du}{8+u^2}$, where $u = \sqrt{2}t$ and $du = \sqrt{2}\,dt$; $t = 0 \Rightarrow u = 0$, $t = 2 \Rightarrow u = 2\sqrt{2}$
$= \left[\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{8}}\tan^{-1}\frac{u}{\sqrt{8}}\right]_0^{2\sqrt{2}} = \frac{1}{4}\left(\tan^{-1}\frac{2\sqrt{2}}{\sqrt{8}} - \tan^{-1} 0\right) = \frac{1}{4}\left(\tan^{-1} 1 - \tan^{-1} 0\right) = \frac{1}{4}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{16}$

52. $\int_{-2}^{2} \frac{dt}{4+3t^2} = \frac{1}{\sqrt{3}} \int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{du}{4+u^2}$, where $u = \sqrt{3}t$ and $du = \sqrt{3}\,dt$; $t = -2 \Rightarrow u = -2\sqrt{3}$, $t = 2 \Rightarrow u = 2\sqrt{3}$
$= \left[\frac{1}{\sqrt{3}} \cdot \frac{1}{2} \tan^{-1} \frac{u}{2}\right]_{-2\sqrt{3}}^{2\sqrt{3}} = \frac{1}{2\sqrt{3}} \left[\tan^{-1} \sqrt{3} - \tan^{-1}\left(-\sqrt{3}\right)\right] = \frac{1}{2\sqrt{3}} \left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \frac{\pi}{3\sqrt{3}}$

53. $\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2-1}} = \int_{-2}^{-\sqrt{2}} \frac{du}{u\sqrt{u^2-1}}$, where $u = 2y$ and $du = 2\,dy$; $y = -1 \Rightarrow u = -2$, $y = -\frac{\sqrt{2}}{2} \Rightarrow u = -\sqrt{2}$
$= \left[\sec^{-1} |u|\right]_{-2}^{-\sqrt{2}} = \sec^{-1} \left|-\sqrt{2}\right| - \sec^{-1} |-2| = \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$

54. $\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}} = \int_{-2}^{-\sqrt{2}} \frac{du}{u\sqrt{u^2-1}}$, where $u = 3y$ and $du = 3\,dy$; $y = -\frac{2}{3} \Rightarrow u = -2$, $y = -\frac{\sqrt{2}}{3} \Rightarrow u = -\sqrt{2}$
$= \left[\sec^{-1} |u|\right]_{-2}^{-\sqrt{2}} = \sec^{-1} \left|-\sqrt{2}\right| - \sec^{-1} |-2| = \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$

55. $\int \frac{3\,dr}{\sqrt{1-4(r-1)^2}} = \frac{3}{2} \int \frac{du}{\sqrt{1-u^2}}$, where $u = 2(r-1)$ and $du = 2\,dr$
$= \frac{3}{2} \sin^{-1} u + C = \frac{3}{2} \sin^{-1} 2(r-1) + C$

56. $\int \frac{6\,dr}{\sqrt{4-(r+1)^2}} = 6 \int \frac{du}{\sqrt{4-u^2}}$, where $u = r+1$ and $du = dr$
$= 6 \sin^{-1} \frac{u}{2} + C = 6 \sin^{-1} \left(\frac{r+1}{2}\right) + C$

57. $\int \frac{dx}{2+(x-1)^2} = \int \frac{du}{2+u^2}$, where $u = x-1$ and $du = dx$
$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + C = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x-1}{\sqrt{2}}\right) + C$

58. $\int \frac{dx}{1+(3x+1)^2} = \frac{1}{3} \int \frac{du}{1+u^2}$, where $u = 3x+1$ and $du = 3\,dx$
$= \frac{1}{3} \tan^{-1} u + C = \frac{1}{3} \tan^{-1} (3x+1) + C$

59. $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}} = \frac{1}{2}\int \frac{du}{u\sqrt{u^2-4}}$, where $u = 2x-1$ and $du = 2\,dx$
$= \frac{1}{2} \cdot \frac{1}{2} \sec^{-1} \left|\frac{u}{2}\right| + C = \frac{1}{4} \sec^{-1} \left|\frac{2x-1}{2}\right| + C$

60. $\int \frac{dx}{(x+3)\sqrt{(x+3)^2-25}} = \int \frac{du}{u\sqrt{u^2-25}}$, where $u = x+3$ and $du = dx$
$= \frac{1}{5} \sec^{-1} \left|\frac{u}{5}\right| + C = \frac{1}{5} \sec^{-1} \left|\frac{x+3}{5}\right| + C$

61. $\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\,d\theta}{1+(\sin\theta)^2} = 2\int_{-1}^{1} \frac{du}{1+u^2}$, where $u = \sin\theta$ and $du = \cos\theta\,d\theta$; $\theta = -\frac{\pi}{2} \Rightarrow u = -1$, $\theta = \frac{\pi}{2} \Rightarrow u = 1$
$= \left[2\tan^{-1} u\right]_{-1}^{1} = 2\left(\tan^{-1} 1 - \tan^{-1}(-1)\right) = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$

62. $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\,dx}{1+(\cot x)^2} = -\int_{\sqrt{3}}^{1} \frac{du}{1+u^2}$, where $u = \cot x$ and $du = -\csc^2 x\,dx$; $x = \frac{\pi}{6} \Rightarrow u = \sqrt{3}$, $x = \frac{\pi}{4} \Rightarrow u = 1$
$= \left[-\tan^{-1} u\right]_{\sqrt{3}}^{1} = -\tan^{-1} 1 + \tan^{-1} \sqrt{3} = -\frac{\pi}{4} + \frac{\pi}{3} = \frac{\pi}{12}$

63. $\int_{0}^{\ln\sqrt{3}} \frac{e^x\,dx}{1+e^{2x}} = \int_{1}^{\sqrt{3}} \frac{du}{1+u^2}$, where $u = e^x$ and $du = e^x\,dx$; $x = 0 \Rightarrow u = 1$, $x = \ln\sqrt{3} \Rightarrow u = \sqrt{3}$
$= \left[\tan^{-1} u\right]_{1}^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1} 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

64. $\int_1^{e^{\pi/4}} \frac{4\,dt}{t\,(1+\ln^2 t)} = 4\int_0^{\pi/4} \frac{du}{1+u^2}$, where $u = \ln t$ and $du = \frac{1}{t}\,dt$; $t = 1 \Rightarrow u = 0$, $t = e^{\pi/4} \Rightarrow u = \frac{\pi}{4}$
$= \left[4\tan^{-1} u\right]_0^{\pi/4} = 4\left(\tan^{-1}\frac{\pi}{4} - \tan^{-1} 0\right) = 4\tan^{-1}\frac{\pi}{4}$

65. $\int \frac{y\,dy}{\sqrt{1-y^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = y^2$ and $du = 2y\,dy$
$= \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1} y^2 + C$

66. $\int \frac{\sec^2 y\,dy}{\sqrt{1-\tan^2 y}} = \int \frac{du}{\sqrt{1-u^2}}$, where $u = \tan y$ and $du = \sec^2 y\,dy$
$= \sin^{-1} u + C = \sin^{-1}(\tan y) + C$

67. $\int \frac{dx}{\sqrt{-x^2+4x-3}} = \int \frac{dx}{\sqrt{1-(x^2-4x+4)}} = \int \frac{dx}{\sqrt{1-(x-2)^2}} = \sin^{-1}(x-2) + C$

68. $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x^2-2x+1)}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}(x-1) + C$

69. $\int_{-1}^{0} \frac{6\,dt}{\sqrt{3-2t-t^2}} = 6\int_{-1}^{0} \frac{dt}{\sqrt{4-(t^2+2t+1)}} = 6\int_{-1}^{0} \frac{dt}{\sqrt{2^2-(t+1)^2}} = 6\left[\sin^{-1}\left(\frac{t+1}{2}\right)\right]_{-1}^{0}$
$= 6\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0\right] = 6\left(\frac{\pi}{6} - 0\right) = \pi$

70. $\int_{1/2}^{1} \frac{6\,dt}{\sqrt{3+4t-4t^2}} = 3\int_{1/2}^{1} \frac{2\,dt}{\sqrt{4-(4t^2-4t+1)}} = 3\int_{1/2}^{1} \frac{2\,dt}{\sqrt{2^2-(2t-1)^2}} = 3\left[\sin^{-1}\left(\frac{2t-1}{2}\right)\right]_{1/2}^{1}$
$= 3\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0\right] = 3\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{2}$

71. $\int \frac{dy}{y^2-2y+5} = \int \frac{dy}{4+y^2-2y+1} = \int \frac{dy}{2^2+(y-1)^2} = \frac{1}{2}\tan^{-1}\left(\frac{y-1}{2}\right) + C$

72. $\int \frac{dy}{y^2+6y+10} = \int \frac{dy}{1+(y^2+6y+9)} = \int \frac{dy}{1+(y+3)^2} = \tan^{-1}(y+3) + C$

73. $\int_1^2 \frac{8\,dx}{x^2-2x+2} = 8\int_1^2 \frac{dx}{1+(x^2-2x+1)} = 8\int_1^2 \frac{dx}{1+(x-1)^2} = 8\left[\tan^{-1}(x-1)\right]_1^2 = 8\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 8\left(\frac{\pi}{4} - 0\right) = 2\pi$

74. $\int_2^4 \frac{2\,dx}{x^2-6x+10} = 2\int_2^4 \frac{dx}{1+(x^2-6x+9)} = 2\int_2^4 \frac{dx}{1+(x-3)^2} = 2\left[\tan^{-1}(x-3)\right]_2^4 = 2\left[\tan^{-1} 1 - \tan^{-1}(-1)\right] = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$

75. $\int \frac{x+4}{x^2+4}dx = \int \frac{x}{x^2+4}dx + \int \frac{4}{x^2+4}dx$; $\int \frac{x}{x^2+4}dx = \frac{1}{2}\int \frac{1}{u}du$ where $u = x^2+4 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$
$\Rightarrow \int \frac{x+4}{x^2+4}dx = \frac{1}{2}\ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + C$

76. $\int \frac{t-2}{t^2-6t+10}dt = \int \frac{t-2}{(t-3)^2+1}dt$ $\left[\text{Let } w = t-3 \Rightarrow w+3 = t \Rightarrow dw = dt\right] \to \int \frac{w+1}{w^2+1}dw = \int \frac{w}{w^2+1}dw + \int \frac{1}{w^2+1}dw$;
$\int \frac{w}{w^2+1}dw = \frac{1}{2}\int \frac{1}{u}du$ where $u = w^2+1 \Rightarrow du = 2w\,dw \Rightarrow \frac{1}{2}du = w\,dw \Rightarrow \int \frac{w}{w^2+1}dw + \int \frac{1}{w^2+1}dw$
$= \frac{1}{2}\ln(w^2+1) + \tan^{-1}(w) + C = \frac{1}{2}\ln\left((t-3)^2+1\right) + \tan^{-1}(t-3) + C = \frac{1}{2}\ln(t^2-6t+10) + \tan^{-1}(t-3) + C$

77. $\int \frac{x^2+2x-1}{x^2+9}dx = \int\left(1 + \frac{2x-10}{x^2+9}\right)dx = \int dx + \int \frac{2x}{x^2+9}dx - 10\int \frac{1}{x^2+9}dx$; $\int \frac{2x}{x^2+9}dx = \int \frac{1}{u}du$ where $u = x^2+9$
$\Rightarrow du = 2x\,dx \Rightarrow \int dx + \int \frac{2x}{x^2+9}dx - 10\int \frac{1}{x^2+9}dx = x + \ln(x^2+9) - \frac{10}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

78. $\int \frac{t^3-2t^2+3t-4}{t^2+1}dt = \int\left(t-2+\frac{2t-2}{t^2+1}\right)dt = \int(t-2)dt + \int \frac{2t}{t^2+1}dt - 2\int \frac{1}{t^2+1}dt$; $\int \frac{2t}{t^2+1}dt = \int \frac{1}{u}du$ where $u = t^2+1$
$\Rightarrow du = 2t\,dt \Rightarrow \int(t-2)dt + \int \frac{2t}{t^2+1}dt - 2\int \frac{1}{t^2+1}dt = \frac{1}{2}t^2 - 2t + \ln(t^2+1) - 2\tan^{-1}(t) + C$

79. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{x^2+2x+1-1}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}} = \int \frac{du}{u\sqrt{u^2-1}}$, where $u = x+1$ and $du = dx$
$= \sec^{-1}|u| + C = \sec^{-1}|x+1| + C$

80. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{x^2-4x+4-1}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}} = \int \frac{1}{u\sqrt{u^2-1}}\,du$, where $u = x-2$ and $du = dx$
$= \sec^{-1}|u| + C = \sec^{-1}|x-2| + C$

81. $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}\,dx = \int e^u\,du$, where $u = \sin^{-1} x$ and $du = \frac{dx}{\sqrt{1-x^2}}$
$= e^u + C = e^{\sin^{-1} x} + C$

82. $\int \frac{e^{\cos^{-1} x}}{\sqrt{1-x^2}}\,dx = -\int e^u\,du$, where $u = \cos^{-1} x$ and $du = \frac{-dx}{\sqrt{1-x^2}}$
$= -e^u + C = -e^{\cos^{-1} x} + C$

83. $\int \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}}\,dx = \int u^2\,du$, where $u = \sin^{-1} x$ and $du = \frac{dx}{\sqrt{1-x^2}}$
$= \frac{u^3}{3} + C = \frac{(\sin^{-1} x)^3}{3} + C$

84. $\int \frac{\sqrt{\tan^{-1} x}}{1+x^2}\,dx = \int u^{1/2}\,du$, where $u = \tan^{-1} x$ and $du = \frac{dx}{1+x^2}$
$= \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\tan^{-1} x)^{3/2} + C = \frac{2}{3}\sqrt{(\tan^{-1} x)^3} + C$

85. $\int \frac{1}{(\tan^{-1} y)(1+y^2)}\,dy = \int \frac{\left(\frac{1}{1+y^2}\right)}{\tan^{-1} y}\,dy = \int \frac{1}{u}\,du$, where $u = \tan^{-1} y$ and $du = \frac{dy}{1+y^2}$
$= \ln|u| + C = \ln|\tan^{-1} y| + C$

86. $\int \frac{1}{(\sin^{-1} y)\sqrt{1+y^2}}\,dy = \int \frac{\left(\frac{1}{\sqrt{1-y^2}}\right)}{\sin^{-1} y}\,dy = \int \frac{1}{u}\,du$, where $u = \sin^{-1} y$ and $du = \frac{dy}{\sqrt{1-y^2}}$
$= \ln|u| + C = \ln|\sin^{-1} y| + C$

87. $\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1} x)}{x\sqrt{x^2-1}}\,dx = \int_{\pi/4}^{\pi/3} \sec^2 u\,du$, where $u = \sec^{-1} x$ and $du = \frac{dx}{x\sqrt{x^2-1}}$; $x = \sqrt{2} \Rightarrow u = \frac{\pi}{4}$, $x = 2 \Rightarrow u = \frac{\pi}{3}$
$= [\tan u]_{\pi/4}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{4} = \sqrt{3} - 1$

88. $\int_{2/\sqrt{3}}^{2} \frac{\cos(\sec^{-1} x)}{x\sqrt{x^2-1}}\,dx = \int_{\pi/6}^{\pi/3} \cos u\,du$, where $u = \sec^{-1} x$ and $du = \frac{dx}{x\sqrt{x^2-1}}$; $x = \frac{2}{\sqrt{3}} \Rightarrow u = \frac{\pi}{6}$, $x = 2 \Rightarrow u = \frac{\pi}{3}$
$= [\sin u]_{\pi/6}^{\pi/3} = \sin\frac{\pi}{3} - \sin\frac{\pi}{6} = \frac{\sqrt{3}-1}{2}$

89. $\int \frac{1}{\sqrt{x}(x+1)\left[(\tan^{-1}\sqrt{x})^2+9\right]}\,dx = 2\int \frac{1}{u^2+9}du$ where $u = \tan^{-1}\sqrt{x} \Rightarrow du = \frac{1}{1+(\sqrt{x})^2}\frac{1}{2\sqrt{x}}dx \Rightarrow 2du = \frac{1}{(1+x)\sqrt{x}}dx$
$= \frac{2}{3}\tan^{-1}\left(\frac{\tan^{-1}\sqrt{x}}{3}\right) + C$

90. $\int \frac{e^x \sin^{-1} e^x}{\sqrt{1-e^{2x}}}\,dx = \int u\,du$ where $u = \sin^{-1} e^x \Rightarrow du = \frac{1}{\sqrt{1-e^{2x}}}e^x dx$
$= \frac{1}{2}(\sin^{-1} e^x)^2 + C$

91. $\lim\limits_{x \to 0} \frac{\sin^{-1} 5x}{x} = \lim\limits_{x \to 0} \frac{\left(\frac{5}{\sqrt{1-25x^2}}\right)}{1} = 5$

92. $\lim\limits_{x \to 1^+} \frac{\sqrt{x^2-1}}{\sec^{-1} x} = \lim\limits_{x \to 1^+} \frac{(x^2-1)^{1/2}}{\sec^{-1} x} = \lim\limits_{x \to 1^+} \frac{\left(\frac{1}{2}\right)(x^2-1)^{-1/2}(2x)}{\left(\frac{1}{|x|\sqrt{x^2-1}}\right)} = \lim\limits_{x \to 1^+} x\,|x| = 1$

93. $\lim\limits_{x \to \infty} x \tan^{-1}\left(\frac{2}{x}\right) = \lim\limits_{x \to \infty} \frac{\tan^{-1}(2x^{-1})}{x^{-1}} = \lim\limits_{x \to \infty} \frac{\left(\frac{-2x^{-2}}{1+4x^{-2}}\right)}{-x^{-2}} = \lim\limits_{x \to \infty} \frac{2}{1+4x^{-2}} = 2$

94. $\lim\limits_{x \to 0} \frac{2\tan^{-1} 3x^2}{7x^2} = \lim\limits_{x \to 0} \frac{\left(\frac{12x}{1+9x^4}\right)}{14x} = \lim\limits_{x \to 0} \frac{6}{7(1+9x^4)} = \frac{6}{7}$

95. $\lim\limits_{x\to0} \frac{\tan^{-1} x^2}{x \sin^{-1} x} = \lim\limits_{x\to0} \left(\frac{\frac{2x}{1+x^4}}{x\frac{1}{\sqrt{1-x^2}} + \sin^{-1} x}\right) = \lim\limits_{x\to0} \left(\frac{\frac{-2\left(3x^4-1\right)}{\left(1+x^4\right)^2}}{\frac{-x^2+2}{\left(1-x^2\right)^{3/2}}}\right) = \frac{\frac{-2(0-1)}{1^2}}{\frac{-0+2}{(1-0)^{3/2}}} = \frac{2}{2} = 1$

96. $\lim\limits_{x\to\infty} \frac{e^x \tan^{-1} e^x}{e^{2x}+x} = \lim\limits_{x\to\infty} \frac{e^x \tan^{-1} e^x + \frac{e^{2x}}{e^{2x}+1}}{2e^{2x}+1} = \lim\limits_{x\to\infty} \frac{e^x \tan^{-1} e^x + \frac{e^{2x}}{e^{2x}+1} + \frac{2e^{2x}}{\left(e^{2x}+1\right)^2}}{4e^{2x}} = \lim\limits_{x\to\infty} \frac{e^x \tan^{-1} e^x + \frac{e^{2x}\left(e^{2x}+3\right)}{\left(e^{2x}+1\right)^2}}{4e^{2x}}$

$= \lim\limits_{x\to\infty}\left[\frac{\tan^{-1} e^x}{4e^x} + \frac{(e^{2x}+3)}{4(e^{2x}+1)^2}\right] = \lim\limits_{x\to\infty}\left[\frac{\tan^{-1} e^x}{4e^x} + \frac{(1+3e^{-2x})}{4(e^x+e^{-x})^2}\right] = 0+0=0$

97. $\lim\limits_{x\to0^+} \frac{\left[\tan^{-1}\left(\sqrt{x}\right)\right]^2}{x\sqrt{x+1}} = \lim\limits_{x\to0^+} \frac{\tan^{-1}\left(\sqrt{x}\right)\frac{1}{\sqrt{x}(1+x)}}{\frac{x}{2\sqrt{x+1}} + \sqrt{x+1}} = \lim\limits_{x\to0^+} \frac{\frac{\tan^{-1}\left(\sqrt{x}\right)}{\sqrt{x}(1+x)}}{\frac{3x+2}{2\sqrt{x+1}}} = \lim\limits_{x\to0^+}\left(\frac{2\tan^{-1}\left(\sqrt{x}\right)}{(3x+2)\sqrt{x}\sqrt{x+1}}\right) = \lim\limits_{x\to0^+}\left(\frac{\frac{1}{\sqrt{x}(1+x)}}{\frac{12x^2+13x+2}{2\sqrt{x}\sqrt{x+1}}}\right)$

$= \lim\limits_{x\to0^+}\left(\frac{2}{(12x^2+13x+2)\sqrt{x+1}}\right) = \frac{2}{2} = 1$

98. $\lim\limits_{x\to0^+} \frac{\sin^{-1}(x^2)}{(\sin^{-1} x)^2} = \lim\limits_{x\to0^+}\left(\frac{\frac{2x}{\sqrt{1-x^4}}}{2(\sin^{-1} x)\frac{1}{\sqrt{1-x^2}}}\right) = \lim\limits_{x\to0^+}\left(\frac{x}{\sin^{-1} x\sqrt{1+x^2}}\right) = \lim\limits_{x\to0^+}\left(\frac{1}{\sin^{-1} x\cdot\frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1-x^2}}\sqrt{1+x^2}}\right) =$

$= \lim\limits_{x\to0^+}\left(\frac{\sqrt{1+x^2}\sqrt{1-x^2}}{1+x^2+x\sqrt{1-x^2}\sin^{-1} x}\right) = \frac{1}{1} = 1$

99. If $y = \ln x - \frac{1}{2}\ln(1+x^2) - \frac{\tan^{-1} x}{x} + C$, then $dy = \left[\frac{1}{x} - \frac{x}{1+x^2} - \frac{\left(\frac{x}{1+x^2}\right) - \tan^{-1} x}{x^2}\right] dx$

$= \left(\frac{1}{x} - \frac{x}{1+x^2} - \frac{1}{x(1+x^2)} + \frac{\tan^{-1} x}{x^2}\right) dx = \frac{x(1+x^2) - x^3 - x + (\tan^{-1} x)(1+x^2)}{x^2(1+x^2)}\, dx = \frac{\tan^{-1} x}{x^2}\, dx$,

which verifies the formula

100. If $y = \frac{x^4}{4}\cos^{-1} 5x + \frac{5}{4}\int \frac{x^4}{\sqrt{1-25x^2}}\, dx$, then $dy = \left[x^3\cos^{-1} 5x + \left(\frac{x^4}{4}\right)\left(\frac{-5}{\sqrt{1-25x^2}}\right) + \frac{5}{4}\left(\frac{x^4}{\sqrt{1-25x^2}}\right)\right] dx$

$= (x^3\cos^{-1} 5x)\, dx$, which verifies the formula

101. If $y = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1-x^2}\sin^{-1} x + C$, then

$dy = \left[(\sin^{-1} x)^2 + \frac{2x(\sin^{-1} x)}{\sqrt{1-x^2}} - 2 + \frac{-2x}{\sqrt{1-x^2}}\sin^{-1} x + 2\sqrt{1-x^2}\left(\frac{1}{\sqrt{1-x^2}}\right)\right] dx = (\sin^{-1} x)^2\, dx$, which verifies the formula

102. If $y = x\ln(a^2+x^2) - 2x + 2a\tan^{-1}\left(\frac{x}{a}\right) + C$, then $dy = \left[\ln(a^2+x^2) + \frac{2x^2}{a^2+x^2} - 2 + \frac{2}{1+\left(\frac{x^2}{a^2}\right)}\right] dx$

$= \left[\ln(a^2+x^2) + 2\left(\frac{a^2+x^2}{a^2+x^2}\right) - 2\right] dx = \ln(a^2+x^2)\, dx$, which verifies the formula

103. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow dy = \frac{dx}{\sqrt{1-x^2}} \Rightarrow y = \sin^{-1} x + C$; $x = 0$ and $y = 0 \Rightarrow 0 = \sin^{-1} 0 + C \Rightarrow C = 0 \Rightarrow y = \sin^{-1} x$

104. $\frac{dy}{dx} = \frac{1}{x^2+1} - 1 \Rightarrow dy = \left(\frac{1}{1+x^2} - 1\right) dx \Rightarrow y = \tan^{-1}(x) - x + C$; $x = 0$ and $y = 1 \Rightarrow 1 = \tan^{-1} 0 - 0 + C$
$\Rightarrow C = 1 \Rightarrow y = \tan^{-1}(x) - x + 1$

105. $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}} \Rightarrow dy = \frac{dx}{x\sqrt{x^2-1}} \Rightarrow y = \sec^{-1}|x| + C$; $x = 2$ and $y = \pi \Rightarrow \pi = \sec^{-1} 2 + C \Rightarrow C = \pi - \sec^{-1} 2$
$= \pi - \frac{\pi}{3} = \frac{2\pi}{3} \Rightarrow y = \sec^{-1}(x) + \frac{2\pi}{3}, x > 1$

106. $\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} \Rightarrow dy = \left(\frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}}\right) dx \Rightarrow y = \tan^{-1} x - 2\sin^{-1} x + C$; $x = 0$ and $y = 2$
$\Rightarrow 2 = \tan^{-1} 0 - 2\sin^{-1} 0 + C \Rightarrow C = 2 \Rightarrow y = \tan^{-1} x - 2\sin^{-1} x + 2$

107. (a) The angle α is the large angle between the wall and the right end of the blackboard minus the small angle between the left end of the blackboard and the wall $\Rightarrow \alpha = \cot^{-1}\left(\frac{x}{15}\right) - \cot^{-1}\left(\frac{x}{3}\right)$.

(b) $\frac{d\alpha}{dt} = -\frac{\frac{1}{15}}{1+\left(\frac{x}{15}\right)^2} + \frac{\frac{1}{3}}{1+\left(\frac{x}{3}\right)^2} = -\frac{15}{225+x^2} + \frac{3}{9+x^2} = \frac{540-12x^2}{(225+x^2)(9+x^2)}$; $\frac{d\alpha}{dt} = 0 \Rightarrow 540 - 12x^2 = 0 \Rightarrow x = \pm 3\sqrt{5}$
Since $x > 0$, consider only $x = 3\sqrt{5} \Rightarrow \alpha\left(3\sqrt{5}\right) = \cot^{-1}\left(\frac{3\sqrt{5}}{15}\right) - \cot^{-1}\left(\frac{3\sqrt{5}}{3}\right) \approx 0.729728 \approx 41.8103°$. Using the first derivative test, $\left.\frac{d\alpha}{dt}\right|_{x=1} = \frac{132}{565} > 0$ and $\left.\frac{d\alpha}{dt}\right|_{x=10} = -\frac{132}{7085} < 0 \Rightarrow$ local maximum of 41.8103° when $x = 3\sqrt{5} \approx 6.7082$ ft.

108. $V = \pi\int_0^{\pi/3}[2^2 - (\sec y)^2]\, dy = \pi\,[4y - \tan y]_0^{\pi/3} = \pi\left(\frac{4\pi}{3} - \sqrt{3}\right)$

109. $V = \left(\frac{1}{3}\right)\pi r^2 h = \left(\frac{1}{3}\right)\pi(3\sin\theta)^2(3\cos\theta) = 9\pi(\cos\theta - \cos^3\theta)$, where $0 \le \theta \le \frac{\pi}{2}$
$\Rightarrow \frac{dV}{d\theta} = -9\pi(\sin\theta)(1 - 3\cos^2\theta) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = \pm\frac{1}{\sqrt{3}} \Rightarrow$ the critical points are: 0, $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, and $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$; but $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is not in the domain. When $\theta = 0$, we have a minimum and when $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ $\approx 54.7°$, we have a maximum volume.

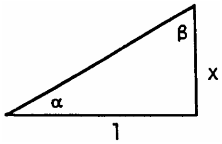
110. $65° + (90° - \beta) + (90° - \alpha) = 180° \Rightarrow \alpha = 65° - \beta = 65° - \tan^{-1}\left(\frac{21}{50}\right) \approx 65° - 22.78° \approx 42.22°$

111. Take each square as a unit square. From the diagram we have the following: the smallest angle α has a tangent of 1 $\Rightarrow \alpha = \tan^{-1} 1$; the middle angle β has a tangent of 2 $\Rightarrow \beta = \tan^{-1} 2$; and the largest angle γ has a tangent of 3 $\Rightarrow \gamma = \tan^{-1} 3$. The sum of these three angles is $\pi \Rightarrow \alpha + \beta + \gamma = \pi$
$\Rightarrow \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$.

112. (a) From the symmetry of the diagram, we see that $\pi - \sec^{-1} x$ is the vertical distance from the graph of $y = \sec^{-1} x$ to the line $y = \pi$ and this distance is the same as the height of $y = \sec^{-1} x$ above the x-axis at $-x$; i.e., $\pi - \sec^{-1} x = \sec^{-1}(-x)$.

(b) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, where $-1 \le x \le 1 \Rightarrow \cos^{-1}\left(-\frac{1}{x}\right) = \pi - \cos^{-1}\left(\frac{1}{x}\right)$, where $x \ge 1$ or $x \le -1$
$\Rightarrow \sec^{-1}(-x) = \pi - \sec^{-1} x$

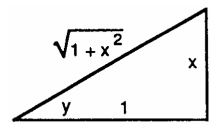
113. $\sin^{-1}(1) + \cos^{-1}(1) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$; $\sin^{-1}(0) + \cos^{-1}(0) = 0 + \frac{\pi}{2} = \frac{\pi}{2}$; and $\sin^{-1}(-1) + \cos^{-1}(-1) = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$.
If $x \in (-1, 0)$ and $x = -a$, then $\sin^{-1}(x) + \cos^{-1}(x) = \sin^{-1}(-a) + \cos^{-1}(-a) = -\sin^{-1} a + (\pi - \cos^{-1} a)$
$= \pi - (\sin^{-1} a + \cos^{-1} a) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$ from Equations (3) and (4) in the text.

114. 

$x \Rightarrow \tan\alpha = x$ and $\tan\beta = \frac{1}{x} \Rightarrow \frac{\pi}{2} = \alpha + \beta = \tan^{-1} x + \tan^{-1}\frac{1}{x}$.

115. $\csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u \Rightarrow \frac{d}{dx}(\csc^{-1} u) = \frac{d}{dx}\left(\frac{\pi}{2} - \sec^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}} = -\frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}}, |u| > 1$

116. $y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow \frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$
$\Rightarrow (\sec^2 y)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\left(\sqrt{1+x^2}\right)^2}$
$= \frac{1}{1+x^2}$, as indicated by the triangle



117. $f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}} = \frac{1}{\sec(\sec^{-1} b)\tan(\sec^{-1} b)} = \frac{1}{b\left(\pm\sqrt{b^2-1}\right)}$.
Since the slope of $\sec^{-1} x$ is always positive, we the right sign by writing $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$.

118. $\cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u \Rightarrow \frac{d}{dx}(\cot^{-1} u) = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{1+u^2} = -\frac{\frac{du}{dx}}{1+u^2}$

119. The functions f and g have the same derivative (for $x \ge 0$), namely $\frac{1}{\sqrt{x}(x+1)}$. The functions therefore differ by a constant. To identify the constant we can set x equal to 0 in the equation $f(x) = g(x) + C$, obtaining $\sin^{-1}(-1) = 2\tan^{-1}(0) + C \Rightarrow -\frac{\pi}{2} = 0 + C \Rightarrow C = -\frac{\pi}{2}$. For $x \ge 0$, we have $\sin^{-1}\left(\frac{x-1}{x+1}\right) = 2\tan^{-1}\sqrt{x} - \frac{\pi}{2}$.

120. The functions f and g have the same derivative for $x > 0$, namely $\frac{-1}{1+x^2}$. The functions therefore differ by a constant for $x > 0$. To identify the constant we can set x equal to 1 in the equation $f(x) = g(x) + C$, obtaining $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \tan^{-1} 1 + C \Rightarrow \frac{\pi}{4} = \frac{\pi}{4} + C \Rightarrow C = 0$. For $x > 0$, we have $\sin^{-1}\frac{1}{\sqrt{x^2+1}} = \tan^{-1}\frac{1}{x}$.

121. $V = \pi\int_{-\sqrt{3}/3}^{\sqrt{3}}\left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx = \pi\int_{-\sqrt{3}/3}^{\sqrt{3}}\frac{1}{1+x^2}\,dx = \pi\left[\tan^{-1} x\right]_{-\sqrt{3}/3}^{\sqrt{3}} = \pi\left[\tan^{-1}\sqrt{3} - \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right]$
$= \pi\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi^2}{2}$

122. Consider $y = \sqrt{r^2 - x^2} \Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$; Since $\frac{dy}{dx}$ is undefined at $x = r$ and $x = -r$, we will find the length from $x = 0$ to $x = \frac{r}{\sqrt{2}}$ (in other words, the length of $\frac{1}{8}$ of a circle) $\Rightarrow L = \int_0^{r/\sqrt{2}}\sqrt{1 + \left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}dx = \int_0^{r/\sqrt{2}}\sqrt{1 + \frac{x^2}{r^2-x^2}}dx$
$= \int_0^{r/\sqrt{2}}\sqrt{\frac{r^2}{r^2-x^2}}dx = \int_0^{r/\sqrt{2}}\frac{r}{\sqrt{r^2-x^2}}dx = \left[r\sin^{-1}\left(\frac{x}{r}\right)\right]_0^{r/\sqrt{2}} = r\sin^{-1}\left(\frac{r/\sqrt{2}}{r}\right) - r\sin^{-1}(0)$
$= r\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 0 = r\left(\frac{\pi}{4}\right) = \frac{\pi r}{4}$. The total circumference of the circle is $C = 8L = 8\left(\frac{\pi r}{4}\right) = 2\pi r$.

123. (a) $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left[\frac{1}{\sqrt{1+x^2}} - \left(-\frac{1}{\sqrt{1+x^2}}\right)\right]^2 = \frac{\pi}{1+x^2} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-1}^1 \frac{\pi\,dx}{1+x^2}$
$= \pi\left[\tan^{-1} x\right]_{-1}^1 = (\pi)(2)\left(\frac{\pi}{4}\right) = \frac{\pi^2}{2}$

(b) $A(x) = (\text{edge})^2 = \left[\frac{1}{\sqrt{1+x^2}} - \left(-\frac{1}{\sqrt{1+x^2}}\right)\right]^2 = \frac{4}{1+x^2} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-1}^1 \frac{4\,dx}{1+x^2}$
$= 4\left[\tan^{-1} x\right]_{-1}^1 = 4\left[\tan^{-1}(1) - \tan^{-1}(-1)\right] = 4\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 2\pi$

124. (a) $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left(\frac{2}{\sqrt[4]{1-x^2}} - 0\right)^2 = \frac{\pi}{4}\left(\frac{4}{\sqrt{1-x^2}}\right) = \frac{\pi}{\sqrt{1-x^2}} \Rightarrow V = \int_a^b A(x)\,dx$

$= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{\pi}{\sqrt{1-x^2}}\,dx = \pi\left[\sin^{-1} x\right]_{-\sqrt{2}/2}^{\sqrt{2}/2} = \pi\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right] = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}$

(b) $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{1}{2}\left(\frac{2}{\sqrt[4]{1-x^2}} - 0\right)^2 = \frac{2}{\sqrt{1-x^2}} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{2}{\sqrt{1-x^2}}\,dx$

$= 2\left[\sin^{-1} x\right]_{-\sqrt{2}/2}^{\sqrt{2}/2} = 2\left(\frac{\pi}{4} \cdot 2\right) = \pi$

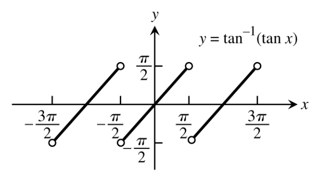
125. (a) $\sec^{-1} 1.5 = \cos^{-1} \frac{1}{1.5} \approx 0.84107$ (b) $\csc^{-1}(-1.5) = \sin^{-1}\left(-\frac{1}{1.5}\right) \approx -0.72973$

(c) $\cot^{-1} 2 = \frac{\pi}{2} - \tan^{-1} 2 \approx 0.46365$

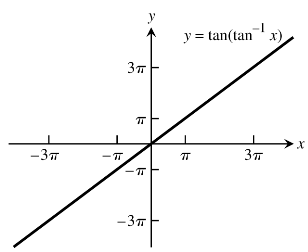
126. (a) $\sec^{-1}(-3) = \cos^{-1}\left(-\frac{1}{3}\right) \approx 1.91063$ (b) $\csc^{-1} 1.7 = \sin^{-1}\left(\frac{1}{1.7}\right) \approx 0.62887$

(c) $\cot^{-1}(-2) = \frac{\pi}{2} - \tan^{-1}(-2) \approx 2.67795$

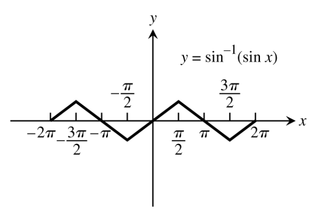
127. (a) Domain: all real numbers except those having the form $\frac{\pi}{2} + k\pi$ where k is an integer.
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



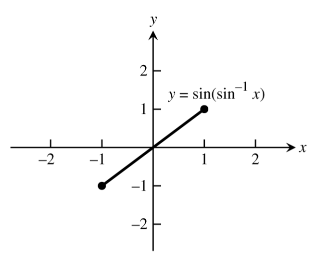
(b) Domain: $-\infty < x < \infty$; Range: $-\infty < y < \infty$
The graph of $y = \tan^{-1}(\tan x)$ is periodic, the graph of $y = \tan(\tan^{-1} x) = x$ for $-\infty \le x < \infty$.



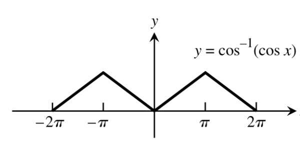
128. (a) Domain: $-\infty < x < \infty$; Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



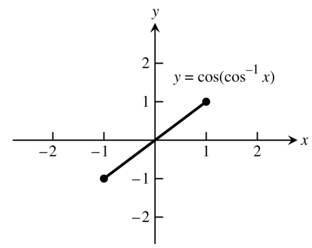
(b) Domain: $-1 \le x \le 1$; Range: $-1 \le y \le 1$
The graph of $y = \sin^{-1}(\sin x)$ is periodic; the graph of $y = \sin(\sin^{-1} x) = x$ for $-1 \le x \le 1$.



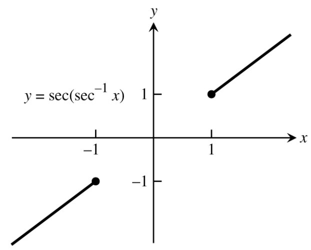
129. (a) Domain: $-\infty < x < \infty$; Range: $0 \le y \le \pi$



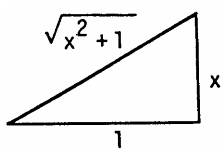
(b) Domain: $-1 \le x \le 1$; Range: $-1 \le y \le 1$
The graph of $y = \cos^{-1}(\cos x)$ is periodic; the graph of $y = \cos(\cos^{-1} x) = x$ for $-1 \le x \le 1$.

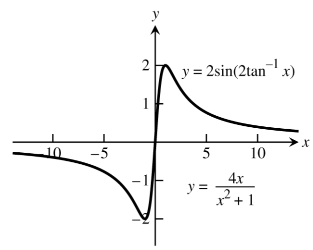


130. Since the domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$, we have $\sec(\sec^{-1} x) = x$ for $|x| \ge 1$. The graph of $y = \sec(\sec^{-1} x)$ is the line $y = x$ with the open line segment from $(-1, -1)$ to $(1, 1)$ removed.

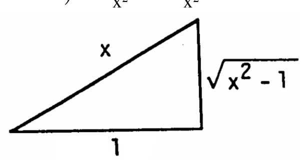


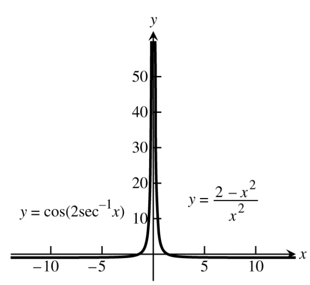
131. The graphs are identical for $y = 2\sin(2\tan^{-1} x)$
$= 4[\sin(\tan^{-1} x)][\cos(\tan^{-1} x)] = 4\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right)$
$= \frac{4x}{x^2+1}$ from the triangle



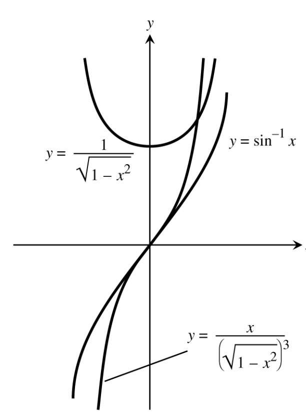


132. The graphs are identical for $y = \cos(2\sec^{-1} x)$
$= \cos^2(\sec^{-1} x) - \sin^2(\sec^{-1} x) = \frac{1}{x^2} - \frac{x^2-1}{x^2}$
$= \frac{2-x^2}{x^2}$ from the triangle

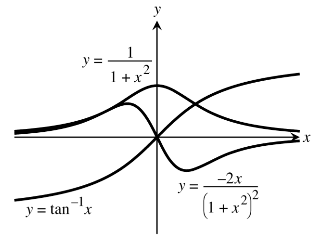




133. The values of f increase over the interval $[-1, 1]$ because $f' > 0$, and the graph of f steepens as the values of f' increase towards the ends of the interval. The graph of f is concave down to the left of the origin where $f'' < 0$, and concave up to the right of the origin where $f'' > 0$. There is an inflection point at $x = 0$ where $f'' = 0$ and f' has a local minimum value.



134. The values of f increase throughout the interval $(-\infty, \infty)$ because $f' > 0$, and they increase most rapidly near the origin where the values of f' are relatively large. The graph of f is concave up to the left of the origin where $f'' > 0$, and concave down to the right of the origin where $f'' < 0$. There is an inflection point at $x = 0$ where $f'' = 0$ and f' has a local maximum value.



7.7 HYPERBOLIC FUNCTIONS

1. $\sinh x = -\frac{3}{4} \Rightarrow \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\left(-\frac{3}{4}\right)^2} = \sqrt{1+\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(-\frac{3}{4}\right)}{\left(\frac{5}{4}\right)} = -\frac{3}{5}$, $\coth x = \frac{1}{\tanh x} = -\frac{5}{3}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{4}{5}$, and $\operatorname{csch} x = \frac{1}{\sin x} = -\frac{4}{3}$

2. $\sinh x = \frac{4}{3} \Rightarrow \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{4}{3}\right)}{\left(\frac{5}{3}\right)} = \frac{4}{5}$, $\coth x = \frac{1}{\tanh x} = \frac{5}{4}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{3}{5}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{3}{4}$

3. $\cosh x = \frac{17}{15}$, $x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\left(\frac{17}{15}\right)^2 - 1} = \sqrt{\frac{289}{225} - 1} = \sqrt{\frac{64}{225}} = \frac{8}{15}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{8}{15}\right)}{\left(\frac{17}{15}\right)}$ $= \frac{8}{17}$, $\coth x = \frac{1}{\tanh x} = \frac{17}{8}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{15}{17}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{15}{8}$

4. $\cosh x = \frac{13}{5}$, $x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\frac{169}{25} - 1} = \sqrt{\frac{144}{25}} = \frac{12}{5}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{12}{5}\right)}{\left(\frac{13}{5}\right)} = \frac{12}{13}$, $\coth x = \frac{1}{\tanh x} = \frac{13}{12}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{5}{13}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{5}{12}$

5. $2\cosh(\ln x) = 2\left(\frac{e^{\ln x} + e^{-\ln x}}{2}\right) = e^{\ln x} + \frac{1}{e^{\ln x}} = x + \frac{1}{x}$

6. $\sinh(2\ln x) = \frac{e^{2\ln x} - e^{-2\ln x}}{2} = \frac{e^{\ln x^2} - e^{\ln x^{-2}}}{2} = \frac{\left(x^2 - \frac{1}{x^2}\right)}{2} = \frac{x^4 - 1}{2x^2}$

7. $\cosh 5x + \sinh 5x = \frac{e^{5x} + e^{-5x}}{2} + \frac{e^{5x} - e^{-5x}}{2} = e^{5x}$

8. $\cosh 3x - \sinh 3x = \frac{e^{3x} + e^{-3x}}{2} - \frac{e^{3x} - e^{-3x}}{2} = e^{-3x}$

9. $(\sinh x + \cosh x)^4 = \left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}\right)^4 = (e^x)^4 = e^{4x}$

10. $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) = \ln(\cosh^2 x - \sinh^2 x) = \ln 1 = 0$

11. (a) $\sinh 2x = \sinh(x + x) = \sinh x \cosh x + \cosh x \sinh x = 2\sinh x \cosh x$
    (b) $\cosh 2x = \cosh(x + x) = \cosh x \cosh x + \sinh x \sin x = \cosh^2 x + \sinh^2 x$

12. $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left[(e^x + e^{-x}) + (e^x - e^{-x})\right]\left[(e^x + e^{-x}) - (e^x - e^{-x})\right] = \frac{1}{4}(2e^x)(2e^{-x})$ $= \frac{1}{4}(4e^0) = \frac{1}{4}(4) = 1$

13. $y = 6\sinh\frac{x}{3} \Rightarrow \frac{dy}{dx} = 6\left(\cosh\frac{x}{3}\right)\left(\frac{1}{3}\right) = 2\cosh\frac{x}{3}$

14. $y = \frac{1}{2}\sinh(2x+1) \Rightarrow \frac{dy}{dx} = \frac{1}{2}[\cosh(2x+1)](2) = \cosh(2x+1)$

15. $y = 2\sqrt{t}\tanh\sqrt{t} = 2t^{1/2}\tanh t^{1/2} \Rightarrow \frac{dy}{dt} = \left[\operatorname{sech}^2\left(t^{1/2}\right)\right]\left(\frac{1}{2}t^{-1/2}\right)\left(2t^{1/2}\right) + \left(\tanh t^{1/2}\right)\left(t^{-1/2}\right) = \operatorname{sech}^2\sqrt{t} + \frac{\tanh\sqrt{t}}{\sqrt{t}}$

16. $y = t^2 \tanh \frac{1}{t} = t^2 \tanh t^{-1} \Rightarrow \frac{dy}{dt} = [\operatorname{sech}^2(t^{-1})](-t^{-2})(t^2) + (2t)(\tanh t^{-1}) = -\operatorname{sech}^2 \frac{1}{t} + 2t \tanh \frac{1}{t}$

17. $y = \ln(\sinh z) \Rightarrow \frac{dy}{dz} = \frac{\cosh z}{\sinh z} = \coth z$

18. $y = \ln(\cosh z) \Rightarrow \frac{dy}{dz} = \frac{\sinh z}{\cosh z} = \tanh z$

19. $y = (\operatorname{sech}\theta)(1 - \ln \operatorname{sech}\theta) \Rightarrow \frac{dy}{d\theta} = \left(-\frac{-\operatorname{sech}\theta \tanh\theta}{\operatorname{sech}\theta}\right)(\operatorname{sech}\theta) + (-\operatorname{sech}\theta\tanh\theta)(1 - \ln\operatorname{sech}\theta)$
$= \operatorname{sech}\theta\tanh\theta - (\operatorname{sech}\theta\tanh\theta)(1 - \ln\operatorname{sech}\theta) = (\operatorname{sech}\theta\tanh\theta)[1 - (1 - \ln\operatorname{sech}\theta)] = (\operatorname{sech}\theta\tanh\theta)(\ln\operatorname{sech}\theta)$

20. $y = (\operatorname{csch}\theta)(1 - \ln\operatorname{csch}\theta) \Rightarrow \frac{dy}{d\theta} = (\operatorname{csch}\theta)\left(-\frac{-\operatorname{csch}\theta\coth\theta}{\operatorname{csch}\theta}\right) + (1 - \ln\operatorname{csch}\theta)(-\operatorname{csch}\theta\coth\theta)$
$= \operatorname{csch}\theta\coth\theta - (1 - \ln\operatorname{csch}\theta)(\operatorname{csch}\theta\coth\theta) = (\operatorname{csch}\theta\coth\theta)(1 - 1 + \ln\operatorname{csch}\theta) = (\operatorname{csch}\theta\coth\theta)(\ln\operatorname{csch}\theta)$

21. $y = \ln\cosh v - \frac{1}{2}\tanh^2 v \Rightarrow \frac{dy}{dv} = \frac{\sinh v}{\cosh v} - \left(\frac{1}{2}\right)(2\tanh v)(\operatorname{sech}^2 v) = \tanh v - (\tanh v)(\operatorname{sech}^2 v)$
$= (\tanh v)(1 - \operatorname{sech}^2 v) = (\tanh v)(\tanh^2 v) = \tanh^3 v$

22. $y = \ln\sinh v - \frac{1}{2}\coth^2 v \Rightarrow \frac{dy}{dv} = \frac{\cosh v}{\sinh v} - \left(\frac{1}{2}\right)(2\coth v)(-\operatorname{csch}^2 v) = \coth v + (\coth v)(\operatorname{csch}^2 v)$
$= (\coth v)(1 + \operatorname{csch}^2 v) = (\coth v)(\coth^2 v) = \coth^3 v$

23. $y = (x^2+1)\operatorname{sech}(\ln x) = (x^2+1)\left(\frac{2}{e^{\ln x} + e^{-\ln x}}\right) = (x^2+1)\left(\frac{2}{x + x^{-1}}\right) = (x^2+1)\left(\frac{2x}{x^2+1}\right) = 2x \Rightarrow \frac{dy}{dx} = 2$

24. $y = (4x^2-1)\operatorname{csch}(\ln 2x) = (4x^2-1)\left(\frac{2}{e^{\ln 2x} - e^{-\ln 2x}}\right) = (4x^2-1)\left(\frac{2}{2x - (2x)^{-1}}\right) = (4x^2-1)\left(\frac{4x}{4x^2-1}\right) = 4x \Rightarrow \frac{dy}{dx} = 4$

25. $y = \sinh^{-1}\sqrt{x} = \sinh^{-1}\left(x^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)x^{-1/2}}{\sqrt{1 + (x^{1/2})^2}} = \frac{1}{2\sqrt{x}\sqrt{1+x}} = \frac{1}{2\sqrt{x(1+x)}}$

26. $y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1}\left(2(x+1)^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{(2)\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{\left[2(x+1)^{1/2}\right]^2 - 1}} = \frac{1}{\sqrt{x+1}\sqrt{4x+3}} = \frac{1}{\sqrt{4x^2+7x+3}}$

27. $y = (1-\theta)\tanh^{-1}\theta \Rightarrow \frac{dy}{d\theta} = (1-\theta)\left(\frac{1}{1-\theta^2}\right) + (-1)\tanh^{-1}\theta = \frac{1}{1+\theta} - \tanh^{-1}\theta$

28. $y = (\theta^2 + 2\theta)\tanh^{-1}(\theta+1) \Rightarrow \frac{dy}{d\theta} = (\theta^2+2\theta)\left[\frac{1}{1-(\theta+1)^2}\right] + (2\theta+2)\tanh^{-1}(\theta+1) = \frac{\theta^2+2\theta}{-\theta^2-2\theta} + (2\theta+2)\tanh^{-1}(\theta+1)$
$= (2\theta+2)\tanh^{-1}(\theta+1) - 1$

29. $y = (1-t)\coth^{-1}\sqrt{t} = (1-t)\coth^{-1}\left(t^{1/2}\right) \Rightarrow \frac{dy}{dt} = (1-t)\left[\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1 - \left(t^{1/2}\right)^2}\right] + (-1)\coth^{-1}\left(t^{1/2}\right) = \frac{1}{2\sqrt{t}} - \coth^{-1}\sqrt{t}$

30. $y = (1-t^2)\coth^{-1} t \Rightarrow \frac{dy}{dt} = (1-t^2)\left(\frac{1}{1-t^2}\right) + (-2t)\coth^{-1} t = 1 - 2t\coth^{-1} t$

31. $y = \cos^{-1} x - x\operatorname{sech}^{-1} x \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} - \left[x\left(\frac{-1}{x\sqrt{1-x^2}}\right) + (1)\operatorname{sech}^{-1} x\right] = \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} - \operatorname{sech}^{-1} x = -\operatorname{sech}^{-1} x$

32. $y = \ln x + \sqrt{1-x^2}\operatorname{sech}^{-1} x = \ln x + (1-x^2)^{1/2}\operatorname{sech}^{-1} x$
$\Rightarrow \frac{dy}{dx} = \frac{1}{x} + (1-x^2)^{1/2}\left(\frac{-1}{x\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)\operatorname{sech}^{-1} x = \frac{1}{x} - \frac{1}{x} - \frac{x}{\sqrt{1-x^2}}\operatorname{sech}^{-1} x = \frac{-x}{\sqrt{1-x^2}}\operatorname{sech}^{-1} x$

33. $y = \operatorname{csch}^{-1}\left(\frac{1}{2}\right)^\theta \Rightarrow \frac{dy}{d\theta} = -\frac{\left[\ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^\theta}{\left(\frac{1}{2}\right)^\theta\sqrt{1 + \left[\left(\frac{1}{2}\right)^\theta\right]^2}} = -\frac{\ln(1) - \ln(2)}{\sqrt{1 + \left(\frac{1}{2}\right)^{2\theta}}} = \frac{\ln 2}{\sqrt{1 + \left(\frac{1}{2}\right)^{2\theta}}}$

34. $y = \operatorname{csch}^{-1} 2^{\theta} \Rightarrow \frac{dy}{d\theta} = -\frac{(\ln 2)\, 2^{\theta}}{2^{\theta}\sqrt{1+(2^{\theta})^2}} = \frac{-\ln 2}{\sqrt{1+2^{2\theta}}}$

35. $y = \sinh^{-1}(\tan x) \Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{1+(\tan x)^2}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{|\sec x|} = \frac{|\sec x|\,|\sec x|}{|\sec x|} = |\sec x|$

36. $y = \cosh^{-1}(\sec x) \Rightarrow \frac{dy}{dx} = \frac{(\sec x)(\tan x)}{\sqrt{\sec^2 x - 1}} = \frac{(\sec x)(\tan x)}{\sqrt{\tan^2 x}} = \frac{(\sec x)(\tan x)}{|\tan x|} = \sec x,\ 0 < x < \frac{\pi}{2}$

37. (a) If $y = \tan^{-1}(\sinh x) + C$, then $\frac{dy}{dx} = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x$, which verifies the formula

(b) If $y = \sin^{-1}(\tanh x) + C$, then $\frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\operatorname{sech} x} = \operatorname{sech} x$, which verifies the formula

38. If $y = \frac{x^2}{2}\operatorname{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$, then $\frac{dy}{dx} = x\operatorname{sech}^{-1} x + \frac{x^2}{2}\left(\frac{-1}{x\sqrt{1-x^2}}\right) + \frac{2x}{4\sqrt{1-x^2}} = x\operatorname{sech}^{-1} x$, which verifies the formula

39. If $y = \frac{x^2-1}{2}\coth^{-1} x + \frac{x}{2} + C$, then $\frac{dy}{dx} = x\coth^{-1} x + \left(\frac{x^2-1}{2}\right)\left(\frac{1}{1-x^2}\right) + \frac{1}{2} = x\coth^{-1} x$, which verifies the formula

40. If $y = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2) + C$, then $\frac{dy}{dx} = \tanh^{-1} x + x\left(\frac{1}{1-x^2}\right) + \frac{1}{2}\left(\frac{-2x}{1-x^2}\right) = \tanh^{-1} x$, which verifies the formula

41. $\int \sinh 2x\, dx = \frac{1}{2}\int \sinh u\, du$, where $u = 2x$ and $du = 2\, dx$

$= \frac{\cosh u}{2} + C = \frac{\cosh 2x}{2} + C$

42. $\int \sinh \frac{x}{5}\, dx = 5\int \sinh u\, du$, where $u = \frac{x}{5}$ and $du = \frac{1}{5}\, dx$

$= 5\cosh u + C = 5\cosh \frac{x}{5} + C$

43. $\int 6\cosh\left(\frac{x}{2} - \ln 3\right) dx = 12\int \cosh u\, du$, where $u = \frac{x}{2} - \ln 3$ and $du = \frac{1}{2}\, dx$

$= 12\sinh u + C = 12\sinh\left(\frac{x}{2} - \ln 3\right) + C$

44. $\int 4\cosh(3x - \ln 2)\, dx = \frac{4}{3}\int \cosh u\, du$, where $u = 3x - \ln 2$ and $du = 3\, dx$

$= \frac{4}{3}\sinh u + C = \frac{4}{3}\sinh(3x - \ln 2) + C$

45. $\int \tanh \frac{x}{7}\, dx = 7\int \frac{\sinh u}{\cosh u}\, du$, where $u = \frac{x}{7}$ and $du = \frac{1}{7}\, dx$

$= 7\ln|\cosh u| + C_1 = 7\ln\left|\cosh \frac{x}{7}\right| + C_1 = 7\ln\left|\frac{e^{x/7}+e^{-x/7}}{2}\right| + C_1 = 7\ln\left|e^{x/7}+e^{-x/7}\right| - 7\ln 2 + C_1$

$= 7\ln\left|e^{x/7}+e^{-x/7}\right| + C$

46. $\int \coth \frac{\theta}{\sqrt{3}}\, d\theta = \sqrt{3}\int \frac{\cosh u}{\sinh u}\, du$, where $u = \frac{\theta}{\sqrt{3}}$ and $du = \frac{d\theta}{\sqrt{3}}$

$= \sqrt{3}\ln|\sinh u| + C_1 = \sqrt{3}\ln\left|\sinh \frac{\theta}{\sqrt{3}}\right| + C_1 = \sqrt{3}\ln\left|\frac{e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}}{2}\right| + C_1$

$= \sqrt{3}\ln\left|e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}\right| - \sqrt{3}\ln 2 + C_1 = \sqrt{3}\ln\left|e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}\right| + C$

47. $\int \operatorname{sech}^2\left(x - \frac{1}{2}\right) dx = \int \operatorname{sech}^2 u\, du$, where $u = \left(x - \frac{1}{2}\right)$ and $du = dx$

$= \tanh u + C = \tanh\left(x - \frac{1}{2}\right) + C$

48. $\int \text{csch}^2(5-x)\,dx = -\int \text{csch}^2 u\,du$, where $u = (5-x)$ and $du = -\,dx$
$= -(-\coth u) + C = \coth u + C = \coth(5-x) + C$

49. $\int \frac{\text{sech}\sqrt{t}\tanh\sqrt{t}}{\sqrt{t}}\,dt = 2\int \text{sech}\,u \tanh u\,du$, where $u = \sqrt{t} = t^{1/2}$ and $du = \frac{dt}{2\sqrt{t}}$,
$= 2(-\text{sech}\,u) + C = -2\,\text{sech}\sqrt{t} + C$

50. $\int \frac{\text{csch}(\ln t)\coth(\ln t)}{t}\,dt = \int \text{csch}\,u \coth u\,du$, where $u = \ln t$ and $du = \frac{dt}{t}$
$= -\text{csch}\,u + C = -\text{csch}(\ln t) + C$

51. $\int_{\ln 2}^{\ln 4} \coth x\,dx = \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x}\,dx = \int_{3/4}^{15/8} \frac{1}{u}\,du = [\ln|u|]_{3/4}^{15/8} = \ln\left|\frac{15}{8}\right| - \ln\left|\frac{3}{4}\right| = \ln\left|\frac{15}{8}\cdot\frac{4}{3}\right| = \ln\frac{5}{2}$,
where $u = \sinh x$, $du = \cosh x\,dx$, the lower limit is $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \left(\frac{1}{2}\right)}{2} = \frac{3}{4}$ and the upper limit is $\sinh(\ln 4) = \frac{e^{\ln 4} - e^{-\ln 4}}{2} = \frac{4 - \left(\frac{1}{4}\right)}{2} = \frac{15}{8}$

52. $\int_0^{\ln 2} \tanh 2x\,dx = \int_0^{\ln 2} \frac{\sinh 2x}{\cosh 2x}\,dx = \frac{1}{2}\int_1^{17/8} \frac{1}{u}\,du = \frac{1}{2}[\ln|u|]_1^{17/8} = \frac{1}{2}\left[\ln\left(\frac{17}{8}\right) - \ln 1\right] = \frac{1}{2}\ln\frac{17}{8}$, where $u = \cosh 2x$, $du = 2\sinh(2x)\,dx$, the lower limit is $\cosh 0 = 1$ and the upper limit is $\cosh(2\ln 2) = \cosh(\ln 4)$
$= \frac{e^{\ln 4} + e^{-\ln 4}}{2} = \frac{4 + \left(\frac{1}{4}\right)}{2} = \frac{17}{8}$

53. $\int_{-\ln 4}^{-\ln 2} 2e^\theta \cosh\theta\,d\theta = \int_{-\ln 4}^{-\ln 2} 2e^\theta\left(\frac{e^\theta + e^{-\theta}}{2}\right)d\theta = \int_{-\ln 4}^{-\ln 2}(e^{2\theta} + 1)\,d\theta = \left[\frac{e^{2\theta}}{2} + \theta\right]_{-\ln 4}^{-\ln 2}$
$= \left(\frac{e^{-2\ln 2}}{2} - \ln 2\right) - \left(\frac{e^{-2\ln 4}}{2} - \ln 4\right) = \left(\frac{1}{8} - \ln 2\right) - \left(\frac{1}{32} - \ln 4\right) = \frac{3}{32} - \ln 2 + 2\ln 2 = \frac{3}{32} + \ln 2$

54. $\int_0^{\ln 2} 4e^{-\theta}\sinh\theta\,d\theta = \int_0^{\ln 2} 4e^{-\theta}\left(\frac{e^\theta - e^{-\theta}}{2}\right)d\theta = 2\int_0^{\ln 2}(1 - e^{-2\theta})\,d\theta = 2\left[\theta + \frac{e^{-2\theta}}{2}\right]_0^{\ln 2}$
$= 2\left[\left(\ln 2 + \frac{e^{-2\ln 2}}{2}\right) - \left(0 + \frac{e^0}{2}\right)\right] = 2\left(\ln 2 + \frac{1}{8} - \frac{1}{2}\right) = 2\ln 2 + \frac{1}{4} - 1 = \ln 4 - \frac{3}{4}$

55. $\int_{-\pi/4}^{\pi/4} \cosh(\tan\theta)\sec^2\theta\,d\theta = \int_{-1}^{1} \cosh u\,du = [\sinh u]_{-1}^{1} = \sinh(1) - \sinh(-1) = \left(\frac{e^1 - e^{-1}}{2}\right) - \left(\frac{e^{-1} - e^1}{2}\right)$
$= \frac{e - e^{-1} - e^{-1} + e}{2} = e - e^{-1}$, where $u = \tan\theta$, $du = \sec^2\theta\,d\theta$, the lower limit is $\tan\left(-\frac{\pi}{4}\right) = -1$ and the upper limit is $\tan\left(\frac{\pi}{4}\right) = 1$

56. $\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta\,d\theta = 2\int_0^1 \sinh u\,du = 2[\cosh u]_0^1 = 2(\cosh 1 - \cosh 0) = 2\left(\frac{e + e^{-1}}{2} - 1\right)$
$= e + e^{-1} - 2$, where $u = \sin\theta$, $du = \cos\theta\,d\theta$, the lower limit is $\sin 0 = 0$ and the upper limit is $\sin\left(\frac{\pi}{2}\right) = 1$

57. $\int_1^2 \frac{\cosh(\ln t)}{t}\,dt = \int_0^{\ln 2} \cosh u\,du = [\sinh u]_0^{\ln 2} = \sinh(\ln 2) - \sinh(0) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} - 0 = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$, where $u = \ln t$, $du = \frac{1}{t}\,dt$, the lower limit is $\ln 1 = 0$ and the upper limit is $\ln 2$

58. $\int_1^4 \frac{8\cosh\sqrt{x}}{\sqrt{x}}\,dx = 16\int_1^2 \cosh u\,du = 16[\sinh u]_1^2 = 16(\sinh 2 - \sinh 1) = 16\left[\left(\frac{e^2 - e^{-2}}{2}\right) - \left(\frac{e - e^{-1}}{2}\right)\right]$
$= 8(e^2 - e^{-2} - e + e^{-1})$, where $u = \sqrt{x} = x^{1/2}$, $du = \frac{1}{2}x^{-1/2}dx = \frac{dx}{2\sqrt{x}}$, the lower limit is $\sqrt{1} = 1$ and the upper limit is $\sqrt{4} = 2$

59. $\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right) dx = \int_{-\ln 2}^{0} \frac{\cosh x + 1}{2}\, dx = \frac{1}{2}\int_{-\ln 2}^{0} (\cosh x + 1)\, dx = \frac{1}{2}\left[\sinh x + x\right]_{-\ln 2}^{0}$
$= \frac{1}{2}\left[(\sinh 0 + 0) - (\sinh(-\ln 2) - \ln 2)\right] = \frac{1}{2}\left[(0+0) - \left(\frac{e^{-\ln 2} - e^{\ln 2}}{2} - \ln 2\right)\right] = \frac{1}{2}\left[-\frac{\left(\frac{1}{2}\right) - 2}{2} + \ln 2\right]$
$= \frac{1}{2}\left(1 - \frac{1}{4} + \ln 2\right) = \frac{3}{8} + \frac{1}{2}\ln 2 = \frac{3}{8} + \ln\sqrt{2}$

60. $\int_{0}^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right) dx = \int_{0}^{\ln 10} 4\left(\frac{\cosh x - 1}{2}\right) dx = 2\int_{0}^{\ln 10} (\cosh x - 1)\, dx = 2\left[\sinh x - x\right]_{0}^{\ln 10}$
$= 2[(\sinh(\ln 10) - \ln 10) - (\sinh 0 - 0)] = e^{\ln 10} - e^{-\ln 10} - 2\ln 10 = 10 - \frac{1}{10} - 2\ln 10 = 9.9 - 2\ln 10$

61. $\sinh^{-1}\left(\frac{-5}{12}\right) = \ln\left(-\frac{5}{12} + \sqrt{\frac{25}{144} + 1}\right) = \ln\left(\frac{2}{3}\right)$

62. $\cosh^{-1}\left(\frac{5}{3}\right) = \ln\left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1}\right) = \ln 3$

63. $\tanh^{-1}\left(-\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1-(1/2)}{1+(1/2)}\right) = -\frac{\ln 3}{2}$

64. $\coth^{-1}\left(\frac{5}{4}\right) = \frac{1}{2}\ln\left(\frac{(9/4)}{(1/4)}\right) = \frac{1}{2}\ln 9 = \ln 3$

65. $\operatorname{sech}^{-1}\left(\frac{3}{5}\right) = \ln\left(\frac{1+\sqrt{1-(9/25)}}{(3/5)}\right) = \ln 3$

66. $\operatorname{csch}^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \ln\left(-\sqrt{3} + \frac{\sqrt{4/3}}{\left(1/\sqrt{3}\right)}\right) = \ln\left(-\sqrt{3} + 2\right)$

67. (a) $\int_{0}^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}} = \left[\sinh^{-1}\frac{x}{2}\right]_{0}^{2\sqrt{3}} = \sinh^{-1}\sqrt{3} - \sinh 0 = \sinh^{-1}\sqrt{3}$
(b) $\sinh^{-1}\sqrt{3} = \ln\left(\sqrt{3} + \sqrt{3+1}\right) = \ln\left(\sqrt{3} + 2\right)$

68. (a) $\int_{0}^{1/3} \frac{6\, dx}{\sqrt{1+9x^2}} = 2\int_{0}^{1} \frac{dx}{\sqrt{a^2+u^2}}$, where $u = 3x$, $du = 3\, dx$, $a = 1$
$= \left[2\sinh^{-1} u\right]_{0}^{1} = 2\left(\sinh^{-1} 1 - \sinh^{-1} 0\right) = 2\sinh^{-1} 1$
(b) $2\sinh^{-1} 1 = 2\ln\left(1 + \sqrt{1^2+1}\right) = 2\ln\left(1+\sqrt{2}\right)$

69. (a) $\int_{5/4}^{2} \frac{1}{1-x^2}\, dx = \left[\coth^{-1} x\right]_{5/4}^{2} = \coth^{-1} 2 - \coth^{-1}\frac{5}{4}$
(b) $\coth^{-1} 2 - \coth^{-1}\frac{5}{4} = \frac{1}{2}\left[\ln 3 - \ln\left(\frac{9/4}{1/4}\right)\right] = \frac{1}{2}\ln\frac{1}{3}$

70. (a) $\int_{0}^{1/2} \frac{1}{1-x^2}\, dx = \left[\tanh^{-1} x\right]_{0}^{1/2} = \tanh^{-1}\frac{1}{2} - \tanh^{-1} 0 = \tanh^{-1}\frac{1}{2}$
(b) $\tanh^{-1}\frac{1}{2} = \frac{1}{2}\ln\left(\frac{1+(1/2)}{1-(1/2)}\right) = \frac{1}{2}\ln 3$

71. (a) $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}} = \int_{4/5}^{12/13} \frac{du}{u\sqrt{a^2-u^2}}$, where $u = 4x$, $du = 4\, dx$, $a = 1$
$= \left[-\operatorname{sech}^{-1} u\right]_{4/5}^{12/13} = -\operatorname{sech}^{-1}\frac{12}{13} + \operatorname{sech}^{-1}\frac{4}{5}$
(b) $-\operatorname{sech}^{-1}\frac{12}{13} + \operatorname{sech}^{-1}\frac{4}{5} = -\ln\left(\frac{1+\sqrt{1-(12/13)^2}}{(12/13)}\right) + \ln\left(\frac{1+\sqrt{1-(4/5)^2}}{(4/5)}\right)$
$= -\ln\left(\frac{13+\sqrt{169-144}}{12}\right) + \ln\left(\frac{5+\sqrt{25-16}}{4}\right) = \ln\left(\frac{5+3}{4}\right) - \ln\left(\frac{13+5}{12}\right) = \ln 2 - \ln\frac{3}{2} = \ln\left(2\cdot\frac{2}{3}\right) = \ln\frac{4}{3}$

72. (a) $\int_{1}^{2} \frac{dx}{x\sqrt{4+x^2}} = \left[-\frac{1}{2}\operatorname{csch}^{-1}\left|\frac{x}{2}\right|\right]_{1}^{2} = -\frac{1}{2}\left(\operatorname{csch}^{-1} 1 - \operatorname{csch}^{-1}\frac{1}{2}\right) = \frac{1}{2}\left(\operatorname{csch}^{-1}\frac{1}{2} - \operatorname{csch}^{-1} 1\right)$
(b) $\frac{1}{2}\left(\operatorname{csch}^{-1}\frac{1}{2} - \operatorname{csch}^{-1} 1\right) = \frac{1}{2}\left[\ln\left(2 + \frac{\sqrt{5/4}}{(1/2)}\right) - \ln\left(1+\sqrt{2}\right)\right] = \frac{1}{2}\ln\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right)$

73. (a) $\int_{0}^{\pi} \frac{\cos x}{\sqrt{1+\sin^2 x}}\, dx = \int_{0}^{0} \frac{1}{\sqrt{1+u^2}}\, du = \left[\sinh^{-1} u\right]_{0}^{0} = \sinh^{-1} 0 - \sinh^{-1} 0 = 0$, where $u = \sin x$, $du = \cos x\, dx$

(b) $\sinh^{-1} 0 - \sinh^{-1} 0 = \ln\left(0 + \sqrt{0+1}\right) - \ln\left(0 + \sqrt{0+1}\right) = 0$

74. (a) $\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}} = \int_0^1 \frac{du}{\sqrt{a^2+u^2}}$, where $u = \ln x$, $du = \frac{1}{x}\, dx$, $a = 1$

$= [\sinh^{-1} u]_0^1 = \sinh^{-1} 1 - \sinh^{-1} 0 = \sinh^{-1} 1$

(b) $\sinh^{-1} 1 - \sinh^{-1} 0 = \ln\left(1 + \sqrt{1^2+1}\right) - \ln\left(0 + \sqrt{0^2+1}\right) = \ln\left(1 + \sqrt{2}\right)$

75. Let $E(x) = \frac{f(x)+f(-x)}{2}$ and $O(x) = \frac{f(x)-f(-x)}{2}$. Then $E(x) + O(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = \frac{2f(x)}{2} = f(x)$. Also, $E(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(x)+f(-x)}{2} = E(x) \Rightarrow E(x)$ is even, and $O(-x) = \frac{f(-x)-f(-(-x))}{2} = -\frac{f(x)-f(-x)}{2} = -O(x)$ $\Rightarrow O(x)$ is odd. Consequently, f(x) can be written as a sum of an even and an odd function.
$f(x) = \frac{f(x)+f(-x)}{2}$ because $\frac{f(x)-f(-x)}{2} = 0$ if f is even and $f(x) = \frac{f(x)-f(-x)}{2}$ because $\frac{f(x)+f(-x)}{2} = 0$ if f is odd.
Thus, if f is even $f(x) = \frac{2f(x)}{2} + 0$ and if f is odd, $f(x) = 0 + \frac{2f(x)}{2}$

76. $y = \sinh^{-1} x \Rightarrow x = \sinh y \Rightarrow x = \frac{e^y - e^{-y}}{2} \Rightarrow 2x = e^y - \frac{1}{e^y} \Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow e^{2y} - 2xe^y - 1 = 0$ $\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} \Rightarrow e^y = x + \sqrt{x^2+1} \Rightarrow \sinh^{-1} x = y = \ln\left(x + \sqrt{x^2+1}\right)$. Since $e^y > 0$, we cannot choose $e^y = x - \sqrt{x^2+1}$ because $x - \sqrt{x^2+1} < 0$.

77. (a) $v = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}}\, t\right) \Rightarrow \frac{dv}{dt} = \sqrt{\frac{mg}{k}}\left[\operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}\, t\right)\right]\left(\sqrt{\frac{gk}{m}}\right) = g \operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}\, t\right)$.

Thus $m\frac{dv}{dt} = mg \operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}\, t\right) = mg\left(1 - \tanh^2\left(\sqrt{\frac{gk}{m}}\, t\right)\right) = mg - kv^2$. Also, since $\tanh x = 0$ when $x = 0$, $v = 0$ when $t = 0$.

(b) $\lim_{t\to\infty} v = \lim_{t\to\infty} \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}\, t\right) = \sqrt{\frac{mg}{k}} \lim_{t\to\infty} \tanh\left(\sqrt{\frac{kg}{m}}\, t\right) = \sqrt{\frac{mg}{k}}(1) = \sqrt{\frac{mg}{k}}$

(c) $\sqrt{\frac{160}{0.005}} = \sqrt{\frac{160{,}000}{5}} = \frac{400}{\sqrt{5}} = 80\sqrt{5} \approx 178.89$ ft/sec

78. (a) $s(t) = a \cos kt + b \sin kt \Rightarrow \frac{ds}{dt} = -ak \sin kt + bk \cos kt \Rightarrow \frac{d^2s}{dt^2} = -ak^2 \cos kt - bk^2 \sin kt$ $= -k^2 (a \cos kt + b \sin kt) = -k^2 s(t) \Rightarrow$ acceleration is proportional to s. The negative constant $-k^2$ implies that the acceleration is directed toward the origin.

(b) $s(t) = a \cosh kt + b \sinh kt \Rightarrow \frac{ds}{dt} = ak \sinh kt + bk \cosh kt \Rightarrow \frac{d^2s}{dt^2} = ak^2 \cosh kt + bk^2 \sinh kt$ $= k^2 (a \cosh kt + b \sinh kt) = k^2 s(t) \Rightarrow$ acceleration is proportional to s. The positive constant k^2 implies that the acceleration is directed away from the origin.

79. $V = \pi\int_0^2 (\cosh^2 x - \sinh^2 x)\, dx = \pi\int_0^2 1\, dx = 2\pi$

80. $V = 2\pi \int_0^{\ln\sqrt{3}} \operatorname{sech}^2 x\, dx = 2\pi\, [\tanh x]_0^{\ln\sqrt{3}} = 2\pi\left[\frac{\sqrt{3} - \left(1/\sqrt{3}\right)}{\sqrt{3} + \left(1/\sqrt{3}\right)}\right] = \pi$

81. $y = \frac{1}{2}\cosh 2x \Rightarrow y' = \sinh 2x \Rightarrow L = \int_0^{\ln\sqrt{5}} \sqrt{1 + (\sinh 2x)^2}\, dx = \int_0^{\ln\sqrt{5}} \cosh 2x\, dx = \left[\frac{1}{2}\sinh 2x\right]_0^{\ln\sqrt{5}}$ $= \left[\frac{1}{2}\left(\frac{e^{2x} - e^{-2x}}{2}\right)\right]_0^{\ln\sqrt{5}} = \frac{1}{4}\left(5 - \frac{1}{5}\right) = \frac{6}{5}$

82. (a) $\lim\limits_{x \to \infty} \tanh x = \lim\limits_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x \to \infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim\limits_{x \to \infty} \frac{\left(e^x - \frac{1}{e^x}\right)}{\left(e^x + \frac{1}{e^x}\right)} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim\limits_{x \to \infty} \frac{1 - \frac{1}{e^{2x}}}{1 + \frac{1}{e^{2x}}} = \frac{1-0}{1+0} = 1$

(b) $\lim\limits_{x \to -\infty} \tanh x = \lim\limits_{x \to -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x \to -\infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim\limits_{x \to -\infty} \frac{\left(e^x - \frac{1}{e^x}\right)}{\left(e^x + \frac{1}{e^x}\right)} \cdot \frac{e^x}{e^x} = \lim\limits_{x \to -\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0-1}{0+1} = -1$

(c) $\lim\limits_{x \to \infty} \sinh x = \lim\limits_{x \to \infty} \frac{e^x - e^{-x}}{2} = \lim\limits_{x \to \infty} \frac{e^x - \frac{1}{e^x}}{2} = \lim\limits_{x \to \infty} \left(\frac{e^x}{2} - \frac{1}{2e^x}\right) = \infty - 0 = \infty$

(d) $\lim\limits_{x \to -\infty} \sinh x = \lim\limits_{x \to -\infty} \frac{e^x - e^{-x}}{2} = \lim\limits_{x \to -\infty} \left(\frac{e^x}{2} - \frac{e^{-x}}{2}\right) = 0 - \infty = -\infty$

(e) $\lim\limits_{x \to \infty} \operatorname{sech} x = \lim\limits_{x \to \infty} \frac{2}{e^x + e^{-x}} = \lim\limits_{x \to \infty} \frac{2}{e^x + \frac{1}{e^x}} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim\limits_{x \to \infty} \frac{\frac{2}{e^x}}{1 + \frac{1}{e^{2x}}} = \frac{0}{1+0} = 0$

(f) $\lim\limits_{x \to \infty} \coth x = \lim\limits_{x \to \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim\limits_{x \to \infty} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} = \lim\limits_{x \to \infty} \frac{\left(e^x + \frac{1}{e^x}\right)}{\left(e^x - \frac{1}{e^x}\right)} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim\limits_{x \to \infty} \frac{1 + \frac{1}{e^{2x}}}{1 - \frac{1}{e^{2x}}} = \frac{1+0}{1-0} = 1$

(g) $\lim\limits_{x \to 0^+} \coth x = \lim\limits_{x \to 0^+} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim\limits_{x \to 0^+} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} = \lim\limits_{x \to 0^+} \frac{e^{2x} + 1}{e^{2x} - 1} = +\infty$

(h) $\lim\limits_{x \to 0^-} \coth x = \lim\limits_{x \to 0^-} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim\limits_{x \to 0^-} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} = \lim\limits_{x \to 0^-} \frac{e^{2x} + 1}{e^{2x} - 1} = -\infty$

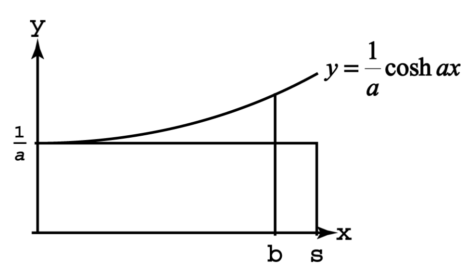
(i) $\lim\limits_{x \to -\infty} \operatorname{csch} x = \lim\limits_{x \to -\infty} \frac{2}{e^x - e^{-x}} = \lim\limits_{x \to -\infty} \frac{2}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} = \lim\limits_{x \to -\infty} \frac{2e^x}{e^{2x} - 1} = \frac{0}{0-1} = 0$

83. (a) $y = \frac{H}{w} \cosh\left(\frac{w}{H} x\right) \Rightarrow \tan \phi = \frac{dy}{dx} = \left(\frac{H}{w}\right)\left[\frac{w}{H} \sinh\left(\frac{w}{H} x\right)\right] = \sinh\left(\frac{w}{H} x\right)$

(b) The tension at P is given by $T \cos \phi = H \Rightarrow T = H \sec \phi = H\sqrt{1 + \tan^2 \phi} = H\sqrt{1 + \left(\sinh \frac{w}{H} x\right)^2}$
$= H \cosh\left(\frac{w}{H} x\right) = w\left(\frac{H}{w}\right) \cosh\left(\frac{w}{H} x\right) = wy$

84. $s = \frac{1}{a} \sinh ax \Rightarrow \sinh ax = as \Rightarrow ax = \sinh^{-1} as \Rightarrow x = \frac{1}{a} \sinh^{-1} as$; $y = \frac{1}{a} \cosh ax = \frac{1}{a}\sqrt{\cosh^2 ax}$
$= \frac{1}{a}\sqrt{\sinh^2 ax + 1} = \frac{1}{a}\sqrt{a^2 s^2 + 1} = \sqrt{s^2 + \frac{1}{a^2}}$

85. To find the length of the curve: $y = \frac{1}{a} \cosh ax \Rightarrow y' = \sinh ax \Rightarrow L = \int_0^b \sqrt{1 + (\sinh ax)^2}\, dx$
$\Rightarrow L = \int_0^b \cosh ax\, dx = \left[\frac{1}{a} \sinh ax\right]_0^b = \frac{1}{a} \sinh ab$. The area under the curve is $A = \int_0^b \frac{1}{a} \cosh ax\, dx$
$= \left[\frac{1}{a^2} \sinh ax\right]_0^b = \frac{1}{a^2} \sinh ab = \left(\frac{1}{a}\right)\left(\frac{1}{a} \sinh ab\right)$ which is the area of the rectangle of height $\frac{1}{a}$ and length L as claimed, and which is illustrated below.



86. (a) Let the point located at $(\cosh u, 0)$ be called T. Then $A(u)$ = area of the triangle ΔOTP minus the area under the curve $y = \sqrt{x^2 - 1}$ from A to T $\Rightarrow A(u) = \frac{1}{2} \cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1}\, dx$.

(b) $A(u) = \frac{1}{2} \cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1}\, dx \Rightarrow A'(u) = \frac{1}{2}\left(\cosh^2 u + \sinh^2 u\right) - \left(\sqrt{\cosh^2 u - 1}\right)(\sinh u)$
$= \frac{1}{2} \cosh^2 u + \frac{1}{2} \sinh^2 u - \sinh^2 u = \frac{1}{2}\left(\cosh^2 u - \sinh^2 u\right) = \left(\frac{1}{2}\right)(1) = \frac{1}{2}$

(c) $A'(u) = \frac{1}{2} \Rightarrow A(u) = \frac{u}{2} + C$, and from part (a) we have $A(0) = 0 \Rightarrow C = 0 \Rightarrow A(u) = \frac{u}{2} \Rightarrow u = 2A$

7.8 RELATIVE RATES OF GROWTH

1. (a) slower, $\lim_{x \to \infty} \frac{x+3}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$

   (b) slower, $\lim_{x \to \infty} \frac{x^3 + \sin^2 x}{e^x} = \lim_{x \to \infty} \frac{3x^2 + 2\sin x \cos x}{e^x} = \lim_{x \to \infty} \frac{6x + 2\cos 2x}{e^x} = \lim_{x \to \infty} \frac{6 - 4\sin 2x}{e^x} = 0$ by the Sandwich Theorem because $\frac{2}{e^x} \le \frac{6 - 4\sin 2x}{e^x} \le \frac{10}{e^x}$ for all reals and $\lim_{x \to \infty} \frac{2}{e^x} = 0 = \lim_{x \to \infty} \frac{10}{e^x}$

   (c) slower, $\lim_{x \to \infty} \frac{\sqrt{x}}{e^x} = \lim_{x \to \infty} \frac{x^{1/2}}{e^x} = \lim_{x \to \infty} \frac{\left(\frac{1}{2}\right) x^{-1/2}}{e^x} = \lim_{x \to \infty} \frac{1}{2\sqrt{x}\, e^x} = 0$

   (d) faster, $\lim_{x \to \infty} \frac{4^x}{e^x} = \lim_{x \to \infty} \left(\frac{4}{e}\right)^x = \infty$ since $\frac{4}{e} > 1$

   (e) slower, $\lim_{x \to \infty} \frac{\left(\frac{3}{2}\right)^x}{e^x} = \lim_{x \to \infty} \left(\frac{3}{2e}\right)^x = 0$ since $\frac{3}{2e} < 1$

   (f) slower, $\lim_{x \to \infty} \frac{e^{x/2}}{e^x} = \lim_{x \to \infty} \frac{1}{e^{x/2}} = 0$

   (g) same, $\lim_{x \to \infty} \frac{\left(\frac{e^x}{2}\right)}{e^x} = \lim_{x \to \infty} \frac{1}{2} = \frac{1}{2}$

   (h) slower, $\lim_{x \to \infty} \frac{\log_{10} x}{e^x} = \lim_{x \to \infty} \frac{\ln x}{(\ln 10)\, e^x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{(\ln 10)\, e^x} = \lim_{x \to \infty} \frac{1}{(\ln 10) x e^x} = 0$

2. (a) slower, $\lim_{x \to \infty} \frac{10x^4 + 30x + 1}{e^x} = \lim_{x \to \infty} \frac{40x^3 + 30}{e^x} = \lim_{x \to \infty} \frac{120x^2}{e^x} = \lim_{x \to \infty} \frac{240x}{e^x} = \lim_{x \to \infty} \frac{240}{e^x} = 0$

   (b) slower, $\lim_{x \to \infty} \frac{x \ln x - x}{e^x} = \lim_{x \to \infty} \frac{x(\ln x - 1)}{e^x} = \lim_{x \to \infty} \frac{\ln x - 1 + x\left(\frac{1}{x}\right)}{e^x} = \lim_{x \to \infty} \frac{\ln x - 1 + 1}{e^x} = \lim_{x \to \infty} \frac{\ln x}{e^x}$
   $= \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{e^x} = \lim_{x \to \infty} \frac{1}{xe^x} = 0$

   (c) slower, $\lim_{x \to \infty} \frac{\sqrt{1+x^4}}{e^x} = \sqrt{\lim_{x \to \infty} \frac{1+x^4}{e^{2x}}} = \sqrt{\lim_{x \to \infty} \frac{4x^3}{2e^{2x}}} = \sqrt{\lim_{x \to \infty} \frac{12x^2}{4e^{2x}}} = \sqrt{\lim_{x \to \infty} \frac{24x}{8e^{2x}}} = \sqrt{\lim_{x \to \infty} \frac{24}{16e^{2x}}}$
   $= \sqrt{0} = 0$

   (d) slower, $\lim_{x \to \infty} \frac{\left(\frac{5}{2}\right)^x}{e^x} = \lim_{x \to \infty} \left(\frac{5}{2e}\right)^x = 0$ since $\frac{5}{2e} < 1$

   (e) slower, $\lim_{x \to \infty} \frac{e^{-x}}{e^x} = \lim_{x \to \infty} \frac{1}{e^{2x}} = 0$

   (f) faster, $\lim_{x \to \infty} \frac{xe^x}{e^x} = \lim_{x \to \infty} x = \infty$

   (g) slower, since for all reals we have $-1 \le \cos x \le 1 \Rightarrow e^{-1} \le e^{\cos x} \le e^1 \Rightarrow \frac{e^{-1}}{e^x} \le \frac{e^{\cos x}}{e^x} \le \frac{e^1}{e^x}$ and also $\lim_{x \to \infty} \frac{e^{-1}}{e^x} = 0 = \lim_{x \to \infty} \frac{e^1}{e^x}$, so by the Sandwich Theorem we conclude that $\lim_{x \to \infty} \frac{e^{\cos x}}{e^x} = 0$

   (h) same, $\lim_{x \to \infty} \frac{e^{x-1}}{e^x} = \lim_{x \to \infty} \frac{1}{e^{(x-x+1)}} = \lim_{x \to \infty} \frac{1}{e} = \frac{1}{e}$

3. (a) same, $\lim_{x \to \infty} \frac{x^2 + 4x}{x^2} = \lim_{x \to \infty} \frac{2x+4}{2x} = \lim_{x \to \infty} \frac{2}{2} = 1$

   (b) faster, $\lim_{x \to \infty} \frac{x^5 - x^2}{x^2} = \lim_{x \to \infty} (x^3 - 1) = \infty$

   (c) same, $\lim_{x \to \infty} \frac{\sqrt{x^4 + x^3}}{x^2} = \sqrt{\lim_{x \to \infty} \frac{x^4 + x^3}{x^4}} = \sqrt{\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)} = \sqrt{1} = 1$

   (d) same, $\lim_{x \to \infty} \frac{(x+3)^2}{x^2} = \lim_{x \to \infty} \frac{2(x+3)}{2x} = \lim_{x \to \infty} \frac{2}{2} = 1$

   (e) slower, $\lim_{x \to \infty} \frac{x \ln x}{x^2} = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$

   (f) faster, $\lim_{x \to \infty} \frac{2^x}{x^2} = \lim_{x \to \infty} \frac{(\ln 2)\, 2^x}{2x} = \lim_{x \to \infty} \frac{(\ln 2)^2\, 2^x}{2} = \infty$

   (g) slower, $\lim_{x \to \infty} \frac{x^3 e^{-x}}{x^2} = \lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$

   (h) same, $\lim_{x \to \infty} \frac{8x^2}{x^2} = \lim_{x \to \infty} 8 = 8$

4. (a) same, $\lim_{x \to \infty} \frac{x^2 + \sqrt{x}}{x^2} = \lim_{x \to \infty} \left(1 + \frac{1}{x^{3/2}}\right) = 1$

   (b) same, $\lim_{x \to \infty} \frac{10x^2}{x^2} = \lim_{x \to \infty} 10 = 10$

   (c) slower, $\lim_{x \to \infty} \frac{x^2 e^{-x}}{x^2} = \lim_{x \to \infty} \frac{1}{e^x} = 0$

(d) slower, $\lim\limits_{x\to\infty} \frac{\log_{10} x^2}{x^2} = \lim\limits_{x\to\infty} \frac{\left(\frac{\ln x^2}{\ln 10}\right)}{x^2} = \frac{1}{\ln 10} \lim\limits_{x\to\infty} \frac{2 \ln x}{x^2} = \frac{2}{\ln 10} \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{2x} = \frac{1}{\ln 10} \lim\limits_{x\to\infty} \frac{1}{x^2} = 0$

(e) faster, $\lim\limits_{x\to\infty} \frac{x^3-x^2}{x^2} = \lim\limits_{x\to\infty} (x-1) = \infty$

(f) slower, $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{10}\right)^x}{x^2} = \lim\limits_{x\to\infty} \frac{1}{10^x x^2} = 0$

(g) faster, $\lim\limits_{x\to\infty} \frac{(1.1)^x}{x^2} = \lim\limits_{x\to\infty} \frac{(\ln 1.1)(1.1)^x}{2x} = \lim\limits_{x\to\infty} \frac{(\ln 1.1)^2(1.1)^x}{2} = \infty$

(h) same, $\lim\limits_{x\to\infty} \frac{x^2+100x}{x^2} = \lim\limits_{x\to\infty} \left(1 + \frac{100}{x}\right) = 1$

5. (a) same, $\lim\limits_{x\to\infty} \frac{\log_3 x}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{\ln x}{\ln 3}\right)}{\ln x} = \lim\limits_{x\to\infty} \frac{1}{\ln 3} = \frac{1}{\ln 3}$

(b) same, $\lim\limits_{x\to\infty} \frac{\ln 2x}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{2}{2x}\right)}{\left(\frac{1}{x}\right)} = 1$

(c) same, $\lim\limits_{x\to\infty} \frac{\ln \sqrt{x}}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{2}\right) \ln x}{\ln x} = \lim\limits_{x\to\infty} \frac{1}{2} = \frac{1}{2}$

(d) faster, $\lim\limits_{x\to\infty} \frac{\sqrt{x}}{\ln x} = \lim\limits_{x\to\infty} \frac{x^{1/2}}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{2}\right) x^{-1/2}}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{x}{2\sqrt{x}} = \lim\limits_{x\to\infty} \frac{\sqrt{x}}{2} = \infty$

(e) faster, $\lim\limits_{x\to\infty} \frac{x}{\ln x} = \lim\limits_{x\to\infty} \frac{1}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} x = \infty$

(f) same, $\lim\limits_{x\to\infty} \frac{5 \ln x}{\ln x} = \lim\limits_{x\to\infty} 5 = 5$

(g) slower, $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\ln x} = \lim\limits_{x\to\infty} \frac{1}{x \ln x} = 0$

(h) faster, $\lim\limits_{x\to\infty} \frac{e^x}{\ln x} = \lim\limits_{x\to\infty} \frac{e^x}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} xe^x = \infty$

6. (a) same, $\lim\limits_{x\to\infty} \frac{\log_2 x^2}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{\ln x^2}{\ln 2}\right)}{\ln x} = \frac{1}{\ln 2} \lim\limits_{x\to\infty} \frac{\ln x^2}{\ln x} = \frac{1}{\ln 2} \lim\limits_{x\to\infty} \frac{2 \ln x}{\ln x} = \frac{1}{\ln 2} \lim\limits_{x\to\infty} 2 = \frac{2}{\ln 2}$

(b) same, $\lim\limits_{x\to\infty} \frac{\log_{10} 10x}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{\ln 10x}{\ln 10}\right)}{\ln x} = \frac{1}{\ln 10} \lim\limits_{x\to\infty} \frac{\ln 10x}{\ln x} = \frac{1}{\ln 10} \lim\limits_{x\to\infty} \frac{\left(\frac{10}{10x}\right)}{\left(\frac{1}{x}\right)} = \frac{1}{\ln 10} \lim\limits_{x\to\infty} 1 = \frac{1}{\ln 10}$

(c) slower, $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x}}\right)}{\ln x} = \lim\limits_{x\to\infty} \frac{1}{\left(\sqrt{x}\right)(\ln x)} = 0$

(d) slower, $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{x^2}\right)}{\ln x} = \lim\limits_{x\to\infty} \frac{1}{x^2 \ln x} = 0$

(e) faster, $\lim\limits_{x\to\infty} \frac{x-2\ln x}{\ln x} = \lim\limits_{x\to\infty} \left(\frac{x}{\ln x} - 2\right) = \left(\lim\limits_{x\to\infty} \frac{x}{\ln x}\right) - 2 = \left(\lim\limits_{x\to\infty} \frac{1}{\left(\frac{1}{x}\right)}\right) - 2 = \left(\lim\limits_{x\to\infty} x\right) - 2 = \infty$

(f) slower, $\lim\limits_{x\to\infty} \frac{e^{-x}}{\ln x} = \lim\limits_{x\to\infty} \frac{1}{e^x \ln x} = 0$

(g) slower, $\lim\limits_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{1/x}{\ln x}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{1}{\ln x} = 0$

(h) same, $\lim\limits_{x\to\infty} \frac{\ln(2x+5)}{\ln x} = \lim\limits_{x\to\infty} \frac{\left(\frac{2}{2x+5}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{2x}{2x+5} = \lim\limits_{x\to\infty} \frac{2}{2} = \lim\limits_{x\to\infty} 1 = 1$

7. $\lim\limits_{x\to\infty} \frac{e^x}{e^{x/2}} = \lim\limits_{x\to\infty} e^{x/2} = \infty \Rightarrow e^x$ grows faster than $e^{x/2}$; since for $x > e^e$ we have $\ln x > e$ and $\lim\limits_{x\to\infty} \frac{(\ln x)^x}{e^x} = \lim\limits_{x\to\infty} \left(\frac{\ln x}{e}\right)^x = \infty \Rightarrow (\ln x)^x$ grows faster than e^x; since $x > \ln x$ for all $x > 0$ and $\lim\limits_{x\to\infty} \frac{x^x}{(\ln x)^x} = \lim\limits_{x\to\infty} \left(\frac{x}{\ln x}\right)^x = \infty \Rightarrow x^x$ grows faster than $(\ln x)^x$. Therefore, slowest to fastest are: $e^{x/2}$, e^x, $(\ln x)^x$, x^x so the order is d, a, c, b

8. $\lim\limits_{x\to\infty} \frac{(\ln 2)^x}{x^2} = \lim\limits_{x\to\infty} \frac{(\ln(\ln 2))(\ln 2)^x}{2x} = \lim\limits_{x\to\infty} \frac{(\ln(\ln 2))^2(\ln 2)^x}{2} = \frac{(\ln(\ln 2))^2}{2} \lim\limits_{x\to\infty} (\ln 2)^x = 0$ $\Rightarrow (\ln 2)^x$ grows slower than x^2; $\lim\limits_{x\to\infty} \frac{x^2}{2^x} = \lim\limits_{x\to\infty} \frac{2x}{(\ln 2)2^x} = \lim\limits_{x\to\infty} \frac{2}{(\ln 2)^2 2^x} = 0 \Rightarrow x^2$ grows slower than 2^x; $\lim\limits_{x\to\infty} \frac{2^x}{e^x} = \lim\limits_{x\to\infty} \left(\frac{2}{e}\right)^x = 0 \Rightarrow 2^x$ grows slower than e^x. Therefore, the slowest to the fastest is: $(\ln 2)^x$, x^2, 2^x and e^x so the order is c, b, a, d

9. (a) false; $\lim_{x \to \infty} \frac{x}{x} = 1$

(b) false; $\lim_{x \to \infty} \frac{x}{x+5} = \frac{1}{1} = 1$

(c) true; $x < x + 5 \Rightarrow \frac{x}{x+5} < 1$ if $x > 1$ (or sufficiently large)

(d) true; $x < 2x \Rightarrow \frac{x}{2x} < 1$ if $x > 1$ (or sufficiently large)

(e) true; $\lim_{x \to \infty} \frac{e^x}{e^{2x}} = \lim_{x \to 0} \frac{1}{e^x} = 0$

(f) true; $\frac{x + \ln x}{x} = 1 + \frac{\ln x}{x} < 1 + \frac{\sqrt{x}}{x} = 1 + \frac{1}{\sqrt{x}} < 2$ if $x > 1$ (or sufficiently large)

(g) false; $\lim_{x \to \infty} \frac{\ln x}{\ln 2x} = \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{2}{2x}\right)} = \lim_{x \to \infty} 1 = 1$

(h) true; $\frac{\sqrt{x^2+5}}{x} < \frac{\sqrt{(x+5)^2}}{x} < \frac{x+5}{x} = 1 + \frac{5}{x} < 6$ if $x > 1$ (or sufficiently large)

10. (a) true; $\frac{\left(\frac{1}{x+3}\right)}{\left(\frac{1}{x}\right)} = \frac{x}{x+3} < 1$ if $x > 1$ (or sufficiently large)

(b) true; $\frac{\left(\frac{1}{x} + \frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)} = 1 + \frac{1}{x} < 2$ if $x > 1$ (or sufficiently large)

(c) false; $\lim_{x \to \infty} \frac{\left(\frac{1}{x} - \frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to \infty} \left(1 - \frac{1}{x}\right) = 1$

(d) true; $2 + \cos x \le 3 \Rightarrow \frac{2 + \cos x}{2} \le \frac{3}{2}$ if x is sufficiently large

(e) true; $\frac{e^x + x}{e^x} = 1 + \frac{x}{e^x}$ and $\frac{x}{e^x} \to 0$ as $x \to \infty \Rightarrow 1 + \frac{x}{e^x} < 2$ if x is sufficiently large

(f) true; $\lim_{x \to \infty} \frac{x \ln x}{x^2} = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$

(g) true; $\frac{\ln(\ln x)}{\ln x} < \frac{\ln x}{\ln x} = 1$ if x is sufficiently large

(h) false; $\lim_{x \to \infty} \frac{\ln x}{\ln(x^2+1)} = \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{2x}{x^2+1}\right)} = \lim_{x \to \infty} \frac{x^2+1}{2x^2} = \lim_{x \to \infty} \left(\frac{1}{2} + \frac{1}{2x^2}\right) = \frac{1}{2}$

11. If f(x) and g(x) grow at the same rate, then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \neq 0 \Rightarrow \lim_{x \to \infty} \frac{g(x)}{f(x)} = \frac{1}{L} \neq 0$. Then $\left|\frac{f(x)}{g(x)} - L\right| < 1$ if x is sufficiently large $\Rightarrow L - 1 < \frac{f(x)}{g(x)} < L + 1 \Rightarrow \frac{f(x)}{g(x)} \le |L| + 1$ if x is sufficiently large $\Rightarrow f = O(g)$. Similarly, $\frac{g(x)}{f(x)} \le \left|\frac{1}{L}\right| + 1 \Rightarrow g = O(f)$.

12. When the degree of f is less than the degree of g since in that case $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$.

13. When the degree of f is less than or equal to the degree of g since $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$ when the degree of f is smaller than the degree of g, and $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{a}{b}$ (the ratio of the leading coefficients) when the degrees are the same.

14. Polynomials of a greater degree grow at a greater rate than polynomials of a lesser degree. Polynomials of the same degree grow at the same rate.

15. $\lim_{x \to \infty} \frac{\ln(x+1)}{\ln x} = \lim_{x \to \infty} \frac{\left(\frac{1}{x+1}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to \infty} \frac{x}{x+1} = \lim_{x \to \infty} \frac{1}{1} = 1$ and $\lim_{x \to \infty} \frac{\ln(x+999)}{\ln x} = \lim_{x \to \infty} \frac{\left(\frac{1}{x+999}\right)}{\left(\frac{1}{x}\right)}$ $= \lim_{x \to \infty} \frac{x}{x+999} = 1$

16. $\lim_{x \to \infty} \frac{\ln(x+a)}{\ln x} = \lim_{x \to \infty} \frac{\left(\frac{1}{x+a}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to \infty} \frac{x}{x+a} = \lim_{x \to \infty} \frac{1}{1} = 1$. Therefore, the relative rates are the same.

17. $\lim_{x \to \infty} \frac{\sqrt{10x+1}}{\sqrt{x}} = \sqrt{\lim_{x \to \infty} \frac{10x+1}{x}} = \sqrt{10}$ and $\lim_{x \to \infty} \frac{\sqrt{x+1}}{\sqrt{x}} = \sqrt{\lim_{x \to \infty} \frac{x+1}{x}} = \sqrt{1} = 1$. Since the growth rate is transitive, we conclude that $\sqrt{10x+1}$ and $\sqrt{x+1}$ have the same growth rate $\left(\text{that of } \sqrt{x}\right)$.

18. $\lim_{x \to \infty} \frac{\sqrt{x^4+x}}{x^2} = \sqrt{\lim_{x \to \infty} \frac{x^4+x}{x^4}} = 1$ and $\lim_{x \to \infty} \frac{\sqrt{x^4-x^3}}{x^2} = \sqrt{\lim_{x \to \infty} \frac{x^4-x^3}{x^4}} = 1$. Since the growth rate is transitive, we conclude that $\sqrt{x^4+x}$ and $\sqrt{x^4-x^3}$ have the same growth rate $(\text{that of } x^2)$.

19. $\lim_{x \to \infty} \frac{x^n}{e^x} = \lim_{x \to \infty} \frac{nx^{n-1}}{e^x} = \ldots = \lim_{x \to \infty} \frac{n!}{e^x} = 0 \Rightarrow x^n = o\left(e^x\right)$ for any non-negative integer n

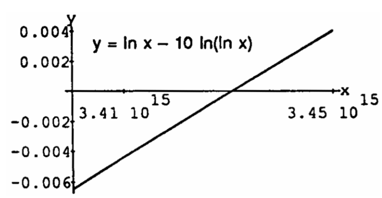
20. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then $\lim_{x \to \infty} \frac{p(x)}{e^x} = a_n \lim_{x \to \infty} \frac{x^n}{e^x} + a_{n-1} \lim_{x \to \infty} \frac{x^{n-1}}{e^x} + \ldots$ $+ a_1 \lim_{x \to \infty} \frac{x}{e^x} + a_0 \lim_{x \to \infty} \frac{1}{e^x}$ where each limit is zero (from Exercise 19). Therefore, $\lim_{x \to \infty} \frac{p(x)}{e^x} = 0$ $\Rightarrow e^x$ grows faster than any polynomial.

21. (a) $\lim_{x \to \infty} \frac{x^{1/n}}{\ln x} = \lim_{x \to \infty} \frac{x^{(1-n)/n}}{n\left(\frac{1}{x}\right)} = \left(\frac{1}{n}\right) \lim_{x \to \infty} x^{1/n} = \infty \Rightarrow \ln x = o\left(x^{1/n}\right)$ for any positive integer n

(b) $\ln\left(e^{17{,}000{,}000}\right) = 17{,}000{,}000 < \left(e^{17 \times 10^6}\right)^{1/10^6} = e^{17} \approx 24{,}154{,}952.75$

(c) $x \approx 3.430631121 \times 10^{15}$

(d) In the interval $\left[3.41 \times 10^{15}, 3.45 \times 10^{15}\right]$ we have $\ln x = 10 \ln(\ln x)$. The graphs cross at about 3.4306311×10^{15}.

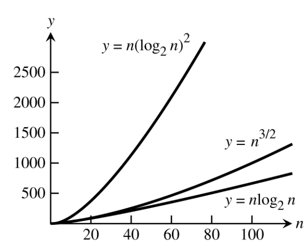


22. $\lim_{x \to \infty} \frac{\ln x}{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0} = \frac{\lim_{x \to \infty} \left(\frac{\ln x}{x^n}\right)}{\lim_{x \to \infty} \left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)} = \frac{\lim_{x \to \infty} \left[\frac{1/x}{nx^{n-1}}\right]}{a_n} = \lim_{x \to \infty} \frac{1}{(a_n)(nx^n)} = 0$

$\Rightarrow \ln x$ grows slower than any non-constant polynomial $(n \geq 1)$

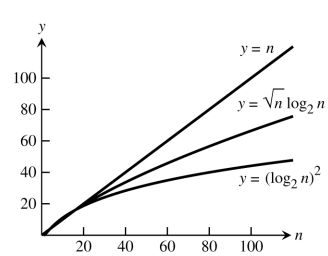
23. (a) $\lim_{n \to \infty} \frac{n \log_2 n}{n (\log_2 n)^2} = \lim_{n \to \infty} \frac{1}{\log_2 n} = 0 \Rightarrow n \log_2 n$ grows slower than $n (\log_2 n)^2$; $\lim_{n \to \infty} \frac{n \log_2 n}{n^{3/2}} = \lim_{n \to \infty} \frac{\left(\frac{\ln n}{\ln 2}\right)}{n^{1/2}}$ $= \frac{1}{\ln 2} \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{2}\right) n^{-1/2}} = \frac{2}{\ln 2} \lim_{n \to \infty} \frac{1}{n^{1/2}} = 0$ $\Rightarrow n \log_2 n$ grows slower than $n^{3/2}$. Therefore, $n \log_2 n$ grows at the slowest rate $\Rightarrow$ the algorithm that takes $O(n \log_2 n)$ steps is the most efficient in the long run.

(b)



24. (a) $\lim_{n \to \infty} \frac{(\log_2 n)^2}{n} = \lim_{n \to \infty} \frac{\left(\frac{\ln n}{\ln 2}\right)^2}{n} = \lim_{n \to \infty} \frac{(\ln n)^2}{n (\ln 2)^2}$ $= \lim_{n \to \infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{(\ln 2)^2} = \frac{2}{(\ln 2)^2} \lim_{n \to \infty} \frac{\ln n}{n}$ $= \frac{2}{(\ln 2)^2} \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{1} = 0 \Rightarrow (\log_2 n)^2$ grows slower than n; $\lim_{n \to \infty} \frac{(\log_2 n)^2}{\sqrt{n} \log_2 n} = \lim_{n \to \infty} \frac{\log_2 n}{\sqrt{n}}$ $= \lim_{n \to \infty} \frac{\left(\frac{\ln n}{\ln 2}\right)}{n^{1/2}} = \frac{1}{\ln 2} \lim_{n \to \infty} \frac{\ln n}{n^{1/2}}$ $= \frac{1}{\ln 2} \lim_{x \to \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{2}\right) n^{-1/2}} = \frac{2}{\ln 2} \lim_{n \to \infty} \frac{1}{n^{1/2}} = 0 \Rightarrow (\log_2 n)^2$ grows slower than $\sqrt{n} \log_2 n$. Therefore $(\log_2 n)^2$ grows at the slowest rate $\Rightarrow$ the algorithm that takes $O\left((\log_2 n)^2\right)$ steps is the most efficient in the long run.

(b)



25. It could take one million steps for a sequential search, but at most 20 steps for a binary search because $2^{19} = 524{,}288 < 1{,}000{,}000 < 1{,}048{,}576 = 2^{20}$.

26. It could take 450,000 steps for a sequential search, but at most 19 steps for a binary search because $2^{18} = 262{,}144 < 450{,}000 < 524{,}288 = 2^{19}$.

CHAPTER 7 PRACTICE EXERCISES

1. $y = 10e^{-x/5} \Rightarrow \frac{dy}{dx} = (10)\left(-\frac{1}{5}\right)e^{-x/5} = -2e^{-x/5}$

2. $y = \sqrt{2}\,e^{\sqrt{2}x} \Rightarrow \frac{dy}{dx} = \left(\sqrt{2}\right)\left(\sqrt{2}\right)e^{\sqrt{2}x} = 2e^{\sqrt{2}x}$

3. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} \Rightarrow \frac{dy}{dx} = \frac{1}{4}\left[x\left(4e^{4x}\right) + e^{4x}(1)\right] - \frac{1}{16}\left(4e^{4x}\right) = xe^{4x} + \frac{1}{4}e^{4x} - \frac{1}{4}e^{4x} = xe^{4x}$

4. $y = x^2e^{-2/x} = x^2e^{-2x^{-1}} \Rightarrow \frac{dy}{dx} = x^2\left[\left(2x^{-2}\right)e^{-2x^{-1}}\right] + e^{-2x^{-1}}(2x) = (2+2x)e^{-2x^{-1}} = 2e^{-2/x}(1+x)$

5. $y = \ln\left(\sin^2\theta\right) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sin\theta)(\cos\theta)}{\sin^2\theta} = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta$

6. $y = \ln\left(\sec^2\theta\right) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sec\theta)(\sec\theta\tan\theta)}{\sec^2\theta} = 2\tan\theta$

7. $y = \log_2\left(\frac{x^2}{2}\right) = \frac{\ln\left(\frac{x^2}{2}\right)}{\ln 2} \Rightarrow \frac{dy}{dx} = \frac{1}{\ln 2}\left(\frac{x}{\left(\frac{x^2}{2}\right)}\right) = \frac{2}{(\ln 2)x}$

8. $y = \log_5(3x-7) = \frac{\ln(3x-7)}{\ln 5} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\ln 5}\right)\left(\frac{3}{3x-7}\right) = \frac{3}{(\ln 5)(3x-7)}$

9. $y = 8^{-t} \Rightarrow \frac{dy}{dt} = 8^{-t}(\ln 8)(-1) = -8^{-t}(\ln 8)$

10. $y = 9^{2t} \Rightarrow \frac{dy}{dt} = 9^{2t}(\ln 9)(2) = 9^{2t}(2\ln 9)$

11. $y = 5x^{3.6} \Rightarrow \frac{dy}{dx} = 5(3.6)x^{2.6} = 18x^{2.6}$

12. $y = \sqrt{2}\,x^{-\sqrt{2}} \Rightarrow \frac{dy}{dx} = \left(\sqrt{2}\right)\left(-\sqrt{2}\right)x^{\left(-\sqrt{2}-1\right)} = -2x^{\left(-\sqrt{2}-1\right)}$

13. $y = (x+2)^{x+2} \Rightarrow \ln y = \ln(x+2)^{x+2} = (x+2)\ln(x+2) \Rightarrow \frac{y'}{y} = (x+2)\left(\frac{1}{x+2}\right) + (1)\ln(x+2)$
$\Rightarrow \frac{dy}{dx} = (x+2)^{x+2}\left[\ln(x+2) + 1\right]$

14. $y = 2(\ln x)^{x/2} \Rightarrow \ln y = \ln\left[2(\ln x)^{x/2}\right] = \ln(2) + \left(\frac{x}{2}\right)\ln(\ln x) \Rightarrow \frac{y'}{y} = 0 + \left(\frac{x}{2}\right)\left[\frac{\left(\frac{1}{x}\right)}{\ln x}\right] + (\ln(\ln x))\left(\frac{1}{2}\right)$
$\Rightarrow y' = \left[\frac{1}{2\ln x} + \left(\frac{1}{2}\right)\ln(\ln x)\right]2(\ln x)^{x/2} = (\ln x)^{x/2}\left[\ln(\ln x) + \frac{1}{\ln x}\right]$

15. $y = \sin^{-1}\sqrt{1-u^2} = \sin^{-1}\left(1-u^2\right)^{1/2} \Rightarrow \frac{dy}{du} = \frac{\frac{1}{2}\left(1-u^2\right)^{-1/2}(-2u)}{\sqrt{1-\left[\left(1-u^2\right)^{1/2}\right]^2}} = \frac{-u}{\sqrt{1-u^2}\sqrt{1-\left(1-u^2\right)}} = \frac{-u}{|u|\sqrt{1-u^2}}$
$= \frac{-u}{u\sqrt{1-u^2}} = \frac{-1}{\sqrt{1-u^2}}, \ 0 < u < 1$

16. $y = \sin^{-1}\left(\frac{1}{\sqrt{v}}\right) = \sin^{-1}v^{-1/2} \Rightarrow \frac{dy}{dv} = \frac{-\frac{1}{2}v^{-3/2}}{\sqrt{1-\left(v^{-1/2}\right)^2}} = \frac{-1}{2v^{3/2}\sqrt{1-v^{-1}}} = \frac{-1}{2v^{3/2}\sqrt{\frac{v-1}{v}}} = \frac{-\sqrt{v}}{2v^{3/2}\sqrt{v-1}} = \frac{-1}{2v\sqrt{v-1}}$

17. $y = \ln\left(\cos^{-1}x\right) \Rightarrow y' = \frac{\left(\frac{-1}{\sqrt{1-x^2}}\right)}{\cos^{-1}x} = \frac{-1}{\sqrt{1-x^2}\cos^{-1}x}$

18. $y = z\cos^{-1} z - \sqrt{1-z^2} = z\cos^{-1} z - (1-z^2)^{1/2} \Rightarrow \frac{dy}{dz} = \cos^{-1} z - \frac{z}{\sqrt{1-z^2}} - \left(\frac{1}{2}\right)(1-z^2)^{-1/2}(-2z)$
$= \cos^{-1} z - \frac{z}{\sqrt{1-z^2}} + \frac{z}{\sqrt{1-z^2}} = \cos^{-1} z$

19. $y = t\tan^{-1} t - \left(\frac{1}{2}\right)\ln t \Rightarrow \frac{dy}{dt} = \tan^{-1} t + t\left(\frac{1}{1+t^2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{t}\right) = \tan^{-1} t + \frac{t}{1+t^2} - \frac{1}{2t}$

20. $y = (1+t^2)\cot^{-1} 2t \Rightarrow \frac{dy}{dt} = 2t\cot^{-1} 2t + (1+t^2)\left(\frac{-2}{1+4t^2}\right)$

21. $y = z\sec^{-1} z - \sqrt{z^2-1} = z\sec^{-1} z - (z^2-1)^{1/2} \Rightarrow \frac{dy}{dz} = z\left(\frac{1}{|z|\sqrt{z^2-1}}\right) + (\sec^{-1} z)(1) - \frac{1}{2}(z^2-1)^{-1/2}(2z)$
$= \frac{z}{|z|\sqrt{z^2-1}} - \frac{z}{\sqrt{z^2-1}} + \sec^{-1} z = \frac{1-z}{\sqrt{z^2-1}} + \sec^{-1} z,\ z > 1$

22. $y = 2\sqrt{x-1}\sec^{-1}\sqrt{x} = 2(x-1)^{1/2}\sec^{-1}\left(x^{1/2}\right)$
$\Rightarrow \frac{dy}{dx} = 2\left[\left(\frac{1}{2}\right)(x-1)^{-1/2}\sec^{-1}\left(x^{1/2}\right) + (x-1)^{1/2}\left(\frac{\left(\frac{1}{2}\right)x^{-1/2}}{\sqrt{x}\sqrt{x-1}}\right)\right] = 2\left(\frac{\sec^{-1}\sqrt{x}}{2\sqrt{x-1}} + \frac{1}{2x}\right) = \frac{\sec^{-1}\sqrt{x}}{\sqrt{x-1}} + \frac{1}{x}$

23. $y = \csc^{-1}(\sec\theta) \Rightarrow \frac{dy}{d\theta} = \frac{-\sec\theta\tan\theta}{|\sec\theta|\sqrt{\sec^2\theta-1}} = -\frac{\tan\theta}{|\tan\theta|} = -1,\ 0 < \theta < \frac{\pi}{2}$

24. $y = (1+x^2)e^{\tan^{-1} x} \Rightarrow y' = 2xe^{\tan^{-1} x} + (1+x^2)\left(\frac{e^{\tan^{-1} x}}{1+x^2}\right) = 2xe^{\tan^{-1} x} + e^{\tan^{-1} x}$

25. $y = \frac{2(x^2+1)}{\sqrt{\cos 2x}} \Rightarrow \ln y = \ln\left(\frac{2(x^2+1)}{\sqrt{\cos 2x}}\right) = \ln(2) + \ln(x^2+1) - \frac{1}{2}\ln(\cos 2x) \Rightarrow \frac{y'}{y} = 0 + \frac{2x}{x^2+1} - \left(\frac{1}{2}\right)\frac{(-2\sin 2x)}{\cos 2x}$
$\Rightarrow y' = \left(\frac{2x}{x^2+1} + \tan 2x\right)y = \frac{2(x^2+1)}{\sqrt{\cos 2x}}\left(\frac{2x}{x^2+1} + \tan 2x\right)$

26. $y = \sqrt[10]{\frac{3x+4}{2x-4}} \Rightarrow \ln y = \ln\sqrt[10]{\frac{3x+4}{2x-4}} = \frac{1}{10}[\ln(3x+4) - \ln(2x-4)] \Rightarrow \frac{y'}{y} = \frac{1}{10}\left(\frac{3}{3x+4} - \frac{2}{2x-4}\right)$
$\Rightarrow y' = \frac{1}{10}\left(\frac{3}{3x+4} - \frac{1}{x-2}\right)y = \sqrt[10]{\frac{3x+4}{2x-4}}\left(\frac{1}{10}\right)\left(\frac{3}{3x+4} - \frac{1}{x-2}\right)$

27. $y = \left[\frac{(t+1)(t-1)}{(t-2)(t+3)}\right]^5 \Rightarrow \ln y = 5[\ln(t+1) + \ln(t-1) - \ln(t-2) - \ln(t+3)] \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{dt}\right)$
$= 5\left(\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t+3}\right) \Rightarrow \frac{dy}{dt} = 5\left[\frac{(t+1)(t-1)}{(t-2)(t+3)}\right]^5\left(\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t+3}\right)$

28. $y = \frac{2u2^u}{\sqrt{u^2+1}} \Rightarrow \ln y = \ln 2 + \ln u + u\ln 2 - \frac{1}{2}\ln(u^2+1) \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{du}\right) = \frac{1}{u} + \ln 2 - \frac{1}{2}\left(\frac{2u}{u^2+1}\right)$
$\Rightarrow \frac{dy}{du} = \frac{2u2^u}{\sqrt{u^2+1}}\left(\frac{1}{u} + \ln 2 - \frac{u}{u^2+1}\right)$

29. $y = (\sin\theta)^{\sqrt{\theta}} \Rightarrow \ln y = \sqrt{\theta}\ln(\sin\theta) \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{d\theta}\right) = \sqrt{\theta}\left(\frac{\cos\theta}{\sin\theta}\right) + \frac{1}{2}\theta^{-1/2}\ln(\sin\theta)$
$\Rightarrow \frac{dy}{d\theta} = (\sin\theta)^{\sqrt{\theta}}\left(\sqrt{\theta}\cot\theta + \frac{\ln(\sin\theta)}{2\sqrt{\theta}}\right)$

30. $y = (\ln x)^{1/\ln x} \Rightarrow \ln y = \left(\frac{1}{\ln x}\right)\ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{\ln x}\right)\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) + \ln(\ln x)\left[\frac{-1}{(\ln x)^2}\right]\left(\frac{1}{x}\right)$
$\Rightarrow y' = (\ln x)^{1/\ln x}\left[\frac{1-\ln(\ln x)}{x(\ln x)^2}\right]$

31. $\int e^x\sin(e^x)\,dx = \int \sin u\,du$, where $u = e^x$ and $du = e^x\,dx$
$= -\cos u + C = -\cos(e^x) + C$

32. $\int e^t \cos(3e^t - 2)\,dt = \frac{1}{3}\int \cos u\,du$, where $u = 3e^t - 2$ and $du = 3e^t\,dt$
$= \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3e^t - 2) + C$

33. $\int e^x \sec^2(e^x - 7)\,dx = \int \sec^2 u\,du$, where $u = e^x - 7$ and $du = e^x\,dx$
$= \tan u + C = \tan(e^x - 7) + C$

34. $\int e^y \csc(e^y + 1)\cot(e^y + 1)\,dy = \int \csc u \cot u\,du$, where $u = e^y + 1$ and $du = e^y\,dy$
$= -\csc u + C = -\csc(e^y + 1) + C$

35. $\int (\sec^2 x)\,e^{\tan x}\,dx = \int e^u\,du$, where $u = \tan x$ and $du = \sec^2 x\,dx$
$= e^u + C = e^{\tan x} + C$

36. $\int (\csc^2 x)\,e^{\cot x}\,dx = -\int e^u\,du$, where $u = \cot x$ and $du = -\csc^2 x\,dx$
$= -e^u + C = -e^{\cot x} + C$

37. $\int_{-1}^{1} \frac{1}{3x-4}\,dx = \frac{1}{3}\int_{-7}^{-1} \frac{1}{u}\,du$, where $u = 3x - 4$, $du = 3\,dx$; $x = -1 \Rightarrow u = -7$, $x = 1 \Rightarrow u = -1$
$= \frac{1}{3}\left[\ln|u|\right]_{-7}^{-1} = \frac{1}{3}\left[\ln|-1| - \ln|-7|\right] = \frac{1}{3}\left[0 - \ln 7\right] = -\frac{\ln 7}{3}$

38. $\int_{1}^{e} \frac{\sqrt{\ln x}}{x}\,dx = \int_{0}^{1} u^{1/2}\,du$, where $u = \ln x$, $du = \frac{1}{x}\,dx$; $x = 1 \Rightarrow u = 0$, $x = e \Rightarrow u = 1$
$= \left[\frac{2}{3}u^{3/2}\right]_0^1 = \left[\frac{2}{3}1^{3/2} - \frac{2}{3}0^{3/2}\right] = \frac{2}{3}$

39. $\int_{0}^{\pi} \tan\left(\frac{x}{3}\right)dx = \int_{0}^{\pi} \frac{\sin\left(\frac{x}{3}\right)}{\cos\left(\frac{x}{3}\right)}\,dx = -3\int_{1}^{1/2} \frac{1}{u}\,du$, where $u = \cos\left(\frac{x}{3}\right)$, $du = -\frac{1}{3}\sin\left(\frac{x}{3}\right)dx$; $x = 0 \Rightarrow u = 1$, $x = \pi$
$\Rightarrow u = \frac{1}{2}$
$= -3\left[\ln|u|\right]_1^{1/2} = -3\left[\ln\left|\frac{1}{2}\right| - \ln|1|\right] = -3\ln\frac{1}{2} = \ln 2^3 = \ln 8$

40. $\int_{1/6}^{1/4} 2\cot \pi x\,dx = 2\int_{1/6}^{1/4} \frac{\cos \pi x}{\sin \pi x}\,dx = \frac{2}{\pi}\int_{1/2}^{1/\sqrt{2}} \frac{1}{u}\,du$, where $u = \sin \pi x$, $du = \pi \cos \pi x\,dx$; $x = \frac{1}{6} \Rightarrow u = \frac{1}{2}$, $x = \frac{1}{4}$
$\Rightarrow u = \frac{1}{\sqrt{2}}$
$= \frac{2}{\pi}\left[\ln|u|\right]_{1/2}^{1/\sqrt{2}} = \frac{2}{\pi}\left[\ln\left|\frac{1}{\sqrt{2}}\right| - \ln\left|\frac{1}{2}\right|\right] = \frac{2}{\pi}\left[\ln 1 - \frac{1}{2}\ln 2 - \ln 1 + \ln 2\right] = \frac{2}{\pi}\left[\frac{1}{2}\ln 2\right] = \frac{\ln 2}{\pi}$

41. $\int_{0}^{4} \frac{2t}{t^2 - 25}\,dt = \int_{-25}^{-9} \frac{1}{u}\,du$, where $u = t^2 - 25$, $du = 2t\,dt$; $t = 0 \Rightarrow u = -25$, $t = 4 \Rightarrow u = -9$
$= \left[\ln|u|\right]_{-25}^{-9} = \ln|-9| - \ln|-25| = \ln 9 - \ln 25 = \ln\frac{9}{25}$

42. $\int_{-\pi/2}^{\pi/6} \frac{\cos t}{1 - \sin t}\,dt = -\int_{2}^{1/2} \frac{1}{u}\,du$, where $u = 1 - \sin t$, $du = -\cos t\,dt$; $t = -\frac{\pi}{2} \Rightarrow u = 2$, $t = \frac{\pi}{6} \Rightarrow u = \frac{1}{2}$
$= -\left[\ln|u|\right]_2^{1/2} = -\left[\ln\left|\frac{1}{2}\right| - \ln|2|\right] = -\ln 1 + \ln 2 + \ln 2 = 2\ln 2 = \ln 4$

43. $\int \frac{\tan(\ln v)}{v}\,dv = \int \tan u\,du = \int \frac{\sin u}{\cos u}\,du$, where $u = \ln v$ and $du = \frac{1}{v}\,dv$
$= -\ln|\cos u| + C = -\ln|\cos(\ln v)| + C$

44. $\int \frac{1}{v \ln v}\,dv = \int \frac{1}{u}\,du$, where $u = \ln v$ and $du = \frac{1}{v}\,dv$
$= \ln|u| + C = \ln|\ln v| + C$

45. $\int \frac{(\ln x)^{-3}}{x}\,dx = \int u^{-3}\,du$, where $u = \ln x$ and $du = \frac{1}{x}\,dx$
$= \frac{u^{-2}}{-2} + C = -\frac{1}{2}(\ln x)^{-2} + C$

46. $\int \frac{\ln(x-5)}{x-5}\,dx = \int u\,du$, where $u = \ln(x-5)$ and $du = \frac{1}{x-5}\,dx$
$= \frac{u^2}{2} + C = \frac{[\ln(x-5)]^2}{2} + C$

47. $\int \frac{1}{r}\csc^2(1+\ln r)\,dr = \int \csc^2 u\,du$, where $u = 1+\ln r$ and $du = \frac{1}{r}\,dr$
$= -\cot u + C = -\cot(1+\ln r) + C$

48. $\int \frac{\cos(1-\ln v)}{v}\,dv = -\int \cos u\,du$, where $u = 1-\ln v$ and $du = -\frac{1}{v}\,dv$
$= -\sin u + C = -\sin(1-\ln v) + C$

49. $\int x3^{x^2}\,dx = \frac{1}{2}\int 3^u\,du$, where $u = x^2$ and $du = 2x\,dx$
$= \frac{1}{2\ln 3}(3^u) + C = \frac{1}{2\ln 3}\left(3^{x^2}\right) + C$

50. $\int 2^{\tan x}\sec^2 x\,dx = \int 2^u\,du$, where $u = \tan x$ and $du = \sec^2 x\,dx$
$= \frac{1}{\ln 2}(2^u) + C = \frac{2^{\tan x}}{\ln 2} + C$

51. $\int_1^7 \frac{3}{x}\,dx = 3\int_1^7 \frac{1}{x}\,dx = 3[\ln|x|]_1^7 = 3(\ln 7 - \ln 1) = 3\ln 7$

52. $\int_1^{32} \frac{1}{5x}\,dx = \frac{1}{5}\int_1^{32} \frac{1}{x}\,dx = \frac{1}{5}[\ln|x|]_1^{32} = \frac{1}{5}(\ln 32 - \ln 1) = \frac{1}{5}\ln 32 = \ln\left(\sqrt[5]{32}\right) = \ln 2$

53. $\int_1^4 \left(\frac{x}{8} + \frac{1}{2x}\right)dx = \frac{1}{2}\int_1^4 \left(\frac{1}{4}x + \frac{1}{x}\right)dx = \frac{1}{2}\left[\frac{1}{8}x^2 + \ln|x|\right]_1^4 = \frac{1}{2}\left[\left(\frac{16}{8} + \ln 4\right) - \left(\frac{1}{8} + \ln 1\right)\right] = \frac{15}{16} + \frac{1}{2}\ln 4$
$= \frac{15}{16} + \ln\sqrt{4} = \frac{15}{16} + \ln 2$

54. $\int_1^8 \left(\frac{2}{3x} - \frac{8}{x^2}\right)dx = \frac{2}{3}\int_1^8 \left(\frac{1}{x} - 12x^{-2}\right)dx = \frac{2}{3}\left[\ln|x| + 12x^{-1}\right]_1^8 = \frac{2}{3}\left[\left(\ln 8 + \frac{12}{8}\right) - (\ln 1 + 12)\right]$
$= \frac{2}{3}\left(\ln 8 + \frac{3}{2} - 12\right) = \frac{2}{3}\left(\ln 8 - \frac{21}{2}\right) = \frac{2}{3}(\ln 8) - 7 = \ln\left(8^{2/3}\right) - 7 = \ln 4 - 7$

55. $\int_{-2}^{-1} e^{-(x+1)}\,dx = -\int_1^0 e^u\,du$, where $u = -(x+1)$, $du = -dx$; $x = -2 \Rightarrow u = 1$, $x = -1 \Rightarrow u = 0$
$= -[e^u]_1^0 = -(e^0 - e^1) = e - 1$

56. $\int_{-\ln 2}^0 e^{2w}\,dw = \frac{1}{2}\int_{\ln(1/4)}^0 e^u\,du$, where $u = 2w$, $du = 2\,dw$; $w = -\ln 2 \Rightarrow u = \ln\frac{1}{4}$, $w = 0 \Rightarrow u = 0$
$= \frac{1}{2}[e^u]_{\ln(1/4)}^0 = \frac{1}{2}\left[e^0 - e^{\ln(1/4)}\right] = \frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{8}$

57. $\int_1^{\ln 5} e^r(3e^r + 1)^{-3/2}\,dr = \frac{1}{3}\int_4^{16} u^{-3/2}\,du$, where $u = 3e^r + 1$, $du = 3e^r dr$; $r = 0 \Rightarrow u = 4$, $r = \ln 5 \Rightarrow u = 16$
$= -\frac{2}{3}\left[u^{-1/2}\right]_4^{16} = -\frac{2}{3}\left(16^{-1/2} - 4^{-1/2}\right) = \left(-\frac{2}{3}\right)\left(\frac{1}{4} - \frac{1}{2}\right) = \left(-\frac{2}{3}\right)\left(-\frac{1}{4}\right) = \frac{1}{6}$

58. $\int_0^{\ln 9} e^\theta(e^\theta - 1)^{1/2}\,d\theta = \int_0^8 u^{1/2}\,du$, where $u = e^\theta - 1$, $du = e^\theta\,d\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \ln 9 \Rightarrow u = 8$
$= \frac{2}{3}\left[u^{3/2}\right]_0^8 = \frac{2}{3}\left(8^{3/2} - 0^{3/2}\right) = \frac{2}{3}\left(2^{9/2} - 0\right) = \frac{2^{11/2}}{3} = \frac{32\sqrt{2}}{3}$

59. $\int_1^e \frac{1}{x}(1+7\ln x)^{-1/3}\,dx = \frac{1}{7}\int_1^8 u^{-1/3}\,du$, where $u = 1 + 7\ln x$, $du = \frac{7}{x}\,dx$, $x = 1 \Rightarrow u = 1$, $x = e \Rightarrow u = 8$
$= \frac{3}{14}\left[u^{2/3}\right]_1^8 = \frac{3}{14}\left(8^{2/3} - 1^{2/3}\right) = \left(\frac{3}{14}\right)(4-1) = \frac{9}{14}$

60. $\int_e^{e^2} \frac{1}{x\sqrt{\ln x}}\,dx = \int_e^{e^2} (\ln x)^{-1/2}\,\frac{1}{x}\,dx = \int_1^2 u^{-1/2}\,du$, where $u = \ln x$, $du = \frac{1}{x}\,dx$; $x = e \Rightarrow u = 1$, $x = e^2 \Rightarrow u = 2$
$= 2\left[u^{1/2}\right]_1^2 = 2\left(\sqrt{2} - 1\right) = 2\sqrt{2} - 2$

61. $\int_1^3 \frac{[\ln(v+1)]^2}{v+1}\,dv = \int_1^3 [\ln(v+1)]^2\,\frac{1}{v+1}\,dv = \int_{\ln 2}^{\ln 4} u^2\,du$, where $u = \ln(v+1)$, $du = \frac{1}{v+1}\,dv$;
$v = 1 \Rightarrow u = \ln 2$, $v = 3 \Rightarrow u = \ln 4$;
$= \frac{1}{3}\left[u^3\right]_{\ln 2}^{\ln 4} = \frac{1}{3}\left[(\ln 4)^3 - (\ln 2)^3\right] = \frac{1}{3}\left[(2\ln 2)^3 - (\ln 2)^3\right] = \frac{(\ln 2)^3}{3}(8-1) = \frac{7}{3}(\ln 2)^3$

62. $\int_2^4 (1+\ln t)(t\ln t)\,dt = \int_2^4 (t\ln t)(1+\ln t)\,dt = \int_{2\ln 2}^{4\ln 4} u\,du$, where $u = t\ln t$, $du = \left((t)\left(\frac{1}{t}\right) + (\ln t)(1)\right)dt$
$= (1+\ln t)\,dt$; $t = 2 \Rightarrow u = 2\ln 2$, $t = 4$
$\Rightarrow u = 4\ln 4$
$= \frac{1}{2}\left[u^2\right]_{2\ln 2}^{4\ln 4} = \frac{1}{2}\left[(4\ln 4)^2 - (2\ln 2)^2\right] = \frac{1}{2}\left[(8\ln 2)^2 - (2\ln 2)^2\right] = \frac{(2\ln 2)^2}{2}(16-1) = 30(\ln 2)^2$

63. $\int_1^8 \frac{\log_4 \theta}{\theta}\,d\theta = \frac{1}{\ln 4}\int_1^8 (\ln\theta)\left(\frac{1}{\theta}\right)d\theta = \frac{1}{\ln 4}\int_0^{\ln 8} u\,du$, where $u = \ln\theta$, $du = \frac{1}{\theta}\,d\theta$, $\theta = 1 \Rightarrow u = 0$, $\theta = 8 \Rightarrow u = \ln 8$
$= \frac{1}{2\ln 4}\left[u^2\right]_0^{\ln 8} = \frac{1}{\ln 16}\left[(\ln 8)^2 - 0^2\right] = \frac{(3\ln 2)^2}{4\ln 2} = \frac{9\ln 2}{4}$

64. $\int_1^e \frac{8(\ln 3)(\log_3 \theta)}{\theta}\,d\theta = \int_1^e \frac{8(\ln 3)(\ln\theta)}{\theta(\ln 3)}\,d\theta = 8\int_1^e (\ln\theta)\left(\frac{1}{\theta}\right)d\theta = 8\int_0^1 u\,du$, where $u = \ln\theta$, $du = \frac{1}{\theta}\,d\theta$;
$\theta = 1 \Rightarrow u = 0$, $\theta = e \Rightarrow u = 1$
$= 4\left[u^2\right]_0^1 = 4\left(1^2 - 0^2\right) = 4$

65. $\int_{-3/4}^{3/4} \frac{6}{\sqrt{9-4x^2}}\,dx = 3\int_{-3/4}^{3/4} \frac{2}{\sqrt{3^2-(2x)^2}}\,dx = 3\int_{-3/2}^{3/2} \frac{1}{\sqrt{3^2-u^2}}\,du$, where $u = 2x$, $du = 2\,dx$;
$x = -\frac{3}{4} \Rightarrow u = -\frac{3}{2}$, $x = \frac{3}{4} \Rightarrow u = \frac{3}{2}$
$= 3\left[\sin^{-1}\left(\frac{u}{3}\right)\right]_{-3/2}^{3/2} = 3\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right] = 3\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = 3\left(\frac{\pi}{3}\right) = \pi$

66. $\int_{-1/5}^{1/5} \frac{6}{\sqrt{4-25x^2}}\,dx = \frac{6}{5}\int_{-1/5}^{1/5} \frac{5}{\sqrt{2^2-(5x)^2}}\,dx = \frac{6}{5}\int_{-1}^{1} \frac{1}{\sqrt{2^2-u^2}}\,du$, where $u = 5x$, $du = 5\,dx$;
$x = -\frac{1}{5} \Rightarrow u = -1$, $x = \frac{1}{5} \Rightarrow u = 1$
$= \frac{6}{5}\left[\sin^{-1}\left(\frac{u}{2}\right)\right]_{-1}^{1} = \frac{6}{5}\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \frac{6}{5}\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = \frac{6}{5}\left(\frac{\pi}{3}\right) = \frac{2\pi}{5}$

67. $\int_{-2}^{2} \frac{3}{4+3t^2}\,dt = \sqrt{3}\int_{-2}^{2} \frac{\sqrt{3}}{2^2+\left(\sqrt{3}t\right)^2}\,dt = \sqrt{3}\int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{1}{2^2+u^2}\,du$, where $u = \sqrt{3}t$, $du = \sqrt{3}\,dt$;
$t = -2 \Rightarrow u = -2\sqrt{3}$, $t = 2 \Rightarrow u = 2\sqrt{3}$
$= \sqrt{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right]_{-2\sqrt{3}}^{2\sqrt{3}} = \frac{\sqrt{3}}{2}\left[\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}\left(-\sqrt{3}\right)\right] = \frac{\sqrt{3}}{2}\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \frac{\pi}{\sqrt{3}}$

68. $\int_{\sqrt{3}}^{3} \frac{1}{3+t^2}\,dt = \int_{\sqrt{3}}^{3} \frac{1}{\left(\sqrt{3}\right)^2+t^2}\,dt = \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right]_{\sqrt{3}}^{3} = \frac{1}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1} 1\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\sqrt{3}\pi}{36}$

69. $\int \frac{1}{y\sqrt{4y^2-1}}\,dy = \int \frac{2}{(2y)\sqrt{(2y)^2-1}}\,dy = \int \frac{1}{u\sqrt{u^2-1}}\,du$, where $u = 2y$ and $du = 2\,dy$
$= \sec^{-1}|u| + C = \sec^{-1}|2y| + C$

70. $\int \frac{24}{y\sqrt{y^2-16}}\,dy = 24\int \frac{1}{y\sqrt{y^2-4^2}}\,dy = 24\left(\frac{1}{4}\sec^{-1}\left|\frac{y}{4}\right|\right) + C = 6\sec^{-1}\left|\frac{y}{4}\right| + C$

71. $\int_{\sqrt{2}/3}^{2/3} \frac{1}{|y|\sqrt{9y^2-1}}\,dy = \int_{\sqrt{2}/3}^{2/3} \frac{3}{|3y|\sqrt{(3y)^2-1}}\,dy = \int_{\sqrt{2}}^{2} \frac{1}{|u|\sqrt{u^2-1}}\,du$, where $u = 3y$, $du = 3\,dy$;

$y = \frac{\sqrt{2}}{3} \Rightarrow u = \sqrt{2}$, $y = \frac{2}{3} \Rightarrow u = 2$

$= \left[\sec^{-1} u\right]_{\sqrt{2}}^{2} = \left[\sec^{-1} 2 - \sec^{-1}\sqrt{2}\right] = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

72. $\int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{1}{|y|\sqrt{5y^2-3}}\,dy = \int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{\sqrt{5}}{-\sqrt{5}y\sqrt{\left(\sqrt{5}y\right)^2-\left(\sqrt{3}\right)^2}}\,dy = \int_{-2}^{-\sqrt{6}} \frac{1}{-u\sqrt{u^2-\left(\sqrt{3}\right)^2}}\,du$,

where $u = \sqrt{5}y$, $du = \sqrt{5}\,dy$; $y = -\frac{2}{\sqrt{5}} \Rightarrow u = -2$, $y = -\frac{\sqrt{6}}{\sqrt{5}} \Rightarrow u = -\sqrt{6}$

$= \left[-\frac{1}{\sqrt{3}}\sec^{-1}\left|\frac{u}{\sqrt{3}}\right|\right]_{-2}^{-\sqrt{6}} = \frac{-1}{\sqrt{3}}\left[\sec^{-1}\sqrt{2} - \sec^{-1}\frac{2}{\sqrt{3}}\right] = \frac{-1}{\sqrt{3}}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{-1}{\sqrt{3}}\left[\frac{3\pi}{12} - \frac{2\pi}{12}\right] = \frac{-\pi}{12\sqrt{3}} = \frac{-\sqrt{3}\pi}{36}$

73. $\int \frac{1}{\sqrt{-2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x^2+2x+1)}}\,dx = \int \frac{1}{\sqrt{1-(x+1)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du$, where $u = x + 1$ and

$du = dx$

$= \sin^{-1} u + C = \sin^{-1}(x+1) + C$

74. $\int \frac{1}{\sqrt{-x^2+4x-1}}\,dx = \int \frac{1}{\sqrt{3-(x^2-4x+4)}}\,dx = \int \frac{1}{\sqrt{\left(\sqrt{3}\right)^2-(x-2)^2}}\,dx = \int \frac{1}{\sqrt{\left(\sqrt{3}\right)^2-u^2}}\,du$

where $u = x - 2$ and $du = dx$

$= \sin^{-1}\left(\frac{u}{\sqrt{3}}\right) + C = \sin^{-1}\left(\frac{x-2}{\sqrt{3}}\right) + C$

75. $\int_{-2}^{-1} \frac{2}{v^2+4v+5}\,dv = 2\int_{-2}^{-1} \frac{1}{1+(v^2+4v+4)}\,dv = 2\int_{-2}^{-1} \frac{1}{1+(v+2)^2}\,dv = 2\int_{0}^{1} \frac{1}{1+u^2}\,du$,

where $u = v + 2$, $du = dv$; $v = -2 \Rightarrow u = 0$, $v = -1 \Rightarrow u = 1$

$= 2\left[\tan^{-1} u\right]_0^1 = 2\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 2\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{2}$

76. $\int_{-1}^{1} \frac{3}{4v^2+4v+4}\,dv = \frac{3}{4}\int_{-1}^{1} \frac{1}{\frac{3}{4}+\left(v^2+v+\frac{1}{4}\right)}\,dv = \frac{3}{4}\int_{-1}^{1} \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+\left(v+\frac{1}{2}\right)^2}\,dv = \frac{3}{4}\int_{-1/2}^{3/2} \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+u^2}\,du$

where $u = v + \frac{1}{2}$, $du = dv$; $v = -1 \Rightarrow u = -\frac{1}{2}$, $v = 1 \Rightarrow u = \frac{3}{2}$

$= \frac{3}{4}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)\right]_{-1/2}^{3/2} = \frac{\sqrt{3}}{2}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right] = \frac{\sqrt{3}}{2}\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \frac{\sqrt{3}}{2}\left(\frac{2\pi}{6} + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\cdot\frac{\pi}{2}$

$= \frac{\sqrt{3}\pi}{4}$

77. $\int \frac{1}{(t+1)\sqrt{t^2+2t-8}}\,dt = \int \frac{1}{(t+1)\sqrt{(t^2+2t+1)-9}}\,dt = \int \frac{1}{(t+1)\sqrt{(t+1)^2-3^2}}\,dt = \int \frac{1}{u\sqrt{u^2-3^2}}\,du$

where $u = t + 1$ and $du = dt$

$= \frac{1}{3}\sec^{-1}\left|\frac{u}{3}\right| + C = \frac{1}{3}\sec^{-1}\left|\frac{t+1}{3}\right| + C$

78. $\int \frac{1}{(3t+1)\sqrt{9t^2+6t}}\,dt = \int \frac{1}{(3t+1)\sqrt{(9t^2+6t+1)-1}}\,dt = \int \frac{1}{(3t+1)\sqrt{(3t+1)^2-1^2}}\,dt = \frac{1}{3}\int \frac{1}{u\sqrt{u^2-1}}\,du$

where $u = 3t + 1$ and $du = 3\,dt$

$= \frac{1}{3}\sec^{-1}|u| + C = \frac{1}{3}\sec^{-1}|3t+1| + C$

79. $3^y = 2^{y+1} \Rightarrow \ln 3^y = \ln 2^{y+1} \Rightarrow y(\ln 3) = (y+1)\ln 2 \Rightarrow (\ln 3 - \ln 2)y = \ln 2 \Rightarrow \left(\ln\frac{3}{2}\right)y = \ln 2 \Rightarrow y = \frac{\ln 2}{\ln\left(\frac{3}{2}\right)}$

80. $4^{-y} = 3^{y+2} \Rightarrow \ln 4^{-y} = \ln 3^{y+2} \Rightarrow -y \ln 4 = (y+2)\ln 3 \Rightarrow -2\ln 3 = (\ln 3 + \ln 4)y \Rightarrow (\ln 12)y = -2\ln 3$
$\Rightarrow y = -\frac{\ln 9}{\ln 12}$

81. $9e^{2y} = x^2 \Rightarrow e^{2y} = \frac{x^2}{9} \Rightarrow \ln e^{2y} = \ln\left(\frac{x^2}{9}\right) \Rightarrow 2y(\ln e) = \ln\left(\frac{x^2}{9}\right) \Rightarrow y = \frac{1}{2}\ln\left(\frac{x^2}{9}\right) = \ln\sqrt{\frac{x^2}{9}} = \ln\left|\frac{x}{3}\right| = \ln|x| - \ln 3$

82. $3^y = 3\ln x \Rightarrow \ln 3^y = \ln(3\ln x) \Rightarrow y\ln 3 = \ln(3\ln x) \Rightarrow y = \frac{\ln(3\ln x)}{\ln 3} = \frac{\ln 3 + \ln(\ln x)}{\ln 3}$

83. $\ln(y-1) = x + \ln y \Rightarrow e^{\ln(y-1)} = e^{(x+\ln y)} = e^x e^{\ln y} \Rightarrow y - 1 = ye^x \Rightarrow y - ye^x = 1 \Rightarrow y(1-e^x) = 1 \Rightarrow y = \frac{1}{1-e^x}$

84. $\ln(10\ln y) = \ln 5x \Rightarrow e^{\ln(10\ln y)} = e^{\ln 5x} \Rightarrow 10\ln y = 5x \Rightarrow \ln y = \frac{x}{2} \Rightarrow e^{\ln y} = e^{x/2} \Rightarrow y = e^{x/2}$

85. $\lim_{x \to 1} \frac{x^2+3x-4}{x-1} = \lim_{x \to 1} \frac{2x+3}{1} = 5$

86. $\lim_{x \to 1} \frac{x^a-1}{x^b-1} = \lim_{x \to 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$

87. $\lim_{x \to \pi} \frac{\tan x}{x} = \frac{\tan \pi}{\pi} = 0$

88. $\lim_{x \to 0} \frac{\tan x}{x + \sin x} = \lim_{x \to 0} \frac{\sec^2 x}{1 + \cos x} = \frac{1}{1+1} = \frac{1}{2}$

89. $\lim_{x \to 0} \frac{\sin^2 x}{\tan(x^2)} = \lim_{x \to 0} \frac{2\sin x \cdot \cos x}{2x\sec^2(x^2)} = \lim_{x \to 0} \frac{\sin(2x)}{2x\sec^2(x^2)} = \lim_{x \to 0} \frac{2\cos(2x)}{2x(2\sec^2(x^2)\tan(x^2)\cdot 2x) + 2\sec^2(x^2)} = \frac{2}{0 + 2\cdot 1} = 1$

90. $\lim_{x \to 0} \frac{\sin(mx)}{\sin(nx)} = \lim_{x \to 0} \frac{m\cos(mx)}{n\cos(nx)} = \frac{m}{n}$

91. $\lim_{x \to \pi/2^-} \sec(7x)\cos(3x) = \lim_{x \to \pi/2^-} \frac{\cos(3x)}{\cos(7x)} = \lim_{x \to \pi/2^-} \frac{-3\sin(3x)}{-7\sin(7x)} = \frac{3}{7}$

92. $\lim_{x \to 0^+} \sqrt{x}\sec x = \lim_{x \to 0^+} \frac{\sqrt{x}}{\cos x} = \frac{0}{1} = 0$

93. $\lim_{x \to 0} (\csc x - \cot x) = \lim_{x \to 0} \frac{1 - \cos x}{\sin x} = \lim_{x \to 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0$

94. $\lim_{x \to 0} \left(\frac{1}{x^4} - \frac{1}{x^2}\right) = \lim_{x \to 0} \left(\frac{1-x^2}{x^4}\right) = \lim_{x \to 0} (1-x^2)\cdot\frac{1}{x^4} = \lim_{x \to 0} (1-x^2) = \lim_{x \to 0} \frac{1}{x^4} = 1\cdot\infty = \infty$

95. $\lim_{x \to \infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right) = \lim_{x \to \infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right)\cdot\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$
$= \lim_{x \to \infty} \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$
Notice that $x = \sqrt{x^2}$ for $x > 0$ so this is equivalent to
$= \lim_{x \to \infty} \frac{\frac{2x+1}{x}}{\sqrt{\frac{x^2+x+1}{x^2}}+\sqrt{\frac{x^2-x}{x^2}}} = \lim_{x \to \infty} \frac{2+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}} = \frac{2}{\sqrt{1}+\sqrt{1}} = 1$

96. $\lim_{x \to \infty} \left(\frac{x^3}{x^2-1} - \frac{x^3}{x^2+1}\right) = \lim_{x \to \infty} \frac{x^3(x^2+1) - x^3(x^2-1)}{(x^2-1)(x^2+1)} = \lim_{x \to \infty} \frac{2x^3}{x^4-1} = \lim_{x \to \infty} \frac{6x^2}{4x^3} = \lim_{x \to \infty} \frac{12x}{12x^2}$
$= \lim_{x \to \infty} \frac{12}{24x} = \lim_{x \to \infty} \frac{1}{2x} = 0$

97. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x \to 0} \frac{10^x - 1}{x} = \lim_{x \to 0} \frac{(\ln 10)10^x}{1} = \ln 10$

98. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{\theta \to 0} \frac{3^\theta - 1}{\theta} = \lim_{\theta \to 0} \frac{(\ln 3)3^\theta}{1} = \ln 3$

99. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x \to 0} \frac{2^{\sin x} - 1}{e^x - 1} = \lim_{x \to 0} \frac{2^{\sin x}(\ln 2)(\cos x)}{e^x} = \ln 2$

100. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x \to 0} \frac{2^{-\sin x} - 1}{e^x - 1} = \lim\limits_{x \to 0} \frac{2^{-\sin x}(\ln 2)(-\cos x)}{e^x} = -\ln 2$

101. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x \to 0} \frac{5 - 5\cos x}{e^x - x - 1} = \lim\limits_{x \to 0} \frac{5\sin x}{e^x - 1} = \lim\limits_{x \to 0} \frac{5\cos x}{e^x} = 5$

102. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x \to 0} \frac{x\sin x^2}{\tan^3 x} = \lim\limits_{x \to 0} \frac{2x^2\cos x^2 + \sin x^2}{3\tan^2 x\sec^2 x} = \lim\limits_{x \to 0} \frac{2x^2\cos x^2 + \sin x^2}{3\tan^4 x + 3\tan^2 x}$

$= \lim\limits_{x \to 0} \frac{6x\cos x^2 - 4x^3\sin x^2}{12\tan^3 x\sec^2 x + 6\tan x\sec^2 x} = \lim\limits_{x \to 0} \frac{6x\cos x^2 - 4x^3\sin x^2}{12\tan^5 x + 18\tan^3 x + 6\tan x} = \lim\limits_{x \to 0} \frac{(6 - 8x^4)\cos x^2 - 24x^2\sin x^2}{60\tan^4 x\sec^2 x + 54\tan^2 x\sec^2 x + 6\sec^2 x} = \frac{6}{6} = 1$

103. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{t \to 0^+} \frac{t - \ln(1 + 2t)}{t^2} = \lim\limits_{t \to 0^+} \frac{\left(1 - \frac{2}{1+2t}\right)}{2t} = -\infty$

104. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x \to 4} \frac{\sin^2(\pi x)}{e^{x-4} + 3 - x} = \lim\limits_{x \to 4} \frac{2\pi(\sin \pi x)(\cos \pi x)}{e^{x-4} - 1}$

$= \lim\limits_{x \to 4} \frac{\pi\sin(2\pi x)}{e^{x-4} - 1} = \lim\limits_{x \to 4} \frac{2\pi^2\cos(2\pi x)}{e^{x-4}} = 2\pi^2$

105. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{t \to 0^+} \left(\frac{e^t}{t} - \frac{1}{t}\right) = \lim\limits_{t \to 0^+} \left(\frac{e^t - 1}{t}\right) = \lim\limits_{t \to 0^+} \frac{e^t}{1} = 1$

106. The limit leads to the indeterminate form $\frac{\infty}{\infty}$: $\lim\limits_{y \to 0^+} e^{-1/y}\ln y = \lim\limits_{y \to 0^+} \frac{\ln y}{e^{y^{-1}}} = \lim\limits_{y \to 0^+} \frac{y^{-1}}{-e^{y^{-1}}(y^{-2})}$

$= \lim\limits_{y \to 0^+} \left(-\frac{y}{e^{y^{-1}}}\right) = 0$

107. Let $f(x) = \left(\frac{e^x + 1}{e^x - 1}\right)^{\ln x} \Rightarrow \ln f(x) = \ln x \ln\left(\frac{e^x + 1}{e^x - 1}\right) \Rightarrow \lim\limits_{x \to \infty} \ln f(x) = \lim\limits_{x \to \infty} \ln x \ln\left(\frac{e^x + 1}{e^x - 1}\right)$; this is limit is currently of the form $0 \cdot \infty$. Before we put in one of the indeterminate forms, we rewrite $\frac{e^x + 1}{e^x - 1} = \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} = \coth\left(\frac{x}{2}\right)$; the limit is $\lim\limits_{x \to \infty} \ln x \ln\coth\left(\frac{x}{2}\right) = \lim\limits_{x \to \infty} \frac{\ln\coth\left(\frac{x}{2}\right)}{\frac{1}{\ln x}}$; the limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x \to \infty} \frac{\ln\coth\left(\frac{x}{2}\right)}{\frac{1}{\ln x}}$

$= \lim\limits_{x \to \infty} \left(\frac{\frac{\operatorname{csch}^2\left(\frac{x}{2}\right)}{\coth\left(\frac{x}{2}\right)}\left(-\frac{1}{2}\right)}{-\frac{1}{(\ln x)^2}\left(\frac{1}{x}\right)}\right) = \lim\limits_{x \to \infty} \left(\frac{x(\ln x)^2}{2\sinh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right)}\right) = \lim\limits_{x \to \infty} \left(\frac{x(\ln x)^2}{\sinh x}\right) = \lim\limits_{x \to \infty} \left(\frac{2x(\ln x)\left(\frac{1}{x}\right) + (\ln x)^2}{\cosh x}\right)$

$= \lim\limits_{x \to \infty} \left(\frac{2\ln x + (\ln x)^2}{\cosh x}\right) = \lim\limits_{x \to \infty} \left(\frac{2\left(\frac{1}{x}\right) + 2(\ln x)\left(\frac{1}{x}\right)}{\sinh x}\right) = \lim\limits_{x \to \infty} \left(\frac{2 + 2\ln x}{x\sinh x}\right) = \lim\limits_{x \to \infty} \left(\frac{\frac{2}{x}}{x\cosh x + \sinh x}\right)$

$= \lim\limits_{x \to \infty} \left(\frac{2}{x^2\cosh x + x\sinh x}\right) = 0 \Rightarrow \lim\limits_{x \to \infty} \left(\frac{e^x + 1}{e^x - 1}\right)^{\ln x} = \lim\limits_{x \to \infty} e^{\ln f(x)} = e^0 = 1$

108. Let $f(x) = \left(1 + \frac{3}{x}\right)^x \Rightarrow \ln f(x) = x\ln\left(1 + \frac{3}{x}\right) \Rightarrow \lim\limits_{x \to 0^+} \ln f(x) = \lim\limits_{x \to 0^+} \frac{\ln(1 + 3x^{-1})}{x^{-1}}$; the limit leads to the indeterminate form $\frac{\infty}{\infty}$: $\lim\limits_{x \to 0^+} \frac{\left(\frac{-3x^{-2}}{1 + 3x^{-1}}\right)}{-x^{-2}} = \lim\limits_{x \to 0^+} \frac{3x}{x + 3} = 0 \Rightarrow \lim\limits_{x \to 0^+} \left(1 + \frac{3}{x}\right)^x = \lim\limits_{x \to 0^+} e^{\ln f(x)} = e^0 = 1$

109. (a) $\lim\limits_{x \to \infty} \frac{\log_2 x}{\log_3 x} = \lim\limits_{x \to \infty} \frac{\left(\frac{\ln x}{\ln 2}\right)}{\left(\frac{\ln x}{\ln 3}\right)} = \lim\limits_{x \to \infty} \frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2} \Rightarrow$ same rate

(b) $\lim\limits_{x \to \infty} \frac{x}{x + \left(\frac{1}{x}\right)} = \lim\limits_{x \to \infty} \frac{x^2}{x^2 + 1} = \lim\limits_{x \to \infty} \frac{2x}{2x} = \lim\limits_{x \to \infty} 1 = 1 \Rightarrow$ same rate

(c) $\lim\limits_{x \to \infty} \frac{\left(\frac{x}{100}\right)}{xe^{-x}} = \lim\limits_{x \to \infty} \frac{xe^x}{100x} = \lim\limits_{x \to \infty} \frac{e^x}{100} = \infty \Rightarrow$ faster

(d) $\lim\limits_{x \to \infty} \frac{x}{\tan^{-1} x} = \infty \Rightarrow$ faster

(e) $\lim\limits_{x \to \infty} \frac{\csc^{-1} x}{\left(\frac{1}{x}\right)} = \lim\limits_{x \to \infty} \frac{\sin^{-1}(x^{-1})}{x^{-1}} = \lim\limits_{x \to \infty} \frac{\frac{\left(-x^{-2}\right)}{\sqrt{1 - \left(x^{-1}\right)^2}}}{-x^{-2}} = \lim\limits_{x \to \infty} \frac{1}{\sqrt{1 - \left(\frac{1}{x^2}\right)}} = 1 \Rightarrow$ same rate

(f) $\lim_{x\to\infty} \frac{\sinh x}{e^x} = \lim_{x\to\infty} \frac{(e^x - e^{-x})}{2e^x} = \lim_{x\to\infty} \frac{1-e^{-2x}}{2} = \frac{1}{2} \Rightarrow$ same rate

110. (a) $\lim_{x\to\infty} \frac{3^{-x}}{2^{-x}} = \lim_{x\to\infty} \left(\frac{2}{3}\right)^x = 0 \Rightarrow$ slower

(b) $\lim_{x\to\infty} \frac{\ln 2x}{\ln x^2} = \lim_{x\to\infty} \frac{\ln 2 + \ln x}{2(\ln x)} = \lim_{x\to\infty} \left(\frac{\ln 2}{2\ln x} + \frac{1}{2}\right) = \frac{1}{2} \Rightarrow$ same rate

(c) $\lim_{x\to\infty} \frac{10x^3+2x^2}{e^x} = \lim_{x\to\infty} \frac{30x^2+4x}{e^x} = \lim_{x\to\infty} \frac{60x+4}{e^x} = \lim_{x\to\infty} \frac{60}{e^x} = 0 \Rightarrow$ slower

(d) $\lim_{x\to\infty} \frac{\tan^{-1}\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{\tan^{-1}(x^{-1})}{x^{-1}} = \lim_{x\to\infty} \frac{\left(\frac{-x^{-2}}{1+x^{-2}}\right)}{-x^{-2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = 1 \Rightarrow$ same rate

(e) $\lim_{x\to\infty} \frac{\sin^{-1}\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{\sin^{-1}(x^{-1})}{x^{-2}} = \lim_{x\to\infty} \frac{\left(\frac{-x^{-2}}{\sqrt{1-(x^{-1})^2}}\right)}{-2x^{-3}} = \lim_{x\to\infty} \frac{x}{2\sqrt{1-\frac{1}{x^2}}} = \infty \Rightarrow$ faster

(f) $\lim_{x\to\infty} \frac{\operatorname{sech} x}{e^{-x}} = \lim_{x\to\infty} \frac{\left(\frac{2}{e^x+e^{-x}}\right)}{e^{-x}} = \lim_{x\to\infty} \frac{2}{e^{-x}(e^x+e^{-x})} = \lim_{x\to\infty} \left(\frac{2}{1+e^{-2x}}\right) = 2 \Rightarrow$ same rate

111. (a) $\frac{\left(\frac{1}{x^2}+\frac{1}{x^4}\right)}{\left(\frac{1}{x^2}\right)} = 1 + \frac{1}{x^2} \le 2$ for x sufficiently large $\Rightarrow$ true

(b) $\frac{\left(\frac{1}{x^2}+\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)} = x^2 + 1 > M$ for any positive integer M whenever $x > \sqrt{M} \Rightarrow$ false

(c) $\lim_{x\to\infty} \frac{x}{x+\ln x} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1 \Rightarrow$ the same growth rate $\Rightarrow$ false

(d) $\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x\to\infty} \frac{\left[\frac{\left(\frac{1}{x}\right)}{\ln x}\right]}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{1}{\ln x} = 0 \Rightarrow$ grows slower $\Rightarrow$ true

(e) $\frac{\tan^{-1} x}{1} \le \frac{\pi}{2}$ for all x $\Rightarrow$ true

(f) $\frac{\cosh x}{e^x} = \frac{1}{2}\left(1+e^{-2x}\right) \le \frac{1}{2}(1+1) = 1$ if $x > 0 \Rightarrow$ true

112. (a) $\frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^2}+\frac{1}{x^4}\right)} = \frac{1}{x^2+1} \le 1$ if $x > 0 \Rightarrow$ true

(b) $\lim_{x\to\infty} \frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^2}+\frac{1}{x^4}\right)} = \lim_{x\to\infty} \left(\frac{1}{x^2+1}\right) = 0 \Rightarrow$ true

(c) $\lim_{x\to\infty} \frac{\ln x}{x+1} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} = 0 \Rightarrow$ true

(d) $\frac{\ln 2x}{\ln x} = \frac{\ln 2}{\ln x} + 1 \le 1 + 1 = 2$ if $x \ge 2 \Rightarrow$ true

(e) $\frac{\sec^{-1} x}{1} = \frac{\cos^{-1}\left(\frac{1}{x}\right)}{1} \le \frac{\left(\frac{\pi}{2}\right)}{1} = \frac{\pi}{2}$ if $x > 1 \Rightarrow$ true

(f) $\frac{\sinh x}{e^x} = \frac{1}{2}\left(1-e^{-2x}\right) \le \frac{1}{2}$ if $x > 0 \Rightarrow$ true

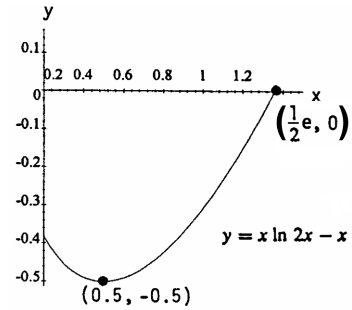
113. $\frac{df}{dx} = e^x + 1 \Rightarrow \left(\frac{df^{-1}}{dx}\right)_{x=f(\ln 2)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=\ln 2}} \Rightarrow \left(\frac{df^{-1}}{dx}\right)_{x=f(\ln 2)} = \frac{1}{(e^x+1)_{x=\ln 2}} = \frac{1}{2+1} = \frac{1}{3}$

114. $y = f(x) \Rightarrow y = 1 + \frac{1}{x} \Rightarrow \frac{1}{x} = y - 1 \Rightarrow x = \frac{1}{y-1} \Rightarrow f^{-1}(x) = \frac{1}{x-1}$; $f^{-1}(f(x)) = \frac{1}{\left(1+\frac{1}{x}\right)-1} = \frac{1}{\left(\frac{1}{x}\right)} = x$ and

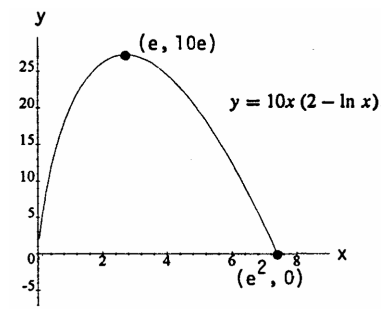
$f\left(f^{-1}(x)\right) = 1 + \frac{1}{\left(\frac{1}{x-1}\right)} = 1 + (x-1) = x$; $\left.\frac{df^{-1}}{dx}\right|_{f(x)} = \left.\frac{-1}{(x-1)^2}\right|_{f(x)} = \frac{-1}{\left[\left(1+\frac{1}{x}\right)-1\right]^2} = -x^2$;

$f'(x) = -\frac{1}{x^2} \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{f(x)} = \frac{1}{f'(x)}$

115. $y = x \ln 2x - x \Rightarrow y' = x\left(\frac{2}{2x}\right) + \ln(2x) - 1 = \ln 2x$; solving $y' = 0 \Rightarrow x = \frac{1}{2}$; $y' > 0$ for $x > \frac{1}{2}$ and $y' < 0$ for $x < \frac{1}{2} \Rightarrow$ relative minimum of $-\frac{1}{2}$ at $x = \frac{1}{2}$; $f\left(\frac{1}{2e}\right) = -\frac{1}{e}$ and $f\left(\frac{e}{2}\right) = 0 \Rightarrow$ absolute minimum is $-\frac{1}{2}$ at $x = \frac{1}{2}$ and the absolute maximum is 0 at $x = \frac{e}{2}$



116. $y = 10x(2 - \ln x) \Rightarrow y' = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right)$ $= 20 - 10 \ln x - 10 = 10(1 - \ln x)$; solving $y' = 0$ $\Rightarrow x = e$; $y' < 0$ for $x > e$ and $y' > 0$ for $x < e$ $\Rightarrow$ relative maximum at $x = e$ of $10e$; $y \ge 0$ on $(0, e^2]$ and $y\left(e^2\right) = 10e^2(2 - 2\ln e) = 0 \Rightarrow$ absolute minimum is 0 at $x = e^2$ and the absolute maximum is $10e$ at $x = e$



117. $A = \int_1^e \frac{2 \ln x}{x}\, dx = \int_0^1 2u\, du = \left[u^2\right]_0^1 = 1$, where $u = \ln x$ and $du = \frac{1}{x}\, dx$; $x = 1 \Rightarrow u = 0$, $x = e \Rightarrow u = 1$

118. (a) $A_1 = \int_{10}^{20} \frac{1}{x}\, dx = [\ln|x|]_{10}^{20} = \ln 20 - \ln 10 = \ln \frac{20}{10} = \ln 2$, and $A_2 = \int_1^2 \frac{1}{x}\, dx = [\ln|x|]_1^2 = \ln 2 - \ln 1 = \ln 2$

(b) $A_1 = \int_{ka}^{kb} \frac{1}{x}\, dx = [\ln|x|]_{ka}^{kb} = \ln kb - \ln ka = \ln \frac{kb}{ka} = \ln \frac{b}{a} = \ln b - \ln a$, and $A_2 = \int_a^b \frac{1}{x}\, dx = [\ln|x|]_a^b = \ln b - \ln a$

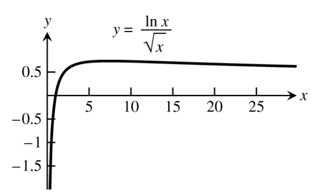
119. $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$; $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \left(\frac{1}{x}\right)\sqrt{x} = \frac{1}{\sqrt{x}} \Rightarrow \left.\frac{dy}{dt}\right|_{e^2} = \frac{1}{e}$ m/sec

120. $y = 9e^{-x/3} \Rightarrow \frac{dy}{dx} = -3e^{-x/3}$; $\frac{dx}{dt} = \frac{(dy/dt)}{(dy/dx)} \Rightarrow \frac{dx}{dt} = \frac{\left(-\frac{1}{4}\right)\sqrt{9-y}}{-3e^{-x/3}}$; $x = 9 \Rightarrow y = 9e^{-3}$ $\Rightarrow \left.\frac{dx}{dt}\right|_{x=9} = \frac{\left(-\frac{1}{4}\right)\sqrt{9 - \frac{9}{e^3}}}{\left(-\frac{3}{e^3}\right)} = \frac{1}{4}\sqrt{e^3}\sqrt{e^3 - 1} \approx 5$ ft/sec

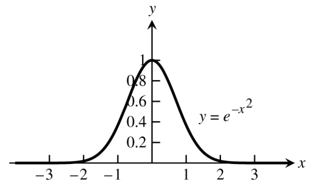
121. $A = xy = xe^{-x^2} \Rightarrow \frac{dA}{dx} = e^{-x^2} + (x)(-2x)e^{-x^2} = e^{-x^2}\left(1 - 2x^2\right)$. Solving $\frac{dA}{dx} = 0 \Rightarrow 1 - 2x^2 = 0$ $\Rightarrow x = \frac{1}{\sqrt{2}}$; $\frac{dA}{dx} < 0$ for $x > \frac{1}{\sqrt{2}}$ and $\frac{dA}{dx} > 0$ for $0 < x < \frac{1}{\sqrt{2}} \Rightarrow$ absolute maximum of $\frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{\sqrt{2e}}$ at $x = \frac{1}{\sqrt{2}}$ units long by $y = e^{-1/2} = \frac{1}{\sqrt{e}}$ units high.

122. $A = xy = x\left(\frac{\ln x}{x^2}\right) = \frac{\ln x}{x} \Rightarrow \frac{dA}{dx} = \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1 - \ln x}{x^2}$. Solving $\frac{dA}{dx} = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow x = e$; $\frac{dA}{dx} < 0$ for $x > e$ and $\frac{dA}{dx} > 0$ for $x < e \Rightarrow$ absolute maximum of $\frac{\ln e}{e} = \frac{1}{e}$ at $x = e$ units long and $y = \frac{1}{e^2}$ units high.

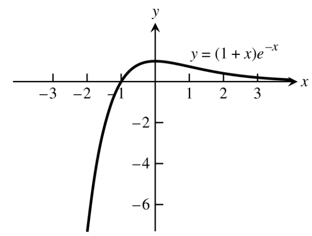
123. (a) $y = \frac{\ln x}{\sqrt{x}} \Rightarrow y' = \frac{1}{x\sqrt{x}} - \frac{\ln x}{2x^{3/2}} = \frac{2 - \ln x}{2x\sqrt{x}}$ $\Rightarrow y'' = -\frac{3}{4}x^{-5/2}(2 - \ln x) - \frac{1}{2}x^{-5/2} = x^{-5/2}\left(\frac{3}{4}\ln x - 2\right)$; solving $y' = 0 \Rightarrow \ln x = 2 \Rightarrow x = e^2$; $y' < 0$ for $x > e^2$ and and $y' > 0$ for $x < e^2 \Rightarrow$ a maximum of $\frac{2}{e}$; $y'' = 0$ $\Rightarrow \ln x = \frac{8}{3} \Rightarrow x = e^{8/3}$; the curve is concave down on $\left(0, e^{8/3}\right)$ and concave up on $\left(e^{8/3}, \infty\right)$; so there is an inflection point at $\left(e^{8/3}, \frac{8}{3e^{4/3}}\right)$.



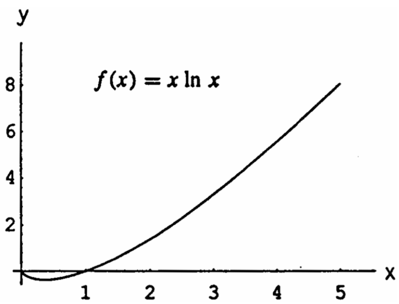
(b) $y = e^{-x^2} \Rightarrow y' = -2xe^{-x^2} \Rightarrow y'' = -2e^{-x^2} + 4x^2e^{-x^2} = (4x^2 - 2)e^{-x^2}$; solving $y' = 0 \Rightarrow x = 0$; $y' < 0$ for $x > 0$ and $y' > 0$ for $x < 0 \Rightarrow$ a maximum at $x = 0$ of $e^0 = 1$; there are points of inflection at $x = \pm\frac{1}{\sqrt{2}}$; the curve is concave down for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and concave up otherwise.



(c) $y = (1 + x)e^{-x} \Rightarrow y' = e^{-x} - (1 + x)e^{-x} = -xe^{-x} \Rightarrow y'' = -e^{-x} + xe^{-x} = (x - 1)e^{-x}$; solving $y' = 0 \Rightarrow -xe^{-x} = 0 \Rightarrow x = 0$; $y' < 0$ for $x > 0$ and $y' > 0$ for $x < 0 \Rightarrow$ a maximum at $x = 0$ of $(1 + 0)e^0 = 1$; there is a point of inflection at $x = 1$ and the curve is concave up for $x > 1$ and concave down for $x < 1$.



124. $y = x \ln x \Rightarrow y' = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$; solving $y' = 0 \Rightarrow \ln x + 1 = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1}$; $y' > 0$ for $x > e^{-1}$ and $y' < 0$ for $x < e^{-1} \Rightarrow$ a minimum of $e^{-1} \ln e^{-1} = -\frac{1}{e}$ at $x = e^{-1}$. This minimum is an absolute minimum since $y'' = \frac{1}{x}$ is positive for all $x > 0$.



125. $\frac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y} \Rightarrow \frac{dy}{\sqrt{y}\cos^2\sqrt{y}} = dx \Rightarrow 2\tan\sqrt{y} = x + C \Rightarrow y = \left(\tan^{-1}\left(\frac{x+C}{2}\right)\right)^2$

126. $y' = \frac{3y(x+1)^2}{y-1} \Rightarrow \frac{(y-1)}{y}dy = 3(x + 1)^2dx \Rightarrow y - \ln y = (x + 1)^3 + C$

127. $yy' = \sec(y^2)\sec^2 x \Rightarrow \frac{y\,dy}{\sec(y^2)} = \sec^2 x\,dx \Rightarrow \frac{\sin(y^2)}{2} = \tan x + C \Rightarrow \sin(y^2) = 2\tan x + C_1$

128. $y\cos^2(x)\,dy + \sin x\,dx = 0 \Rightarrow y\,dy = -\frac{\sin x}{\cos^2(x)}dx \Rightarrow \frac{y^2}{2} = -\frac{1}{\cos(x)} + C \Rightarrow y = \pm\sqrt{\frac{-2}{\cos(x)} + C_1}$

129. $\frac{dy}{dx} = e^{-x-y-2} \Rightarrow e^y dy = e^{-(x+2)}dx \Rightarrow e^y = -e^{-(x+2)} + C$. We have $y(0) = -2$, so $e^{-2} = -e^{-2} + C \Rightarrow C = 2e^{-2}$ and $e^y = -e^{-(x+2)} + 2e^{-2} \Rightarrow y = \ln\left(-e^{-(x+2)} + 2e^{-2}\right)$

130. $\frac{dy}{dx} = \frac{y \ln y}{1 + x^2} \Rightarrow \frac{dy}{y \ln y} = \frac{dx}{1 + x^2} \Rightarrow \ln(\ln y) = \tan^{-1}(x) + C \Rightarrow y = e^{e^{\tan^{-1}(x)+C}}$. We have $y(0) = e^2 \Rightarrow e^2 = e^{e^{\tan^{-1}(0)+C}}$ $\Rightarrow e^{\tan^{-1}(0)+C} = 2 \Rightarrow \tan^{-1}(0) + C = \ln 2 \Rightarrow 0 + C = \ln 2 \Rightarrow C = \ln 2 \Rightarrow y = e^{e^{\tan^{-1}(x)+\ln 2}}$

131. $x\,dy - \left(y + \sqrt{y}\right)dx = 0 \Rightarrow \frac{dy}{(y+\sqrt{y})} = \frac{dx}{x} \Rightarrow 2\ln\left(\sqrt{y} + 1\right) = \ln x + C$. We have $y(1) = 1 \Rightarrow 2\ln\left(\sqrt{1} + 1\right) = \ln 1 + C$ $\Rightarrow 2\ln 2 = C = \ln 2^2 = \ln 4$. So $2\ln\left(\sqrt{y} + 1\right) = \ln x + \ln 4 = \ln(4x) \Rightarrow \ln\left(\sqrt{y} + 1\right) = \frac{1}{2}\ln(4x) = \ln(4x)^{1/2}$ $\Rightarrow e^{\ln\left(\sqrt{y}+1\right)} = e^{\ln(4x)^{1/2}} \Rightarrow \sqrt{y} + 1 = 2\sqrt{x} \Rightarrow y = \left(2\sqrt{x} - 1\right)^2$

132. $y^{-2}\frac{dx}{dy} = \frac{e^x}{e^{2x} + 1} \Rightarrow \frac{e^{2x} + 1}{e^x}dx = \frac{dy}{y^{-2}} \Rightarrow \frac{y^3}{3} = e^x - e^{-x} + C$. We have $y(0) = 1 \Rightarrow \frac{(1)^3}{3} = e^0 - e^0 + C \Rightarrow C = \frac{1}{3}$. So $\frac{y^3}{3} = e^x - e^{-x} + \frac{1}{3} \Rightarrow y^3 = 3(e^x - e^{-x}) + 1 \Rightarrow y = [3(e^x - e^{-x}) + 1\,]^{1/3}$

133. Since the half life is 5700 years and $A(t) = A_0e^{kt}$ we have $\frac{A_0}{2} = A_0e^{5700k} \Rightarrow \frac{1}{2} = e^{5700k} \Rightarrow \ln(0.5) = 5700k$ $\Rightarrow k = \frac{\ln(0.5)}{5700}$. With 10% of the original carbon-14 remaining we have $0.1A_0 = A_0e^{\frac{\ln(0.5)}{5700}t} \Rightarrow 0.1 = e^{\frac{\ln(0.5)}{5700}t}$ $\Rightarrow \ln(0.1) = \frac{\ln(0.5)}{5700}t \Rightarrow t = \frac{(5700)\ln(0.1)}{\ln(0.5)} \approx 18{,}935$ years (rounded to the nearest year).

134. $T - T_s = (T_o - T_s)e^{-kt} \Rightarrow 180 - 40 = (220 - 40)e^{-k/4}$, time in hours, $\Rightarrow k = -4\ln\left(\frac{7}{9}\right) = 4\ln\left(\frac{9}{7}\right) \Rightarrow 70 - 40$ $= (220 - 40)e^{-4\ln(9/7)t} \Rightarrow t = \frac{\ln 6}{4\ln\left(\frac{9}{7}\right)} \approx 1.78$ hr ≈ 107 min, the total time $\Rightarrow$ the time it took to cool from 180° F to 70° F was $107 - 15 = 92$ min

135. $\theta = \pi - \cot^{-1}\left(\frac{x}{60}\right) - \cot^{-1}\left(\frac{5}{3} - \frac{x}{30}\right), 0 < x < 50 \Rightarrow \frac{d\theta}{dx} = \frac{\left(\frac{1}{60}\right)}{1+\left(\frac{x}{60}\right)^2} + \frac{\left(-\frac{1}{30}\right)}{1+\left(\frac{50-x}{30}\right)^2}$ $= 30\left[\frac{2}{60^2+x^2} - \frac{1}{30^2+(50-x)^2}\right]$; solving $\frac{d\theta}{dx} = 0 \Rightarrow x^2 - 200x + 3200 = 0 \Rightarrow x = 100 \pm 20\sqrt{17}$, but $100 + 20\sqrt{17}$ is not in the domain; $\frac{d\theta}{dx} > 0$ for $x < 20\left(5 - \sqrt{17}\right)$ and $\frac{d\theta}{dx} < 0$ for $20\left(5 - \sqrt{17}\right) < x < 50$ $\Rightarrow x = 20\left(5 - \sqrt{17}\right) \approx 17.54$ m maximizes θ

136. $v = x^2\ln\left(\frac{1}{x}\right) = x^2(\ln 1 - \ln x) = -x^2\ln x \Rightarrow \frac{dv}{dx} = -2x\ln x - x^2\left(\frac{1}{x}\right) = -x(2\ln x + 1)$; solving $\frac{dv}{dx} = 0$ $\Rightarrow 2\ln x + 1 = 0 \Rightarrow \ln x = -\frac{1}{2} \Rightarrow x = e^{-1/2}$; $\frac{dv}{dx} < 0$ for $x > e^{-1/2}$ and $\frac{dv}{dx} > 0$ for $x < e^{-1/2} \Rightarrow$ a relative maximum at $x = e^{-1/2}$; $\frac{r}{h} = x$ and $r = 1 \Rightarrow h = e^{1/2} = \sqrt{e} \approx 1.65$ cm

CHAPTER 7 ADDITIONAL AND ADVANCED EXERCISES

1. $\lim\limits_{b \to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim\limits_{b \to 1^-} \left[\sin^{-1} x\right]_0^b = \lim\limits_{b \to 1^-} \left(\sin^{-1} b - \sin^{-1} 0\right) = \lim\limits_{b \to 1^-} \left(\sin^{-1} b - 0\right) = \lim\limits_{b \to 1^-} \sin^{-1} b = \frac{\pi}{2}$

2. $\lim\limits_{x \to \infty} \frac{1}{x}\int_0^x \tan^{-1} t\,dt = \lim\limits_{x \to \infty} \frac{\int_0^x \tan^{-1} t\,dt}{x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$
$= \lim\limits_{x \to \infty} \frac{\tan^{-1} x}{1} = \frac{\pi}{2}$

3. $y = \left(\cos\sqrt{x}\right)^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln\left(\cos\sqrt{x}\right)$ and $\lim\limits_{x \to 0^+} \frac{\ln\left(\cos\sqrt{x}\right)}{x} = \lim\limits_{x \to 0^+} \frac{-\sin\sqrt{x}}{2\sqrt{x}\cos\sqrt{x}} = \frac{-1}{2}\lim\limits_{x \to 0^+} \frac{\tan\sqrt{x}}{\sqrt{x}}$ $= -\frac{1}{2}\lim\limits_{x \to 0^+} \frac{\frac{1}{2}x^{-1/2}\sec^2\sqrt{x}}{\frac{1}{2}x^{-1/2}} = -\frac{1}{2} \Rightarrow \lim\limits_{x \to 0^+} \left(\cos\sqrt{x}\right)^{1/x} = e^{-1/2} = \frac{1}{\sqrt{e}}$

4. $y = (x + e^x)^{2/x} \Rightarrow \ln y = \frac{2\ln(x+e^x)}{x} \Rightarrow \lim\limits_{x \to \infty} \ln y = \lim\limits_{x \to \infty} \frac{2(1+e^x)}{x+e^x} = \lim\limits_{x \to \infty} \frac{2e^x}{1+e^x} = \lim\limits_{x \to \infty} \frac{2e^x}{e^x} = 2$ $\Rightarrow \lim\limits_{x \to \infty} (x + e^x)^{2/x} = \lim\limits_{x \to \infty} e^y = e^2$

5. $\lim\limits_{x \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right) = \lim\limits_{x \to \infty} \left(\left(\frac{1}{n}\right)\left[\frac{1}{1+\left(\frac{1}{n}\right)}\right] + \left(\frac{1}{n}\right)\left[\frac{1}{1+2\left(\frac{1}{n}\right)}\right] + \ldots + \left(\frac{1}{n}\right)\left[\frac{1}{1+n\left(\frac{1}{n}\right)}\right]\right)$ which can be interpreted as a Riemann sum with partitioning $\Delta x = \frac{1}{n} \Rightarrow \lim\limits_{x \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right)$ $= \int_0^1 \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_0^1 = \ln 2$

6. $\lim\limits_{x \to \infty} \frac{1}{n}\left[e^{1/n} + e^{2/n} + \ldots + e\right] = \lim\limits_{x \to \infty} \left[\left(\frac{1}{n}\right)e^{(1/n)} + \left(\frac{1}{n}\right)e^{2(1/n)} + \ldots + \left(\frac{1}{n}\right)e^{n(1/n)}\right]$ which can be interpreted as a Riemann sum with partitioning $\Delta x = \frac{1}{n} \Rightarrow \lim\limits_{x \to \infty} \frac{1}{n}\left[e^{1/n} + e^{2/n} + \ldots + e\right] = \int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e - 1$

7. $A(t) = \int_0^t e^{-x}\,dx = \left[-e^{-x}\right]_0^t = 1 - e^{-t}$, $V(t) = \pi\int_0^t e^{-2x}\,dx = \left[-\frac{\pi}{2}e^{-2x}\right]_0^t = \frac{\pi}{2}\left(1 - e^{-2t}\right)$

(a) $\lim\limits_{t \to \infty} A(t) = \lim\limits_{t \to \infty} \left(1 - e^{-t}\right) = 1$

(b) $\lim\limits_{t \to \infty} \frac{V(t)}{A(t)} = \lim\limits_{t \to \infty} \frac{\frac{\pi}{2}\left(1 - e^{-2t}\right)}{1 - e^{-t}} = \frac{\pi}{2}$

(c) $\lim\limits_{t \to 0^+} \frac{V(t)}{A(t)} = \lim\limits_{t \to 0^+} \frac{\frac{\pi}{2}\left(1 - e^{-2t}\right)}{1 - e^{-t}} = \lim\limits_{t \to 0^+} \frac{\frac{\pi}{2}\left(1 - e^{-t}\right)\left(1 + e^{-t}\right)}{\left(1 - e^{-t}\right)} = \lim\limits_{t \to 0^+} \frac{\pi}{2}\left(1 + e^{-t}\right) = \pi$

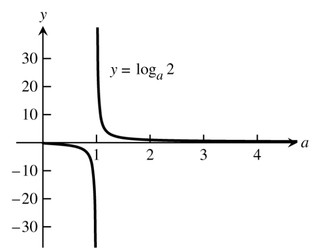
8. (a) $\lim\limits_{a \to 0^+} \log_a 2 = \lim\limits_{a \to 0^+} \frac{\ln 2}{\ln a} = 0$;

$\lim\limits_{a \to 1^-} \log_a 2 = \lim\limits_{a \to 1^-} \frac{\ln 2}{\ln a} = -\infty$;

$\lim\limits_{a \to 1^+} \log_a 2 = \lim\limits_{a \to 1^+} \frac{\ln 2}{\ln 1} = \infty$;

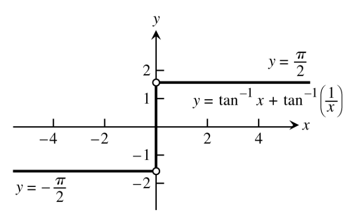
$\lim\limits_{a \to \infty} \log_a 2 = \lim\limits_{a \to \infty} \frac{\ln 2}{\ln a} = 0$

(b)



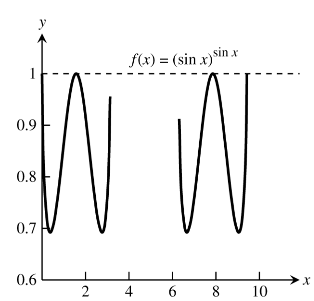
9. $A_1 = \int_1^e \frac{2\log_2 x}{x}\,dx = \frac{2}{\ln 2}\int_1^e \frac{\ln x}{x}\,dx = \left[\frac{(\ln x)^2}{\ln 2}\right]_1^e = \frac{1}{\ln 2}$; $A_2 = \int_1^e \frac{2\log_4 x}{4}\,dx = \frac{2}{\ln 4}\int_1^e \frac{\ln x}{x}\,dx$
$= \left[\frac{(\ln x)^2}{2\ln 2}\right]_1^e = \frac{1}{2\ln 2} \Rightarrow A_1 : A_2 = 2 : 1$

10. $y = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) \Rightarrow y' = \frac{1}{1+x^2} + \frac{\left(-\frac{1}{x^2}\right)}{\left(1+\frac{1}{x^2}\right)}$
$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \Rightarrow \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ is a constant and the constant is $\frac{\pi}{2}$ for $x > 0$; it is $-\frac{\pi}{2}$ for $x < 0$ since $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ is odd. Next the
$\lim\limits_{x \to 0^+} \left[\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right] = 0 + \frac{\pi}{2} = \frac{\pi}{2}$
and $\lim\limits_{x \to 0^-} \left(\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right) = 0 + \left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}$



11. $\ln x^{(x^x)} = x^x \ln x$ and $\ln\left(x^x\right)^x = x \ln x^x = x^2 \ln x$; then, $x^x \ln x = x^2 \ln x \Rightarrow \left(x^x - x^2\right)\ln x = 0 \Rightarrow x^x = x^2$ or $\ln x = 0$.
$\ln x = 0 \Rightarrow x = 1$; $x^x = x^2 \Rightarrow x \ln x = 2\ln x \Rightarrow x = 2$. Therefore, $x^{(x^x)} = \left(x^x\right)^x$ when $x = 2$ or $x = 1$.

12. In the interval $\pi < x < 2\pi$ the function $\sin x < 0$ $\Rightarrow (\sin x)^{\sin x}$ is not defined for all values in that interval or its translation by 2π.



13. $f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)}\,g'(x)$, where $g'(x) = \frac{x}{1+x^4} \Rightarrow f'(2) = e^0\left(\frac{2}{1+16}\right) = \frac{2}{17}$

14. (a) $\frac{df}{dx} = \frac{2\ln e^x}{e^x} \cdot e^x = 2x$ (b) $f(0) = \int_1^1 \frac{2\ln t}{t}\,dt = 0$

(c) $\frac{df}{dx} = 2x \Rightarrow f(x) = x^2 + C$; $f(0) = 0 \Rightarrow C = 0 \Rightarrow f(x) = x^2 \Rightarrow$ the graph of $f(x)$ is a parabola

15. (a) $g(x) + h(x) = 0 \Rightarrow g(x) = -h(x)$; also $g(x) + h(x) = 0 \Rightarrow g(-x) + h(-x) = 0 \Rightarrow g(x) - h(x) = 0$
$\Rightarrow g(x) = h(x)$; therefore $-h(x) = h(x) \Rightarrow h(x) = 0 \Rightarrow g(x) = 0$

(b) $\frac{f(x)+f(-x)}{2} = \frac{[f_E(x)+f_O(x)]+[f_E(-x)+f_O(-x)]}{2} = \frac{f_E(x)+f_O(x)+f_E(x)-f_O(x)}{2} = f_E(x)$;
$\frac{f(x)-f(-x)}{2} = \frac{[f_E(x)+f_O(x)]-[f_E(-x)+f_O(-x)]}{2} = \frac{f_E(x)+f_O(x)-f_E(x)+f_O(x)}{2} = f_O(x)$

(c) Part b $\Rightarrow$ such a decomposition is unique.

16. (a) $g(0+0) = \frac{g(0)+g(0)}{1-g(0)\,g(0)} \Rightarrow [1-g^2(0)]\,g(0) = 2g(0) \Rightarrow g(0)-g^3(0) = 2g(0) \Rightarrow g^3(0)+g(0) = 0$
$\Rightarrow g(0)\,[g^2(0)+1] = 0 \Rightarrow g(0) = 0$

(b) $g'(x) = \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h \to 0} \frac{\left[\frac{g(x)+g(h)}{1-g(x)\,g(h)}\right]-g(x)}{h} = \lim_{h \to 0} \frac{g(x)+g(h)-g(x)+g^2(x)\,g(h)}{h\,[1-g(x)\,g(h)]}$
$= \lim_{h \to 0} \left[\frac{g(h)}{h}\right]\left[\frac{1+g^2(x)}{1-g(x)\,g(h)}\right] = 1 \cdot [1+g^2(x)] = 1+g^2(x) = 1+[g(x)]^2$

(c) $\frac{dy}{dx} = 1+y^2 \Rightarrow \frac{dy}{1+y^2} = dx \Rightarrow \tan^{-1} y = x+C \Rightarrow \tan^{-1}(g(x)) = x+C$; $g(0) = 0 \Rightarrow \tan^{-1} 0 = 0+C$
$\Rightarrow C = 0 \Rightarrow \tan^{-1}(g(x)) = x \Rightarrow g(x) = \tan x$

17. $M = \int_0^1 \frac{2}{1+x^2}\,dx = 2\left[\tan^{-1} x\right]_0^1 = \frac{\pi}{2}$ and $M_y = \int_0^1 \frac{2x}{1+x^2}\,dx = \left[\ln(1+x^2)\right]_0^1 = \ln 2 \Rightarrow \overline{x} = \frac{M_y}{M}$
$= \frac{\ln 2}{\left(\frac{\pi}{2}\right)} = \frac{\ln 4}{\pi}$; $\overline{y} = 0$ by symmetry

18. (a) $V = \pi \int_{1/4}^4 \left(\frac{1}{2\sqrt{x}}\right)^2 dx = \frac{\pi}{4}\int_{1/4}^4 \frac{1}{x}\,dx = \frac{\pi}{4}\left[\ln|x|\right]_{1/4}^4 = \frac{\pi}{4}\left(\ln 4 - \ln \frac{1}{4}\right) = \frac{\pi}{4}\ln 16 = \frac{\pi}{4}\ln(2^4) = \pi \ln 2$

(b) $M_y = \int_{1/4}^4 x\left(\frac{1}{2\sqrt{x}}\right)dx = \frac{1}{2}\int_{1/4}^4 x^{1/2}\,dx = \left[\frac{1}{3}x^{3/2}\right]_{1/4}^4 = \left(\frac{8}{3}-\frac{1}{24}\right) = \frac{64-1}{24} = \frac{63}{24}$;
$M_x = \int_{1/4}^4 \frac{1}{2}\left(\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{2\sqrt{x}}\right)dx = \frac{1}{8}\int_{1/4}^4 \frac{1}{x}\,dx = \left[\frac{1}{8}\ln|x|\right]_{1/4}^4 = \frac{1}{8}\ln 16 = \frac{1}{2}\ln 2$;
$M = \int_{1/4}^4 \frac{1}{2\sqrt{x}}\,dx = \int_{1/4}^4 \frac{1}{2}x^{-1/2}\,dx = \left[x^{1/2}\right]_{1/4}^4 = 2-\frac{1}{2} = \frac{3}{2}$; therefore, $\overline{x} = \frac{M_y}{M} = \left(\frac{63}{24}\right)\left(\frac{2}{3}\right) = \frac{21}{12} = \frac{7}{4}$ and
$\overline{y} = \frac{M_x}{M} = \left(\frac{1}{2}\ln 2\right)\left(\frac{2}{3}\right) = \frac{\ln 2}{3}$

19. (a) $L = k\left(\frac{a-b\cot\theta}{R^4}+\frac{b\csc\theta}{r^4}\right) \Rightarrow \frac{dL}{d\theta} = k\left(\frac{b\csc^2\theta}{R^4}-\frac{b\csc\theta\cot\theta}{r^4}\right)$; solving $\frac{dL}{d\theta} = 0$
$\Rightarrow r^4 b\csc^2\theta - bR^4\csc\theta\cot\theta = 0 \Rightarrow (b\csc\theta)(r^4\csc\theta - R^4\cot\theta) = 0$; but $b\csc\theta \neq 0$ since
$\theta \neq \frac{\pi}{2} \Rightarrow r^4\csc\theta - R^4\cot\theta = 0 \Rightarrow \cos\theta = \frac{r^4}{R^4} \Rightarrow \theta = \cos^{-1}\left(\frac{r^4}{R^4}\right)$, the critical value of θ

(b) $\theta = \cos^{-1}\left(\frac{5}{6}\right)^4 \approx \cos^{-1}(0.48225) \approx 61°$

20. In order to maximize the amount of sunlight, we need to maximize the angle θ formed by extending the two red line segments to their vertex. The angle between the two lines is given by $\theta = \pi - (\theta_1 + (\pi - \theta_2))$. From trig we have $\tan\theta_1 = \frac{350}{450-x} \Rightarrow \theta_1 = \tan^{-1}\left(\frac{350}{450-x}\right)$ and $\tan(\pi-\theta_2) = \frac{200}{x} \Rightarrow (\pi-\theta_2) = \tan^{-1}\left(\frac{200}{x}\right)$
$\Rightarrow \theta = \pi - (\theta_1 + (\pi-\theta_2)) = \pi - \tan^{-1}\left(\frac{350}{450-x}\right) - \tan^{-1}\left(\frac{200}{x}\right)$
$\Rightarrow \frac{d\theta}{dx} = -\frac{1}{1+\left(\frac{350}{450-x}\right)^2}\cdot\frac{350}{(450-x)^2} - \frac{1}{1+\left(\frac{200}{x}\right)^2}\cdot\left(-\frac{200}{x^2}\right) = \frac{-350}{(450-x)^2+122500} + \frac{200}{x^2+40000}$
$\frac{d\theta}{dx} = 0 \Rightarrow \frac{-350}{(450-x)^2+122500} + \frac{200}{x^2+40000} = 0 \Rightarrow 200\Big((450-x)^2+122500\Big) = 350(x^2+40000)$
$\Rightarrow 3x^2 + 3600x - 1020000 = 0 \Rightarrow x = -600 \pm 100\sqrt{70}$. Since $x > 0$, consider only $x = -600+100\sqrt{70}$.
Using the first derivative test, $\left.\frac{d\theta}{dx}\right|_{x=100} = \frac{9}{3500} > 0$ and $\left.\frac{d\theta}{dx}\right|_{x=400} = \frac{-9}{5000} < 0 \Rightarrow$ local max when
$x = -600+100\sqrt{70} \approx 236.67$ ft.

CHAPTER 8 TECHNIQUES OF INTEGRATION

8.1 INTEGRATION BY PARTS

1. $u = x$, $du = dx$; $dv = \sin\frac{x}{2}\,dx$, $v = -2\cos\frac{x}{2}$;
$\int x\sin\frac{x}{2}\,dx = -2x\cos\frac{x}{2} - \int\left(-2\cos\frac{x}{2}\right)dx = -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C$

2. $u = \theta$, $du = d\theta$; $dv = \cos\pi\theta\,d\theta$, $v = \frac{1}{\pi}\sin\pi\theta$;
$\int\theta\cos\pi\theta\,d\theta = \frac{\theta}{\pi}\sin\pi\theta - \int\frac{1}{\pi}\sin\pi\theta\,d\theta = \frac{\theta}{\pi}\sin\pi\theta + \frac{1}{\pi^2}\cos\pi\theta + C$

3.
		$\cos t$
t^2	$\xrightarrow{(+)}$	$\sin t$
$2t$	$\xrightarrow{(-)}$	$-\cos t$
2	$\xrightarrow{(+)}$	$-\sin t$
0		

$\int t^2\cos t\,dt = t^2\sin t + 2t\cos t - 2\sin t + C$

4.
		$\sin x$
x^2	$\xrightarrow{(+)}$	$-\cos x$
$2x$	$\xrightarrow{(-)}$	$-\sin x$
2	$\xrightarrow{(+)}$	$\cos x$
0		

$\int x^2\sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$

5. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x\,dx$, $v = \frac{x^2}{2}$;
$\int_1^2 x\ln x\,dx = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2\frac{x^2}{2}\,\frac{dx}{x} = 2\ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2\ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$

6. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x^3\,dx$, $v = \frac{x^4}{4}$;
$\int_1^e x^3\ln x\,dx = \left[\frac{x^4}{4}\ln x\right]_1^e - \int_1^e\frac{x^4}{4}\,\frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4+1}{16}$

7. $u = x$, $du = dx$; $dv = e^x dx$, $v = e^x$;
$\int x\,e^x dx = x\,e^x - \int e^x dx = x\,e^x - e^x + C$

8. $u = x$, $du = dx$; $dv = e^{3x}dx$, $v = \frac{1}{3}e^{3x}$;
$\int x\,e^{3x}dx = \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x}dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C$

9.

		e^{-x}
x^2	$\xrightarrow{(+)}$	$-e^{-x}$
$2x$	$\xrightarrow{(-)}$	e^{-x}
2	$\xrightarrow{(+)}$	$-e^{-x}$
0		

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} - 2x\,e^{-x} - 2\,e^{-x} + C$$

10.

		e^{2x}
$x^2 - 2x + 1$	$\xrightarrow{(+)}$	$\frac{1}{2}e^{2x}$
$2x - 2$	$\xrightarrow{(-)}$	$\frac{1}{4}e^{2x}$
2	$\xrightarrow{(+)}$	$\frac{1}{8}e^{2x}$
0		

$$\int (x^2 - 2x + 1)e^{2x}\,dx = \tfrac{1}{2}(x^2 - 2x + 1)e^{2x} - \tfrac{1}{4}(2x - 2)e^{2x} + \tfrac{1}{4}e^{2x} + C$$
$$= \left(\tfrac{1}{2}x^2 - \tfrac{3}{2}x + \tfrac{5}{4}\right)e^{2x} + C$$

11. $u = \tan^{-1} y$, $du = \frac{dy}{1+y^2}$; $dv = dy$, $v = y$;
$$\int \tan^{-1} y\,dy = y\tan^{-1} y - \int \frac{y\,dy}{(1+y^2)} = y\tan^{-1} y - \tfrac{1}{2}\ln(1+y^2) + C = y\tan^{-1} y - \ln\sqrt{1+y^2} + C$$

12. $u = \sin^{-1} y$, $du = \frac{dy}{\sqrt{1-y^2}}$; $dv = dy$, $v = y$;
$$\int \sin^{-1} y\,dy = y\sin^{-1} y - \int \frac{y\,dy}{\sqrt{1-y^2}} = y\sin^{-1} y + \sqrt{1-y^2} + C$$

13. $u = x$, $du = dx$; $dv = \sec^2 x\,dx$, $v = \tan x$;
$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C$$

14. $\int 4x\sec^2 2x\,dx$; $[y = 2x]$ $\rightarrow$ $\int y\sec^2 y\,dy = y\tan y - \int \tan y\,dy = y\tan y - \ln|\sec y| + C$
$= 2x\tan 2x - \ln|\sec 2x| + C$

15.

		e^x
x^3	$\xrightarrow{(+)}$	e^x
$3x^2$	$\xrightarrow{(-)}$	e^x
$6x$	$\xrightarrow{(+)}$	e^x
6	$\xrightarrow{(-)}$	e^x
0		

$$\int x^3e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = (x^3 - 3x^2 + 6x - 6)\,e^x + C$$

16.

		e^{-p}
p^4	$\xrightarrow{(+)}$	$-e^{-p}$
$4p^3$	$\xrightarrow{(-)}$	e^{-p}
$12p^2$	$\xrightarrow{(+)}$	$-e^{-p}$
$24p$	$\xrightarrow{(-)}$	e^{-p}
24	$\xrightarrow{(+)}$	$-e^{-p}$
0		

$$\int p^4e^{-p}\,dp = -p^4e^{-p} - 4p^3e^{-p} - 12p^2e^{-p} - 24pe^{-p} - 24e^{-p} + C$$
$$= \left(-p^4 - 4p^3 - 12p^2 - 24p - 24\right) e^{-p} + C$$

17.

		e^x
$x^2 - 5x$	$\xrightarrow{(+)}$	e^x
$2x - 5$	$\xrightarrow{(-)}$	e^x
2	$\xrightarrow{(+)}$	e^x
0		

$$\int \left(x^2 - 5x\right) e^x\,dx = \left(x^2 - 5x\right) e^x - (2x - 5)e^x + 2e^x + C = x^2e^x - 7xe^x + 7e^x + C$$
$$= \left(x^2 - 7x + 7\right) e^x + C$$

18.

		e^r
$r^2 + r + 1$	$\xrightarrow{(+)}$	e^r
$2r + 1$	$\xrightarrow{(-)}$	e^r
2	$\xrightarrow{(+)}$	e^r
0		

$$\int \left(r^2 + r + 1\right) e^r\,dr = \left(r^2 + r + 1\right) e^r - (2r + 1)\, e^r + 2e^r + C$$
$$= \left[\left(r^2 + r + 1\right) - (2r + 1) + 2\right] e^r + C = \left(r^2 - r + 2\right) e^r + C$$

19.

		e^x
x^5	$\xrightarrow{(+)}$	e^x
$5x^4$	$\xrightarrow{(-)}$	e^x
$20x^3$	$\xrightarrow{(+)}$	e^x
$60x^2$	$\xrightarrow{(-)}$	e^x
$120x$	$\xrightarrow{(+)}$	e^x
120	$\xrightarrow{(-)}$	e^x
0		

$$\int x^5e^x\,dx = x^5e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + C$$
$$= \left(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120\right) e^x + C$$

20.

		e^{4t}
t^2	$\xrightarrow{(+)}$	$\frac{1}{4}e^{4t}$
$2t$	$\xrightarrow{(-)}$	$\frac{1}{16}e^{4t}$
2	$\xrightarrow{(+)}$	$\frac{1}{64}e^{4t}$
0		

$$\int t^2e^{4t}\,dt = \frac{t^2}{4}e^{4t} - \frac{2t}{16}e^{4t} + \frac{2}{64}e^{4t} + C = \frac{t^2}{4}e^{4t} - \frac{t}{8}e^{4t} + \frac{1}{32}e^{4t} + C$$
$$= \left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32}\right)e^{4t} + C$$

21. $I = \int e^\theta \sin\theta\,d\theta$; $[u = \sin\theta,\ du = \cos\theta\,d\theta;\ dv = e^\theta\,d\theta,\ v = e^\theta] \Rightarrow I = e^\theta\sin\theta - \int e^\theta\cos\theta\,d\theta$;
$[u = \cos\theta,\ du = -\sin\theta\,d\theta;\ dv = e^\theta\,d\theta,\ v = e^\theta] \Rightarrow I = e^\theta\sin\theta - \left(e^\theta\cos\theta + \int e^\theta\sin\theta\,d\theta\right)$
$= e^\theta\sin\theta - e^\theta\cos\theta - I + C' \Rightarrow 2I = (e^\theta\sin\theta - e^\theta\cos\theta) + C' \Rightarrow I = \frac{1}{2}(e^\theta\sin\theta - e^\theta\cos\theta) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

22. $I = \int e^{-y}\cos y\,dy$; $[u = \cos y,\ du = -\sin y\,dy;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$
$\Rightarrow I = -e^{-y}\cos y - \int(-e^{-y})(-\sin y)\,dy = -e^{-y}\cos y - \int e^{-y}\sin y\,dy$; $[u = \sin y,\ du = \cos y\,dy;$
$dv = e^{-y}\,dy,\ v = -e^{-y}] \Rightarrow I = -e^{-y}\cos y - \left(-e^{-y}\sin y - \int(-e^{y})\cos y\,dy\right) = -e^{-y}\cos y + e^{-y}\sin y - I + C'$
$\Rightarrow 2I = e^{-y}(\sin y - \cos y) + C' \Rightarrow I = \frac{1}{2}(e^{-y}\sin y - e^{-y}\cos y) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

23. $I = \int e^{2x}\cos 3x\,dx$; $\left[u = \cos 3x;\ du = -3\sin 3x\,dx,\ dv = e^{2x}\,dx;\ v = \frac{1}{2}e^{2x}\right]$
$\Rightarrow I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx$; $\left[u = \sin 3x,\ du = 3\cos 3x,\ dv = e^{2x}\,dx;\ v = \frac{1}{2}e^{2x}\right]$
$\Rightarrow I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left(\frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx\right) = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}I + C'$
$\Rightarrow \frac{13}{4}I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C' \Rightarrow \frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$, where $C = \frac{4}{13}C'$

24. $\int e^{-2x}\sin 2x\,dx$; $[y = 2x] \to \frac{1}{2}\int e^{-y}\sin y\,dy = I$; $[u = \sin y,\ du = \cos y\,dy;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$
$\Rightarrow I = \frac{1}{2}\left(-e^{-y}\sin y + \int e^{-y}\cos y\,dy\right)$ $[u = \cos y,\ du = -\sin y;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$
$\Rightarrow I = -\frac{1}{2}e^{-y}\sin y + \frac{1}{2}\left(-e^{-y}\cos y - \int(-e^{-y})(-\sin y)\,dy\right) = -\frac{1}{2}e^{-y}(\sin y + \cos y) - I + C'$
$\Rightarrow 2I = -\frac{1}{2}e^{-y}(\sin y + \cos y) + C' \Rightarrow I = -\frac{1}{4}e^{-y}(\sin y + \cos y) + C = -\frac{e^{-2x}}{4}(\sin 2x + \cos 2x) + C$, where $C = \frac{C'}{2}$

25. $\int e^{\sqrt{3s+9}}\,ds$; $\begin{bmatrix} 3s + 9 = x^2 \\ ds = \frac{2}{3}x\,dx \end{bmatrix} \to \int e^x \cdot \frac{2}{3}x\,dx = \frac{2}{3}\int xe^x\,dx$; $[u = x,\ du = dx;\ dv = e^x\,dx,\ v = e^x]$;
$\frac{2}{3}\int xe^x\,dx = \frac{2}{3}\left(xe^x - \int e^x\,dx\right) = \frac{2}{3}(xe^x - e^x) + C = \frac{2}{3}\left(\sqrt{3s+9}\,e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$

26. $u = x,\ du = dx;\ dv = \sqrt{1-x}\,dx,\ v = -\frac{2}{3}\sqrt{(1-x)^3}$;
$\int_0^1 x\sqrt{1-x}\,dx = \left[-\frac{2}{3}\sqrt{(1-x)^3}\,x\right]_0^1 + \frac{2}{3}\int_0^1\sqrt{(1-x)^3}\,dx = \frac{2}{3}\left[-\frac{2}{5}(1-x)^{5/2}\right]_0^1 = \frac{4}{15}$

27. $u = x,\ du = dx;\ dv = \tan^2 x\,dx,\ v = \int\tan^2 x\,dx = \int\frac{\sin^2 x}{\cos^2 x}\,dx = \int\frac{1-\cos^2 x}{\cos^2 x}\,dx = \int\frac{dx}{\cos^2 x} - \int dx$
$= \tan x - x$; $\int_0^{\pi/3} x\tan^2 x\,dx = [x(\tan x - x)]_0^{\pi/3} - \int_0^{\pi/3}(\tan x - x)\,dx = \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \left[\ln|\cos x| + \frac{x^2}{2}\right]_0^{\pi/3}$
$= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \ln\frac{1}{2} + \frac{\pi^2}{18} = \frac{\pi\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18}$

28. $u = \ln(x + x^2)$, $du = \frac{(2x+1)\,dx}{x+x^2}$; $dv = dx$, $v = x$; $\int \ln(x + x^2)\,dx = x\ln(x + x^2) - \int \frac{2x+1}{x(x+1)} \cdot x\,dx$
$= x\ln(x + x^2) - \int \frac{(2x+1)\,dx}{x+1} = x\ln(x + x^2) - \int \frac{2(x+1)-1}{x+1}\,dx = x\ln(x + x^2) - 2x + \ln|x+1| + C$

29. $\int \sin(\ln x)\,dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ dx = e^u\,du \end{bmatrix} \to \int (\sin u)\,e^u\,du$. From Exercise 21, $\int (\sin u)\,e^u\,du = e^u\left(\frac{\sin u - \cos u}{2}\right) + C$
$= \frac{1}{2}[-x\cos(\ln x) + x\sin(\ln x)] + C$

30. $\int z(\ln z)^2\,dz$; $\begin{bmatrix} u = \ln z \\ du = \frac{1}{z}\,dz \\ dz = e^u\,du \end{bmatrix} \to \int e^u \cdot u^2 \cdot e^u\,du = \int e^{2u} \cdot u^2\,du$;

e^{2u}

$u^2 \xrightarrow{(+)} \frac{1}{2}e^{2u}$

$2u \xrightarrow{(-)} \frac{1}{4}e^{2u}$

$2 \xrightarrow{(+)} \frac{1}{8}e^{2u}$

0

$\int u^2 e^{2u}\,du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}[2u^2 - 2u + 1] + C$
$= \frac{z^2}{4}[2(\ln z)^2 - 2\ln z + 1] + C$

31. $\int x\sec x^2\,dx$ $\left[\text{Let } u = x^2,\ du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx\right] \to \int x\sec x^2\,dx = \frac{1}{2}\int \sec u\,du = \frac{1}{2}\ln|\sec u + \tan u| + C$
$= \frac{1}{2}\ln|\sec x^2 + \tan x^2| + C$

32. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ $\left[\text{Let } u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2du = \frac{1}{\sqrt{x}}\,dx\right] \to \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$

33. $\int x(\ln x)^2\,dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ dx = e^u\,du \end{bmatrix} \to \int e^u \cdot u^2 \cdot e^u\,du = \int e^{2u} \cdot u^2\,du$;

e^{2u}

$u^2 \xrightarrow{(+)} \frac{1}{2}e^{2u}$

$2u \xrightarrow{(-)} \frac{1}{4}e^{2u}$

$2 \xrightarrow{(+)} \frac{1}{8}e^{2u}$

0

$\int u^2 e^{2u}\,du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}[2u^2 - 2u + 1] + C$
$= \frac{x^2}{4}[2(\ln x)^2 - 2\ln x + 1] + C = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C$

34. $\int \frac{1}{x(\ln x)^2}\,dx$ $\left[\text{Let } u = \ln x,\ du = \frac{1}{x}\,dx\right] \to \int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$

35. $u = \ln x$, $du = \frac{1}{x}\,dx$; $dv = \frac{1}{x^2}\,dx$, $v = -\frac{1}{x}$;
$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C$

36. $\int \frac{(\ln x)^3}{x}\,dx$ $\left[\text{Let } u = \ln x,\ du = \frac{1}{x}\,dx\right] \to \int \frac{(\ln x)^3}{x}\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}(\ln x)^4 + C$

37. $\int x^3 e^{x^4}\,dx\ \left[\text{Let } u = x^4,\ du = 4x^3\,dx \Rightarrow \frac{1}{4}du = x^3\,dx\right] \to \int x^3 e^{x^4}\,dx = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C = \frac{1}{4}e^{x^4} + C$

38. $u = x^3,\ du = 3x^2\,dx;\ dv = x^2 e^{x^3}\,dx,\ v = \frac{1}{3}e^{x^3}$;
$\int x^5 e^{x^3}\,dx = \int x^3 e^{x^3}\,x^2 dx = \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}\int e^{x^3}\,3x^2 dx = \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3} + C$

39. $u = x^2,\ du = 2x\,dx;\ dv = \sqrt{x^2+1}\,x\,dx,\ v = \frac{1}{3}(x^2+1)^{3/2}$;
$\int x^3\sqrt{x^2+1}\,dx = \frac{1}{3}x^2(x^2+1)^{3/2} - \frac{1}{3}\int (x^2+1)^{3/2} 2x\,dx = \frac{1}{3}x^2(x^2+1)^{3/2} - \frac{2}{15}(x^2+1)^{5/2} + C$

40. $\int x^2 \sin x^3\,dx\ \left[\text{Let } u = x^3,\ du = 3x^2\,dx \Rightarrow \frac{1}{3}du = x^2\,dx\right] \to \int x^2 \sin x^3\,dx = \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C$
$= -\frac{1}{3}\cos x^3 + C$

41. $u = \sin 3x,\ du = 3\cos 3x\,dx;\ dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x$;
$\int \sin 3x \cos 2x\,dx = \frac{1}{2}\sin 3x \sin 2x - \frac{3}{2}\int \cos 3x \sin 2x\,dx$
$u = \cos 3x,\ du = -3\sin 3x\,dx;\ dv = \sin 2x\,dx,\ v = -\frac{1}{2}\cos 2x$;
$\int \sin 3x \cos 2x\,dx = \frac{1}{2}\sin 3x \sin 2x - \frac{3}{2}\left[-\frac{1}{2}\cos 3x \cos 2x - \frac{3}{2}\int \sin 3x \cos 2x\,dx\right]$
$= \frac{1}{2}\sin 3x \sin 2x + \frac{3}{4}\cos 3x \cos 2x + \frac{9}{4}\int \sin 3x \cos 2x\,dx \Rightarrow -\frac{5}{4}\int \sin 3x \cos 2x\,dx = \frac{1}{2}\sin 3x \sin 2x + \frac{3}{4}\cos 3x \cos 2x$
$\Rightarrow \int \sin 3x \cos 2x\,dx = -\frac{2}{5}\sin 3x \sin 2x - \frac{3}{5}\cos 3x \cos 2x + C$

42. $u = \sin 2x,\ du = 2\cos 2x\,dx;\ dv = \cos 4x\,dx,\ v = \frac{1}{4}\sin 4x$;
$\int \sin 2x \cos 4x\,dx = \frac{1}{4}\sin 2x \sin 4x - \frac{1}{2}\int \cos 2x \sin 4x\,dx$
$u = \cos 2x,\ du = -2\sin 2x\,dx;\ dv = \sin 4x\,dx,\ v = -\frac{1}{4}\cos 4x$;
$\int \sin 2x \cos 4x\,dx = \frac{1}{4}\sin 2x \sin 4x - \frac{1}{2}\left[-\frac{1}{4}\cos 2x \cos 4x - \frac{1}{2}\int \sin 2x \cos 4x\,dx\right]$
$= \frac{1}{4}\sin 2x \sin 4x + \frac{1}{8}\cos 2x \cos 4x + \frac{1}{4}\int \sin 2x \cos 4x\,dx \Rightarrow \frac{3}{4}\int \sin 2x \cos 4x\,dx = \frac{1}{4}\sin 2x \sin 4x + \frac{1}{8}\cos 2x \cos 4x$
$\Rightarrow \int \sin 2x \cos 4x\,dx = \frac{1}{3}\sin 2x \sin 4x + \frac{1}{6}\cos 2x \cos 4x + C$

43. $\int e^x \sin e^x\,dx\ \left[\text{Let } u = e^x,\ du = e^x\,dx\right] \to \int e^x \sin e^x\,dx = \int \sin u\,du = -\cos u + C = -\cos e^x + C$

44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx\ \left[\text{Let } u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2du = \frac{1}{\sqrt{x}}\,dx\right] \to \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$

45. $\int \cos\sqrt{x}\,dx;\ \begin{bmatrix} y = \sqrt{x} \\ dy = \frac{1}{2\sqrt{x}}\,dx \\ dx = 2y\,dy \end{bmatrix} \to \int \cos y\,2y\,dy = \int 2y \cos y\,dy;$
$u = 2y,\ du = 2\,dy;\ dv = \cos y\,dy,\ v = \sin y$;
$\int 2y \cos y\,dy = 2y \sin y - \int 2\sin y\,dy = 2y \sin y + 2\cos y + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$

46. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$; $\begin{bmatrix} y = \sqrt{x} \\ dy = \frac{1}{2\sqrt{x}}\, dx \\ dx = 2y\, dy \end{bmatrix} \rightarrow \int y\, e^y\, 2y\, dy = \int 2y^2\, e^y\, dy$;

$$\begin{array}{lcl} & & e^y \\ 2y^2 & \xrightarrow{(+)} & e^y \\ 4y & \xrightarrow{(-)} & e^y \\ 4 & \xrightarrow{(+)} & e^y \\ 0 & & \end{array}$$

$$\int 2y^2\, e^y\, dy = 2y^2 e^y - 4y\, e^y + 4\, e^y + C = 2x\, e^{\sqrt{x}} - 4\sqrt{x}\, e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$$

47.

$$\begin{array}{lcl} & & \sin 2\theta \\ \theta^2 & \xrightarrow{(+)} & -\frac{1}{2}\cos 2\theta \\ 2\theta & \xrightarrow{(-)} & -\frac{1}{4}\sin 2\theta \\ 2 & \xrightarrow{(+)} & \frac{1}{8}\cos 2\theta \\ 0 & & \end{array}$$

$$\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta = \left[-\frac{\theta^2}{2}\cos 2\theta + \frac{\theta}{2}\sin 2\theta + \frac{1}{4}\cos 2\theta\right]_0^{\pi/2}$$
$$= \left[-\frac{\pi^2}{8}\cdot(-1) + \frac{\pi}{4}\cdot 0 + \frac{1}{4}\cdot(-1)\right] - \left[0 + 0 + \frac{1}{4}\cdot 1\right] = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2 - 4}{8}$$

48.

$$\begin{array}{lcl} & & \cos 2x \\ x^3 & \xrightarrow{(+)} & \frac{1}{2}\sin 2x \\ 3x^2 & \xrightarrow{(-)} & -\frac{1}{4}\cos 2x \\ 6x & \xrightarrow{(+)} & -\frac{1}{8}\sin 2x \\ 6 & \xrightarrow{(-)} & \frac{1}{16}\cos 2x \\ 0 & & \end{array}$$

$$\int_0^{\pi/2} x^3 \cos 2x\, dx = \left[\frac{x^3}{2}\sin 2x + \frac{3x^2}{4}\cos 2x - \frac{3x}{4}\sin 2x - \frac{3}{8}\cos 2x\right]_0^{\pi/2}$$
$$= \left[\frac{\pi^3}{16}\cdot 0 + \frac{3\pi^2}{16}\cdot(-1) - \frac{3\pi}{8}\cdot 0 - \frac{3}{8}\cdot(-1)\right] - \left[0 + 0 - 0 - \frac{3}{8}\cdot 1\right] = -\frac{3\pi^2}{16} + \frac{3}{4} = \frac{3(4-\pi^2)}{16}$$

49. $u = \sec^{-1} t$, $du = \frac{dt}{t\sqrt{t^2-1}}$; $dv = t\, dt$, $v = \frac{t^2}{2}$;

$$\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\, dt = \left[\frac{t^2}{2}\sec^{-1} t\right]_{2/\sqrt{3}}^{2} - \int_{2/\sqrt{3}}^{2}\left(\frac{t^2}{2}\right)\frac{dt}{t\sqrt{t^2-1}} = \left(2\cdot\frac{\pi}{3} - \frac{2}{3}\cdot\frac{\pi}{6}\right) - \int_{2/\sqrt{3}}^{2}\frac{t\, dt}{2\sqrt{t^2-1}}$$
$$= \frac{5\pi}{9} - \left[\frac{1}{2}\sqrt{t^2-1}\right]_{2/\sqrt{3}}^{2} = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \sqrt{\frac{4}{3}-1}\right) = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = \frac{5\pi}{9} - \frac{\sqrt{3}}{3} = \frac{5\pi - 3\sqrt{3}}{9}$$

50. $u = \sin^{-1}(x^2)$, $du = \frac{2x\, dx}{\sqrt{1-x^4}}$; $dv = 2x\, dx$, $v = x^2$;

$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx = \left[x^2 \sin^{-1}(x^2)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \cdot \frac{2x\, dx}{\sqrt{1-x^4}} = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) + \int_0^{1/\sqrt{2}} \frac{d(1-x^4)}{2\sqrt{1-x^4}}$$
$$= \frac{\pi}{12} + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

51. (a) $u = x$, $du = dx$; $dv = \sin x\, dx$, $v = -\cos x$;

$$S_1 = \int_0^{\pi} x \sin x\, dx = [-x\cos x]_0^{\pi} + \int_0^{\pi} \cos x\, dx = \pi + [\sin x]_0^{\pi} = \pi$$

(b) $S_2 = -\int_{\pi}^{2\pi} x \sin x\, dx = -\left[[-x\cos x]_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos x\, dx\right] = -[-3\pi + [\sin x]_{\pi}^{2\pi}] = 3\pi$

(c) $S_3 = \int_{2\pi}^{3\pi} x \sin x\, dx = [-x\cos x]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} \cos x\, dx = 5\pi + [\sin x]_{2\pi}^{3\pi} = 5\pi$

(d) $S_{n+1} = (-1)^{n+1}\int_{n\pi}^{(n+1)\pi} x \sin x\, dx = (-1)^{n+1}\left[[-x\cos x]_{n\pi}^{(n+1)\pi} + [\sin x]_{n\pi}^{(n+1)\pi}\right]$
$= (-1)^{n+1}\left[-(n+1)\pi(-1)^n + n\pi(-1)^{n+1}\right] + 0 = (2n+1)\pi$

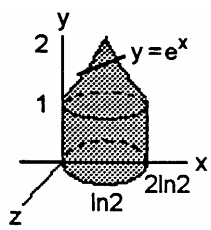
52. (a) $u = x,\ du = dx;\ dv = \cos x\, dx,\ v = \sin x;$
$S_1 = -\int_{\pi/2}^{3\pi/2} x\cos x\, dx = -\left[[x\sin x]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2}\sin x\, dx\right] = -\left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) - [\cos x]_{\pi/2}^{3\pi/2} = 2\pi$

(b) $S_2 = \int_{3\pi/2}^{5\pi/2} x\cos x\, dx = [x\sin x]_{3\pi/2}^{5\pi/2} - \int_{3\pi/2}^{5\pi/2}\sin x\, dx = \left[\frac{5\pi}{2} - \left(-\frac{3\pi}{2}\right)\right] - [\cos x]_{3\pi/2}^{5\pi/2} = 4\pi$

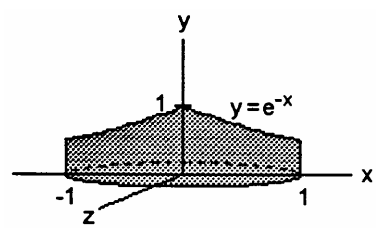
(c) $S_3 = -\int_{5\pi/2}^{7\pi/2} x\cos x\, dx = -\left[[x\sin x]_{5\pi/2}^{7\pi/2} - \int_{5\pi/2}^{7\pi/2}\sin x\, dx\right] = -\left(-\frac{7\pi}{2} - \frac{5\pi}{2}\right) - [\cos x]_{5\pi/2}^{7\pi/2} = 6\pi$

(d) $S_n = (-1)^n\int_{(2n-1)\pi/2}^{(2n+1)\pi/2} x\cos x\, dx = (-1)^n\left[[x\sin x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} - {}^n\int_{(2n-1)\pi/2}^{(2n+1)\pi/2}\sin x\, dx\right]$
$= (-1)^n\left[\frac{(2n+1)\pi}{2}(-1)^n - \frac{(2n-1)\pi}{2}(-1)^{n-1}\right] - [\cos x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} = \frac{1}{2}(2n\pi + \pi + 2n\pi - \pi) = 2n\pi$

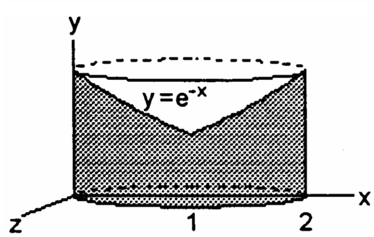
53. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x)e^x\, dx = 2\pi\ln 2\int_0^{\ln 2} e^x\, dx - 2\pi\int_0^{\ln 2} xe^x\, dx$
$= (2\pi\ln 2)[e^x]_0^{\ln 2} - 2\pi\left([xe^x]_0^{\ln 2} - \int_0^{\ln 2} e^x\, dx\right)$
$= 2\pi\ln 2 - 2\pi\left(2\ln 2 - [e^x]_0^{\ln 2}\right) = -2\pi\ln 2 + 2\pi = 2\pi(1 - \ln 2)$



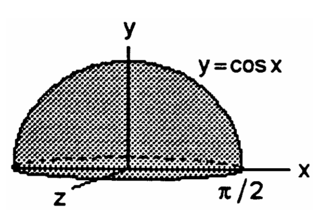
54. (a) $V = \int_0^1 2\pi xe^{-x}\, dx = 2\pi\left([-xe^{-x}]_0^1 + \int_0^1 e^{-x}\, dx\right)$
$= 2\pi\left(-\frac{1}{e} + [-e^{-x}]_0^1\right) = 2\pi\left(-\frac{1}{e} - \frac{1}{e} + 1\right)$
$= 2\pi - \frac{4\pi}{e}$



(b) $V = \int_0^1 2\pi(1-x)e^{-x}\, dx;\ u = 1 - x,\ du = -dx;\ dv = e^{-x}\, dx,$
$v = -e^{-x};\ V = 2\pi\left[[(1-x)(-e^{-x})]_0^1 - \int_0^1 e^{-x}\, dx\right]$
$= 2\pi\left[[0 - 1(-1)] + [e^{-x}]_0^1\right] = 2\pi\left(1 + \frac{1}{e} - 1\right) = \frac{2\pi}{e}$



55. (a) $V = \int_0^{\pi/2} 2\pi x\cos x\, dx = 2\pi\left([x\sin x]_0^{\pi/2} - \int_0^{\pi/2}\sin x\, dx\right)$
$= 2\pi\left(\frac{\pi}{2} + [\cos x]_0^{\pi/2}\right) = 2\pi\left(\frac{\pi}{2} + 0 - 1\right) = \pi(\pi - 2)$



(b) $V = \int_0^{\pi/2} 2\pi\left(\frac{\pi}{2} - x\right)\cos x\, dx;\ u = \frac{\pi}{2} - x,\ du = -dx;\ dv = \cos x\, dx,\ v = \sin x;$
$V = 2\pi\left[\left(\frac{\pi}{2} - x\right)\sin x\right]_0^{\pi/2} + 2\pi\int_0^{\pi/2}\sin x\, dx = 0 + 2\pi[-\cos x]_0^{\pi/2} = 2\pi(0 + 1) = 2\pi$

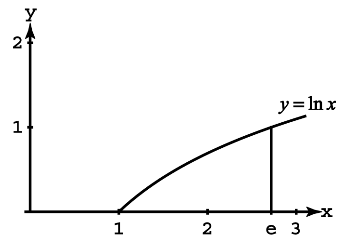
56. (a) $V = \int_0^\pi 2\pi x(x \sin x)\, dx$;

$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$\Rightarrow V = 2\pi \int_0^\pi x^2 \sin x\, dx = 2\pi \left[-x^2 \cos x + 2x \sin x + 2 \cos x\right]_0^\pi = 2\pi\left(\pi^2 - 4\right)$

(b) $V = \int_0^\pi 2\pi(\pi - x)x \sin x\, dx = 2\pi^2 \int_0^\pi x \sin x\, dx - 2\pi \int_0^\pi x^2 \sin x\, dx = 2\pi^2 \left[-x \cos x + \sin x\right]_0^\pi - \left(2\pi^3 - 8\pi\right)$
$= 8\pi$

57. (a) $A = \int_1^e \ln x\, dx = \left[x \ln x\right]_1^e - \int_1^e dx$
$= (e \ln e - 1 \ln 1) - \left[x\right]_1^e = e - (e - 1) = 1$

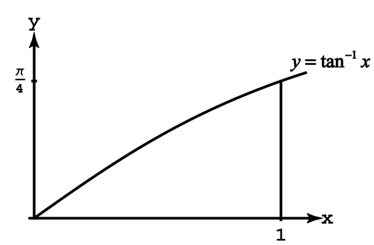


(b) $V = \int_1^e \pi(\ln x)^2\, dx = \pi\left(\left[x(\ln x)^2\right]_1^e - \int_1^e 2 \ln x\, dx\right)$
$= \pi\left[\left(e(\ln e)^2 - 1(\ln 1)^2\right) - \left(\left[2x \ln x\right]_1^e - \int_1^e 2\, dx\right)\right]$
$= \pi\left[e - \left((2e \ln e - 2(1) \ln 1) - \left[2x\right]_1^e\right)\right]$
$= \pi\left[e - (2e - (2e - 2))\right] = \pi(e - 2)$

(c) $V = \int_1^e 2\pi(x + 2) \ln x\, dx = 2\pi \int_1^e (x + 2) \ln x\, dx = 2\pi\left(\left[\left(\frac{1}{2}x^2 + 2x\right)\ln x\right]_1^e - \int_1^e \left(\frac{1}{2}x + 2\right) dx\right)$
$= 2\pi\left(\left(\frac{1}{2}e^2 + 2e\right)\ln e - \left(\frac{1}{2} + 2\right)\ln 1 - \left[\left(\frac{1}{4}x^2 + 2x\right)\right]_1^e\right) = 2\pi\left(\left(\frac{1}{2}e^2 + 2e\right) - \left(\left(\frac{1}{4}e^2 + 2e\right) - \frac{9}{4}\right)\right) = \frac{\pi}{2}\left(e^2 + 9\right)$

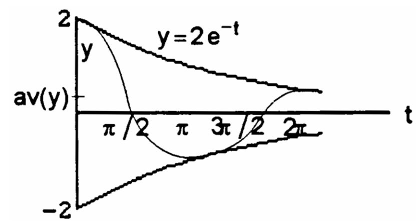
(d) $M = \int_1^e \ln x\, dx = 1$ (from part (a)); $\overline{x} = \frac{1}{1}\int_1^e x \ln x\, dx = \left[\frac{1}{2}x^2 \ln x\right]_1^e - \int_1^e \frac{1}{2}x\, dx = \left(\frac{1}{2}e^2 \ln e - \frac{1}{2}(1)^2 \ln 1\right) - \left[\frac{1}{4}x^2\right]_1^e$
$= \frac{1}{2}e^2 - \left(\frac{1}{4}e^2 - \frac{1}{4}(1)^2\right) = \frac{1}{4}\left(e^2 + 1\right)$; $\overline{y} = \frac{1}{1}\int_1^e \frac{1}{2}(\ln x)^2\, dx = \frac{1}{2}\left(\left[x(\ln x)^2\right]_1^e - \int_1^e 2 \ln x\, dx\right)$
$= \frac{1}{2}\left(\left(e(\ln e)^2 - 1 \cdot (\ln 1)^2\right) - \left(\left[2x \ln x\right]_1^e - \int_1^e 2\, dx\right)\right) = \frac{1}{2}\left(e - \left((2e \ln e - 2(1) \ln 1) - \left[2x\right]_1^e\right)\right)$
$= \frac{1}{2}(e - 2e + 2e - 2) = \frac{1}{2}(e - 2) \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{e^2 + 1}{4}, \frac{e - 2}{2}\right)$ is the centroid.

58. (a) $A = \int_0^1 \tan^{-1} x\, dx = \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1 + x^2} dx$
$= \left(\tan^{-1} 1 - 0\right) - \frac{1}{2}\left[\ln\left(1 + x^2\right)\right]_0^1$
$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \ln 1) = \frac{\pi}{4} - \frac{1}{2}\ln 2$

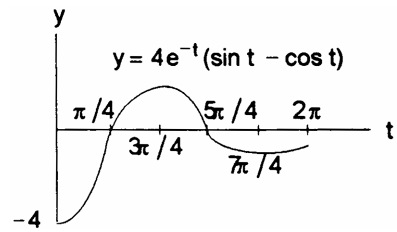


(b) $V = \int_0^1 2\pi x \tan^{-1} x\, dx$
$= 2\pi\left(\left[\frac{x^2}{2} \tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1 + x^2} dx\right)$
$= 2\pi\left(\frac{1}{2}\tan^{-1} 1 - 0 - \frac{1}{2}\int_0^1 \left(1 - \frac{1}{1 + x^2}\right) dx\right) = 2\pi\left(\frac{\pi}{8} - \frac{1}{2}\left[x - \tan^{-1} x\right]_0^1\right) = 2\pi\left(\frac{\pi}{8} - \frac{1}{2}\left(1 - \tan^{-1} 1 - (0 - 0)\right)\right)$
$= 2\pi\left(\frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right)\right) = \frac{\pi(\pi - 2)}{2}$

59. $\text{av}(y) = \frac{1}{2\pi}\int_0^{2\pi} 2e^{-t}\cos t\,dt$
$= \frac{1}{\pi}\left[e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi}$
(see Exercise 22) $\Rightarrow \text{av}(y) = \frac{1}{2\pi}\left(1 - e^{-2\pi}\right)$



60. $\text{av}(y) = \frac{1}{2\pi}\int_0^{2\pi} 4e^{-t}(\sin t - \cos t)\,dt$
$= \frac{2}{\pi}\int_0^{2\pi} e^{-t}\sin t\,dt - \frac{2}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt$
$= \frac{2}{\pi}\left[e^{-t}\left(\frac{-\sin t - \cos t}{2}\right) - e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi}$
$= \frac{2}{\pi}\left[-e^{-t}\sin t\right]_0^{2\pi} = 0$



61. $I = \int x^n \cos x\,dx$; $[u = x^n,\ du = nx^{n-1}\,dx;\ dv = \cos x\,dx,\ v = \sin x]$
$\Rightarrow I = x^n \sin x - \int nx^{n-1}\sin x\,dx$

62. $I = \int x^n \sin x\,dx$; $[u = x^n,\ du = nx^{n-1}\,dx;\ dv = \sin x\,dx,\ v = -\cos x]$
$\Rightarrow I = -x^n \cos x + \int nx^{n-1}\cos x\,dx$

63. $I = \int x^n e^{ax}\,dx$; $\left[u = x^n,\ du = nx^{n-1}\,dx;\ dv = e^{ax}\,dx,\ v = \frac{1}{a}e^{ax}\right]$
$\Rightarrow I = \frac{x^n e^{ax}}{a}e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx,\ a \neq 0$

64. $I = \int (\ln x)^n\,dx$; $\left[u = (\ln x)^n,\ du = \frac{n(\ln x)^{n-1}}{x}\,dx;\ dv = 1\,dx,\ v = x\right]$
$\Rightarrow I = x(\ln x)^n - \int n(\ln x)^{n-1}\,dx$

65. $\int_a^b (x-a)\,f(x)\,dx$; $\left[u = x - a,\ du = dx;\ dv = f(x)\,dx,\ v = \int_b^x f(t)\,dt = -\int_x^b f(t)\,dt\right]$
$= \left[(x-a)\int_b^x f(t)\,dt\right]_a^b - \int_a^b\left(\int_b^x f(t)\,dt\right)dx = \left((b-a)\int_b^b f(t)\,dt - (a-a)\int_b^a f(t)\,dt\right) - \int_a^b\left(-\int_x^b f(t)\,dt\right)dx$
$= 0 + \int_a^b\left(\int_x^b f(t)\,dt\right)dx = \int_a^b\left(\int_x^b f(t)\,dt\right)dx$

66. $\int \sqrt{1-x^2}\,dx$; $\left[u = \sqrt{1-x^2},\ du = \frac{-x}{\sqrt{1-x^2}}dx;\ dv = dx,\ v = x\right]$
$= x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\,dx = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx = x\sqrt{1-x^2} - \left(\int \frac{1-x^2}{\sqrt{1-x^2}}\,dx - \int \frac{1}{\sqrt{1-x^2}}\,dx\right)$
$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$
$\Rightarrow \int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \sqrt{1-x^2}\,dx \Rightarrow 2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx$
$\Rightarrow \int \sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx + C$

67. $\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \sin y\,dy = x\sin^{-1} x + \cos y + C = x\sin^{-1} x + \cos\left(\sin^{-1} x\right) + C$

68. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \tan y\,dy = x\tan^{-1} x + \ln|\cos y| + C = x\tan^{-1} x + \ln\left|\cos\left(\tan^{-1} x\right)\right| + C$

69. $\int \sec^{-1} x\, dx = x \sec^{-1} x - \int \sec y\, dy = x \sec^{-1} x - \ln|\sec y + \tan y| + C$
$= x \sec^{-1} x - \ln|\sec(\sec^{-1} x) + \tan(\sec^{-1} x)| + C = x \sec^{-1} x - \ln\left|x + \sqrt{x^2 - 1}\right| + C$

70. $\int \log_2 x\, dx = x \log_2 x - \int 2^y\, dy = x \log_2 x - \frac{2^y}{\ln 2} + C = x \log_2 x - \frac{x}{\ln 2} + C$

71. Yes, $\cos^{-1} x$ is the angle whose cosine is x which implies $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$.

72. Yes, $\tan^{-1} x$ is the angle whose tangent is x which implies $\sec(\tan^{-1} x) = \sqrt{1 + x^2}$.

73. (a) $\int \sinh^{-1} x\, dx = x \sinh^{-1} x - \int \sinh y\, dy = x \sinh^{-1} x - \cosh y + C = x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C$;
check: $d[x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \sinh(\sinh^{-1} x)\, \frac{1}{\sqrt{1+x^2}}\right] dx$
$= \sinh^{-1} x\, dx$
(b) $\int \sinh^{-1} x\, dx = x \sinh^{-1} x - \int x \left(\frac{1}{\sqrt{1+x^2}}\right) dx = x \sinh^{-1} x - \frac{1}{2} \int (1 + x^2)^{-1/2} 2x\, dx$
$= x \sinh^{-1} x - (1 + x^2)^{1/2} + C$
check: $d\left[x \sinh^{-1} x - (1 + x^2)^{1/2} + C\right] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}\right] dx = \sinh^{-1} x\, dx$

74. (a) $\int \tanh^{-1} x\, dx = x \tanh^{-1} x - \int \tanh y\, dy = x \tanh^{-1} x - \ln|\cosh y| + C = x \tanh^{-1} x - \ln|\cosh(\tanh^{-1} x)| + C$;
check: $d[x \tanh^{-1} x - \ln|\cosh(\tanh^{-1} x)| + C] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{\sinh(\tanh^{-1} x)}{\cosh(\tanh^{-1} x)}\, \frac{1}{1-x^2}\right] dx$
$= \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\right] dx = \tanh^{-1} x\, dx$
(b) $\int \tanh^{-1} x\, dx = x \tanh^{-1} x - \int \frac{x}{1-x^2}\, dx = x \tanh^{-1} x - \frac{1}{2} \int \frac{2x}{1-x^2}\, dx = x \tanh^{-1} x + \frac{1}{2} \ln|1 - x^2| + C$
check: $d\left[x \tanh^{-1} x + \frac{1}{2} \ln|1 - x^2| + C\right] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\right] dx = \tanh^{-1} x\, dx$

8.2 TRIGONOMETRIC INTEGRALS

1. $\int \cos 2x\, dx = \frac{1}{2} \int \cos 2x \cdot 2dx = \frac{1}{2} \sin 2x + C$

2. $\int_0^{\pi} 3 \sin \frac{x}{3}\, dx = 9 \int_0^{\pi} \sin \frac{x}{3} \cdot \frac{1}{3} dx = 9\left[-\cos \frac{x}{3}\right]_0^{\pi} = 9\left(-\cos \frac{\pi}{3} + \cos 0\right) = 9\left(-\frac{1}{2} + 1\right) = \frac{9}{2}$

3. $\int \cos^3 x \sin x\, dx = -\int \cos^3 x\, (-\sin x) dx = -\frac{1}{4} \cos^4 x + C$

4. $\int \sin^4 2x \cos 2x\, dx = \frac{1}{2} \int \sin^4 2x \cos 2x \cdot 2dx = \frac{1}{10} \sin^5 2x + C$

5. $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = \int \left(1 - \cos^2 x\right) \sin x\, dx = \int \sin x\, dx - \int \cos^2 x \sin x\, dx = -\cos x + \frac{1}{3} \cos^3 x + C$

6. $\int \cos^3 4x\, dx = \int \cos^2 4x \cos 4x\, dx = \frac{1}{4} \int \left(1 - \sin^2 4x\right) \cos 4x \cdot 4dx = \frac{1}{4} \int \cos 4x \cdot 4dx - \frac{1}{4} \int \sin^2 4x \cos 4x \cdot 4dx$
$= \frac{1}{4} \sin 4x - \frac{1}{12} \sin^3 4x + C$

7. $\int \sin^5 x\, dx = \int (\sin^2 x)^2 \sin x\, dx = \int (1 - \cos^2 x)^2 \sin x\, dx = \int (1 - 2\cos^2 x + \cos^4 x) \sin x\, dx$
$= \int \sin x\, dx - \int 2\cos^2 x \sin x\, dx + \int \cos^4 x \sin x\, dx = -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C$

8. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)dx$ (using Exercise 7) $= \int_0^{\pi} \sin\left(\frac{x}{2}\right)dx - \int_0^{\pi} 2\cos^2\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx + \int_0^{\pi} \cos^4\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx$
$= \left[-2\cos\left(\frac{x}{2}\right) + \frac{4}{3}\cos^3\left(\frac{x}{2}\right) - \frac{2}{5}\cos^5\left(\frac{x}{2}\right)\right]_0^{\pi} = (0) - \left(-2 + \frac{4}{3} - \frac{2}{5}\right) = \frac{16}{15}$

9. $\int \cos^3 x\, dx = \int (\cos^2 x)\cos x\, dx = \int (1 - \sin^2 x)\cos x\, dx = \int \cos x\, dx - \int \sin^2 x \cos x\, dx = \sin x - \frac{1}{3}\sin^3 x + C$

10. $\int_0^{\pi/6} 3\cos^5 3x\, dx = \int_0^{\pi/6} (\cos^2 3x)^2 \cos 3x \cdot 3dx = \int_0^{\pi/6} (1 - \sin^2 3x)^2 \cos 3x \cdot 3dx = \int_0^{\pi/6} (1 - 2\sin^2 3x + \sin^4 3x)\cos 3x \cdot 3dx$
$= \int_0^{\pi/6} \cos 3x \cdot 3dx - 2\int_0^{\pi/6} \sin^2 3x \cos 3x \cdot 3dx + \int_0^{\pi/6} \sin^4 3x \cos 3x \cdot 3dx = \left[\sin 3x - 2\frac{\sin^3 3x}{3} + \frac{\sin^5 3x}{5}\right]_0^{\pi/6}$
$= \left(1 - \frac{2}{3} + \frac{1}{5}\right) - (0) = \frac{8}{15}$

11. $\int \sin^3 x \cos^3 x\, dx = \int \sin^3 x \cos^2 x \cos x dx = \int \sin^3 x\,(1 - \sin^2 x)\cos x\, dx = \int \sin^3 x \cos x\, dx - \int \sin^5 x \cos x\, dx$
$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$

12. $\int \cos^3 2x \sin^5 2x\, dx = \frac{1}{2}\int \cos^3 2x \sin^5 2x \cdot 2dx = \frac{1}{2}\int \cos 2x \cos^2 2x \sin^5 2x \cdot 2dx = \frac{1}{2}\int (1 - \sin^2 2x) \sin^5 2x \cos 2x \cdot 2dx$
$= \frac{1}{2}\int \sin^5 2x \cos 2x \cdot 2dx - \frac{1}{2}\int \sin^7 2x \cos 2x \cdot 2dx = \frac{1}{12}\sin^6 2x - \frac{1}{16}\sin^8 2x + C$

13. $\int \cos^2 x\, dx = \int \frac{1 + \cos 2x}{2}\, dx = \frac{1}{2}\int \left(1 + \cos 2x\right)dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\, dx = \frac{1}{2}\int dx + \frac{1}{4}\int \cos 2x \cdot 2dx$
$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

14. $\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \frac{1 - \cos 2x}{2}\, dx = \frac{1}{2}\int_0^{\pi/2} \left(1 - \cos 2x\right)dx = \frac{1}{2}\int_0^{\pi/2} dx - \frac{1}{2}\int_0^{\pi/2} \cos 2x\, dx = \frac{1}{2}\int_0^{\pi/2} dx - \frac{1}{4}\int_0^{\pi/2} \cos 2x \cdot 2dx$
$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \left(\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)\right) - \left(\frac{1}{2}(0) - \frac{1}{4}\sin 2(0)\right) = \left(\frac{\pi}{4} - 0\right) - (0 - 0) = \frac{\pi}{4}$

15. $\int_0^{\pi/2} \sin^7 y\, dy = \int_0^{\pi/2} \sin^6 y \sin y\, dy = \int_0^{\pi/2} (1 - \cos^2 y)^3 \sin y\, dy = \int_0^{\pi/2} \sin y\, dy - 3\int_0^{\pi/2} \cos^2 y \sin y\, dy$
$+ 3\int_0^{\pi/2} \cos^4 y \sin y\, dy - \int_0^{\pi/2} \cos^6 y \sin y\, dy = \left[-\cos y + 3\frac{\cos^3 y}{3} - 3\frac{\cos^5 y}{5} + \frac{\cos^7 y}{7}\right]_0^{\pi/2} = (0) - \left(-1 + 1 - \frac{3}{5} + \frac{1}{7}\right) = \frac{16}{35}$

16. $\int 7\cos^7 t\, dt$ (using Exercise 15) $= 7\left[\int \cos t\, dt - 3\int \sin^2 t \cos t\, dt + 3\int \sin^4 t \cos t\, dt - \int \sin^6 t \cos t\, dt\right]$
$= 7\left(\sin t - 3\frac{\sin^3 t}{3} + 3\frac{\sin^5 t}{5} - \frac{\sin^7 t}{7}\right) + C = 7\sin t - 7\sin^3 t + \frac{21}{5}\sin^5 t - \sin^7 t + C$

17. $\int_0^{\pi} 8\sin^4 x\, dx = 8\int_0^{\pi} \left(\frac{1 - \cos 2x}{2}\right)^2 dx = 2\int_0^{\pi} (1 - 2\cos 2x + \cos^2 2x)dx = 2\int_0^{\pi} dx - 2\int_0^{\pi} \cos 2x \cdot 2dx + 2\int_0^{\pi} \frac{1 + \cos 4x}{2}\, dx$
$= [2x - 2\sin 2x]_0^{\pi} + \int_0^{\pi} dx + \int_0^{\pi} \cos 4x\, dx = 2\pi + \left[x + \frac{1}{2}\sin 4x\right]_0^{\pi} = 2\pi + \pi = 3\pi$

18. $\int 8\cos^4 2\pi x\, dx = 8\int \left(\frac{1 + \cos 4\pi x}{2}\right)^2 dx = 2\int (1 + 2\cos 4\pi x + \cos^2 4\pi x)dx = 2\int dx + 4\int \cos 4\pi x\, dx + 2\int \frac{1 + \cos 8\pi x}{2}\, dx$
$= 3\int dx + 4\int \cos 4\pi x\, dx + \int \cos 8\pi x\, dx = 3x + \frac{1}{\pi}\sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C$

19. $\int 16 \sin^2 x \cos^2 x\, dx = 16\int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)dx = 4\int (1 - \cos^2 2x)dx = 4\int dx - 4\int \left(\frac{1 + \cos 4x}{2}\right)dx$
$= 4x - 2\int dx - 2\int \cos 4x\, dx = 4x - 2x - \frac{1}{2}\sin 4x + C = 2x - \frac{1}{2}\sin 4x + C = 2x - \sin 2x \cos 2x + C$
$= 2x - 2\sin x \cos x\,(2\cos^2 x - 1) + C = 2x - 4\sin x \cos^3 x + 2\sin x \cos x + C$

20. $\int_0^{\pi} 8\sin^4 y\cos^2 y\,dy = 8\int_0^{\pi}\left(\frac{1-\cos 2y}{2}\right)^2\left(\frac{1+\cos 2y}{2}\right)dy = \int_0^{\pi} dy - \int_0^{\pi}\cos 2y\,dy - \int_0^{\pi}\cos^2 2y\,dy + \int_0^{\pi}\cos^3 2y\,dy$
$= \left[y - \frac{1}{2}\sin 2y\right]_0^{\pi} - \int_0^{\pi}\left(\frac{1+\cos 4y}{2}\right)dy + \int_0^{\pi}(1-\sin^2 2y)\cos 2y\,dy = \pi - \frac{1}{2}\int_0^{\pi} dy - \frac{1}{2}\int_0^{\pi}\cos 4y\,dy + \int_0^{\pi}\cos 2y\,dy$
$- \int_0^{\pi}\sin^2 2y\cos 2y\,dy = \pi + \left[-\frac{1}{2}y - \frac{1}{8}\sin 4y + \frac{1}{2}\sin 2y - \frac{1}{2}\cdot\frac{\sin^3 2y}{3}\right]_0^{\pi} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

21. $\int 8\cos^3 2\theta\sin 2\theta\,d\theta = 8\left(-\frac{1}{2}\right)\frac{\cos^4 2\theta}{4} + C = -\cos^4 2\theta + C$

22. $\int_0^{\pi/2}\sin^2 2\theta\cos^3 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta(1-\sin^2 2\theta)\cos 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta\cos 2\theta\,d\theta - \int_0^{\pi/2}\sin^4 2\theta\cos 2\theta\,d\theta$
$= \left[\frac{1}{2}\cdot\frac{\sin^3 2\theta}{3} - \frac{1}{2}\cdot\frac{\sin^5 2\theta}{5}\right]_0^{\pi/2} = 0$

23. $\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}\,dx = \int_0^{2\pi}\left|\sin\frac{x}{2}\right|dx = \int_0^{2\pi}\sin\frac{x}{2}\,dx = \left[-2\cos\frac{x}{2}\right]_0^{2\pi} = 2+2 = 4$

24. $\int_0^{\pi}\sqrt{1-\cos 2x}\,dx = \int_0^{\pi}\sqrt{2}\,|\sin 2x|\,dx = \int_0^{\pi}\sqrt{2}\sin 2x\,dx = \left[-\sqrt{2}\cos 2x\right]_0^{\pi} = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

25. $\int_0^{\pi}\sqrt{1-\sin^2 t}\,dt = \int_0^{\pi}|\cos t|\,dt = \int_0^{\pi/2}\cos t\,dt - \int_{\pi/2}^{\pi}\cos t\,dt = [\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^{\pi} = 1-0-0+1 = 2$

26. $\int_0^{\pi}\sqrt{1-\cos^2\theta}\,d\theta = \int_0^{\pi}|\sin\theta|\,d\theta = \int_0^{\pi}\sin\theta\,d\theta = [-\cos\theta]_0^{\pi} = 1+1 = 2$

27. $\int_{\pi/3}^{\pi/2}\frac{\sin^2 x}{\sqrt{1-\cos x}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sin^2 x}{\sqrt{1-\cos x}}\frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sin^2 x\sqrt{1+\cos x}}{\sqrt{1-\cos^2 x}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sin^2 x\sqrt{1+\cos x}}{\sqrt{\sin^2 x}}\,dx$
$= \int_{\pi/3}^{\pi/2}\sin x\sqrt{1+\cos x}\,dx = \left[-\frac{2}{3}(1+\cos x)^{3/2}\right]_{\pi/3}^{\pi/2} = -\frac{2}{3}\left(1+\cos\left(\frac{\pi}{2}\right)\right)^{3/2} + \frac{2}{3}\left(1+\cos\left(\frac{\pi}{3}\right)\right)^{3/2} = -\frac{2}{3} + \frac{2}{3}\left(\frac{3}{2}\right)^{3/2}$
$= \sqrt{\frac{3}{2}} - \frac{2}{3}$

28. $\int_0^{\pi/6}\sqrt{1+\sin x}\,dx = \int_0^{\pi/6}\frac{\sqrt{1+\sin x}}{1}\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6}\frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6}\frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6}\frac{\cos x}{\sqrt{1-\sin x}}\,dx$
$= \left[-2(1-\sin x)^{1/2}\right]_0^{\pi/6} = -2\sqrt{1-\sin\left(\frac{\pi}{6}\right)} + 2\sqrt{1-\sin 0} = -2\sqrt{\frac{1}{2}} + 2\sqrt{1} = 2-\sqrt{2}$

29. $\int_{5\pi/6}^{\pi}\frac{\cos^4 x}{\sqrt{1-\sin x}}\,dx = \int_{5\pi/6}^{\pi}\frac{\cos^4 x}{\sqrt{1-\sin x}}\frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}\,dx = \int_{5\pi/6}^{\pi}\frac{\cos^4 x\sqrt{1+\sin x}}{\sqrt{1-\sin^2 x}}\,dx = \int_{5\pi/6}^{\pi}\frac{\cos^4 x\sqrt{1+\sin x}}{\sqrt{\cos^2 x}}\,dx$
$= \int_{5\pi/6}^{\pi}\frac{\cos^4 x\sqrt{1+\sin x}}{-\cos x}\,dx = -\int_{5\pi/6}^{\pi}\cos^3 x\sqrt{1+\sin x}\,dx = -\int_{5\pi/6}^{\pi}\cos x(1-\sin^2 x)\sqrt{1+\sin x}\,dx$
$= -\int_{5\pi/6}^{\pi}\cos x\sqrt{1+\sin x}\,dx + \int_{5\pi/6}^{\pi}\cos x\sin^2 x\sqrt{1+\sin x}\,dx;\ u^2\sqrt{u}\,du$
$\left[\text{Let } u = 1+\sin x \Rightarrow u-1 = \sin x \Rightarrow du = \cos x\,dx,\ x = \frac{5\pi}{6} \Rightarrow u = 1+\sin\left(\frac{5\pi}{6}\right) = \frac{3}{2},\ x = \pi \Rightarrow u = 1+\sin\pi = 1\right]$
$= \left[-\frac{2}{3}(1+\sin x)^{3/2}\right]_{5\pi/6}^{\pi} + \int_{3/2}^{1}(u-1)^2\sqrt{u}\,du = \left[-\frac{2}{3}(1+\sin x)^{3/2}\right]_{5\pi/6}^{\pi} + \int_{3/2}^{1}\left(u^{5/2} - 2u^{3/2} + \sqrt{u}\right)du$
$= \left(-\frac{2}{3}(1+\sin\pi)^{3/2} + \frac{2}{3}\left(1+\sin\left(\frac{5\pi}{6}\right)\right)^{3/2}\right) + \left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_{3/2}^{1}$
$= \left(-\frac{2}{3} + \frac{2}{3}\left(\frac{3}{2}\right)^{3/2}\right) + \left(\frac{2}{7} - \frac{4}{5} + \frac{2}{3}\right) - \left(\frac{2}{7}\left(\frac{3}{2}\right)^{7/2} - \frac{4}{5}\left(\frac{3}{2}\right)^{5/2} + \frac{2}{3}\left(\frac{3}{2}\right)^{3/2}\right) = \frac{4}{5}\left(\frac{3}{2}\right)^{5/2} - \frac{2}{7}\left(\frac{3}{2}\right)^{7/2} - \frac{18}{35}$

30. $\int_{\pi/2}^{7\pi/12}\sqrt{1-\sin 2x}\,dx = \int_{\pi/2}^{7\pi/12}\frac{\sqrt{1-\sin 2x}}{1}\frac{\sqrt{1+\sin 2x}}{\sqrt{1+\sin 2x}}\,dx = \int_{\pi/2}^{7\pi/12}\frac{\sqrt{1-\sin^2 2x}}{\sqrt{1+\sin 2x}}\,dx = \int_{\pi/2}^{7\pi/12}\frac{\sqrt{\cos^2 2x}}{\sqrt{1+\sin 2x}}\,dx$
$= \int_{\pi/2}^{7\pi/12}\frac{-\cos 2x}{\sqrt{1+\sin 2x}}\,dx = \left[-\sqrt{1+\sin 2x}\right]_{\pi/2}^{7\pi/12} = -\sqrt{1+\sin 2\left(\frac{7\pi}{12}\right)} + \sqrt{1+\sin 2\left(\frac{\pi}{2}\right)} = -\sqrt{\frac{1}{2}} + 1 = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$

31. $\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta}\, d\theta = \int_0^{\pi/2} \theta\sqrt{2}\,|\sin\theta|\, d\theta = \sqrt{2}\int_0^{\pi/2} \theta\sin\theta\, d\theta = \sqrt{2}\left[-\theta\cos\theta + \sin\theta\right]_0^{\pi/2} = \sqrt{2}(1) = \sqrt{2}$

32. $\int_{-\pi}^{\pi} (1-\cos^2 t)^{3/2}\, dt = \int_{-\pi}^{\pi} (\sin^2 t)^{3/2}\, dt = \int_{-\pi}^{\pi} |\sin^3 t|\, dt = -\int_{-\pi}^{0} \sin^3 t\, dt + \int_0^{\pi} \sin^3 t\, dt = -\int_{-\pi}^{0} (1-\cos^2 t)\sin t\, dt$
$+ \int_0^{\pi} (1-\cos^2 t)\sin t\, dt = -\int_{-\pi}^{0} \sin t\, dt + \int_{-\pi}^{0} \cos^2 t\sin t\, dt + \int_0^{\pi} \sin t\, dt - \int_0^{\pi} \cos^2 t\sin t\, dt = \left[\cos t - \frac{\cos^3 t}{3}\right]_{-\pi}^{0}$
$+ \left[-\cos t + \frac{\cos^3 t}{3}\right]_0^{\pi} = \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) + \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = \frac{8}{3}$

33. $\int \sec^2 x\tan x\, dx = \int \tan x\sec^2 x\, dx = \frac{1}{2}\tan^2 x + C$

34. $\int \sec x\tan^2 x\, dx = \int \sec x\tan x\tan x dx$; $u = \tan x$, $du = \sec^2 x\, dx$, $dv = \sec x\tan x\, dx$, $v = \sec x$;
$= \sec x\tan x - \int \sec^3 x\, dx = \sec x\tan x - \int \sec^2 x\sec x dx = \sec x\tan x - \int \left(\tan^2 x + 1\right)\sec x dx$
$= \sec x\tan x - \left(\int \tan^2 x\sec x dx + \int \sec x dx\right) = \sec x\tan x - \ln|\sec x + \tan x| - \int \tan^2 x\sec x dx$
$\Rightarrow \int \sec x\tan^2 x\, dx = \sec x\tan x - \ln|\sec x + \tan x| - \int \tan^2 x\sec x dx$
$\Rightarrow 2\int \tan^2 x\sec x dx = \sec x\tan x - \ln|\sec x + \tan x| \Rightarrow \int \tan^2 x\sec x dx = \frac{1}{2}\sec x\tan x - \frac{1}{2}\ln|\sec x + \tan x| + C$

35. $\int \sec^3 x\tan x\, dx = \int \sec^2 x\sec x\tan x\, dx = \frac{1}{3}\sec^3 x + C$

36. $\int \sec^3 x\tan^3 x\, dx = \int \sec^2 x\tan^2 x\sec x\tan x\, dx = \int \sec^2 x(\sec^2 x - 1)\sec x\tan x\, dx$
$= \int \sec^4 x\sec x\tan x\, dx - \int \sec^2 x\sec x\tan x\, dx = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$

37. $\int \sec^2 x\tan^2 x\, dx = \int \tan^2 x\sec^2 x\, dx = \frac{1}{3}\tan^3 x + C$

38. $\int \sec^4 x\tan^2 x\, dx = \int \sec^2 x\tan^2 x\sec^2 x\, dx = \int \left(\tan^2 x + 1\right)\tan^2 x\sec^2 x\, dx = \int \tan^4 x\sec^2 x\, dx + \int \tan^2 x\sec^2 x\, dx$
$= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$

39. $\int_{-\pi/3}^{0} 2\sec^3 x\, dx$; $u = \sec x$, $du = \sec x\tan x\, dx$, $dv = \sec^2 x\, dx$, $v = \tan x$;
$\int_{-\pi/3}^{0} 2\sec^3 x\, dx = [2\sec x\tan x]_{-\pi/3}^{0} - 2\int_{-\pi/3}^{0} \sec x\tan^2 x\, dx = 2\cdot 1\cdot 0 - 2\cdot 2\cdot\sqrt{3} - 2\int_{-\pi/3}^{0} \sec x\,(\sec^2 x - 1)dx$
$= 4\sqrt{3} - 2\int_{-\pi/3}^{0} \sec^3 x\, dx + 2\int_{-\pi/3}^{0} \sec x\, dx$; $2\int_{-\pi/3}^{0} 2\sec^3 x\, dx = 4\sqrt{3} + [2\ln|\sec x + \tan x|]_{-\pi/3}^{0}$
$2\int_{-\pi/3}^{0} 2\sec^3 x\, dx = 4\sqrt{3} + 2\ln|1 + 0| - 2\ln|2 - \sqrt{3}| = 4\sqrt{3} - 2\ln\left(2 - \sqrt{3}\right)$
$\int_{-\pi/3}^{0} 2\sec^3 x\, dx = 2\sqrt{3} - \ln\left(2 - \sqrt{3}\right)$

40. $\int e^x\sec^3(e^x)dx$; $u = \sec(e^x)$, $du = \sec(e^x)\tan(e^x)e^x dx$, $dv = \sec^2(e^x)e^x dx$, $v = \tan(e^x)$.
$\int e^x\sec^3(e^x)\, dx = \sec(e^x)\tan(e^x) - \int \sec(e^x)\tan^2(e^x)e^x dx$
$= \sec(e^x)\tan(e^x) - \int \sec(e^x)(\sec^2(e^x) - 1)e^x dx$
$= \sec(e^x)\tan(e^x) - \int \sec^3(e^x)e^x dx + \int \sec(e^x)e^x dx$
$2\int e^x\sec^3(e^x)\, dx = \sec(e^x)\tan(e^x) + \ln\left|\sec(e^x) + \tan(e^x)\right| + C$
$\int e^x\sec^3(e^x)\, dx = \frac{1}{2}\left(\sec(e^x)\tan(e^x) + \ln\left|\sec(e^x) + \tan(e^x)\right|\right) + C$

41. $\int \sec^4\theta\, d\theta = \int (1+\tan^2\theta)\sec^2\theta\, d\theta = \int \sec^2\theta\, d\theta + \int \tan^2\theta \sec^2\theta\, d\theta = \tan\theta + \frac{1}{3}\tan^3\theta + C$
$= \tan\theta + \frac{1}{3}\tan\theta(\sec^2\theta - 1) + C = \frac{1}{3}\tan\theta\sec^2\theta + \frac{2}{3}\tan\theta + C$

42. $\int 3\sec^4(3x)\, dx = \int (1+\tan^2(3x))\sec^2(3x)3dx = \int \sec^2(3x)3dx + \int \tan^2(3x)\sec^2(3x)3dx = \tan(3x) + \frac{1}{3}\tan^3(3x) + C$

43. $\int_{\pi/4}^{\pi/2} \csc^4\theta\, d\theta = \int_{\pi/4}^{\pi/2} (1+\cot^2\theta)\csc^2\theta\, d\theta = \int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} \cot^2\theta\csc^2\theta\, d\theta = \left[-\cot\theta - \frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2}$
$= (0) - \left(-1 - \frac{1}{3}\right) = \frac{4}{3}$

44. $\int \sec^6 x\, dx = \int \sec^4 x \sec^2 x\, dx = \int \left(\tan^2 x + 1\right)^2 \sec^2 x\, dx = \int \left(\tan^4 x + 2\tan^2 x + 1\right)\sec^2 x\, dx$
$= \int \tan^4 x \sec^2 x\, dx + 2\int \tan^2 x \sec^2 x\, dx + \int \sec^2 x\, dx = \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

45. $\int 4\tan^3 x\, dx = 4\int (\sec^2 x - 1)\tan x\, dx = 4\int \sec^2 x \tan x\, dx - 4\int \tan x\, dx = 4\frac{\tan^2 x}{2} - 4\ln|\sec x| + C$
$= 2\tan^2 x - 4\ln|\sec x| + C = 2\tan^2 x - 2\ln|\sec^2 x| + C = 2\tan^2 x - 2\ln(1+\tan^2 x) + C$

46. $\int_{-\pi/4}^{\pi/4} 6\tan^4 x\, dx = 6\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)\tan^2 x\, dx = 6\int_{-\pi/4}^{\pi/4} \sec^2 x\tan^2 x\, dx - 6\int_{-\pi/4}^{\pi/4} \tan^2 x\, dx$
$= 6\int_{-\pi/4}^{\pi/4} \sec^2 x\tan^2 x\, dx - 6\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)dx = \left[6\frac{\tan^3 x}{3}\right]_{-\pi/4}^{\pi/4} - 6\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx + 6\int_{-\pi/4}^{\pi/4} dx$
$= 2(1-(-1)) - [6\tan x]_{-\pi/4}^{\pi/4} + [6x]_{-\pi/4}^{\pi/4} = 4 - 6(1-(-1)) + \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi - 8$

47. $\int \tan^5 x\, dx = \int \tan^4 x \tan x\, dx = \int \left(\sec^2 x - 1\right)^2 \tan x\, dx = \int \left(\sec^4 x - 2\sec^2 x + 1\right)\tan x\, dx$
$= \int \sec^4 x \tan x\, dx - 2\int \sec^2 x \tan x\, dx + \int \tan x\, dx = \int \sec^3 x \sec x \tan x\, dx - 2\int \sec x \sec x \tan x\, dx + \int \tan x\, dx$
$= \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C = \frac{1}{4}\left(\tan^2 x + 1\right)^2 - \left(\tan^2 x + 1\right) + \ln|\sec x| + C = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C$

48. $\int \cot^6 2x\, dx = \int \cot^4 2x \cot^2 2x\, dx = \int \cot^4 2x\,(\csc^2 2x - 1)\, dx = \int \cot^4 2x \csc^2 2x\, dx - \int \cot^4 2x\, dx$
$= \int \cot^4 2x \csc^2 2x\, dx - \int \cot^2 2x \cot^2 2x\, dx = \int \cot^4 2x \csc^2 2x\, dx - \int \cot^2 2x(\csc^2 2x - 1)dx$
$= \int \cot^4 2x \csc^2 2x\, dx - \int \cot^2 2x \csc^2 2x\, dx + \int \cot^2 2x\, dx$
$= \int \cot^4 2x \csc^2 2x\, dx - \int \cot^2 2x \csc^2 2x dx + \int (\csc^2 2x - 1)dx$
$= \int \cot^4 2x \csc^2 2x\, dx - \int \cot^2 2x \csc^2 2x dx + \int \csc^2 2x\, dx - \int dx = -\frac{1}{10}\cot^5 2x + \frac{1}{6}\cot^3 2x - \frac{1}{2}\cot 2x - x + C$

49. $\int_{\pi/6}^{\pi/3} \cot^3 x\, dx = \int_{\pi/6}^{\pi/3} (\csc^2 x - 1)\cot x\, dx = \int_{\pi/6}^{\pi/3} \csc^2 x \cot x\, dx - \int_{\pi/6}^{\pi/3} \cot x\, dx = \left[-\frac{\cot^2 x}{2} + \ln|\csc x|\right]_{\pi/6}^{\pi/3}$
$= -\frac{1}{2}\left(\frac{1}{3} - 3\right) + \left(\ln\frac{2}{\sqrt{3}} - \ln 2\right) = \frac{4}{3} - \ln\sqrt{3}$

50. $\int 8\cot^4 t\, dt = 8\int (\csc^2 t - 1)\cot^2 t\, dt = 8\int \csc^2 t \cot^2 t\, dt - 8\int \cot^2 t\, dt = -\frac{8}{3}\cot^3 t - 8\int (\csc^2 t - 1)dt$
$= -\frac{8}{3}\cot^3 t + 8\cot t + 8t + C$

51. $\int \sin 3x \cos 2x\, dx = \frac{1}{2}\int (\sin x + \sin 5x)\, dx = -\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$

52. $\int \sin 2x \cos 3x\, dx = \frac{1}{2}\int (\sin(-x) + \sin 5x)\, dx = \frac{1}{2}\int (-\sin x + \sin 5x)\, dx = \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$

53. $\int_{-\pi}^{\pi} \sin 3x \sin 3x\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 0 - \cos 6x)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos 6x\, dx = \frac{1}{2}\left[x - \frac{1}{12}\sin 6x\right]_{-\pi}^{\pi} = \frac{\pi}{2} + \frac{\pi}{2} - 0 = \pi$

54. $\int_{0}^{\pi/2} \sin x \cos x\, dx = \frac{1}{2}\int_{0}^{\pi/2} (\sin 0 + \sin 2x)\, dx = \frac{1}{2}\int_{0}^{\pi/2} \sin 2x\, dx = -\frac{1}{4}[\cos 2x]_{0}^{\pi/2} = -\frac{1}{4}(-1-1) = \frac{1}{2}$

55. $\int \cos 3x \cos 4x\, dx = \frac{1}{2}\int (\cos(-x) + \cos 7x)\, dx = \frac{1}{2}\int (\cos x + \cos 7x)\, dx = \frac{1}{2}\sin x + \frac{1}{14}\sin 7x + C$

56. $\int_{-\pi/2}^{\pi/2} \cos 7x \cos x\, dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 6x + \cos 8x)\, dx = \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x\right]_{-\pi/2}^{\pi/2} = 0$

57. $\int \sin^2\theta \cos 3\theta\, d\theta = \int \frac{1-\cos 2\theta}{2} \cos 3\theta\, d\theta = \frac{1}{2}\int \cos 3\theta\, d\theta - \frac{1}{2}\int \cos 2\theta \cos 3\theta\, d\theta$
$= \frac{1}{2}\int \cos 3\theta\, d\theta - \frac{1}{2}\int \frac{1}{2}(\cos(2-3)\theta + \cos(2+3)\theta)\, d\theta = \frac{1}{2}\int \cos 3\theta\, d\theta - \frac{1}{4}\int (\cos(-\theta) + \cos 5\theta)\, d\theta$
$= \frac{1}{2}\int \cos 3\theta\, d\theta - \frac{1}{4}\int \cos\theta\, d\theta - \frac{1}{4}\int \cos 5\theta\, d\theta = \frac{1}{6}\sin 3\theta - \frac{1}{4}\sin\theta - \frac{1}{20}\sin 5\theta + C$

58. $\int \cos^2 2\theta \sin\theta\, d\theta = \int (2\cos^2\theta - 1)^2 \sin\theta\, d\theta = \int (4\cos^4\theta - 4\cos^2\theta + 1)\sin\theta\, d\theta$
$= \int 4\cos^4\theta \sin\theta\, d\theta - \int 4\cos^2\theta \sin\theta\, d\theta + \int \sin\theta\, d\theta = -\frac{4}{5}\cos^5\theta + \frac{4}{3}\cos^3\theta - \cos\theta + C$

59. $\int \cos^3\theta \sin 2\theta\, d\theta = \int \cos^3\theta\,(2\sin\theta\cos\theta)\, d\theta = 2\int \cos^4\theta \sin\theta\, d\theta = -\frac{2}{5}\cos^5\theta + C$

60. $\int \sin^3\theta \cos 2\theta\, d\theta = \int \sin^2\theta \cos 2\theta \sin\theta\, d\theta = \int \left(1 - \cos^2\theta\right)(2\cos^2\theta - 1)\sin\theta\, d\theta$
$= \int (-2\cos^4\theta + 3\cos^2\theta - 1)\sin\theta\, d\theta = -2\int \cos^4\theta \sin\theta\, d\theta + 3\int \cos^2\theta \sin\theta\, d\theta - \int \sin\theta\, d\theta$
$= \frac{2}{5}\cos^5\theta - \cos^3\theta + \cos\theta + C$

61. $\int \sin\theta \cos\theta \cos 3\theta\, d\theta = \frac{1}{2}\int 2\sin\theta\cos\theta\cos 3\theta\, d\theta = \frac{1}{2}\int \sin 2\theta \cos 3\theta\, d\theta = \frac{1}{2}\int \frac{1}{2}(\sin(2-3)\theta + \sin(2+3)\theta)\, d\theta$
$= \frac{1}{4}\int (\sin(-\theta) + \sin 5\theta)\, d\theta = \frac{1}{4}\int (-\sin\theta + \sin 5\theta)\, d\theta = \frac{1}{4}\cos\theta - \frac{1}{20}\cos 5\theta + C$

62. $\int \sin\theta \sin 2\theta \sin 3\theta\, d\theta = \int \frac{1}{2}\left(\cos(1-2)\theta - \cos(1+2)\theta\right)\sin 3\theta\, d\theta = \frac{1}{2}\int \left(\cos(-\theta) - \cos 3\theta\right)\sin 3\theta\, d\theta$
$= \frac{1}{2}\int \sin 3\theta \cos\theta\, d\theta - \frac{1}{2}\int \sin 3\theta \cos 3\theta\, d\theta = \frac{1}{2}\int \frac{1}{2}(\sin(3-1)\theta + \sin(3+1)\theta)d\theta - \frac{1}{4}\int 2\sin 3\theta \cos 3\theta\, d\theta$
$= \frac{1}{4}\int (\sin 2\theta + \sin 4\theta)d\theta - \frac{1}{4}\int \sin 6\theta\, d\theta = \frac{1}{4}\int (\sin 2\theta + \sin 4\theta)d\theta - \frac{1}{4}\int \sin 6\theta\, d\theta$
$= -\frac{1}{8}\cos 2\theta - \frac{1}{16}\cos 4\theta + \frac{1}{24}\cos 6\theta + C$

63. $\int \frac{\sec^3 x}{\tan x}\, dx = \int \frac{\sec^2 x \sec x}{\tan x}\, dx = \int \frac{\left(\tan^2 x + 1\right)\sec x}{\tan x}\, dx = \int \frac{\tan^2 x \sec x}{\tan x}\, dx + \int \frac{\sec x}{\tan x}\, dx = \int \tan x \sec x\, dx + \int \csc x\, dx$
$= \sec x - \ln|\csc x + \cot x| + C$

64. $\int \frac{\sin^3 x}{\cos^4 x}\, dx = \int \frac{\sin^2 x \sin x}{\cos^4 x}\, dx = \int \frac{\left(1-\cos^2 x\right)\sin x}{\cos^4 x}\, dx = \int \frac{\sin x}{\cos^4 x}\, dx - \int \frac{\cos^2 x \sin x}{\cos^4 x}\, dx = \int \sec^3 x \tan x\, dx - \int \sec x \tan x\, dx$
$= \int \sec^2 x \sec x \tan x\, dx - \int \sec x \tan x\, dx = \frac{1}{3}\sec^3 x - \sec x + C$

65. $\int \frac{\tan^2 x}{\csc x}\, dx = \int \frac{\sin^2 x}{\cos^2 x}\sin x\, dx = \int \frac{\left(1-\cos^2 x\right)}{\cos^2 x}\sin x\, dx = \int \frac{1}{\cos^2 x}\sin x\, dx - \int \frac{\cos^2 x}{\cos^2 x}\sin x\, dx = \int \sec x \tan x\, dx - \int \sin x\, dx$
$= \sec x + \cos x + C$

66. $\int \frac{\cot x}{\cos^2 x}\,dx = \int \frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x}\,dx = \int \frac{2}{2\sin x\cos x}\,dx = \int \frac{2}{\sin 2x}\,dx = \int \csc 2x\,2dx = -\ln|\csc 2x + \cot 2x| + C$

67. $\int x\sin^2 x\,dx = \int x\,\frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx \left[u = x,\ du = dx,\ dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x\right]$
$= \frac{1}{4}x^2 - \frac{1}{2}\left[\frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx\right] = \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$

68. $\int x\cos^3 x\,dx = \int x\cos^2 x\cos x\,dx = \int x(1-\sin^2 x)\cos x\,dx = \int x\cos x\,dx - \int x\sin^2 x\cos x\,dx;$
$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x;$
$\left[u = x,\ du = dx,\ dv = \cos x\,dx,\ v = \sin x\right]$
$\int x\sin^2 x\cos x\,dx = \frac{1}{3}x\sin^3 x - \int \frac{1}{3}\sin^3 x\,dx;$
$\left[u = x,\ du = dx,\ dv = \sin^2 x\cos x\,dx,\ v = \frac{1}{3}\sin^3 x\right]$
$= \frac{1}{3}x\sin^3 x - \frac{1}{3}\int (1-\cos^2 x)\sin x\,dx = \frac{1}{3}x\sin^3 x - \frac{1}{3}\int \sin x\,dx + \frac{1}{3}\int \cos^2 x\sin x\,dx = \frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x;$
$\Rightarrow \int x\cos x\,dx - \int x\sin^2 x\cos x\,dx = (x\sin x + \cos x) - \left(\frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x\right) + C$
$= x\sin x - \frac{1}{3}x\sin^3 x + \frac{2}{3}\cos x + \frac{1}{9}\cos^3 x + C$

69. $y = \ln(\sec x);\ y' = \frac{\sec x\tan x}{\sec x} = \tan x; (y')^2 = \tan^2 x;\ \int_0^{\pi/4}\sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4}|\sec x|\,dx = [\ln|\sec x + \tan x|]_0^{\pi/4}$
$= \ln\left(\sqrt{2}+1\right) - \ln(0+1) = \ln\left(\sqrt{2}+1\right)$

70. $M = \int_{-\pi/4}^{\pi/4}\sec x\,dx = [\ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln|\sqrt{2}-1| = \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$
$\bar{y} = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}\int_{-\pi/4}^{\pi/4}\frac{\sec^2 x}{2}\,dx = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}[\tan x]_{-\pi/4}^{\pi/4} = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}(1-(-1)) = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}$
$\Rightarrow (\bar{x},\bar{y}) = \left(0, \left(\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^{-1}\right)$

71. $V = \pi\int_0^{\pi}\sin^2 x\,dx = \pi\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx = \frac{\pi}{2}\int_0^{\pi}dx - \frac{\pi}{2}\int_0^{\pi}\cos 2x\,dx = \frac{\pi}{2}[x]_0^{\pi} - \frac{\pi}{4}[\sin 2x]_0^{\pi} = \frac{\pi}{2}(\pi-0) - \frac{\pi}{4}(0-0) = \frac{\pi^2}{2}$

72. $A = \int_0^{\pi}\sqrt{1+\cos 4x}\,dx = \int_0^{\pi}\sqrt{2}\,|\cos 2x|dx = \sqrt{2}\int_0^{\pi/4}\cos 2x\,dx - \sqrt{2}\int_{\pi/4}^{3\pi/4}\cos 2x\,dx + \sqrt{2}\int_{3\pi/4}^{\pi}\cos 2x\,dx$
$= \frac{\sqrt{2}}{2}[\sin 2x]_0^{\pi/4} - \frac{\sqrt{2}}{2}[\sin 2x]_{\pi/4}^{3\pi/4} + \frac{\sqrt{2}}{2}[\sin 2x]_{3\pi/4}^{\pi} = \frac{\sqrt{2}}{2}(1-0) - \frac{\sqrt{2}}{2}(-1-1) + \frac{\sqrt{2}}{2}(0+1) = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

73. $M = \int_0^{2\pi}(x+\cos x)dx = \left[\frac{1}{2}x^2 + \sin x\right]_0^{2\pi} = \left(\frac{1}{2}(2\pi)^2 + \sin(2\pi)\right) - \left(\frac{1}{2}(0)^2 + \sin(0)\right) = 2\pi^2;$
$\bar{x} = \frac{1}{2\pi^2}\int_0^{2\pi}x(x+\cos x)dx = \frac{1}{2\pi^2}\int_0^{2\pi}(x^2 + x\cos x)dx = \frac{1}{2\pi^2}\int_0^{2\pi}x^2dx + \frac{1}{2\pi^2}\int_0^{2\pi}x\cos x dx$
$\left[u = x,\ du = dx,\ dv = \cos x\,dx,\ v = \sin x\right]$
$= \frac{1}{6\pi^2}\left[x^3\right]_0^{2\pi} + \frac{1}{2\pi^2}\left(\left[x\sin x\right]_0^{2\pi} - \int_0^{2\pi}\sin x dx\right) = \frac{1}{6\pi^2}(8\pi^3 - 0) + \frac{1}{2\pi^2}\left(2\pi\sin 2\pi - 0 - \int_0^{2\pi}\sin x dx\right)$
$= \frac{4\pi}{3} + \frac{1}{2\pi^2}\left[\cos x\right]_0^{2\pi} = \frac{4\pi}{3} + \frac{1}{2\pi^2}(\cos 2\pi - \cos 0) = \frac{4\pi}{3} + 0 = \frac{4\pi}{3};\ \bar{y} = \frac{1}{2\pi^2}\int_0^{2\pi}\frac{1}{2}(x+\cos x)^2dx$
$= \frac{1}{4\pi^2}\int_0^{2\pi}(x^2 + 2x\cos x + \cos^2 x)dx = \frac{1}{4\pi^2}\int_0^{2\pi}x^2dx + \frac{1}{2\pi^2}\int_0^{2\pi}x\cos x dx + \frac{1}{4\pi^2}\int_0^{2\pi}\cos^2 x\,dx$

$= \frac{1}{12\pi^2}\left[x^3\right]_0^{2\pi} + \frac{1}{2\pi^2}\left[x\sin x + \cos x\right]_0^{2\pi} + \frac{1}{4\pi^2}\int_0^{2\pi} \frac{\cos 2x + 1}{2}\,dx = \frac{2\pi}{3} + 0 + \frac{1}{8\pi^2}\int_0^{2\pi}\cos 2x\,dx + \frac{1}{8\pi^2}\int_0^{2\pi} dx$

$= \frac{2\pi}{3} + \frac{1}{16\pi^2}\left[\sin 2x\right]_0^{2\pi} + \frac{1}{8\pi^2}\left[x\right]_0^{2\pi} = \frac{2\pi}{3} + 0 + \frac{1}{4\pi} = \frac{8\pi^2+3}{12\pi} \Rightarrow$ The centroid is $\left(\frac{4\pi}{3}, \frac{8\pi^2+3}{12\pi}\right)$.

74. $V = \int_0^{\pi/3} \pi(\sin x + \sec x)^2\,dx = \pi\int_0^{\pi/3}(\sin^2 x + 2\sin x\sec x + \sec^2 x)\,dx$

$= \pi\int_0^{\pi/3}\sin^2 x\,dx + \pi\int_0^{\pi/3} 2\tan x\,dx + \pi\int_0^{\pi/3}\sec^2 x\,dx = \pi\int_0^{\pi/3}\frac{1-\cos 2x}{2}\,dx + 2\pi\left[\ln|\sec x|\right]_0^{\pi/3} + \pi\left[\tan x\right]_0^{\pi/3}$

$= \frac{\pi}{2}\int_0^{\pi/3} dx - \frac{\pi}{2}\int_0^{\pi/3}\cos 2x\,dx + 2\pi\left(\ln\left|\sec\frac{\pi}{3}\right| - \ln|\sec 0|\right) + \pi\left(\tan\frac{\pi}{3} - \tan 0\right)$

$= \frac{\pi}{2}\left[x\right]_0^{\pi/3} - \frac{\pi}{4}\left[\sin 2x\right]_0^{\pi/3} + 2\pi\ln 2 + \pi\sqrt{3} = \frac{\pi}{2}\left(\frac{\pi}{3} - 0\right) - \frac{\pi}{4}\left(\sin 2\left(\frac{\pi}{3}\right) - \sin 2(0)\right) + 2\pi\ln 2 + \pi\sqrt{3}$

$= \frac{\pi^2}{6} - \frac{\pi\sqrt{3}}{8} + 2\pi\ln 2 + \pi\sqrt{3} = \frac{\pi\left(4\pi + 21\sqrt{3} - 48\ln 2\right)}{24}$

8.3 TRIGONOMETRIC SUBSTITUTIONS

1. $x = 3\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{3\,d\theta}{\cos^2\theta},\ 9 + x^2 = 9\left(1 + \tan^2\theta\right) = 9\sec^2\theta \Rightarrow \frac{1}{\sqrt{9+x^2}} = \frac{1}{3|\sec\theta|} = \frac{|\cos\theta|}{3} = \frac{\cos\theta}{3}$;
$\left(\text{because } \cos\theta > 0 \text{ when } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$;
$\int \frac{dx}{\sqrt{9+x^2}} = 3\int \frac{\cos\theta\,d\theta}{3\cos^2\theta} = \int \frac{d\theta}{\cos\theta} = \ln|\sec\theta + \tan\theta| + C' = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C' = \ln\left|\sqrt{9+x^2} + x\right| + C$

2. $\int \frac{3\,dx}{\sqrt{1+9x^2}}$; $[3x = u] \rightarrow \int \frac{du}{\sqrt{1+u^2}}$; $u = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2},\ du = \frac{dt}{\cos^2 t},\ \sqrt{1+u^2} = |\sec t| = \sec t$;
$\int \frac{du}{\sqrt{1+u^2}} = \int \frac{dt}{\cos^2 t\,(\sec t)} = \int \sec t\,dt = \ln|\sec t + \tan t| + C = \ln\left|\sqrt{u^2+1} + u\right| + C = \ln\left|\sqrt{1+9x^2} + 3x\right| + C$

3. $\int_{-2}^{2} \frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2} = \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}(-1) = \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) - \left(\frac{1}{2}\right)\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$

4. $\int_0^2 \frac{dx}{8+2x^2} = \frac{1}{2}\int_0^2 \frac{dx}{4+x^2} = \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 = \frac{1}{2}\left(\frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{16}$

5. $\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{3/2} = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}$

6. $\int_0^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$; $[t = 2x] \rightarrow \int_0^{1/2\sqrt{2}} \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1}t\right]_0^{1/\sqrt{2}} = \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

7. $t = 5\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = 5\cos\theta\,d\theta,\ \sqrt{25-t^2} = 5\cos\theta$;
$\int \sqrt{25-t^2}\,dt = \int (5\cos\theta)(5\cos\theta)\,d\theta = 25\int\cos^2\theta\,d\theta = 25\int \frac{1+\cos 2\theta}{2}\,d\theta = 25\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C$
$= \frac{25}{2}(\theta + \sin\theta\cos\theta) + C = \frac{25}{2}\left[\sin^{-1}\left(\frac{t}{5}\right) + \left(\frac{t}{5}\right)\left(\frac{\sqrt{25-t^2}}{5}\right)\right] + C = \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25-t^2}}{2} + C$

8. $t = \frac{1}{3}\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\cos\theta\,d\theta,\ \sqrt{1-9t^2} = \cos\theta$;
$\int \sqrt{1-9t^2}\,dt = \frac{1}{3}\int(\cos\theta)(\cos\theta)\,d\theta = \frac{1}{3}\int\cos^2\theta\,d\theta = \frac{1}{6}(\theta + \sin\theta\cos\theta) + C = \frac{1}{6}\left[\sin^{-1}(3t) + 3t\sqrt{1-9t^2}\right] + C$

9. $x = \frac{7}{2}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{7}{2}\sec\theta\tan\theta\,d\theta,\ \sqrt{4x^2-49} = \sqrt{49\sec^2\theta - 49} = 7\tan\theta$;
$\int \frac{dx}{\sqrt{4x^2-49}} = \int \frac{\left(\frac{7}{2}\sec\theta\tan\theta\right)d\theta}{7\tan\theta} = \frac{1}{2}\int\sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right| + C$

10. $x = \frac{3}{5}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{3}{5}\sec\theta\tan\theta\, d\theta,\ \sqrt{25x^2 - 9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta;$
$\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta\right)d\theta}{3\tan\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$

11. $y = 7\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dy = 7\sec\theta\tan\theta\, d\theta,\ \sqrt{y^2 - 49} = 7\tan\theta;$
$\int \frac{\sqrt{y^2-49}}{y}\,dy = \int \frac{(7\tan\theta)(7\sec\theta\tan\theta)\,d\theta}{7\sec\theta} = 7\int \tan^2\theta\, d\theta = 7\int (\sec^2\theta - 1)\,d\theta = 7(\tan\theta - \theta) + C$
$= 7\left[\frac{\sqrt{y^2-49}}{7} - \sec^{-1}\left(\frac{y}{7}\right)\right] + C$

12. $y = 5\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dy = 5\sec\theta\tan\theta\, d\theta,\ \sqrt{y^2 - 25} = 5\tan\theta;$
$\int \frac{\sqrt{y^2-25}}{y^3}\,dy = \int \frac{(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta}{125\sec^3\theta} = \frac{1}{5}\int \tan^2\theta\cos^2\theta\, d\theta = \frac{1}{5}\int \sin^2\theta\, d\theta = \frac{1}{10}\int (1 - \cos 2\theta)\,d\theta$
$= \frac{1}{10}(\theta - \sin\theta\cos\theta) + C = \frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right) - \left(\frac{\sqrt{y^2-25}}{y}\right)\left(\frac{5}{y}\right)\right] + C = \left[\frac{\sec^{-1}\left(\frac{y}{5}\right)}{10} - \frac{\sqrt{y^2-25}}{2y^2}\right] + C$

13. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ \sqrt{x^2 - 1} = \tan\theta;$
$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec^2\theta\tan\theta} = \int \frac{d\theta}{\sec\theta} = \sin\theta + C = \frac{\sqrt{x^2-1}}{x} + C$

14. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ \sqrt{x^2 - 1} = \tan\theta;$
$\int \frac{2\,dx}{x^3\sqrt{x^2-1}} = \int \frac{2\tan\theta\sec\theta\, d\theta}{\sec^3\theta\tan\theta} = 2\int \cos^2\theta\, d\theta = 2\int \left(\frac{1+\cos 2\theta}{2}\right)d\theta = \theta + \sin\theta\cos\theta + C$
$= \theta + \tan\theta\cos^2\theta + C = \sec^{-1}x + \sqrt{x^2-1}\left(\frac{1}{x}\right)^2 + C = \sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2} + C$

15. $u = 9 - x^2 \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx;$
$\int \frac{x\,dx}{\sqrt{9-x^2}} = -\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -\sqrt{u} + C = -\sqrt{9-x^2} + C$

16. $x = 2\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = 2\sec^2\theta\, d\theta,\ 4 + x^2 = 4\sec^2\theta$
$\int \frac{x^2\,dx}{4+x^2} = \int \frac{(4\tan^2\theta)(2\sec^2\theta)d\theta}{4\sec^2\theta} = \int 2\tan^2\theta\, d\theta = 2\int (\sec^2\theta - 1)d\theta = 2\int \sec^2\theta\, d\theta - 2\int d\theta = 2\tan\theta - 2\theta + C$
$= x - 2\tan^{-1}\left(\frac{x}{2}\right) + C$

17. $x = 2\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{2\,d\theta}{\cos^2\theta},\ \sqrt{x^2+4} = \frac{2}{\cos\theta};$
$\int \frac{x^3\,dx}{\sqrt{x^2+4}} = \int \frac{(8\tan^3\theta)(\cos\theta)\,d\theta}{\cos^2\theta} = 8\int \frac{\sin^3\theta\, d\theta}{\cos^4\theta} = 8\int \frac{(\cos^2\theta - 1)(-\sin\theta)\,d\theta}{\cos^4\theta};$
$[t = \cos\theta] \rightarrow 8\int \frac{t^2-1}{t^4}\,dt = 8\int \left(\frac{1}{t^2} - \frac{1}{t^4}\right)dt = 8\left(-\frac{1}{t} + \frac{1}{3t^3}\right) + C = 8\left(-\sec\theta + \frac{\sec^3\theta}{3}\right) + C$
$= 8\left(-\frac{\sqrt{x^2+4}}{2} + \frac{(x^2+4)^{3/2}}{8\cdot 3}\right) + C = \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C = \frac{1}{3}(x^2-8)\sqrt{x^2+4} + C$

18. $x = \tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \sec^2\theta\, d\theta,\ \sqrt{x^2+1} = \sec\theta;$
$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sec\theta} = \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \frac{-\sqrt{x^2+1}}{x} + C$

19. $w = 2\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 2\cos\theta\, d\theta,\ \sqrt{4-w^2} = 2\cos\theta;$
$\int \frac{8\,dw}{w^2\sqrt{4-w^2}} = \int \frac{8\cdot 2\cos\theta\, d\theta}{4\sin^2\theta\cdot 2\cos\theta} = 2\int \frac{d\theta}{\sin^2\theta} = -2\cot\theta + C = \frac{-2\sqrt{4-w^2}}{w} + C$

20. $w = 3\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 3\cos\theta\, d\theta,\ \sqrt{9 - w^2} = 3\cos\theta;$
$\int \frac{\sqrt{9-w^2}}{w^2}\, dw = \int \frac{3\cos\theta \cdot 3\cos\theta\, d\theta}{9\sin^2\theta} = \int \cot^2\theta\, d\theta = \int \left(\frac{1-\sin^2\theta}{\sin^2\theta}\right) d\theta = \int (\csc^2\theta - 1)\, d\theta$
$= -\cot\theta - \theta + C = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$

21. $u = 5x \Rightarrow du = 5dx,\ a = 6$
$\int \frac{100}{36+25x^2} dx = 20\int \frac{1}{(6)^2+(5x)^2} 5dx = 20\int \frac{1}{a^2+u^2} du = 20 \cdot \frac{1}{6}\tan^{-1}\left(\frac{u}{6}\right) + C = \frac{10}{3}\tan^{-1}\left(\frac{5x}{6}\right) + C$

22. $u = x^2 - 4 \Rightarrow du = 2x\, dx \Rightarrow \frac{1}{2}du = x\, dx$
$\int x\sqrt{x^2-4}\, dx = \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2-4)^{3/2} + C$

23. $x = \sin\theta,\ 0 \le \theta \le \frac{\pi}{3},\ dx = \cos\theta\, d\theta,\ (1-x^2)^{3/2} = \cos^3\theta;$
$\int_0^{\sqrt{3}/2} \frac{4x^2\, dx}{(1-x^2)^{3/2}} = \int_0^{\pi/3} \frac{4\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta} = 4\int_0^{\pi/3}\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) d\theta = 4\int_0^{\pi/3}(\sec^2\theta - 1)\, d\theta$
$= 4\left[\tan\theta - \theta\right]_0^{\pi/3} = 4\sqrt{3} - \frac{4\pi}{3}$

24. $x = 2\sin\theta,\ 0 \le \theta \le \frac{\pi}{6},\ dx = 2\cos\theta\, d\theta,\ (4-x^2)^{3/2} = 8\cos^3\theta;$
$\int_0^1 \frac{dx}{(4-x^2)^{3/2}} = \int_0^{\pi/6} \frac{2\cos\theta\, d\theta}{8\cos^3\theta} = \frac{1}{4}\int_0^{\pi/6} \frac{d\theta}{\cos^2\theta} = \frac{1}{4}\left[\tan\theta\right]_0^{\pi/6} = \frac{\sqrt{3}}{12} = \frac{1}{4\sqrt{3}}$

25. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ (x^2-1)^{3/2} = \tan^3\theta;$
$\int \frac{dx}{(x^2-1)^{3/2}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\tan^3\theta} = \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = -\frac{x}{\sqrt{x^2-1}} + C$

26. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ (x^2-1)^{5/2} = \tan^5\theta;$
$\int \frac{x^2\, dx}{(x^2-1)^{5/2}} = \int \frac{\sec^2\theta \cdot \sec\theta\tan\theta\, d\theta}{\tan^5\theta} = \int \frac{\cos\theta}{\sin^4\theta}\, d\theta = -\frac{1}{3\sin^3\theta} + C = -\frac{x^3}{3(x^2-1)^{3/2}} + C$

27. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\, d\theta,\ (1-x^2)^{3/2} = \cos^3\theta;$
$\int \frac{(1-x^2)^{3/2}\, dx}{x^6} = \int \frac{\cos^3\theta \cdot \cos\theta\, d\theta}{\sin^6\theta} = \int \cot^4\theta\csc^2\theta\, d\theta = -\frac{\cot^5\theta}{5} + C = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$

28. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\, d\theta,\ (1-x^2)^{1/2} = \cos\theta;$
$\int \frac{(1-x^2)^{1/2}\, dx}{x^4} = \int \frac{\cos\theta \cdot \cos\theta\, d\theta}{\sin^4\theta} = \int \cot^2\theta\csc^2\theta\, d\theta = -\frac{\cot^3\theta}{3} + C = -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C$

29. $x = \frac{1}{2}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{1}{2}\sec^2\theta\, d\theta,\ (4x^2+1)^2 = \sec^4\theta;$
$\int \frac{8\, dx}{(4x^2+1)^2} = \int \frac{8\left(\frac{1}{2}\sec^2\theta\right) d\theta}{\sec^4\theta} = 4\int \cos^2\theta\, d\theta = 2(\theta + \sin\theta\cos\theta) + C = 2\tan^{-1}2x + \frac{4x}{(4x^2+1)} + C$

30. $t = \frac{1}{3}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\sec^2\theta\, d\theta,\ 9t^2 + 1 = \sec^2\theta;$
$\int \frac{6\, dt}{(9t^2+1)^2} = \int \frac{6\left(\frac{1}{3}\sec^2\theta\right) d\theta}{\sec^4\theta} = 2\int \cos^2\theta\, d\theta = \theta + \sin\theta\cos\theta + C = \tan^{-1}3t + \frac{3t}{(9t^2+1)} + C$

31. $u = x^2 - 1 \Rightarrow du = 2x\, dx \Rightarrow \frac{1}{2}du = x\, dx$
$\int \frac{x^3}{x^2-1}dx = \int\left(x + \frac{x}{x^2-1}\right)dx = \int x\, dx + \int \frac{x}{x^2-1}dx = \frac{1}{2}x^2 + \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}x^2 + \frac{1}{2}\ln|u| + C = \frac{1}{2}x^2 + \frac{1}{2}\ln|x^2-1| + C$

32. $u = 25 + 4x^2 \Rightarrow du = 8x\,dx \Rightarrow \frac{1}{8}du = x\,dx$
$\int \frac{x}{25+4x^2}dx = \frac{1}{8}\int \frac{1}{u}du = \frac{1}{8}\ln|u| + C = \frac{1}{8}\ln(25 + 4x^2) + C$

33. $v = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dv = \cos\theta\,d\theta,\ (1-v^2)^{5/2} = \cos^5\theta;$
$\int \frac{v^2\,dv}{(1-v^2)^{5/2}} = \int \frac{\sin^2\theta\cos\theta\,d\theta}{\cos^5\theta} = \int \tan^2\theta\sec^2\theta\,d\theta = \frac{\tan^3\theta}{3} + C = \frac{1}{3}\left(\frac{v}{\sqrt{1-v^2}}\right)^3 + C$

34. $r = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2};$
$\int \frac{(1-r^2)^{5/2}\,dr}{r^8} = \int \frac{\cos^5\theta\cdot\cos\theta\,d\theta}{\sin^8\theta} = \int \cot^6\theta\csc^2\theta\,d\theta = -\frac{\cot^7\theta}{7} + C = -\frac{1}{7}\left[\frac{\sqrt{1-r^2}}{r}\right]^7 + C$

35. Let $e^t = 3\tan\theta,\ t = \ln(3\tan\theta),\ \tan^{-1}\left(\frac{1}{3}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right),\ dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta,\ \sqrt{e^{2t}+9} = \sqrt{9\tan^2\theta+9} = 3\sec\theta;$
$\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3\tan\theta\cdot\sec^2\theta\,d\theta}{\tan\theta\cdot 3\sec\theta} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec\theta\,d\theta = [\ln|\sec\theta + \tan\theta|]_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)}$
$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3} + \frac{1}{3}\right) = \ln 9 - \ln\left(1 + \sqrt{10}\right)$

36. Let $e^t = \tan\theta,\ t = \ln(\tan\theta),\ \tan^{-1}\left(\frac{3}{4}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right),\ dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta,\ 1 + e^{2t} = 1 + \tan^2\theta = \sec^2\theta;$
$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{(\tan\theta)\left(\frac{\sec^2\theta}{\tan\theta}\right)d\theta}{\sec^3\theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta\,d\theta = [\sin\theta]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$

37. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}};\ \left[u = 2\sqrt{t},\ du = \frac{1}{\sqrt{t}}\,dt\right] \to \int_{1/\sqrt{3}}^{1} \frac{2\,du}{1+u^2};\ u = \tan\theta,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{4},\ du = \sec^2\theta\,d\theta,\ 1 + u^2 = \sec^2\theta;$
$\int_{1/\sqrt{3}}^{1} \frac{2\,du}{1+u^2} = \int_{\pi/6}^{\pi/4} \frac{2\sec^2\theta\,d\theta}{\sec^2\theta} = [2\theta]_{\pi/6}^{\pi/4} = 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi}{6}$

38. $y = e^{\tan\theta},\ 0 \le \theta \le \frac{\pi}{4},\ dy = e^{\tan\theta}\sec^2\theta\,d\theta,\ \sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2\theta} = \sec\theta;$
$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}\,d\theta = \int_0^{\pi/4} \sec\theta\,d\theta = [\ln|\sec\theta + \tan\theta|]_0^{\pi/4} = \ln\left(1 + \sqrt{2}\right)$

39. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta;$
$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec\theta\tan\theta} = \theta + C = \sec^{-1}x + C$

40. $x = \tan\theta,\ dx = \sec^2\theta\,d\theta,\ 1 + x^2 = \sec^2\theta;$
$\int \frac{dx}{x^2+1} = \int \frac{\sec^2\theta\,d\theta}{\sec^2\theta} = \theta + C = \tan^{-1}x + C$

41. $x = \sec\theta,\ dx = \sec\theta\tan\theta\,d\theta,\ \sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta;$
$\int \frac{x\,dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\cdot\sec\theta\tan\theta\,d\theta}{\tan\theta} = \int \sec^2\theta\,d\theta = \tan\theta + C = \sqrt{x^2-1} + C$

42. $x = \sin\theta,\ dx = \cos\theta\,d\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2};$
$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\,d\theta}{\cos\theta} = \theta + C = \sin^{-1}x + C$

43. Let $x^2 = \tan\theta,\ 0 \le \theta < \frac{\pi}{2},\ 2x\,dx = \sec^2\theta\,d\theta \Rightarrow x\,dx = \frac{1}{2}\sec^2\theta\,d\theta;\ \sqrt{1+x^4} = \sqrt{1+\tan^2\theta} = \sec\theta$
$\int \frac{x}{\sqrt{1+x^4}}dx = \frac{1}{2}\int \frac{\sec^2\theta}{\sec\theta}d\theta = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln|\sqrt{1+x^4} + x^2| + C$

44. Let $\ln x = \sin\theta,\ -\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$, $\frac{1}{x}dx = \cos\theta\, d\theta$, $\sqrt{1-(\ln x)^2} = \cos\theta$

$$\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx = \int \frac{\cos^2\theta}{\sin\theta}d\theta = \int \frac{1-\sin^2\theta}{\sin\theta}d\theta = \int \csc\theta\, d\theta - \int \sin\theta\, d\theta = -\ln|\csc\theta + \cot\theta| + \cos\theta + C$$

$$= -\ln\left|\frac{1}{\ln x} + \frac{\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C = -\ln\left|\frac{1+\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C$$

45. Let $u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u\, du \Rightarrow \int \sqrt{\frac{4-x}{x}}\, dx = \int \sqrt{\frac{4-u^2}{u^2}}\, 2u\, du = 2\int \sqrt{4-u^2}\, du$;

$u = 2\sin\theta$, $du = 2\cos\theta\, d\theta$, $0 < \theta \le \frac{\pi}{2}$, $\sqrt{4-u^2} = 2\cos\theta$

$$2\int \sqrt{4-u^2}\, du = 2\int (2\cos\theta)(2\cos\theta)\, d\theta = 8\int \cos^2\theta\, d\theta = 8\int \frac{1+\cos 2\theta}{2}\, d\theta = 4\int d\theta + 4\int \cos 2\theta\, d\theta$$

$$= 4\theta + 2\sin 2\theta + C = 4\theta + 4\sin\theta\cos\theta + C = 4\sin^{-1}\left(\frac{u}{2}\right) + 4\left(\frac{u}{2}\right)\left(\frac{\sqrt{4-u^2}}{2}\right) + C = 4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sqrt{x}\sqrt{4-x} + C$$

$$= 4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sqrt{4x-x^2} + C$$

46. Let $u = x^{3/2} \Rightarrow x = u^{2/3} \Rightarrow dx = \frac{2}{3}u^{-1/3}du$

$$\int \sqrt{\frac{x}{1-x^3}}dx = \int \sqrt{\frac{u^{2/3}}{1-(u^{2/3})^3}}\left(\frac{2}{3}u^{-1/3}\right)du = \int \frac{u^{1/3}}{\sqrt{1-u^2}}\left(\frac{2}{3u^{1/3}}\right)du = \frac{2}{3}\int \frac{1}{\sqrt{1-u^2}}du = \frac{2}{3}\sin^{-1}u + C = \frac{2}{3}\sin^{-1}\left(x^{3/2}\right) + C$$

47. Let $u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u\, du \Rightarrow \int \sqrt{x}\sqrt{1-x}\, dx = \int u\sqrt{1-u^2}\, 2u\, du = 2\int u^2\sqrt{1-u^2}\, du$;

$u = \sin\theta$, $du = \cos\theta\, d\theta$, $-\frac{\pi}{2} < \theta \le \frac{\pi}{2}$, $\sqrt{1-u^2} = \cos\theta$

$$2\int u^2\sqrt{1-u^2}\, du = 2\int \sin^2\theta\cos\theta\cos\theta\, d\theta = 2\int \sin^2\theta\cos^2\theta\, d\theta = \frac{1}{2}\int \sin^2 2\theta\, d\theta = \frac{1}{2}\int \frac{1-\cos 4\theta}{2}\, d\theta$$

$$= \frac{1}{4}\int d\theta - \frac{1}{4}\int \cos 4\theta\, d\theta = \frac{1}{4}\theta - \frac{1}{16}\sin 4\theta + C = \frac{1}{4}\theta - \frac{1}{8}\sin 2\theta\cos 2\theta + C = \frac{1}{4}\theta - \frac{1}{4}\sin\theta\cos\theta\,(2\cos^2\theta - 1) + C$$

$$= \frac{1}{4}\theta - \frac{1}{2}\sin\theta\cos^3\theta + \frac{1}{4}\sin\theta\cos\theta + C = \frac{1}{4}\sin^{-1}u - \frac{1}{2}u\,(1-u^2)^{3/2} - \frac{1}{4}u\sqrt{1-u^2} + C$$

$$= \frac{1}{4}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x}\,(1-x)^{3/2} - \frac{1}{4}\sqrt{x}\sqrt{1-x} + C$$

48. Let $w = \sqrt{x-1} \Rightarrow w^2 = x-1 \Rightarrow 2w\, dw = dx \Rightarrow \int \frac{\sqrt{x-2}}{\sqrt{x-1}}dx = \int \frac{\sqrt{w^2-1}}{w}2w\, dw = 2\int \sqrt{w^2-1}\, dw$

$w = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$, $0 < \theta < \frac{\pi}{2}$, $\sqrt{w^2-1} = \tan\theta$

$2\int \sqrt{w^2-1}\, dw = 2\int \tan\theta\sec\theta\tan\theta\, d\theta$; $u = \tan\theta$, $du = \sec^2\theta\, d\theta$, $dv = \sec\theta\tan\theta\, d\theta$, $v = \sec\theta$

$$2\int \tan\theta\sec\theta\tan\theta\, d\theta = 2\sec\theta\tan\theta - 2\int \sec^3\theta\, d\theta = 2\sec\theta\tan\theta - 2\int \sec^2\theta\sec\theta d\theta$$

$$= 2\sec\theta\tan\theta - 2\int (\tan^2\theta + 1)\sec\theta\, d\theta = 2\sec\theta\tan\theta - 2\left(\int \tan^2\theta\sec\theta\, d\theta + \int \sec\theta\, d\theta\right)$$

$$= 2\sec\theta\tan\theta - 2\ln|\sec\theta + \tan\theta| - 2\int \tan^2\theta\sec\theta\, d\theta \Rightarrow 2\int \tan^2\theta\sec\theta\, d\theta = \sec\theta\tan\theta - \ln|\sec\theta + \tan\theta| + C$$

$$= w\sqrt{w^2-1} - \ln|w + \sqrt{w^2-1}| + C = \sqrt{x-1}\sqrt{x-2} - \ln|\sqrt{x-1} + \sqrt{x-2}| + C$$

49. $x\frac{dy}{dx} = \sqrt{x^2-4}$; $dy = \sqrt{x^2-4}\,\frac{dx}{x}$; $y = \int \frac{\sqrt{x^2-4}}{x}\, dx$; $\begin{bmatrix} x = 2\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 2\sec\theta\tan\theta\, d\theta \\ \sqrt{x^2-4} = 2\tan\theta \end{bmatrix}$

$$\rightarrow\ y = \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)\, d\theta}{2\sec\theta} = 2\int \tan^2\theta\, d\theta = 2\int (\sec^2\theta - 1)\, d\theta = 2(\tan\theta - \theta) + C$$

$= 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right] + C$; $x = 2$ and $y = 0 \Rightarrow 0 = 0 + C \Rightarrow C = 0 \Rightarrow y = 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\frac{x}{2}\right]$

50. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1$, $dy = \frac{dx}{\sqrt{x^2-9}}$; $y = \int \frac{dx}{\sqrt{x^2-9}}$; $\begin{bmatrix} x = 3\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 3\sec\theta\tan\theta\,d\theta \\ \sqrt{x^2-9} = 3\tan\theta \end{bmatrix} \rightarrow y = \int \frac{3\sec\theta\tan\theta\,d\theta}{3\tan\theta}$

$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$; $x = 5$ and $y = \ln 3 \Rightarrow \ln 3 = \ln 3 + C \Rightarrow C = 0$

$\Rightarrow y = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|$

51. $(x^2+4)\frac{dy}{dx} = 3$, $dy = \frac{3\,dx}{x^2+4}$; $y = 3\int \frac{dx}{x^2+4} = \frac{3}{2}\tan^{-1}\frac{x}{2} + C$; $x = 2$ and $y = 0 \Rightarrow 0 = \frac{3}{2}\tan^{-1} 1 + C$

$\Rightarrow C = -\frac{3\pi}{8} \Rightarrow y = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{3\pi}{8}$

52. $(x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}$, $dy = \frac{dx}{(x^2+1)^{3/2}}$; $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $(x^2+1)^{3/2} = \sec^3\theta$;

$y = \int \frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \int \cos\theta\,d\theta = \sin\theta + C = \tan\theta\cos\theta + C = \frac{\tan\theta}{\sec\theta} + C = \frac{x}{\sqrt{x^2+1}} + C$; $x = 0$ and $y = 1$

$\Rightarrow 1 = 0 + C \Rightarrow y = \frac{x}{\sqrt{x^2+1}} + 1$

53. $A = \int_0^3 \frac{\sqrt{9-x^2}}{3}\,dx$; $x = 3\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$, $dx = 3\cos\theta\,d\theta$, $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\cos\theta$;

$A = \int_0^{\pi/2} \frac{3\cos\theta\cdot 3\cos\theta\,d\theta}{3} = 3\int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{3}{2}\left[\theta + \sin\theta\cos\theta\right]_0^{\pi/2} = \frac{3\pi}{4}$

54. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \pm b\sqrt{1-\frac{x^2}{a^2}}$; $A = 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = 4b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx$

$\left[x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ dx = a\cos\theta\,d\theta,\ \sqrt{1-\frac{x^2}{a^2}} = \cos\theta,\ x = 0 = a\sin\theta \Rightarrow \theta = 0,\ x = a = a\sin\theta \Rightarrow \theta = \frac{\pi}{2}\right]$

$4b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx = 4b\int_0^{\pi/2} \cos\theta\,(a\cos\theta)\,d\theta = 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta = 4ab\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta$

$= 2ab\int_0^{\pi/2} d\theta + 2ab\int_0^{\pi/2} \cos 2\theta\,d\theta = 2ab\Big[\theta\Big]_0^{\pi/2} + ab\Big[\sin 2\theta\Big]_0^{\pi/2} = 2ab\left(\frac{\pi}{2} - 0\right) + ab(\sin\pi - \sin 0) = \pi ab$

55. (a) $A = \int_0^{1/2} \sin^{-1}x\,dx$ $\left[u = \sin^{-1}x,\ du = \frac{1}{\sqrt{1-x^2}}dx,\ dv = dx,\ v = x\right]$

$= \Big[x\sin^{-1}x\Big]_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx = = \left(\frac{1}{2}\sin^{-1}\frac{1}{2} - 0\right) + \Big[\sqrt{1-x^2}\Big]_0^{1/2} = \frac{\pi + 6\sqrt{3} - 12}{12}$

(b) $M = \int_0^{1/2} \sin^{-1}x\,dx = \frac{\pi + 6\sqrt{3} - 12}{12}$; $\overline{x} = \frac{1}{\frac{\pi+6\sqrt{3}-12}{12}}\int_0^{1/2} x\sin^{-1}x\,dx = \frac{12}{\pi+6\sqrt{3}-12}\int_0^{1/2} x\sin^{-1}x\,dx$

$\left[u = \sin^{-1}x,\ du = \frac{1}{\sqrt{1-x^2}}dx,\ dv = x\,dx,\ v = \frac{1}{2}x^2\right]$

$= \frac{12}{\pi+6\sqrt{3}-12}\left(\Big[\frac{1}{2}x^2\sin^{-1}x\Big]_0^{1/2} - \frac{1}{2}\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx\right)$

$\left[x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ \sqrt{1-x^2} = \cos\theta,\ x = 0 = \sin\theta \Rightarrow \theta = 0,\ x = \frac{1}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{6}\right]$

$= \frac{12}{\pi+6\sqrt{3}-12}\left(\left(\frac{1}{2}\left(\frac{1}{2}\right)^2\sin^{-1}\left(\frac{1}{2}\right) - 0\right) - \frac{1}{2}\int_0^{\pi/6} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta\right) = \frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48} - \frac{1}{2}\int_0^{\pi/6} \sin^2\theta\,d\theta\right)$

$= \frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48} - \frac{1}{2}\int_0^{\pi/6} \frac{1-\cos 2\theta}{2}\,d\theta\right) = \frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48} - \frac{1}{4}\int_0^{\pi/6} d\theta + \frac{1}{4}\int_0^{\pi/6} \cos 2\theta\,d\theta\right)$

$= \frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48} + \left[-\frac{\theta}{4} + \frac{1}{8}\sin 2\theta\right]_0^{\pi/6}\right) = \frac{3\sqrt{3}-\pi}{4\left(\pi+6\sqrt{3}-12\right)}$; $\overline{y} = \frac{1}{\frac{\pi+6\sqrt{3}-12}{12}}\int_0^{1/2} \frac{1}{2}(\sin^{-1}x)^2\,dx$

$\left[u = (\sin^{-1}x)^2,\ du = \frac{2\sin^{-1}x}{\sqrt{1-x^2}}dx,\ dv = dx,\ v = x\right]$

$$= \frac{6}{\pi + 6\sqrt{3} - 12}\left(\left[x(\sin^{-1}x\ dx)^2\right]_0^{1/2} - \int_0^{1/2} \frac{2x\sin^{-1}x}{\sqrt{1-x^2}}dx\right)$$

$$\left[u = \sin^{-1}x,\ du = \frac{1}{\sqrt{1-x^2}}dx,\ dv = \frac{2x}{\sqrt{1-x^2}}dx,\ v = -2\sqrt{1-x^2}\right]$$

$$= \frac{6}{\pi + 6\sqrt{3} - 12}\left(\left(\frac{1}{2}\left(\sin^{-1}\left(\frac{1}{2}\right)\right)^2 - 0\right) + \left[2\sqrt{1-x^2}\sin^{-1}x\right]_0^{1/2} - \int_0^{1/2}\frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}}dx\right)$$

$$= \frac{6}{\pi + 6\sqrt{3} - 12}\left(\frac{\pi^2}{72} + \left(2\sqrt{1-\left(\frac{1}{2}\right)^2}\sin^{-1}\left(\frac{1}{2}\right) - 0\right) - \left[2x\right]_0^{1/2}\right) = \frac{6}{\pi + 6\sqrt{3} - 12}\left(\frac{\pi^2}{72} + \frac{\pi\sqrt{3}}{6} - 1\right) = \frac{\pi^2 + 12\pi\sqrt{3} - 72}{12\left(\pi + 6\sqrt{3} - 12\right)}$$

56. $V = \int_0^1 \pi\left(\sqrt{x\tan^{-1}x}\right)^2 dx = \pi\int_0^1 x\tan^{-1}x\ dx \qquad \left[u = \tan^{-1}x,\ du = \frac{1}{1+x^2}dx,\ dv = x\,dx,\ v = \frac{1}{2}x^2\right]$

$$= \pi\left(\left[\frac{1}{2}x^2\tan^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}dx\right) = \pi\left(\left(\frac{1}{2}\tan^{-1}1 - 0\right) - \frac{1}{2}\int_0^1\left(1 - \frac{1}{1+x^2}\right)dx\right) = \pi\left(\frac{\pi}{8} - \frac{1}{2}\int_0^1\left(1 - \frac{1}{1+x^2}\right)dx\right)$$

$$= \pi\left(\frac{\pi}{8} - \frac{1}{2}\int_0^1 dx + \frac{1}{2}\int_0^1\frac{1}{1+x^2}dx\right) = \pi\left(\frac{\pi}{8} + \left[-\frac{1}{2}x + \frac{1}{2}\tan^{-1}x\right]_0^1\right) = \pi\left(\frac{\pi}{8} + \left(-\frac{1}{2} + \frac{1}{2}\tan^{-1}1 + 0 - 0\right)\right) = \frac{\pi(\pi-2)}{4}$$

57. (a) Integration by parts: $u = x^2,\ du = 2x\,dx,\ dv = x\sqrt{1-x^2}\,dx,\ v = -\frac{1}{3}(1-x^2)^{3/2}$

$$\int x^3\sqrt{1-x^2}\,dx = -\frac{1}{3}x^2(1-x^2)^{3/2} + \frac{1}{3}\int(1-x^2)^{3/2}\,2x\,dx = -\frac{1}{3}x^2(1-x^2)^{3/2} - \frac{2}{15}(1-x^2)^{5/2} + C$$

(b) Substitution: $u = 1 - x^2 \Rightarrow x^2 = 1 - u \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx$

$$\int x^3\sqrt{1-x^2}\,dx = \int x^2\sqrt{1-x^2}\,x\,dx = -\frac{1}{2}\int(1-u)\sqrt{u}\,du = -\frac{1}{2}\int\left(\sqrt{u} - u^{3/2}\right)du = -\frac{1}{3}u^{3/2} + \frac{1}{5}u^{5/2} + C$$

$$= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C$$

(c) Trig substitution: $x = \sin\theta,\ \frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ \sqrt{1-x^2} = \cos\theta$

$$\int x^3\sqrt{1-x^2}\,dx = \int\sin^3\theta\cos\theta\cos\theta\,d\theta = \int\sin^2\theta\cos^2\theta\sin\theta\,d\theta = \int(1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta$$

$$= \int\cos^2\theta\sin\theta\,d\theta - \int\cos^4\theta\sin\theta\,d\theta = -\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C = -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C$$

58. (a) The slope of the line tangent to $y = f(x)$ is given by $f'(x)$. Consider the triangle whose hypotenuse is the 30 ft rope, the length of the base is x and the height $h = \sqrt{900 - x^2}$. The slope of the tangent line is also $-\frac{\sqrt{900-x^2}}{x}$, thus $f'(x) = -\frac{\sqrt{900-x^2}}{x}$.

(b) $f(x) = \int -\frac{\sqrt{900-x^2}}{x}dx \qquad \left[x = 30\sin\theta,\ 0 < \theta \le \frac{\pi}{2},\ dx = 30\cos\theta\,d\theta,\ \sqrt{900 - x^2} = 30\cos\theta\right]$

$$= -\int\frac{30\cos\theta}{30\sin\theta}30\cos\theta\,d\theta = -30\int\frac{\cos^2\theta}{\sin\theta}d\theta = -30\int\frac{(1-\sin^2\theta)}{\sin\theta}d\theta = -30\int\csc\theta\,d\theta + 30\int\sin\theta\,d\theta$$

$$= 30\ln|\csc\theta + \cot\theta| - 30\cos\theta + C = 30\ln\left|\frac{30}{x} + \frac{\sqrt{900-x^2}}{x}\right| - \sqrt{900 - x^2} + C;\ f(30) = 0$$

$$\Rightarrow 0 = 30\ln\left|\frac{30}{30} + \frac{\sqrt{900-30^2}}{30}\right| - \sqrt{900 - 30^2} + C = C \Rightarrow f(x) = 30\ln\left|\frac{30}{x} + \frac{\sqrt{900-x^2}}{x}\right| - \sqrt{900 - x^2}$$

8.4 INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

1. $\frac{5x-13}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow 5x - 13 = A(x-2) + B(x-3) = (A+B)x - (2A+3B)$

$$\Rightarrow \left.\begin{array}{c} A + B = 5 \\ 2A + 3B = 13 \end{array}\right\} \Rightarrow -B = (10 - 13) \Rightarrow B = 3 \Rightarrow A = 2;\ \text{thus, } \frac{5x-13}{(x-3)(x-2)} = \frac{2}{x-3} + \frac{3}{x-2}$$

2. $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 5x - 7 = A(x-1) + B(x-2) = (A+B)x - (A+2B)$

$$\Rightarrow \left.\begin{array}{c} A + B = 5 \\ A + 2B = 7 \end{array}\right\} \Rightarrow B = 2 \Rightarrow A = 3;\ \text{thus, } \frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$$

3. $\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x+4 = A(x+1)+B = Ax+(A+B) \Rightarrow \left.\begin{array}{c} A=1 \\ A+B=4 \end{array}\right\} \Rightarrow A=1$ and $B=3$; thus, $\frac{x+4}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}$

4. $\frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 2x+2 = A(x-1)+B = Ax+(-A+B) \Rightarrow \left.\begin{array}{c} A=2 \\ -A+B=2 \end{array}\right\}$
$\Rightarrow A=2$ and $B=4$; thus, $\frac{2x+2}{x^2-2x+1} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$

5. $\frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} \Rightarrow z+1 = Az(z-1)+B(z-1)+Cz^2 \Rightarrow z+1 = (A+C)z^2+(-A+B)z-B$
$\Rightarrow \left.\begin{array}{c} A+C=0 \\ -A+B=1 \\ -B=1 \end{array}\right\} \Rightarrow B=-1 \Rightarrow A=-2 \Rightarrow C=2$; thus, $\frac{z+1}{z^2(z-1)} = \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1}$

6. $\frac{z}{z^3-z^2-6z} = \frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2} \Rightarrow 1 = A(z+2)+B(z-3) = (A+B)z+(2A-3B)$
$\Rightarrow \left.\begin{array}{c} A+B=0 \\ 2A-3B=1 \end{array}\right\} \Rightarrow -5B=1 \Rightarrow B=-\frac{1}{5} \Rightarrow A=\frac{1}{5}$; thus, $\frac{z}{z^3-z^2-6z} = \frac{\frac{1}{5}}{z-3} + \frac{-\frac{1}{5}}{z+2}$

7. $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2}$
$\Rightarrow 5t+2 = A(t-2)+B(t-3) = (A+B)t+(-2A-3B) \Rightarrow \left.\begin{array}{c} A+B=5 \\ -2A-3B=2 \end{array}\right\} \Rightarrow -B=(10+2)=12$
$\Rightarrow B=-12 \Rightarrow A=17$; thus, $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$

8. $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^2(t^2+9)}$ (after long division); $\frac{-9t^2+9}{t^2(t^2+9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}$
$\Rightarrow -9t^2+9 = At(t^2+9)+B(t^2+9)+(Ct+D)t^2 = (A+C)t^3+(B+D)t^2+9At+9B$
$\Rightarrow \left.\begin{array}{c} A+C=0 \\ B+D=-9 \\ 9A=0 \\ 9B=9 \end{array}\right\} \Rightarrow A=0 \Rightarrow C=0; B=1 \Rightarrow D=-10$; thus, $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{1}{t^2} + \frac{-10}{t^2+9}$

9. $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x)+B(1-x); x=1 \Rightarrow A=\frac{1}{2}; x=-1 \Rightarrow B=\frac{1}{2}$;
$\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x} = \frac{1}{2}[\ln|1+x| - \ln|1-x|] + C$

10. $\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2)+Bx; x=0 \Rightarrow A=\frac{1}{2}; x=-2 \Rightarrow B=-\frac{1}{2}$;
$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}[\ln|x| - \ln|x+2|] + C$

11. $\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1)+B(x+6); x=1 \Rightarrow B=\frac{5}{7}; x=-6 \Rightarrow A=\frac{-2}{-7}=\frac{2}{7}$;
$\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C = \frac{1}{7}\ln|(x+6)^2(x-1)^5| + C$

12. $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3)+B(x-4); x=3 \Rightarrow B=\frac{7}{-1}=-7; x=4 \Rightarrow A=\frac{9}{1}=9$;
$\int \frac{2x+1}{x^2-7x+12}\,dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3} = 9\ln|x-4| - 7\ln|x-3| + C = \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$

13. $\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1)+B(y-3); y=-1 \Rightarrow B=\frac{-1}{-4}=\frac{1}{4}; y=3 \Rightarrow A=\frac{3}{4}$;
$\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8 = \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right)$
$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 = \frac{\ln 15}{2}$

14. $\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By;\ y=0 \Rightarrow A=4;\ y=-1 \Rightarrow B = \frac{3}{-1} = -3;$
$\int_{1/2}^{1} \frac{y+4}{y^2+y}\,dy = 4\int_{1/2}^{1}\frac{dy}{y} - 3\int_{1/2}^{1}\frac{dy}{y+1} = [4\ln|y| - 3\ln|y+1|]_{1/2}^{1} = (4\ln 1 - 3\ln 2) - \left(4\ln\frac{1}{2} - 3\ln\frac{3}{2}\right)$
$= \ln\frac{1}{8} - \ln\frac{1}{16} + \ln\frac{27}{8} = \ln\left(\frac{27}{8}\cdot\frac{1}{8}\cdot 16\right) = \ln\frac{27}{4}$

15. $\frac{1}{t^3+t^2-2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Rightarrow 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2);\ t=0 \Rightarrow A = -\frac{1}{2};\ t=-2$
$\Rightarrow B = \frac{1}{6};\ t=1 \Rightarrow C = \frac{1}{3};\ \int\frac{dt}{t^3+t^2-2t} = -\frac{1}{2}\int\frac{dt}{t} + \frac{1}{6}\int\frac{dt}{t+2} + \frac{1}{3}\int\frac{dt}{t-1}$
$= -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| + C$

16. $\frac{x+3}{2x^3-8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \Rightarrow \frac{1}{2}(x+3) = A(x+2)(x-2) + Bx(x-2) + Cx(x+2);\ x=0 \Rightarrow A = \frac{3}{-8};\ x=-2$
$\Rightarrow B = \frac{1}{16};\ x=2 \Rightarrow C = \frac{5}{16};\ \int\frac{x+3}{2x^3-8x}\,dx = -\frac{3}{8}\int\frac{dx}{x} + \frac{1}{16}\int\frac{dx}{x+2} + \frac{5}{16}\int\frac{dx}{x-2}$
$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + C = \frac{1}{16}\ln\left|\frac{(x-2)^5(x+2)}{x^6}\right| + C$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1) + B$
$= Ax + (A+B) \Rightarrow A=3,\ A+B=2 \Rightarrow A=3,\ B=-1;\ \int_0^1 \frac{x^3\,dx}{x^2+2x+1}$
$= \int_0^1 (x-2)\,dx + 3\int_0^1\frac{dx}{x+1} - \int_0^1\frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$
$= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$

18. $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{(x-1)^2}$ (after long division); $\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1) + B$
$= Ax + (-A+B) \Rightarrow A=3,\ -A+B=-2 \Rightarrow A=3,\ B=1;\ \int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$
$= \int_{-1}^0 (x+2)\,dx + 3\int_{-1}^0\frac{dx}{x-1} + \int_{-1}^0\frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}\right]_{-1}^0$
$= \left(0 + 0 + 3\ln 1 - \frac{1}{(-1)}\right) - \left(\frac{1}{2} - 2 + 3\ln 2 - \frac{1}{(-2)}\right) = 2 - 3\ln 2$

19. $\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$
$x=-1 \Rightarrow C = \frac{1}{4};\ x=1 \Rightarrow D = \frac{1}{4}$; coefficient of $x^3 = A+B \Rightarrow A+B = 0$; constant $= A - B + C + D$
$\Rightarrow A - B + C + D = 1 \Rightarrow A - B = \frac{1}{2}$; thus, $A = \frac{1}{4} \Rightarrow B = -\frac{1}{4};\ \int\frac{dx}{(x^2-1)^2}$
$= \frac{1}{4}\int\frac{dx}{x+1} - \frac{1}{4}\int\frac{dx}{x-1} + \frac{1}{4}\int\frac{dx}{(x+1)^2} + \frac{1}{4}\int\frac{dx}{(x-1)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C$

20. $\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1);\ x=-1$
$\Rightarrow C = -\frac{1}{2};\ x=1 \Rightarrow A = \frac{1}{4}$; coefficient of $x^2 = A+B \Rightarrow A+B = 1 \Rightarrow B = \frac{3}{4};\ \int\frac{x^2\,dx}{(x-1)(x^2+2x+1)}$
$= \frac{1}{4}\int\frac{dx}{x-1} + \frac{3}{4}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{dx}{(x+1)^2} = \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C = \frac{\ln|(x-1)(x+1)^3|}{4} + \frac{1}{2(x+1)} + C$

21. $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x+1);\ x=-1 \Rightarrow A = \frac{1}{2}$; coefficient of x^2
$= A+B \Rightarrow A+B = 0 \Rightarrow B = -\frac{1}{2}$; constant $= A+C \Rightarrow A+C = 1 \Rightarrow C = \frac{1}{2};\ \int_0^1\frac{dx}{(x+1)(x^2+1)}$
$= \frac{1}{2}\int_0^1\frac{dx}{x+1} + \frac{1}{2}\int_0^1\frac{(-x+1)}{x^2+1}\,dx = \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x\right]_0^1$
$= \left(\frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\tan^{-1}1\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\ln 1 + \frac{1}{2}\tan^{-1}0\right) = \frac{1}{4}\ln 2 + \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{(\pi + 2\ln 2)}{8}$

22. $\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} \Rightarrow 3t^2+t+4 = A(t^2+1) + (Bt+C)t;\ t=0 \Rightarrow A=4$; coefficient of t^2
$= A+B \Rightarrow A+B = 3 \Rightarrow B = -1$; coefficient of $t = C \Rightarrow C = 1;\ \int_1^{\sqrt{3}}\frac{3t^2+t+4}{t^3+1}\,dt$

$= 4\int_1^{\sqrt{3}} \frac{dt}{t} + \int_1^{\sqrt{3}} \frac{(-t+1)}{t^2+1}\,dt = \left[4\ln|t| - \frac{1}{2}\ln(t^2+1) + \tan^{-1} t\right]_1^{\sqrt{3}}$

$= \left(4\ln\sqrt{3} - \frac{1}{2}\ln 4 + \tan^{-1}\sqrt{3}\right) - \left(4\ln 1 - \frac{1}{2}\ln 2 + \tan^{-1} 1\right) = 2\ln 3 - \ln 2 + \frac{\pi}{3} + \frac{1}{2}\ln 2 - \frac{\pi}{4}$

$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12} = \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}$

23. $\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2} \Rightarrow y^2+2y+1 = (Ay+B)(y^2+1) + Cy + D$

$= Ay^3 + By^2 + (A+C)y + (B+D) \Rightarrow A = 0, B = 1; A + C = 2 \Rightarrow C = 2; B + D = 1 \Rightarrow D = 0;$

$\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy = \int \frac{1}{y^2+1}\,dy + 2\int \frac{y}{(y^2+1)^2}\,dy = \tan^{-1} y - \frac{1}{y^2+1} + C$

24. $\frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2} \Rightarrow 8x^2+8x+2 = (Ax+B)(4x^2+1) + Cx + D$

$= 4Ax^3 + 4Bx^2 + (A+C)x + (B+D); A = 0, B = 2; A + C = 8 \Rightarrow C = 8; B + D = 2 \Rightarrow D = 0;$

$\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx = 2\int \frac{dx}{4x^2+1} + 8\int \frac{x\,dx}{(4x^2+1)^2} = \tan^{-1} 2x - \frac{1}{4x^2+1} + C$

25. $\frac{2s+2}{(s^2+1)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3} \Rightarrow 2s+2$

$= (As+B)(s-1)^3 + C(s^2+1)(s-1)^2 + D(s^2+1)(s-1) + E(s^2+1)$

$= [As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B] + C(s^4 - 2s^3 + 2s^2 - 2s + 1) + D(s^3 - s^2 + s - 1)$
$\quad + E(s^2+1)$

$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$

$$\Rightarrow \left.\begin{array}{l} A + C = 0 \\ -3A + B - 2C + D = 0 \\ 3A - 3B + 2C - D + E = 0 \\ -A + 3B - 2C + D = 2 \\ -B + C - D + E = 2 \end{array}\right\} \text{summing all equations} \Rightarrow 2E = 4 \Rightarrow E = 2;$$

summing eqs (2) and (3) $\Rightarrow -2B + 2 = 0 \Rightarrow B = 1$; summing eqs (3) and (4) $\Rightarrow 2A + 2 = 2 \Rightarrow A = 0; C = 0$ from eq (1); then $-1 + 0 - D + 2 = 2$ from eq (5) $\Rightarrow D = -1$;

$\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1} s + C$

26. $\frac{s^4+81}{s(s^2+9)^2} = \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{(s^2+9)^2} \Rightarrow s^4 + 81 = A(s^2+9)^2 + (Bs+C)s(s^2+9) + (Ds+E)s$

$= A(s^4 + 18s^2 + 81) + (Bs^4 + Cs^3 + 9Bs^2 + 9Cs) + Ds^2 + Es$

$= (A+B)s^4 + Cs^3 + (18A + 9B + D)s^2 + (9C+E)s + 81A \Rightarrow 81A = 81$ or $A = 1; A + B = 1 \Rightarrow B = 0;$

$C = 0; 9C + E = 0 \Rightarrow E = 0; 18A + 9B + D = 0 \Rightarrow D = -18; \int \frac{s^4+81}{s(s^2+9)^2}\,ds = \int \frac{ds}{s} - 18\int \frac{s\,ds}{(s^2+9)^2}$

$= \ln|s| + \frac{9}{(s^2+9)} + C$

27. $\frac{x^2-x+2}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \Rightarrow x^2 - x + 2 = A(x^2+x+1) + (Bx+C)(x-1) = (A+B)x^2 + (A-B+C)x + (A-C)$

$\Rightarrow A + B = 1, A - B + C = -1, A - C = 2 \Rightarrow$ adding eq(2) and eq(3) $\Rightarrow 2A - B = 1$, add this equation to eq(1)

$\Rightarrow 3A = 2 \Rightarrow A = \frac{2}{3} \Rightarrow B = 1 - A = \frac{1}{3} \Rightarrow C = -1 - A + B = -\frac{4}{3}; \int \frac{x^2-x+2}{x^3-1}dx = \int\left(\frac{2/3}{x-1} + \frac{(1/3)x - 4/3}{x^2+x+1}\right)dx$

$= \frac{2}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{x-4}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}dx \quad \left[u = x + \frac{1}{2} \Rightarrow u - \frac{1}{2} = x \Rightarrow du = dx\right]$

$= \frac{2}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{u - \frac{9}{2}}{u^2 + \frac{3}{4}}du = \frac{2}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}du - \frac{3}{2}\int \frac{1}{u^2+\frac{3}{4}}du$

$= \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln\left|\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right| - \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + C = \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| - \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

28. $\frac{1}{x^4+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1} \Rightarrow 1 = A(x+1)(x^2-x+1) + Bx(x^2-x+1) + (Cx+D)x(x+1)$
$= (A+B+C)x^3 + (-B+C+D)x^2 + (B+D)x + A \Rightarrow A = 1, B+D = 0 \Rightarrow D = -B, -B+C+D = 0$
$\Rightarrow -2B + C = 0 \Rightarrow C = 2B, A+B+C = 0 \Rightarrow 1 + B + 2B = 0 \Rightarrow B = -\frac{1}{3} \Rightarrow C = -\frac{2}{3} \Rightarrow D = \frac{1}{3};$
$\int \frac{1}{x^4+x}dx = \int \left(\frac{1}{x} - \frac{1/3}{x+1} + \frac{(-2/3)x+1/3}{x^2-x+1}\right)dx = \int \frac{1}{x}dx - \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{2x-1}{x^2-x+1}dx$
$= \ln|x| - \frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x^2-x+1| + C$

29. $\frac{x^2}{x^4-1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \Rightarrow x^2 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x-1)(x+1)$
$= (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x - A + B - D \Rightarrow A+B+C = 0, -A+B+D = 1,$
$A+B-C = 0, -A+B-D = 0 \Rightarrow$ adding eq(1) to eq (3) gives $2A+2B = 0$, adding eq(2) to eq(4) gives
$-2A+2B = 1$, adding these two equations gives $4B = 1 \Rightarrow B = \frac{1}{4}$, using $2A+2B = 0 \Rightarrow A = -\frac{1}{4}$, using
$-A+B-D = 0 \Rightarrow D = \frac{1}{2}$, and using $A+B-C = 0 \Rightarrow C = 0$; $\int \frac{x^2}{x^4-1}dx = \int \left(\frac{-1/4}{x+1} + \frac{1/4}{x-1} + \frac{1/2}{x^2+1}\right)dx$
$= -\frac{1}{4}\int \frac{1}{x+1}dx + \frac{1}{4}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx = -\frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + \frac{1}{2}\tan^{-1}x + C = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\tan^{-1}x + C$

30. $\frac{x^2+x}{x^4-3x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1} \Rightarrow x^2 + x = A(x+2)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x-2)(x+2)$
$= (A+B+C)x^3 + (2A-2B+D)x^2 + (A+B-4C)x + 2A - 2B - 4D \Rightarrow A+B+C = 0, 2A-2B+D = 1,$
$A+B-4C = 1, 2A-2B-4D = 0 \Rightarrow$ subtractin eq(1) from eq (3) gives $-5C = 1 \Rightarrow C = -\frac{1}{5}$, subtacting eq(2) from eq(4) gives $-5D = -1 \Rightarrow D = \frac{1}{5}$, substituting for C in eq(1) gives $A+B = \frac{1}{5}$, and substituting for D in eq(4) gives
$2A-2B = \frac{4}{5} \Rightarrow A-B = \frac{2}{5}$, adding this equation to the previous equatin gives $2A = \frac{3}{5} \Rightarrow A = \frac{3}{10} \Rightarrow B = -\frac{1}{10};$
$\int \frac{x^2+x}{x^4-3x^2-4}dx = \int \left(\frac{3/10}{x-2} - \frac{1/10}{x+2} + \frac{(-1/5)x+1/5}{x^2+1}\right)dx = \frac{3}{10}\int \frac{1}{x-2}dx - \frac{1}{10}\int \frac{1}{x+2}dx - \frac{1}{5}\int \frac{x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx$
$\frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln|x^2+1| + \frac{1}{5}\tan^{-1}x + C$

31. $\frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} = \frac{A\theta+B}{\theta^2+2\theta+2} + \frac{C\theta+D}{(\theta^2+2\theta+2)^2} \Rightarrow 2\theta^3+5\theta^2+8\theta+4 = (A\theta+B)(\theta^2+2\theta+2) + C\theta + D$
$= A\theta^3 + (2A+B)\theta^2 + (2A+2B+C)\theta + (2B+D) \Rightarrow A = 2; 2A+B = 5 \Rightarrow B = 1; 2A+2B+C = 8 \Rightarrow C = 2;$
$2B+D = 4 \Rightarrow D = 2; \int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta = \int \frac{2\theta+1}{(\theta^2+2\theta+2)}\,d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2}\,d\theta$
$= \int \frac{2\theta+2}{\theta^2+2\theta+2}\,d\theta - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{d(\theta^2+2\theta+2)}{(\theta^2+2\theta+2)^2} = \int \frac{d(\theta^2+2\theta+2)}{\theta^2+2\theta+2} - \int \frac{d\theta}{(\theta+1)^2+1} - \frac{1}{\theta^2+2\theta+2}$
$= \frac{-1}{\theta^2+2\theta+2} + \ln(\theta^2+2\theta+2) - \tan^{-1}(\theta+1) + C$

32. $\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{(\theta^2+1)^2} + \frac{E\theta+F}{(\theta^2+1)^3} \Rightarrow \theta^4-4\theta^3+2\theta^2-3\theta+1$
$= (A\theta+B)(\theta^2+1)^2 + (C\theta+D)(\theta^2+1) + E\theta + F = (A\theta+B)(\theta^4+2\theta^2+1) + (C\theta^3+D\theta^2+C\theta+D) + E\theta + F$
$= (A\theta^5+B\theta^4+2A\theta^3+2B\theta^2+A\theta+B) + (C\theta^3+D\theta^2+C\theta+D) + E\theta + F$
$= A\theta^5 + B\theta^4 + (2A+C)\theta^3 + (2B+D)\theta^2 + (A+C+E)\theta + (B+D+F) \Rightarrow A = 0; B = 1; 2A+C = -4$
$\Rightarrow C = -4; 2B+D = 2 \Rightarrow D = 0; A+C+E = -3 \Rightarrow E = 1; B+D+F = 1 \Rightarrow F = 0;$
$\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta = \int \frac{d\theta}{\theta^2+1} - 4\int \frac{\theta\,d\theta}{(\theta^2+1)^2} + \int \frac{\theta\,d\theta}{(\theta^2+1)^3} = \tan^{-1}\theta + 2(\theta^2+1)^{-1} - \frac{1}{4}(\theta^2+1)^{-2} + C$

33. $\frac{2x^3-2x^2+1}{x^2-x} = 2x + \frac{1}{x^2-x} = 2x + \frac{1}{x(x-1)}; \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + Bx; x = 0 \Rightarrow A = -1;$
$x = 1 \Rightarrow B = 1; \int \frac{2x^3-2x^2+1}{x^2-x} = \int 2x\,dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C = x^2 + \ln\left|\frac{x-1}{x}\right| + C$

34. $\frac{x^4}{x^2-1} = (x^2+1) + \frac{1}{x^2-1} = (x^2+1) + \frac{1}{(x+1)(x-1)}; \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x+1);$
$x = -1 \Rightarrow A = -\frac{1}{2}; x = 1 \Rightarrow B = \frac{1}{2}; \int \frac{x^4}{x^2-1}\,dx = \int (x^2+1)\,dx - \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$
$= \frac{1}{3}x^3 + x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C = \frac{x^3}{3} + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

35. $\frac{9x^3-3x+1}{x^3-x^2} = 9 + \frac{9x^2-3x+1}{x^2(x-1)}$ (after long division); $\frac{9x^2-3x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$
$\Rightarrow 9x^2 - 3x + 1 = Ax(x-1) + B(x-1) + Cx^2$; $x = 1 \Rightarrow C = 7$; $x = 0 \Rightarrow B = -1$; $A + C = 9 \Rightarrow A = 2$;
$\int \frac{9x^3-3x+1}{x^3-x^2}\,dx = \int 9\,dx + 2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 7\int \frac{dx}{x-1} = 9x + 2\ln|x| + \frac{1}{x} + 7\ln|x-1| + C$

36. $\frac{16x^3}{4x^2-4x+1} = (4x+4) + \frac{12x-4}{4x^2-4x+1}$; $\frac{12x-4}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} \Rightarrow 12x - 4 = A(2x-1) + B$
$\Rightarrow A = 6$; $-A + B = -4 \Rightarrow B = 2$; $\int \frac{16x^3}{4x^2-4x+1}\,dx = 4\int (x+1)\,dx + 6\int \frac{dx}{2x-1} + 2\int \frac{dx}{(2x-1)^2}$
$= 2(x+1)^2 + 3\ln|2x-1| - \frac{1}{2x-1} + C_1 = 2x^2 + 4x + 3\ln|2x-1| - (2x-1)^{-1} + C$, where $C = 2 + C_1$

37. $\frac{y^4+y^2-1}{y^3+y} = y - \frac{1}{y(y^2+1)}$; $\frac{1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1} \Rightarrow 1 = A(y^2+1) + (By+C)y = (A+B)y^2 + Cy + A$
$7 \Rightarrow A = 1$; $A + B = 0 \Rightarrow B = -1$; $C = 0$; $\int \frac{y^4+y^2-1}{y^3+y}\,dy = \int y\,dy - \int \frac{dy}{y} + \int \frac{y\,dy}{y^2+1}$
$= \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln(1+y^2) + C$

38. $\frac{2y^4}{y^3-y^2+y-1} = 2y + 2 + \frac{2}{y^3-y^2+y-1}$; $\frac{2}{y^3-y^2+y-1} = \frac{2}{(y^2+1)(y-1)} = \frac{A}{y-1} + \frac{By+C}{y^2+1}$
$\Rightarrow 2 = A(y^2+1) + (By+C)(y-1) = (Ay^2+A) + (By^2+Cy-By-C) = (A+B)y^2 + (-B+C)y + (A-C)$
$\Rightarrow A + B = 0$, $-B + C = 0$ or $C = B$, $A - C = A - B = 2 \Rightarrow A = 1, B = -1, C = -1$;
$\int \frac{2y^4}{y^3-y^2+y-1}\,dy = 2\int (y+1)\,dy + \int \frac{dy}{y-1} - \int \frac{y}{y^2+1}\,dy - \int \frac{dy}{y^2+1}$
$= (y+1)^2 + \ln|y-1| - \frac{1}{2}\ln(y^2+1) - \tan^{-1} y + C_1 = y^2 + 2y + \ln|y-1| - \frac{1}{2}\ln(y^2+1) - \tan^{-1} y + C$,
where $C = C_1 + 1$

39. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2} = [e^t = y]\int \frac{dy}{y^2+3y+2} = \int \frac{dy}{y+1} - \int \frac{dy}{y+2} = \ln\left|\frac{y+1}{y+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

40. $\int \frac{e^{4t}+2e^{2t}-e^t}{e^{2t}+1}\,dt = \int \frac{e^{3t}+2e^t-1}{e^{2t}+1}e^t dt$; $\begin{bmatrix} y = e^t \\ dy = e^t\,dt \end{bmatrix} \to \int \frac{y^3+2y-1}{y^2+1}\,dy = \int \left(y + \frac{y-1}{y^2+1}\right)dy = \frac{y^2}{2} + \int \frac{y}{y^2+1}\,dy - \int \frac{dy}{y^2+1}$
$= \frac{y^2}{2} + \frac{1}{2}\ln(y^2+1) - \tan^{-1} y + C = \frac{1}{2}e^{2t} + \frac{1}{2}\ln(e^{2t}+1) - \tan^{-1}(e^t) + C$

41. $\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$; $[\sin y = t, \cos y\,dy = dt] \to \int \frac{dy}{t^2+t-6} = \frac{1}{5}\int \left(\frac{1}{t-2} - \frac{1}{t+3}\right)dt = \frac{1}{5}\ln\left|\frac{t-2}{t+3}\right| + C$
$= \frac{1}{5}\ln\left|\frac{\sin y - 2}{\sin y + 3}\right| + C$

42. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$; $[\cos\theta = y] \to -\int \frac{dy}{y^2+y-2} = \frac{1}{3}\int \frac{dy}{y+2} - \frac{1}{3}\int \frac{dy}{y-1} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C = \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C$
$= \frac{1}{3}\ln\left|\frac{2+\cos\theta}{1-\cos\theta}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

43. $\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx = \int \frac{\tan^{-1}(2x)}{4x^2+1}\,dx - 3\int \frac{x}{(x-2)^2}\,dx$
$= \frac{1}{2}\int \tan^{-1}(2x)\,d(\tan^{-1}(2x)) - 3\int \frac{dx}{x-2} - 6\int \frac{dx}{(x-2)^2} = \frac{(\tan^{-1} 2x)^2}{4} - 3\ln|x-2| + \frac{6}{x-2} + C$

44. $\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\,dx = \int \frac{\tan^{-1}(3x)}{9x^2+1}\,dx + \int \frac{x}{(x+1)^2}\,dx$
$= \frac{1}{3}\int \tan^{-1}(3x)\,d(\tan^{-1}(3x)) + \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \frac{(\tan^{-1} 3x)^2}{6} + \ln|x+1| + \frac{1}{x+1} + C$

45. $\int \frac{1}{x^{3/2}-\sqrt{x}}dx = \int \frac{1}{\sqrt{x}(x-1)}dx$ $\left[\text{Let } u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}dx\right] \to \int \frac{2}{u^2-1}du$;
$\frac{2}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A + B = 0, -A + B = 2$

$\Rightarrow B = 1 \Rightarrow A = -1$; $\int \frac{2}{u^2-1}du = \int\left(\frac{-1}{u+1} + \frac{1}{u-1}\right)du = -\int \frac{1}{u+1}du + \int \frac{1}{u-1}du = -\ln|u+1| + \ln|u-1| + C$
$= \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$

46. $\int \frac{1}{(x^{1/3}-1)\sqrt{x}}dx$ $\left[\text{Let } x = u^6 \Rightarrow dx = 6u^5\,du\right] \to \int \frac{1}{(u^2-1)u^3}6u^5\,du = \int \frac{6u^2}{u^2-1}du = \int\left(6 + \frac{6}{u^2-1}\right)du$
$= 6\int du + \int \frac{6}{u^2-1}du$; $\frac{6}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 6 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A + B = 0,$
$-A + B = 6 \Rightarrow B = 3 \Rightarrow A = -3$; $6\int du + \int \frac{6}{u^2-1}du = 6u + \int\left(\frac{-3}{u+1} + \frac{3}{u-1}\right)du = 6u - 3\int \frac{1}{u+1}du + 3\int \frac{1}{u-1}du$
$= 6u - 3\ln|u+1| + 3\ln|u-1| + C = 6x^{1/6} + 3\ln\left|\frac{x^{1/6}-1}{x^{1/6}+1}\right| + C$

47. $\int \frac{\sqrt{x+1}}{x}\,dx$ $\left[\text{Let } x+1 = u^2 \Rightarrow dx = 2u\,du\right] \to \int \frac{u}{u^2-1}\,2u\,du = \int \frac{2u^2}{u^2-1}\,du = \int\left(2 + \frac{2}{u^2-1}\right)du$
$= 2\int du + \int \frac{2}{u^2-1}du$; $\frac{2}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A + B = 0,$
$-A + B = 2 \Rightarrow B = 1 \Rightarrow A = -1$; $2\int du + \int \frac{2}{u^2-1}du = 2u + \int\left(\frac{-1}{u+1} + \frac{1}{u-1}\right)du = 2u - \int \frac{1}{u+1}du + \int \frac{1}{u-1}du$
$= 2u - \ln|u+1| + \ln|u-1| + C = 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

48. $\int \frac{1}{x\sqrt{x+9}}\,dx$ $\left[\text{Let } x+9 = u^2 \Rightarrow dx = 2u\,du\right] \to \int \frac{1}{(u^2-9)u}\,2u\,du = \int \frac{2}{u^2-9}\,du$; $\frac{2}{u^2-9} = \frac{A}{u-3} + \frac{B}{u+3}$
$\Rightarrow 2 = A(u+3) + B(u-3) = (A+B)u + 3A - 3B \Rightarrow A + B = 0, 3A - 3B = 2 \Rightarrow A = \frac{1}{3} \Rightarrow B = -\frac{1}{3}$;
$\int \frac{2}{u^2-9}\,du = \int\left(\frac{1/3}{u-3} - \frac{1/3}{u+3}\right)du = \frac{1}{3}\int \frac{1}{u-3}\,du - \frac{1}{3}\int \frac{1}{u+3}\,du = \frac{1}{3}\ln|u-3| - \frac{1}{3}\ln|u+3| + C = \frac{1}{3}\ln\left|\frac{\sqrt{x+9}-3}{\sqrt{x+9}+3}\right| + C$

49. $\int \frac{1}{x(x^4+1)}dx = \int \frac{x^3}{x^4(x^4+1)}dx$ $\left[\text{Let } u = x^4 \Rightarrow du = 4x^3\,dx\right] \to \frac{1}{4}\int \frac{1}{u(u+1)}du$; $\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$
$\Rightarrow 1 = A(u+1) + Bu = (A+B)u + A \Rightarrow A = 1 \Rightarrow B = -1$; $\frac{1}{4}\int \frac{1}{u(u+1)}du = \frac{1}{4}\int\left(\frac{1}{u} - \frac{1}{u+1}\right)du$
$= \frac{1}{4}\int \frac{1}{u}du - \frac{1}{4}\int \frac{1}{u+1}du = \frac{1}{4}\ln|u| - \frac{1}{4}\ln|u+1| + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C$

50. $\int \frac{1}{x^6(x^5+4)}dx = \int \frac{x^4}{x^{10}(x^5+4)}dx = \left[\text{Let } u = x^5 \Rightarrow du = 5x^4\,dx\right] \to \frac{1}{5}\int \frac{1}{u^2(u+4)}du$; $\frac{1}{u^2(u+4)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+4}$
$\Rightarrow 1 = Au(u+4) + B(u+4) + Cu^2 = (A+C)u^2 + (4A+B)u + 4B \Rightarrow A + C = 0, 4A + B = 0, 4B = 1 \Rightarrow B = \frac{1}{4}$
$\Rightarrow A = -\frac{1}{16} \Rightarrow C = \frac{1}{16}$; $\frac{1}{5}\int \frac{1}{u^2(u+4)}du = \frac{1}{5}\int\left(-\frac{1/16}{u} + \frac{1/4}{u^2} + \frac{1/16}{u+4}\right)du = -\frac{1}{80}\int \frac{1}{u}du + \frac{1}{20}\int \frac{1}{u^2}du + \frac{1}{80}\int \frac{1}{u+4}du$
$= -\frac{1}{80}\ln|u| - \frac{1}{20u} + \frac{1}{80}\ln|u+4| + C = -\frac{1}{80}\ln|x^5| - \frac{1}{20x^5} + \frac{1}{80}\ln|x^5+4| + C = \frac{1}{80}\ln\left|\frac{x^5+4}{x^5}\right| - \frac{1}{20x^5} + C$

51. $(t^2 - 3t + 2)\frac{dx}{dt} = 1$; $x = \int \frac{dt}{t^2-3t+2} = \int \frac{dt}{t-2} - \int \frac{dt}{t-1} = \ln\left|\frac{t-2}{t-1}\right| + C$; $\frac{t-2}{t-1} = Ce^x$; $t = 3$ and $x = 0$
$\Rightarrow \frac{1}{2} = C \Rightarrow \frac{t-2}{t-1} = \frac{1}{2}e^x \Rightarrow x = \ln\left|2\left(\frac{t-2}{t-1}\right)\right| = \ln|t-2| - \ln|t-1| + \ln 2$

52. $(3t^4 + 4t^2 + 1)\frac{dx}{dt} = 2\sqrt{3}$; $x = 2\sqrt{3}\int \frac{dt}{3t^4+4t^2+1} = \sqrt{3}\int \frac{dt}{t^2+\frac{1}{3}} - \sqrt{3}\int \frac{dt}{t^2+1}$
$= 3\tan^{-1}\left(\sqrt{3}t\right) - \sqrt{3}\tan^{-1}t + C$; $t = 1$ and $x = \frac{-\pi\sqrt{3}}{4} \Rightarrow -\frac{\sqrt{3}\pi}{4} = \pi - \frac{\sqrt{3}}{4}\pi + C \Rightarrow C = -\pi$
$\Rightarrow x = 3\tan^{-1}\left(\sqrt{3}t\right) - \sqrt{3}\tan^{-1}t - \pi$

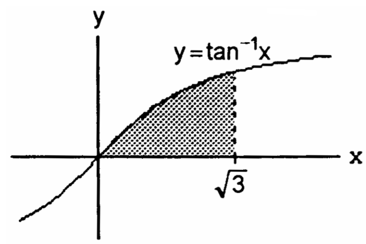
53. $(t^2+2t)\frac{dx}{dt}=2x+2;\ \frac{1}{2}\int\frac{dx}{x+1}=\int\frac{dt}{t^2+2t}\ \Rightarrow\ \frac{1}{2}\ln|x+1|=\frac{1}{2}\int\frac{dt}{t}-\frac{1}{2}\int\frac{dt}{t+2}\ \Rightarrow\ \ln|x+1|=\ln\left|\frac{t}{t+2}\right|+C;$
$t=1$ and $x=1\ \Rightarrow\ \ln 2=\ln\frac{1}{3}+C\ \Rightarrow\ C=\ln 2+\ln 3=\ln 6\ \Rightarrow\ \ln|x+1|=\ln 6\left|\frac{t}{t+2}\right|\ \Rightarrow\ x+1=\frac{6t}{t+2}$
$\Rightarrow\ x=\frac{6t}{t+2}-1,\ t>0$

54. $(t+1)\frac{dx}{dt}=x^2+1\ \Rightarrow\ \int\frac{dx}{x^2+1}=\int\frac{dt}{t+1}\ \Rightarrow\ \tan^{-1}x=\ln|t+1|+C;\ t=0$ and $x=0\ \Rightarrow\ \tan^{-1}0=\ln|1|+C$
$\Rightarrow\ C=\tan^{-1}0=0\ \Rightarrow\ \tan^{-1}x=\ln|t+1|\Rightarrow\ x=\tan(\ln(t+1)),\ t>-1$

55. $V=\pi\int_{0.5}^{2.5}y^2\,dx=\pi\int_{0.5}^{2.5}\frac{9}{3x-x^2}\,dx=3\pi\left(\int_{0.5}^{2.5}\left(-\frac{1}{x-3}+\frac{1}{x}\right)\right)dx=\left[3\pi\ln\left|\frac{x}{x-3}\right|\right]_{0.5}^{2.5}=3\pi\ln 25$

56. $V=2\pi\int_0^1 xy\,dx=2\pi\int_0^1\frac{2x}{(x+1)(2-x)}\,dx=4\pi\int_0^1\left(-\frac{1}{3}\left(\frac{1}{x+1}\right)+\frac{2}{3}\left(\frac{1}{2-x}\right)\right)dx$
$=\left[-\frac{4\pi}{3}(\ln|x+1|+2\ln|2-x|)\right]_0^1=\frac{4\pi}{3}(\ln 2)$

57. $A=\int_0^{\sqrt{3}}\tan^{-1}x\,dx=\left[x\tan^{-1}x\right]_0^{\sqrt{3}}-\int_0^{\sqrt{3}}\frac{x}{1+x^2}\,dx$
$=\frac{\pi\sqrt{3}}{3}-\left[\frac{1}{2}\ln(x^2+1)\right]_0^{\sqrt{3}}=\frac{\pi\sqrt{3}}{3}-\ln 2;$
$\overline{x}=\frac{1}{A}\int_0^{\sqrt{3}}x\tan^{-1}x\,dx$
$=\frac{1}{A}\left(\left[\frac{1}{2}x^2\tan^{-1}x\right]_0^{\sqrt{3}}-\frac{1}{2}\int_0^{\sqrt{3}}\frac{x^2}{1+x^2}\,dx\right)$
$=\frac{1}{A}\left[\frac{\pi}{2}-\left[\frac{1}{2}(x-\tan^{-1}x)\right]_0^{\sqrt{3}}\right]$
$=\frac{1}{A}\left(\frac{\pi}{2}-\frac{\sqrt{3}}{2}+\frac{\pi}{6}\right)=\frac{1}{A}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)\cong 1.10$



58. $A=\int_3^5\frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx=3\int_3^5\frac{dx}{x}-\int_3^5\frac{dx}{x+3}+2\int_3^5\frac{dx}{x-1}=\left[3\ln|x|-\ln|x+3|+2\ln|x-1|\right]_3^5=\ln\frac{125}{9};$
$\overline{x}=\frac{1}{A}\int_3^5\frac{x(4x^2+13x-9)}{x^3+2x^2-3x}\,dx=\frac{1}{A}\left([4x]_3^5+3\int_3^5\frac{dx}{x+3}+2\int_3^5\frac{dx}{x-1}\right)=\frac{1}{A}(8+11\ln 2-3\ln 6)\cong 3.90$

59. (a) $\frac{dx}{dt}=kx(N-x)\ \Rightarrow\int\frac{dx}{x(N-x)}=\int k\,dt\ \Rightarrow\ \frac{1}{N}\int\frac{dx}{x}+\frac{1}{N}\int\frac{dx}{N-x}=\int k\,dt\ \Rightarrow\ \frac{1}{N}\ln\left|\frac{x}{N-x}\right|=kt+C;$
$k=\frac{1}{250},\ N=1000,\ t=0$ and $x=2\ \Rightarrow\ \frac{1}{1000}\ln\left|\frac{2}{998}\right|=C\ \Rightarrow\ \frac{1}{1000}\ln\left|\frac{x}{1000-x}\right|=\frac{t}{250}+\frac{1}{1000}\ln\left(\frac{1}{499}\right)$
$\Rightarrow\ \ln\left|\frac{499x}{1000-x}\right|=4t\ \Rightarrow\ \frac{499x}{1000-x}=e^{4t}\ \Rightarrow\ 499x=e^{4t}(1000-x)\ \Rightarrow\ (499+e^{4t})\,x=1000e^{4t}\ \Rightarrow\ x=\frac{1000e^{4t}}{499+e^{4t}}$
(b) $x=\frac{1}{2}N=500\ \Rightarrow\ 500=\frac{1000e^{4t}}{499+e^{4t}}\ \Rightarrow\ 500\cdot 499+500e^{4t}=1000e^{4t}\ \Rightarrow\ e^{4t}=499\ \Rightarrow\ t=\frac{1}{4}\ln 499\approx 1.55$ days

60. $\frac{dx}{dt}=k(a-x)(b-x)\ \Rightarrow\ \frac{dx}{(a-x)(b-x)}=k\,dt$
(a) $a=b$: $\int\frac{dx}{(a-x)^2}=\int k\,dt\ \Rightarrow\ \frac{1}{a-x}=kt+C;\ t=0$ and $x=0\ \Rightarrow\ \frac{1}{a}=C\ \Rightarrow\ \frac{1}{a-x}=kt+\frac{1}{a}$
$\Rightarrow\ \frac{1}{a-x}=\frac{akt+1}{a}\ \Rightarrow\ a-x=\frac{a}{akt+1}\ \Rightarrow\ x=a-\frac{a}{akt+1}=\frac{a^2kt}{akt+1}$
(b) $a\neq b$: $\int\frac{dx}{(a-x)(b-x)}=\int k\,dt\ \Rightarrow\ \frac{1}{b-a}\int\frac{dx}{a-x}-\frac{1}{b-a}\int\frac{dx}{b-x}=\int k\,dt\ \Rightarrow\ \frac{1}{b-a}\ln\left|\frac{b-x}{a-x}\right|=kt+C;$
$t=0$ and $x=0\ \Rightarrow\ \frac{1}{b-a}\ln\frac{b}{a}=C\ \Rightarrow\ \ln\left|\frac{b-x}{a-x}\right|=(b-a)kt+\ln\left(\frac{b}{a}\right)\ \Rightarrow\ \frac{b-x}{a-x}=\frac{b}{a}e^{(b-a)kt}$
$\Rightarrow\ x=\frac{ab\left[1-e^{(b-a)kt}\right]}{a-be^{(b-a)kt}}$

8.5 INTEGRAL TABLES AND COMPUTER ALGEBRA SYSTEMS

1. $\int\frac{dx}{x\sqrt{x-3}}=\frac{2}{\sqrt{3}}\tan^{-1}\sqrt{\frac{x-3}{3}}+C$
(We used FORMULA 13(a) with $a=1,\ b=3$)

2. $\int \frac{dx}{x\sqrt{x+4}} = \frac{1}{\sqrt{4}} \ln \left| \frac{\sqrt{x+4}-\sqrt{4}}{\sqrt{x+4}+\sqrt{4}} \right| + C = \frac{1}{2} \ln \left| \frac{\sqrt{x+4}-2}{\sqrt{x+4}+2} \right| + C$

(We used FORMULA 13(b) with $a = 1$, $b = 4$)

3. $\int \frac{x\,dx}{\sqrt{x-2}} = \int \frac{(x-2)\,dx}{\sqrt{x-2}} + 2\int \frac{dx}{\sqrt{x-2}} = \int \left(\sqrt{x-2}\right)^1 dx + 2\int \left(\sqrt{x-2}\right)^{-1} dx$

$= \left(\frac{2}{1}\right) \frac{\left(\sqrt{x-2}\right)^3}{3} + 2\left(\frac{2}{1}\right) \frac{\left(\sqrt{x-2}\right)^1}{1} = \sqrt{x-2}\left[\frac{2(x-2)}{3} + 4\right] + C$

(We used FORMULA 11 with $a = 1$, $b = -2$, $n = 1$ and $a = 1$, $b = -2$, $n = -1$)

4. $\int \frac{x\,dx}{(2x+3)^{3/2}} = \frac{1}{2}\int \frac{(2x+3)\,dx}{(2x+3)^{3/2}} - \frac{3}{2}\int \frac{dx}{(2x+3)^{3/2}} = \frac{1}{2}\int \frac{dx}{\sqrt{2x+3}} - \frac{3}{2}\int \frac{dx}{\left(\sqrt{2x+3}\right)^3}$

$= \frac{1}{2}\int \left(\sqrt{2x+3}\right)^{-1} dx - \frac{3}{2}\int \left(\sqrt{2x+3}\right)^{-3} dx = \left(\frac{1}{2}\right)\left(\frac{2}{2}\right) \frac{\left(\sqrt{2x+3}\right)^1}{1} - \left(\frac{3}{2}\right)\left(\frac{2}{2}\right) \frac{\left(\sqrt{2x+3}\right)^{-1}}{(-1)} + C$

$= \frac{1}{2\sqrt{2x+3}}(2x+3+3) + C = \frac{(x+3)}{\sqrt{2x+3}} + C$

(We used FORMULA 11 with $a = 2$, $b = 3$, $n = -1$ and $a = 2$, $b = 3$, $n = -3$)

5. $\int x\sqrt{2x-3}\,dx = \frac{1}{2}\int (2x-3)\sqrt{2x-3}\,dx + \frac{3}{2}\int \sqrt{2x-3}\,dx = \frac{1}{2}\int \left(\sqrt{2x-3}\right)^3 dx + \frac{3}{2}\int \left(\sqrt{2x-3}\right)^1 dx$

$= \left(\frac{1}{2}\right)\left(\frac{2}{2}\right) \frac{\left(\sqrt{2x-3}\right)^5}{5} + \left(\frac{3}{2}\right)\left(\frac{2}{2}\right) \frac{\left(\sqrt{2x-3}\right)^3}{3} + C = \frac{(2x-3)^{3/2}}{2}\left[\frac{2x-3}{5} + 1\right] + C = \frac{(2x-3)^{3/2}(x+1)}{5} + C$

(We used FORMULA 11 with $a = 2$, $b = -3$, $n = 3$ and $a = 2$, $b = -3$, $n = 1$)

6. $\int x(7x+5)^{3/2}\,dx = \frac{1}{7}\int (7x+5)(7x+5)^{3/2}\,dx - \frac{5}{7}\int (7x+5)^{3/2}\,dx = \frac{1}{7}\int \left(\sqrt{7x+5}\right)^5 dx - \frac{5}{7}\int \left(\sqrt{7x+5}\right)^3 dx$

$= \left(\frac{1}{7}\right)\left(\frac{2}{7}\right) \frac{\left(\sqrt{7x+5}\right)^7}{7} - \left(\frac{5}{7}\right)\left(\frac{2}{7}\right) \frac{\left(\sqrt{7x+5}\right)^5}{5} + C = \left[\frac{(7x+5)^{5/2}}{49}\right]\left[\frac{2(7x+5)}{7} - 2\right] + C$

$= \left[\frac{(7x+5)^{5/2}}{49}\right]\left(\frac{14x-4}{7}\right) + C$

(We used FORMULA 11 with $a = 7$, $b = 5$, $n = 5$ and $a = 7$, $b = 5$, $n = 3$)

7. $\int \frac{\sqrt{9-4x}}{x^2}\,dx = -\frac{\sqrt{9-4x}}{x} + \frac{(-4)}{2}\int \frac{dx}{x\sqrt{9-4x}} + C$

(We used FORMULA 14 with $a = -4$, $b = 9$)

$= -\frac{\sqrt{9-4x}}{x} - 2\left(\frac{1}{\sqrt{9}}\right) \ln \left| \frac{\sqrt{9-4x}-\sqrt{9}}{\sqrt{9-4x}+\sqrt{9}} \right| + C$

(We used FORMULA 13(b) with $a = -4$, $b = 9$)

$= \frac{-\sqrt{9-4x}}{x} - \frac{2}{3} \ln \left| \frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3} \right| + C$

8. $\int \frac{dx}{x^2\sqrt{4x-9}} = -\frac{\sqrt{4x-9}}{(-9)x} + \frac{4}{18}\int \frac{dx}{x\sqrt{4x-9}} + C$

(We used FORMULA 15 with $a = 4$, $b = -9$)

$= \frac{\sqrt{4x-9}}{9x} + \left(\frac{2}{9}\right)\left(\frac{2}{\sqrt{9}}\right) \tan^{-1}\sqrt{\frac{4x-9}{9}} + C$

(We used FORMULA 13(a) with $a = 4$, $b = 9$)

$= \frac{\sqrt{4x-9}}{9x} + \frac{4}{27} \tan^{-1}\sqrt{\frac{4x-9}{9}} + C$

9. $\int x\sqrt{4x-x^2}\,dx = \int x\sqrt{2\cdot 2x - x^2}\,dx = \frac{(x+2)(2x-3\cdot 2)\sqrt{2\cdot 2\cdot x - x^2}}{6} + \frac{2^3}{2}\sin^{-1}\left(\frac{x-2}{2}\right) + C$

$= \frac{(x+2)(2x-6)\sqrt{4x-x^2}}{6} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C = \frac{(x+2)(x-3)\sqrt{4x-x^2}}{3} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$

(We used FORMULA 51 with $a = 2$)

10. $\int \frac{\sqrt{x-x^2}}{x}\,dx = \int \frac{\sqrt{2\cdot\frac{1}{2}x - x^2}}{x}\,dx = \sqrt{2\cdot\frac{1}{2}x - x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right) + C = \sqrt{x-x^2} + \frac{1}{2}\sin^{-1}(2x-1) + C$
(We used FORMULA 52 with $a = \frac{1}{2}$)

11. $\int \frac{dx}{x\sqrt{7+x^2}} = \int \frac{dx}{x\sqrt{\left(\sqrt{7}\right)^2 + x^2}} = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{\left(\sqrt{7}\right)^2+x^2}}{x}\right| + C = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{7+x^2}}{x}\right| + C$
(We used FORMULA 26 with $a = \sqrt{7}$)

12. $\int \frac{dx}{x\sqrt{7-x^2}} = \int \frac{dx}{x\sqrt{\left(\sqrt{7}\right)^2 - x^2}} = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{\left(\sqrt{7}\right)^2-x^2}}{x}\right| + C = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{7-x^2}}{x}\right| + C$
(We used FORMULA 34 with $a = \sqrt{7}$)

13. $\int \frac{\sqrt{4-x^2}}{x}\,dx = \int \frac{\sqrt{2^2-x^2}}{x}\,dx = \sqrt{2^2-x^2} - 2\ln\left|\frac{2+\sqrt{2^2-x^2}}{x}\right| + C = \sqrt{4-x^2} - 2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + C$
(We used FORMULA 31 with $a = 2$)

14. $\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{\sqrt{x^2-2^2}}{x}\,dx = \sqrt{x^2-2^2} - 2\sec^{-1}\left|\frac{x}{2}\right| + C = \sqrt{x^2-4} - 2\sec^{-1}\left|\frac{x}{2}\right| + C$
(We used FORMULA 42 with $a = 2$)

15. $\int e^{2t}\cos 3t\,dt = \frac{e^{2t}}{2^2+3^2}(2\cos 3t + 3\sin 3t) + C = \frac{e^{2t}}{13}(2\cos 3t + 3\sin 3t) + C$
(We used FORMULA 108 with $a = 2$, $b = 3$)

16. $\int e^{-3t}\sin 4t\,dt = \frac{e^{-3t}}{(-3)^2+4^2}(-3\sin 4t - 4\cos 4t) + C = \frac{e^{-3t}}{25}(-3\sin 4t - 4\cos 4t) + C$
(We used FORMULA 107 with $a = -3$, $b = 4$)

17. $\int x\cos^{-1}x\,dx = \int x^1\cos^{-1}x\,dx = \frac{x^{1+1}}{1+1}\cos^{-1}x + \frac{1}{1+1}\int \frac{x^{1+1}\,dx}{\sqrt{1-x^2}} = \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\int \frac{x^2\,dx}{\sqrt{1-x^2}}$
(We used FORMULA 100 with $a = 1$, $n = 1$)
$= \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\left(\frac{1}{2}\sin^{-1}x\right) - \frac{1}{2}\left(\frac{1}{2}x\sqrt{1-x^2}\right) + C = \frac{x^2}{2}\cos^{-1}x + \frac{1}{4}\sin^{-1}x - \frac{1}{4}x\sqrt{1-x^2} + C$
(We used FORMULA 33 with $a = 1$)

18. $\int x\tan^{-1}x\,dx = \int x^1\tan^{-1}(1x)\,dx = \frac{x^{1+1}}{1+1}\tan^{-1}(1x) - \frac{1}{1+1}\int \frac{x^{1+1}\,dx}{1+(1)^2x^2} = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2\,dx}{1+x^2}$
(We used FORMULA 101 with $a = 1$, $n = 1$)
$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$ (after long division)
$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2}dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C = \frac{1}{2}\left((x^2+1)\tan^{-1}x - x\right) + C$

19. $\int x^2\tan^{-1}x\,dx = \frac{x^{2+1}}{2+1}\tan^{-1}x - \frac{1}{2+1}\int \frac{x^{2+1}}{1+x^2}\,dx = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$
(We used FORMULA 101 with $a = 1$, $n = 2$);
$\int \frac{x^3}{1+x^2}\,dx = \int x\,dx - \int \frac{x\,dx}{1+x^2} = \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + C \Rightarrow \int x^2\tan^{-1}x\,dx$
$= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C$

20. $\int \frac{\tan^{-1} x}{x^2}\, dx = \int x^{-2} \tan^{-1} x\, dx = \frac{x^{(-2+1)}}{(-2+1)} \tan^{-1} x - \frac{1}{(-2+1)} \int \frac{x^{(-2+1)}}{1+x^2}\, dx = \frac{x^{-1}}{(-1)} \tan^{-1} x + \int \frac{x^{-1}}{(1+x^2)}\, dx$

(We used FORMULA 101 with $a = 1$, $n = -2$);

$\int \frac{x^{-1}\, dx}{1+x^2} = \int \frac{dx}{x(1+x^2)} = \int \frac{dx}{x} - \int \frac{x\, dx}{1+x^2} = \ln|x| - \frac{1}{2}\ln(1+x^2) + C$

$\Rightarrow \int \frac{\tan^{-1} x}{x^2}\, dx = -\frac{1}{x}\tan^{-1} x + \ln|x| - \frac{1}{2}\ln(1+x^2) + C$

21. $\int \sin 3x \cos 2x\, dx = -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$

(We used FORMULA 62(a) with $a = 3$, $b = 2$)

22. $\int \sin 2x \cos 3x\, dx = -\frac{\cos 5x}{10} + \frac{\cos x}{2} + C$

(We used FORMULA 62(a) with $a = 2$, $b = 3$)

23. $\int 8 \sin 4t \sin \frac{t}{2}\, dx = \frac{8}{7}\sin\left(\frac{7t}{2}\right) - \frac{8}{9}\sin\left(\frac{9t}{2}\right) + C = 8\left[\frac{\sin\left(\frac{7t}{2}\right)}{7} - \frac{\sin\left(\frac{9t}{2}\right)}{9}\right] + C$

(We used FORMULA 62(b) with $a = 4$, $b = \frac{1}{2}$)

24. $\int \sin \frac{t}{3} \sin \frac{t}{6}\, dt = 3\sin\left(\frac{t}{6}\right) - \sin\left(\frac{t}{2}\right) + C$

(We used FORMULA 62(b) with $a = \frac{1}{3}$, $b = \frac{1}{6}$)

25. $\int \cos \frac{\theta}{3} \cos \frac{\theta}{4}\, d\theta = 6\sin\left(\frac{\theta}{12}\right) + \frac{6}{7}\sin\left(\frac{7\theta}{12}\right) + C$

(We used FORMULA 62(c) with $a = \frac{1}{3}$, $b = \frac{1}{4}$)

26. $\int \cos \frac{\theta}{2} \cos 7\theta\, d\theta = \frac{1}{13}\sin\left(\frac{13\theta}{2}\right) + \frac{1}{15}\sin\left(\frac{15\theta}{2}\right) + C = \frac{\sin\left(\frac{13\theta}{2}\right)}{13} + \frac{\sin\left(\frac{15\theta}{2}\right)}{15} + C$

(We used FORMULA 62(c) with $a = \frac{1}{2}$, $b = 7$)

27. $\int \frac{x^3+x+1}{(x^2+1)^2}\, dx = \int \frac{x\, dx}{x^2+1} + \int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} + \int \frac{dx}{(x^2+1)^2}$

$= \frac{1}{2}\ln(x^2+1) + \frac{x}{2(1+x^2)} + \frac{1}{2}\tan^{-1} x + C$

(For the second integral we used FORMULA 17 with $a = 1$)

28. $\int \frac{x^2+6x}{(x^2+3)^2}\, dx = \int \frac{dx}{x^2+3} + \int \frac{6x\, dx}{(x^2+3)^2} - \int \frac{3\, dx}{(x^2+3)^2} = \int \frac{dx}{x^2+\left(\sqrt{3}\right)^2} + 3\int \frac{d(x^2+3)}{(x^2+3)^2} - 3\int \frac{dx}{\left[x^2+\left(\sqrt{3}\right)^2\right]^2}$

$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{(x^2+3)} - 3\left(\frac{x}{2\left(\sqrt{3}\right)^2\left(\left(\sqrt{3}\right)^2+x^2\right)} + \frac{1}{2\left(\sqrt{3}\right)^3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right) + C$

(For the first integral we used FORMULA 16 with $a = \sqrt{3}$; for the third integral we used FORMULA 17 with $a = \sqrt{3}$)

$= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{x^2+3} - \frac{x}{2(x^2+3)} + C$

29. $\int \sin^{-1}\sqrt{x}\, dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\, du \end{bmatrix} \rightarrow 2\int u^1 \sin^{-1} u\, du = 2\left(\frac{u^{1+1}}{1+1}\sin^{-1} u - \frac{1}{1+1}\int \frac{u^{1+1}}{\sqrt{1-u^2}}\, du\right)$

$= u^2 \sin^{-1} u - \int \frac{u^2\,du}{\sqrt{1-u^2}}$

(We used FORMULA 99 with a = 1, n = 1)

$= u^2 \sin^{-1} u - \left(\frac{1}{2}\sin^{-1} u - \frac{1}{2} u\sqrt{1-u^2}\right) + C = \left(u^2 - \frac{1}{2}\right)\sin^{-1} u + \frac{1}{2} u\sqrt{1-u^2} + C$

(We used FORMULA 33 with a = 1)

$= \left(x - \frac{1}{2}\right)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x - x^2} + C$

30. $\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\,du \end{bmatrix} \to \int \frac{\cos^{-1} u}{u}\cdot 2u\,du = 2\int \cos^{-1} u\,du = 2\left(u\cos^{-1} u - \frac{1}{1}\sqrt{1-u^2}\right) + C$

(We used FORMULA 97 with a = 1)

$= 2\left(\sqrt{x}\cos^{-1}\sqrt{x} - \sqrt{1-x}\right) + C$

31. $\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\,du \end{bmatrix} \to \int \frac{u\cdot 2u}{\sqrt{1-u^2}}\,du = 2\int \frac{u^2}{\sqrt{1-u^2}}\,du = 2\left(\frac{1}{2}\sin^{-1} u - \frac{1}{2}u\sqrt{1-u^2}\right) + C$

$= \sin^{-1} u - u\sqrt{1-u^2} + C$

(We used FORMULA 33 with a = 1)

$= \sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x} + C = \sin^{-1}\sqrt{x} - \sqrt{x-x^2} + C$

32. $\int \frac{\sqrt{2-x}}{\sqrt{x}}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\,du \end{bmatrix} \to \int \frac{\sqrt{2-u^2}}{u}\cdot 2u\,du = 2\int \sqrt{\left(\sqrt{2}\right)^2 - u^2}\,du$

$= 2\left[\frac{u}{2}\sqrt{\left(\sqrt{2}\right)^2 - u^2} + \frac{\left(\sqrt{2}\right)^2}{2}\sin^{-1}\left(\frac{u}{\sqrt{2}}\right)\right] + C = u\sqrt{2-u^2} + 2\sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$

(We used FORMULA 29 with $a = \sqrt{2}$)

$= \sqrt{2x - x^2} + 2\sin^{-1}\sqrt{\frac{x}{2}} + C$

33. $\int (\cot t)\sqrt{1-\sin^2 t}\,dt = \int \frac{\sqrt{1-\sin^2 t}\,(\cos t)\,dt}{\sin t};\ \begin{bmatrix} u = \sin t \\ du = \cos t\,dt \end{bmatrix} \to \int \frac{\sqrt{1-u^2}\,du}{u}$

$= \sqrt{1-u^2} - \ln\left|\frac{1+\sqrt{1-u^2}}{u}\right| + C$

(We used FORMULA 31 with a = 1)

$= \sqrt{1-\sin^2 t} - \ln\left|\frac{1+\sqrt{1-\sin^2 t}}{\sin t}\right| + C$

34. $\int \frac{dt}{(\tan t)\sqrt{4-\sin^2 t}} = \int \frac{\cos t\,dt}{(\sin t)\sqrt{4-\sin^2 t}};\ \begin{bmatrix} u = \sin t \\ du = \cos t\,dt \end{bmatrix} \to \int \frac{du}{u\sqrt{4-u^2}} = -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-u^2}}{u}\right| + C$

(We used FORMULA 34 with a = 2)

$= -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-\sin^2 t}}{\sin t}\right| + C$

35. $\int \frac{dy}{y\sqrt{3+(\ln y)^2}};\ \begin{bmatrix} u = \ln y \\ y = e^u \\ dy = e^u\,du \end{bmatrix} \to \int \frac{e^u\,du}{e^u\sqrt{3+u^2}} = \int \frac{du}{\sqrt{3+u^2}} = \ln\left|u + \sqrt{3+u^2}\right| + C$

$= \ln\left|\ln y + \sqrt{3+(\ln y)^2}\right| + C$

(We used FORMULA 20 with $a = \sqrt{3}$)

36. $\int \tan^{-1} \sqrt{y}\, dy;$ $\begin{bmatrix} t = \sqrt{y} \\ y = t^2 \\ dy = 2t\, dt \end{bmatrix}$ $\rightarrow\ 2\int t \tan^{-1} t\, dt = 2\left[\frac{t^2}{2} \tan^{-1} t - \frac{1}{2}\int \frac{t^2}{1+t^2}\, dt\right] = t^2 \tan^{-1} t - \int \frac{t^2}{1+t^2}\, dt$

(We used FORMULA 101 with n = 1, a = 1)

$= t^2 \tan^{-1} t - \int \frac{t^2+1}{t^2+1}\, dt + \int \frac{dt}{1+t^2} = t^2 \tan^{-1} t - t + \tan^{-1} t + C = y \tan^{-1} \sqrt{y} + \tan^{-1} \sqrt{y} - \sqrt{y} + C$

37. $\int \frac{1}{\sqrt{x^2+2x+5}} dx = \int \frac{1}{\sqrt{(x+1)^2+4}} dx;$ $\begin{bmatrix} t = x+1 \\ dt = dx \end{bmatrix}$ $\rightarrow \int \frac{1}{\sqrt{t^2+4}} dt$

(We used FORMULA 20 with a = 2)

$= \ln\left|t + \sqrt{t^2+4}\right| + C = \ln\left|(x+1) + \sqrt{(x+1)^2+4} + C = \ln\right|(x+1) + \sqrt{x^2 + 2x + 5}\left| + C$

38. $\int \frac{x^2}{\sqrt{x^2-4x+5}} dx = \int \frac{x^2}{\sqrt{(x-2)^2+1}} dx;$ $\begin{bmatrix} t = x-2 \\ dt = dx \end{bmatrix}$ $\rightarrow \int \frac{(t+2)^2}{\sqrt{t^2+1}} dt = \int \frac{t^2+4t+2}{\sqrt{t^2+1}} dt = \int \frac{t^2}{\sqrt{t^2+1}} dt + \int \frac{4t}{\sqrt{t^2+1}} dt + \int \frac{4}{\sqrt{t^2+1}} dt$

(We used FORMULA 25 with a = 1) (We used FORMULA 20 with a = 1)

$= \left[-\frac{1}{2}\ln\left|t + \sqrt{t^2+1}\right| + \frac{t\sqrt{t^2+1}}{2}\right] + 4\sqrt{t^2+1} + \left[4\ln\left|t + \sqrt{t^2+1}\right|\right] + C$

$= -\frac{1}{2}\ln\left|(x-2) + \sqrt{(x-2)^2+1}\right| + \frac{(x-2)\sqrt{(x-2)^2+1}}{2} + 4\sqrt{(x-2)^2+1} + 4\ln\left|(x-2) + \sqrt{(x-2)^2+1}\right| + C$

$= \frac{7}{2}\ln\left|(x-2) + \sqrt{x^2-4x+5}\right| + \frac{(x+6)\sqrt{x^2-4x+5}}{2} + C$

39. $\int \sqrt{5-4x-x^2} dx = \int \sqrt{9-(x+2)^2} dx;$ $\begin{bmatrix} t = x+2 \\ dt = dx \end{bmatrix}$ $\rightarrow \int \sqrt{9-t^2} dt;$

(We used FORMULA 29 with a = 3)

$= \frac{t}{2}\sqrt{9-t^2} + \frac{3^2}{2}\sin^{-1}\left(\frac{t}{3}\right) + C = \frac{x+2}{2}\sqrt{9-(x+2)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + C = \frac{x+2}{2}\sqrt{5-4x-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + C$

40. $\int x^2\sqrt{2x-x^2}\, dx = \int x^2\sqrt{1-(x-1)^2}\, dx;$ $\begin{bmatrix} t = x-1 \\ dt = dx \end{bmatrix}$ $\rightarrow \int (t+1)^2\sqrt{1-t^2}\, dt = \int \left(t^2+2t+1\right)\sqrt{1-t^2}\, dt$

$= \int t^2\sqrt{1-t^2}\, dt + \int 2t\sqrt{1-t^2}\, dt + \int \sqrt{1-t^2}\, dt$

(We used FORMULA 30 with a = 1) (We used FORMULA 29 with a = 1)

$= \left[\frac{1^4}{8}\sin^{-1}\left(\frac{t}{1}\right) - \frac{1}{8}t\sqrt{1-t^2}(1^2-2t^2)\right] - \frac{2}{3}(1-t^2)^{3/2} + \left[\frac{t}{2}\sqrt{1-t^2} + \frac{1^2}{2}\sin^{-1}\left(\frac{t}{1}\right)\right] + C$

$= \frac{1}{8}\sin^{-1}(x-1) - \frac{1}{8}(x-1)\sqrt{1-(x-1)^2}\left(1^2-2(x-1)^2\right) - \frac{2}{3}\left(1-(x-1)^2\right)^{3/2} + \frac{x-1}{2}\sqrt{1-(x-1)^2}$

$+ \frac{1}{2}\sin^{-1}(x-1) + C = \frac{5}{8}\sin^{-1}(x-1) - \frac{2}{3}(2x-x^2)^{3/2} + \frac{x-1}{8}\sqrt{2x-x^2}(2x^2-4x+5) + C$

41. $\int \sin^5 2x\, dx = -\frac{\sin^4 2x \cos 2x}{5\cdot 2} + \frac{5-1}{5}\int \sin^3 2x\, dx = -\frac{\sin^4 2x \cos 2x}{10} + \frac{4}{5}\left[-\frac{\sin^2 2x \cos 2x}{3\cdot 2} + \frac{3-1}{3}\int \sin 2x\, dx\right]$

(We used FORMULA 60 with a = 2, n = 5 and a = 2, n = 3)

$= -\frac{\sin^4 2x \cos 2x}{10} - \frac{2}{15}\sin^2 2x \cos 2x + \frac{8}{15}\left(-\frac{1}{2}\right)\cos 2x + C = -\frac{\sin^4 2x \cos 2x}{10} - \frac{2\sin^2 2x \cos 2x}{15} - \frac{4\cos 2x}{15} + C$

42. $\int 8\cos^4 2\pi t\, dt = 8\left(\frac{\cos^3 2\pi t \sin 2\pi t}{4\cdot 2\pi} + \frac{4-1}{4}\int \cos^2 2\pi t\, dt\right)$

(We used FORMULA 61 with $a = 2\pi$, n = 4)

$= \frac{\cos^3 2\pi t \sin 2\pi t}{\pi} + 6\left[\frac{t}{2} + \frac{\sin(2\cdot 2\pi\cdot t)}{4\cdot 2\pi}\right] + C$

(We used FORMULA 59 with $a = 2\pi$)

$= \frac{\cos^3 2\pi t \sin 2\pi t}{\pi} + 3t + \frac{3\sin 4\pi t}{4\pi} + C = \frac{\cos^3 2\pi t \sin 2\pi t}{\pi} + \frac{3\cos 2\pi t \sin 2\pi t}{2\pi} + 3t + C$

43. $\int \sin^2 2\theta \cos^3 2\theta \, d\theta = \frac{\sin^3 2\theta \cos^2 2\theta}{2(2+3)} + \frac{3-1}{3+2} \int \sin^2 2\theta \cos 2\theta \, d\theta$

(We used FORMULA 69 with a = 2, m = 3, n = 2)

$= \frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{2}{5} \int \sin^2 2\theta \cos 2\theta \, d\theta = \frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{2}{5}\left[\frac{1}{2}\int \sin^2 2\theta \, d(\sin 2\theta)\right] = \frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{\sin^3 2\theta}{15} + C$

44. $\int 2\sin^2 t \sec^4 t \, dt = \int 2\sin^2 t \cos^{-4} t \, dt = 2\left(-\frac{\sin t \cos^{-3} t}{2-4} + \frac{2-1}{2-4}\int \cos^{-4} t \, dt\right)$

(We used FORMULA 68 with a = 1, n = 2, m = −4)

$= \sin t \cos^{-3} t - \int \cos^{-4} t \, dt = \sin t \cos^{-3} t - \int \sec^4 t \, dt = \sin t \cos^{-3} t - \left(\frac{\sec^2 t \tan t}{4-1} + \frac{4-2}{4-1}\int \sec^2 t \, dt\right)$

(We used FORMULA 92 with a = 1, n = 4)

$= \sin t \cos^{-3} t - \left(\frac{\sec^2 t \tan t}{3}\right) - \frac{2}{3}\tan t + C = \frac{2}{3}\sec^2 t \tan t - \frac{2}{3}\tan t + C = \frac{2}{3}\tan t\,(\sec^2 t - 1) + C$

$= \frac{2}{3}\tan^3 t + C$

An easy way to find the integral using substitution:

$\int 2\sin^2 t \cos^{-4} t \, dt = \int 2\tan^2 t \sec^2 t \, dt = \int 2\tan^2 t \, d(\tan t) = \frac{2}{3}\tan^3 t + C$

45. $\int 4\tan^3 2x \, dx = 4\left(\frac{\tan^2 2x}{2\cdot 2} - \int \tan 2x \, dx\right) = \tan^2 2x - 4\int \tan 2x \, dx$

(We used FORMULA 86 with n = 3, a = 2)

$= \tan^2 2x - \frac{4}{2}\ln|\sec 2x| + C = \tan^2 2x - 2\ln|\sec 2x| + C$

46. $\int 8\cot^4 t \, dt = 8\left(-\frac{\cot^3 t}{3} - \int \cot^2 t \, dt\right)$

(We used FORMULA 87 with a = 1, n = 4)

$= 8\left(-\frac{1}{3}\cot^3 t + \cot t + t\right) + C$

(We used FORMULA 85 with a = 1)

47. $\int 2\sec^3 \pi x \, dx = 2\left[\frac{\sec \pi x \tan \pi x}{\pi(3-1)} + \frac{3-2}{3-1}\int \sec \pi x \, dx\right]$

(We used FORMULA 92 with n = 3, a = π)

$= \frac{1}{\pi}\sec \pi x \tan \pi x + \frac{1}{\pi}\ln|\sec \pi x + \tan \pi x| + C$

(We used FORMULA 88 with a = π)

48. $\int 3\sec^4 3x \, dx = 3\left[\frac{\sec^2 3x \tan 3x}{3(4-1)} + \frac{4-2}{4-1}\int \sec^2 3x \, dx\right]$

(We used FORMULA 92 with n = 4, a = 3)

$= \frac{\sec^2 3x \tan 3x}{3} + \frac{2}{3}\tan 3x + C$

(We used FORMULA 90 with a = 3)

49. $\int \csc^5 x \, dx = -\frac{\csc^3 x \cot x}{5-1} + \frac{5-2}{5-1}\int \csc^3 x \, dx = -\frac{\csc^3 x \cot x}{4} + \frac{3}{4}\left(-\frac{\csc x \cot x}{3-1} + \frac{3-2}{3-1}\int \csc x \, dx\right)$

(We used FORMULA 93 with n = 5, a = 1 and n = 3, a = 1)

$= -\frac{1}{4}\csc^3 x \cot x - \frac{3}{8}\csc x \cot x - \frac{3}{8}\ln|\csc x + \cot x| + C$

(We used FORMULA 89 with a = 1)

50. $\int 16x^3(\ln x)^2 \, dx = 16\left[\frac{x^4(\ln x)^2}{4} - \frac{2}{4}\int x^3 \ln x \, dx\right] = 16\left[\frac{x^4(\ln x)^2}{4} - \frac{1}{2}\left[\frac{x^4(\ln x)}{4} - \frac{1}{4}\int x^3 \, dx\right]\right]$

(We used FORMULA 110 with a = 1, n = 3, m = 2 and a = 1, n = 3, m = 1)

$= 16\left(\frac{x^4(\ln x)^2}{4} - \frac{x^4(\ln x)}{8} + \frac{x^4}{32}\right) + C = 4x^4(\ln x)^2 - 2x^4 \ln x + \frac{x^4}{2} + C$

51. $\int e^t \sec^3(e^t - 1)\,dt;\ \begin{bmatrix} x = e^t - 1 \\ dx = e^t\,dt \end{bmatrix} \to \int \sec^3 x\,dx = \frac{\sec x \tan x}{3-1} + \frac{3-2}{3-1}\int \sec x\,dx$

(We used FORMULA 92 with a = 1, n = 3)

$= \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + C = \frac{1}{2}\left[\sec(e^t - 1)\tan(e^t - 1) + \ln|\sec(e^t - 1) + \tan(e^t - 1)|\right] + C$

52. $\int \frac{\csc^3\sqrt{\theta}}{\sqrt{\theta}}\,d\theta;\ \begin{bmatrix} t = \sqrt{\theta} \\ \theta = t^2 \\ d\theta = 2t\,dt \end{bmatrix} \to 2\int \csc^3 t\,dt = 2\left[-\frac{\csc t \cot t}{3-1} + \frac{3-2}{3-1}\int \csc t\,dt\right]$

(We used FORMULA 93 with a = 1, n = 3)

$= 2\left[-\frac{\csc t \cot t}{2} - \frac{1}{2}\ln|\csc t + \cot t|\right] + C = -\csc\sqrt{\theta}\cot\sqrt{\theta} - \ln\left|\csc\sqrt{\theta} + \cot\sqrt{\theta}\right| + C$

53. $\int_0^1 2\sqrt{x^2+1}\,dx;\ [x = \tan t] \to 2\int_0^{\pi/4} \sec t \cdot \sec^2 t\,dt = 2\int_0^{\pi/4} \sec^3 t\,dt = 2\left[\left[\frac{\sec t \cdot \tan t}{3-1}\right]_0^{\pi/4} + \frac{3-2}{3-1}\int_0^{\pi/4} \sec t\,dt\right]$

(We used FORMULA 92 with n = 3, a = 1)

$= [\sec t \cdot \tan t + \ln|\sec t + \tan t|]_0^{\pi/4} = \sqrt{2} + \ln\left(\sqrt{2}+1\right)$

54. $\int_0^{\sqrt{3}/2} \frac{dy}{(1-y^2)^{5/2}};\ [y = \sin x] \to \int_0^{\pi/3} \frac{\cos x\,dx}{\cos^5 x} = \int_0^{\pi/3} \sec^4 x\,dx = \left[\frac{\sec^2 x \tan x}{4-1}\right]_0^{\pi/3} + \frac{4-2}{4-1}\int_0^{\pi/3} \sec^2 x\,dx$

(We used FORMULA 92 with a = 1, n = 4)

$= \left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x\right]_0^{\pi/3} = \left(\frac{4}{3}\right)\sqrt{3} + \left(\frac{2}{3}\right)\sqrt{3} = 2\sqrt{3}$

55. $\int_1^2 \frac{(r^2-1)^{3/2}}{r}\,dr;\ [r = \sec\theta] \to \int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}(\sec\theta\tan\theta)\,d\theta = \int_0^{\pi/3} \tan^4\theta\,d\theta = \left[\frac{\tan^3\theta}{4-1}\right]_0^{\pi/3} - \int_0^{\pi/3} \tan^2\theta\,d\theta$

$= \left[\frac{\tan^3\theta}{3} - \tan\theta + \theta\right]_0^{\pi/3} = \frac{3\sqrt{3}}{3} - \sqrt{3} + \frac{\pi}{3} = \frac{\pi}{3}$

(We used FORMULA 86 with a = 1, n = 4 and FORMULA 84 with a = 1)

56. $\int_0^{1/\sqrt{3}} \frac{dt}{(t^2+1)^{7/2}};\ [t = \tan\theta] \to \int_0^{\pi/6} \frac{\sec^2\theta\,d\theta}{\sec^7\theta} = \int_0^{\pi/6} \cos^5\theta\,d\theta = \left[\frac{\cos^4\theta\sin\theta}{5}\right]_0^{\pi/6} + \left(\frac{5-1}{5}\right)\int_0^{\pi/6} \cos^3\theta\,d\theta$

$= \left[\frac{\cos^4\theta\sin\theta}{5}\right]_0^{\pi/6} + \frac{4}{5}\left[\left[\frac{\cos^2\theta\sin\theta}{3}\right]_0^{\pi/6} + \left(\frac{3-1}{3}\right)\int_0^{\pi/6} \cos\theta\,d\theta\right] = \left[\frac{\cos^4\theta\sin\theta}{5} + \frac{4}{15}\cos^2\theta\sin\theta + \frac{8}{15}\sin\theta\right]_0^{\pi/6}$

(We used FORMULA 61 with a = 1, n = 5 and a = 1, n = 3)

$= \frac{\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right)}{5} + \left(\frac{4}{15}\right)\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right) + \left(\frac{8}{15}\right)\left(\frac{1}{2}\right) = \frac{9}{160} + \frac{1}{10} + \frac{4}{15} = \frac{3\cdot 9 + 48 + 32\cdot 4}{480} = \frac{203}{480}$

57. $S = \int_0^{\sqrt{2}} 2\pi y\sqrt{1+(y')^2}\,dx$

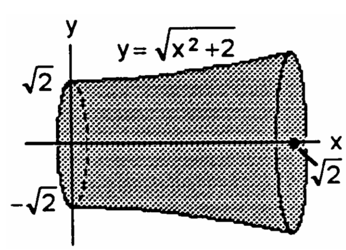
$= 2\pi\int_0^{\sqrt{2}} \sqrt{x^2+2}\sqrt{1+\frac{x^2}{x^2+2}}\,dx$

$= 2\sqrt{2}\pi\int_0^{\sqrt{2}} \sqrt{x^2+1}\,dx$

$= 2\sqrt{2}\pi\left[\frac{x\sqrt{x^2+1}}{2} + \frac{1}{2}\ln\left|x + \sqrt{x^2+1}\right|\right]_0^{\sqrt{2}}$

(We used FORMULA 21 with a = 1)

$= \sqrt{2}\pi\left[\sqrt{6} + \ln\left(\sqrt{2}+\sqrt{3}\right)\right] = 2\pi\sqrt{3} + \pi\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right)$



58. $L = \int_0^{\sqrt{3}/2} \sqrt{1+(2x)^2}\,dx = 2\int_0^{\sqrt{3}/2} \sqrt{\frac{1}{4}+x^2}\,dx = 2\left[\frac{x}{2}\sqrt{\frac{1}{4}+x^2} + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\ln\left(x + \sqrt{\frac{1}{4}+x^2}\right)\right]_0^{\sqrt{3}/2}$

(We used FORMULA 2 with $a = \frac{1}{2}$)

$= \left[\frac{x}{2}\sqrt{1+4x^2} + \frac{1}{4}\ln\left(x + \frac{1}{2}\sqrt{1+4x^2}\right)\right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}}{4}\sqrt{1+4\left(\frac{3}{4}\right)} + \frac{1}{4}\ln\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{3}{4}\right)}\right) - \frac{1}{4}\ln\frac{1}{2}$

$= \frac{\sqrt{3}}{4}(2) + \frac{1}{4}\ln\left(\frac{\sqrt{3}}{2}+1\right) + \frac{1}{4}\ln 2 = \frac{\sqrt{3}}{2} + \frac{1}{4}\ln\left(\sqrt{3}+2\right)$

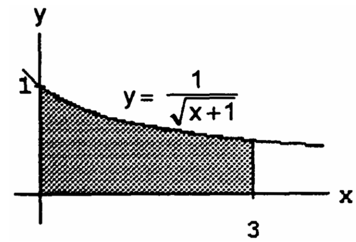
59. $A = \int_0^3 \frac{dx}{\sqrt{x+1}} = \left[2\sqrt{x+1}\right]_0^3 = 2$; $\overline{x} = \frac{1}{A}\int_0^3 \frac{x\,dx}{\sqrt{x+1}}$

$= \frac{1}{A}\int_0^3 \sqrt{x+1}\,dx - \frac{1}{A}\int_0^3 \frac{dx}{\sqrt{x+1}}$

$= \frac{1}{2}\cdot\frac{2}{3}\left[(x+1)^{3/2}\right]_0^3 - 1 = \frac{4}{3}$;

(We used FORMULA 11 with $a = 1, b = 1, n = 1$ and $a = 1, b = 1, n = -1$)

$\overline{y} = \frac{1}{2A}\int_0^3 \frac{dx}{x+1} = \frac{1}{4}\left[\ln(x+1)\right]_0^3 = \frac{1}{4}\ln 4 = \frac{1}{2}\ln 2 = \ln\sqrt{2}$



60. $M_y = \int_0^3 x\left(\frac{36}{2x+3}\right)dx = 18\int_0^3 \frac{2x+3}{2x+3}\,dx - 54\int_0^3 \frac{dx}{2x+3} = \left[18x - 27\ln|2x+3|\right]_0^3$

$= 18\cdot 3 - 27\ln 9 - (-27\ln 3) = 54 - 27\cdot 2\ln 3 + 27\ln 3 = 54 - 27\ln 3$

61. $S = 2\pi\int_{-1}^1 x^2\sqrt{1+4x^2}\,dx$;

$\begin{bmatrix} u = 2x \\ du = 2\,dx \end{bmatrix} \rightarrow \frac{\pi}{4}\int_{-2}^2 u^2\sqrt{1+u^2}\,du$

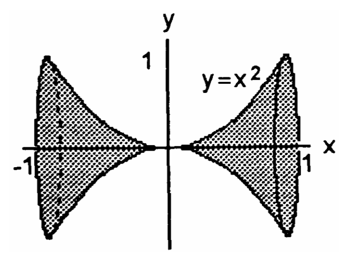
$= \frac{\pi}{4}\left[\frac{u}{8}(1+2u^2)\sqrt{1+u^2} - \frac{1}{8}\ln\left(u+\sqrt{1+u^2}\right)\right]_{-2}^2$

(We used FORMULA 22 with $a = 1$)

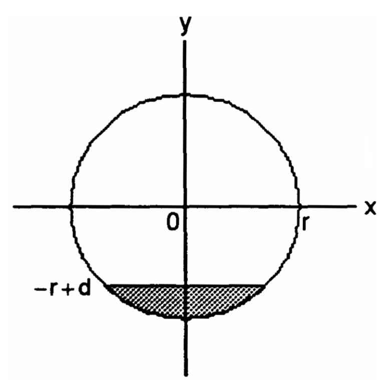
$= \frac{\pi}{4}\left[\frac{2}{8}(1+2\cdot 4)\sqrt{1+4} - \frac{1}{8}\ln\left(2+\sqrt{1+4}\right)\right.$

$\left. + \frac{2}{8}(1+2\cdot 4)\sqrt{1+4} + \frac{1}{8}\ln\left(-2+\sqrt{1+4}\right)\right]$

$= \frac{\pi}{4}\left[\frac{9}{2}\sqrt{5} - \frac{1}{8}\ln\left(\frac{2+\sqrt{5}}{-2+\sqrt{5}}\right)\right] \approx 7.62$



62. (a) The volume of the filled part equals the length of the tank times the area of the shaded region shown in the accompanying figure. Consider a layer of gasoline of thickness dy located at height y where $-r < y < -r + d$. The width of this layer is $2\sqrt{r^2 - y^2}$. Therefore, $A = 2\int_{-r}^{-r+d}\sqrt{r^2-y^2}\,dy$ and $V = L\cdot A = 2L\int_{-r}^{-r+d}\sqrt{r^2-y^2}\,dy$



(b) $2L\int_{-r}^{-r+d}\sqrt{r^2-y^2}\,dy = 2L\left[\frac{y\sqrt{r^2-y^2}}{2} + \frac{r^2}{2}\sin^{-1}\frac{y}{r}\right]_{-r}^{-r+d}$

(We used FORMULA 29 with $a = r$)

$= 2L\left[\frac{(d-r)}{2}\sqrt{2rd-d^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{d-r}{r}\right) + \frac{r^2}{2}\left(\frac{\pi}{2}\right)\right] = 2L\left[\left(\frac{d-r}{2}\right)\sqrt{2rd-d^2} + \left(\frac{r^2}{2}\right)\left(\sin^{-1}\left(\frac{d-r}{r}\right) + \frac{\pi}{2}\right)\right]$

63. The integrand $f(x) = \sqrt{x - x^2}$ is nonnegative, so the integral is maximized by integrating over the function's entire domain, which runs from $x = 0$ to $x = 1$

$$\Rightarrow \int_0^1 \sqrt{x - x^2}\, dx = \int_0^1 \sqrt{2 \cdot \tfrac{1}{2}\, x - x^2}\, dx = \left[\frac{\left(x - \frac{1}{2}\right)}{2} \sqrt{2 \cdot \tfrac{1}{2}\, x - x^2} + \frac{\left(\frac{1}{2}\right)^2}{2} \sin^{-1}\left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

(We used FORMULA 48 with $a = \frac{1}{2}$)

$$= \left[\frac{\left(x - \frac{1}{2}\right)}{2} \sqrt{x - x^2} + \frac{1}{8} \sin^{-1}(2x - 1)\right]_0^1 = \frac{1}{8} \cdot \frac{\pi}{2} - \frac{1}{8}\left(-\frac{\pi}{2}\right) = \frac{\pi}{8}$$

64. The integrand is maximized by integrating $g(x) = x\sqrt{2x - x^2}$ over the largest domain on which g is nonnegative, namely $[0, 2]$

$$\Rightarrow \int_0^2 x\sqrt{2x - x^2}\, dx = \left[\frac{(x+1)(2x-3)\sqrt{2x - x^2}}{6} + \frac{1}{2} \sin^{-1}(x - 1)\right]_0^2$$

(We used FORMULA 51 with $a = 1$)

$$= \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

CAS EXPLORATIONS

65. Example CAS commands:

Maple:

```
q1 := Int( x*ln(x), x );                # (a)
q1 = value( q1 );
q2 := Int( x^2*ln(x), x );              # (b)
q2 = value( q2 );
q3 := Int( x^3*ln(x), x );              # (c)
q3 = value( q3 );
q4 := Int( x^4*ln(x), x );              # (d)
q4 = value( q4 );
q5 := Int( x^n*ln(x), x );              # (e)
q6 := value( q5 );
q7 := simplify(q6) assuming n::integer;
q5 = collect( factor(q7), ln(x) );
```

66. Example CAS commands:

Maple:

```
q1 := Int( ln(x)/x, x );                # (a)
q1 = value( q1 );
q2 := Int( ln(x)/x^2, x );              # (b)
q2 = value( q2 );
q3 := Int( ln(x)/x^3, x );              # (c)
q3 = value( q3 );
q4 := Int( ln(x)/x^4, x );              # (d)
q4 = value( q4 );
q5 := Int( ln(x)/x^n, x );              # (e)
q6 := value( q5 );
q7 := simplify(q6) assuming n::integer;
q5 = collect( factor(q7), ln(x) );
```

67. Example CAS commands:

Maple:

```
q := Int( sin(x)^n/(sin(x)^n+cos(x)^n), x=0..Pi/2 );      # (a)
q = value( q );
q1 := eval( q, n=1 ):                                        # (b)
q1 = value( q1 );
for N in [1,2,3,5,7] do
  q1 := eval( q, n=N );
  print( q1 = evalf(q1) );
end do:
qq1 := PDEtools[dchange]( x=Pi/2-u, q, [u] );             # (c)
qq2 := subs( u=x, qq1 );
qq3 := q + q = q + qq2;
qq4 := combine( qq3 );
qq5 := value( qq4 );
simplify( qq5/2 );
```

65-67. Example CAS commands:

Mathematica: (functions may vary)

In Mathematica, the natural log is denoted by Log rather than Ln, Log base 10 is Log[x,10]

Mathematica does not include an arbitrary constant when computing an indefinite integral,

```
Clear[x, f, n]
f[x_]:=Log[x] / x^n
Integrate[f[x], x]
```

For exercise 67, Mathematica cannot evaluate the integral with arbitrary n. It does evaluate the integral (value is $\pi/4$ in each case) for small values of n, but for large values of n, it identifies this integral as Indeterminate

65. (e) $\int x^n \ln x\, dx = \frac{x^{n+1}\ln x}{n+1} - \frac{1}{n+1}\int x^n\, dx,\ n \neq -1$

(We used FORMULA 110 with $a = 1$, $m = 1$)

$= \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C$

66. (e) $\int x^{-n} \ln x\, dx = \frac{x^{-n+1}\ln x}{-n+1} - \frac{1}{(-n)+1}\int x^{-n}\, dx,\ n \neq 1$

(We used FORMULA 110 with $a = 1$, $m = 1$, $n = -n$)

$= \frac{x^{1-n}\ln x}{1-n} - \frac{1}{1-n}\left(\frac{x^{1-n}}{1-n}\right) + C = \frac{x^{1-n}}{1-n}\left(\ln x - \frac{1}{1-n}\right) + C$

67. (a) Neither MAPLE nor MATHEMATICA can find this integral for arbitrary n.

(b) MAPLE and MATHEMATICA get stuck at about $n = 5$.

(c) Let $x = \frac{\pi}{2} - u \Rightarrow dx = -\,du$; $x = 0 \Rightarrow u = \frac{\pi}{2}$, $x = \frac{\pi}{2} \Rightarrow u = 0$;

$I = \int_0^{\pi/2} \frac{\sin^n x\, dx}{\sin^n x + \cos^n x} = \int_{\pi/2}^0 \frac{-\sin^n\left(\frac{\pi}{2}-u\right) du}{\sin^n\left(\frac{\pi}{2}-u\right)+\cos^n\left(\frac{\pi}{2}-u\right)} = \int_0^{\pi/2} \frac{\cos^n u\, du}{\cos^n u + \sin^n u} = \int_0^{\pi/2} \frac{\cos^n x\, dx}{\cos^n x + \sin^n x}$

$\Rightarrow I + I = \int_0^{\pi/2} \left(\frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\right) dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

8.6 NUMERICAL INTEGRATION

1. $\int_1^2 x\,dx$

I. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;
$\sum mf(x_i) = 12 \Rightarrow T = \frac{1}{8}(12) = \frac{3}{2}$;
$f(x) = x \Rightarrow f'(x) = 1 \Rightarrow f'' = 0 \Rightarrow M = 0$
$\Rightarrow |E_T| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	2	5/2
x_2	3/2	3/2	2	3
x_3	7/4	7/4	2	7/2
x_4	2	2	1	2

(b) $\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2} \Rightarrow |E_T| = \int_1^2 x\,dx - T = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;
$\sum mf(x_i) = 18 \Rightarrow S = \frac{1}{12}(18) = \frac{3}{2}$;
$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	4	5
x_2	3/2	3/2	2	3
x_3	7/4	7/4	4	7
x_4	2	2	1	2

(b) $\int_1^2 x\,dx = \frac{3}{2} \Rightarrow |E_S| = \int_1^2 x\,dx - S = \frac{3}{2} - \frac{3}{2} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

2. $\int_1^3 (2x-1)\,dx$

I. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;
$\sum mf(x_i) = 24 \Rightarrow T = \frac{1}{4}(24) = 6$;
$f(x) = 2x - 1 \Rightarrow f'(x) = 2 \Rightarrow f'' = 0 \Rightarrow M = 0$
$\Rightarrow |E_T| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	2	4
x_2	2	3	2	6
x_3	5/2	4	2	8
x_4	3	5	1	5

(b) $\int_1^3 (2x-1)\,dx = [x^2 - x]_1^3 = (9-3) - (1-1) = 6 \Rightarrow |E_T| = \int_1^3 (2x-1)\,dx - T = 6 - 6 = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
$\sum mf(x_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6$;
$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	4	8
x_2	2	3	2	6
x_3	5/2	4	4	16
x_4	3	5	1	5

(b) $\int_1^3 (2x-1)\,dx = 6 \Rightarrow |E_S| = \int_1^3 (2x-1)\,dx - S$
$= 6 - 6 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

3. $\int_{-1}^1 (x^2+1)\,dx$

I. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;
$\sum mf(x_i) = 11 \Rightarrow T = \frac{1}{4}(11) = 2.75$;
$f(x) = x^2 + 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2 \Rightarrow M = 2$
$\Rightarrow |E_T| \le \frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(2) = \frac{1}{12}$ or 0.08333

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	−1	2	1	2
x_1	−1/2	5/4	2	5/2
x_2	0	1	2	2
x_3	1/2	5/4	2	5/2
x_4	1	2	1	2

(b) $\int_{-1}^1 (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_{-1}^1 = \left(\frac{1}{3}+1\right) - \left(-\frac{1}{3}-1\right) = \frac{8}{3} \Rightarrow E_T = \int_{-1}^1 (x^2+1)\,dx - T = \frac{8}{3} - \frac{11}{4} = -\frac{1}{12}$
$\Rightarrow |E_T| = \left|-\frac{1}{12}\right| \approx 0.08333$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{\frac{1}{12}}{\frac{8}{3}}\right) \times 100 \approx 3\%$

II. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
$\sum mf(x_i) = 16 \Rightarrow S = \frac{1}{6}(16) = \frac{8}{3} = 2.66667$;
$f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	−1	2	1	2
x_1	−1/2	5/4	4	5
x_2	0	1	2	2
x_3	1/2	5/4	4	5
x_4	1	2	1	2

(b) $\int_{-1}^{1}(x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_{-1}^{1} = \frac{8}{3}$
$\Rightarrow |E_S| = \int_{-1}^{1}(x^2+1)\,dx - S = \frac{8}{3} - \frac{8}{3} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

4. $\int_{-2}^{0}(x^2-1)\,dx$

I. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{0-(-2)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$
$\sum mf(x_i) = 3 \Rightarrow T = \frac{1}{4}(3) = \frac{3}{4}$;
$f(x) = x^2 - 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2$
$\Rightarrow M = 2 \Rightarrow |E_T| \le \frac{0-(-2)}{12}\left(\frac{1}{2}\right)^2(2) = \frac{1}{12} = 0.08333$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	−2	3	1	3
x_1	−3/2	5/4	2	5/2
x_2	−1	0	2	0
x_3	− 1/2	−3/4	2	−3/2
x_4	0	−1	1	−1

(b) $\int_{-2}^{0}(x^2-1)\,dx = \left[\frac{x^3}{3}-x\right]_{-2}^{0} = 0 - \left(-\frac{8}{3}+2\right) = \frac{2}{3} \Rightarrow E_T = \int_{-2}^{0}(x^2-1)\,dx - T = \frac{2}{3} - \frac{3}{4} = -\frac{1}{12}$
$\Rightarrow |E_T| = \frac{1}{12}$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{\frac{1}{12}}{\frac{2}{3}}\right) \times 100 \approx 13\%$

II. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{0-(-2)}{4} = \frac{2}{4} = \frac{1}{2}$
$\Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$; $\sum mf(x_i) = 4 \Rightarrow S = \frac{1}{6}(4) = \frac{2}{3}$;
$f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	−2	3	1	3
x_1	−3/2	5/4	4	5
x_2	−1	0	2	0
x_3	− 1/2	−3/4	4	−3
x_4	0	−1	1	−1

(b) $\int_{-2}^{0}(x^2-1)\,dx = \frac{2}{3} \Rightarrow |E_S| = \int_{-2}^{0}(x^2-1)\,dx - S$
$= \frac{2}{3} - \frac{2}{3} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

5. $\int_{0}^{2}(t^3+t)\,dt$

I. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2}$
$\Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$; $\sum mf(t_i) = 25 \Rightarrow T = \frac{1}{4}(25) = \frac{25}{4}$;
$f(t) = t^3 + t \Rightarrow f'(t) = 3t^2 + 1 \Rightarrow f''(t) = 6t$
$\Rightarrow M = 12 = f''(2) \Rightarrow |E_T| \le \frac{2-0}{12}\left(\frac{1}{2}\right)^2(12) = \frac{1}{2}$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/2	5/8	2	5/4
t_2	1	2	2	4
t_3	3/2	39/8	2	39/4
t_4	2	10	1	10

(b) $\int_{0}^{2}(t^3+t)\,dt = \left[\frac{t^4}{4}+\frac{t^2}{2}\right]_{0}^{2} = \left(\frac{2^4}{4}+\frac{2^2}{2}\right) - 0 = 6 \Rightarrow |E_T| = \int_{0}^{2}(t^3+t)\,dt - T = 6 - \frac{25}{4} = -\frac{1}{4} \Rightarrow |E_T| = \frac{1}{4}$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{\left|-\frac{1}{4}\right|}{6} \times 100 \approx 4\%$

II. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
$\sum mf(t_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6$;
$f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/2	5/8	4	5/2
t_2	1	2	2	4
t_3	3/2	39/8	4	39/2
t_4	2	10	1	10

(b) $\int_{0}^{2}(t^3+t)\,dt = 6 \Rightarrow |E_S| = \int_{0}^{2}(t^3+t)\,dt - S$
$= 6 - 6 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

6. $\int_{-1}^{1}(t^3+1)\,dt$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	−1	0	1	0
t_1	−1/2	7/8	2	7/4
t_2	0	1	2	2
t_3	1/2	9/8	2	9/4
t_4	1	2	1	2

I. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{1-(-1)}{4}=\frac{2}{4}=\frac{1}{2}$
$\Rightarrow \frac{\Delta x}{2}=\frac{1}{4}$; $\sum mf(t_i)=8 \Rightarrow T=\frac{1}{4}(8)=2$;
$f(t)=t^3+1 \Rightarrow f'(t)=3t^2 \Rightarrow f''(t)=6t$
$\Rightarrow M=6=f''(1) \Rightarrow |E_T|\le\frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(6)=\frac{1}{4}$

(b) $\int_{-1}^{1}(t^3+1)\,dt=\left[\frac{t^4}{4}+t\right]_{-1}^{1}=\left(\frac{1^4}{4}+1\right)-\left(\frac{(-1)^4}{4}+(-1)\right)=2 \Rightarrow |E_T|=\int_{-1}^{1}(t^3+1)\,dt-T=2-2=0$

(c) $\frac{|E_T|}{\text{True Value}}\times 100=0\%$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	−1	0	1	0
t_1	−1/2	7/8	4	7/2
t_2	0	1	2	2
t_3	1/2	9/8	4	9/2
t_4	1	2	1	2

II. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{1-(-1)}{4}=\frac{2}{4}=\frac{1}{2}$
$\Rightarrow \frac{\Delta x}{3}=\frac{1}{6}$; $\sum mf(t_i)=12 \Rightarrow S=\frac{1}{6}(12)=2$;
$f^{(3)}(t)=6 \Rightarrow f^{(4)}(t)=0 \Rightarrow M=0 \Rightarrow |E_S|=0$

(b) $\int_{-1}^{1}(t^3+1)\,dt=2 \Rightarrow |E_S|=\int_{-1}^{1}(t^3+1)\,dt-S$
$=2-2=0$

(c) $\frac{|E_S|}{\text{True Value}}\times 100=0\%$

7. $\int_{1}^{2}\frac{1}{s^2}\,ds$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	1	1	1	1
s_1	5/4	16/25	2	32/25
s_2	3/2	4/9	2	8/9
s_3	7/4	16/49	2	32/49
s_4	2	1/4	1	1/4

I. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{2-1}{4}=\frac{1}{4} \Rightarrow \frac{\Delta x}{2}=\frac{1}{8}$;
$\sum mf(s_i)=\frac{179{,}573}{44{,}100} \Rightarrow T=\frac{1}{8}\left(\frac{179{,}573}{44{,}100}\right)=\frac{179{,}573}{352{,}800}$
≈ 0.50899; $f(s)=\frac{1}{s^2} \Rightarrow f'(s)=-\frac{2}{s^3}$
$\Rightarrow f''(s)=\frac{6}{s^4} \Rightarrow M=6=f''(1)$
$\Rightarrow |E_T|\le\frac{2-1}{12}\left(\frac{1}{4}\right)^2(6)=\frac{1}{32}=0.03125$

(b) $\int_{1}^{2}\frac{1}{s^2}\,ds=\int_{1}^{2}s^{-2}\,ds=\left[-\frac{1}{s}\right]_{1}^{2}=-\frac{1}{2}-\left(-\frac{1}{1}\right)=\frac{1}{2} \Rightarrow E_T=\int_{1}^{2}\frac{1}{s^2}\,ds-T=\frac{1}{2}-0.50899=-0.00899$
$\Rightarrow |E_T|=0.00899$

(c) $\frac{|E_T|}{\text{True Value}}\times 100=\frac{0.00899}{0.5}\times 100\approx 2\%$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	1	1	1	1
s_1	5/4	16/25	4	64/25
s_2	3/2	4/9	2	8/9
s_3	7/4	16/49	4	64/49
s_4	2	1/4	1	1/4

II. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{2-1}{4}=\frac{1}{4} \Rightarrow \frac{\Delta x}{3}=\frac{1}{12}$;
$\sum mf(s_i)=\frac{264{,}821}{44{,}100} \Rightarrow S=\frac{1}{12}\left(\frac{264{,}821}{44{,}100}\right)=\frac{264{,}821}{529{,}200}$
≈ 0.50042; $f^{(3)}(s)=-\frac{24}{s^5} \Rightarrow f^{(4)}(s)=\frac{120}{s^6}$
$\Rightarrow M=120 \Rightarrow |E_S|\le\left|\frac{2-1}{180}\right|\left(\frac{1}{4}\right)^4(120)$
$=\frac{1}{384}\approx 0.00260$

(b) $\int_{1}^{2}\frac{1}{s^2}\,ds=\frac{1}{2} \Rightarrow E_S=\int_{1}^{2}\frac{1}{s^2}\,ds-S=\frac{1}{2}-0.50042=-0.00042 \Rightarrow |E_S|=0.00042$

(c) $\frac{|E_S|}{\text{True Value}}\times 100=\frac{0.0004}{0.5}\times 100\approx 0.08\%$

8. $\int_{2}^{4}\frac{1}{(s-1)^2}\,ds$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	2	1	1	1
s_1	5/2	4/9	2	8/9
s_2	3	1/4	2	1/2
s_3	7/2	4/25	2	8/25
s_4	4	1/9	1	1/9

I. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{4-2}{4}=\frac{1}{2} \Rightarrow \frac{\Delta x}{2}=\frac{1}{4}$;
$\sum mf(s_i)=\frac{1269}{450}$
$\Rightarrow T=\frac{1}{4}\left(\frac{1269}{450}\right)=\frac{1269}{1800}=0.70500$;
$f(s)=(s-1)^{-2} \Rightarrow f'(s)=-\frac{2}{(s-1)^3}$
$\Rightarrow f''(s)=\frac{6}{(s-1)^4} \Rightarrow M=6$
$\Rightarrow |E_T|\le\frac{4-2}{12}\left(\frac{1}{2}\right)^2(6)=\frac{1}{4}=0.25$

(b) $\int_2^4 \frac{1}{(s-1)^2}\,ds = \left[\frac{-1}{(s-1)}\right]_2^4 = \left(\frac{-1}{4-1}\right) - \left(\frac{-1}{2-1}\right) = \frac{2}{3} \Rightarrow E_T = \int_2^4 \frac{1}{(s-1)^2}\,ds - T = \frac{2}{3} - 0.705 \approx -0.03833$
$\Rightarrow |E_T| \approx 0.03833$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03833}{\left(\frac{2}{3}\right)} \times 100 \approx 6\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
$\sum mf(s_i) = \frac{1813}{450}$
$\Rightarrow S = \frac{1}{6}\left(\frac{1813}{450}\right) = \frac{1813}{2700} \approx 0.67148$;
$f^{(3)}(s) = \frac{-24}{(s-1)^5} \Rightarrow f^{(4)}(s) = \frac{120}{(s-1)^6} \Rightarrow M = 120$
$\Rightarrow |E_S| \le \frac{4-2}{180}\left(\frac{1}{2}\right)^4(120) = \frac{1}{12} \approx 0.08333$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	2	1	1	1
s_1	5/2	4/9	4	16/9
s_2	3	1/4	2	1/2
s_3	7/2	4/25	4	16/25
s_4	4	1/9	1	1/9

(b) $\int_2^4 \frac{1}{(s-1)^2}\,ds = \frac{2}{3} \Rightarrow E_S = \int_2^4 \frac{1}{(s-1)^2}\,ds - S \approx \frac{2}{3} - 0.67148 = -0.00481 \Rightarrow |E_S| \approx 0.00481$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00481}{\left(\frac{2}{3}\right)} \times 100 \approx 1\%$

9. $\int_0^\pi \sin t\,dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{\Delta x}{2} = \frac{\pi}{8}$;
$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.8284$
$\Rightarrow T = \frac{\pi}{8}\left(2 + 2\sqrt{2}\right) \approx 1.89612$;
$f(t) = \sin t \Rightarrow f'(t) = \cos t \Rightarrow f''(t) = -\sin t$
$\Rightarrow M = 1 \Rightarrow |E_T| \le \frac{\pi-0}{12}\left(\frac{\pi}{4}\right)^2(1) = \frac{\pi^3}{192}$
≈ 0.16149

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	$\pi/4$	$\sqrt{2}/2$	2	$\sqrt{2}$
t_2	$\pi/2$	1	2	2
t_3	$3\pi/4$	$\sqrt{2}/2$	2	$\sqrt{2}$
t_4	π	0	1	0

(b) $\int_0^\pi \sin t\,dt = [-\cos t]_0^\pi = (-\cos \pi) - (-\cos 0) = 2 \Rightarrow |E_T| = \int_0^\pi \sin t\,dt - T \approx 2 - 1.89612 = 0.10388$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.10388}{2} \times 100 \approx 5\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{12}$;
$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.6569$
$\Rightarrow S = \frac{\pi}{12}\left(2 + 4\sqrt{2}\right) \approx 2.00456$;
$f^{(3)}(t) = -\cos t \Rightarrow f^{(4)}(t) = \sin t$
$\Rightarrow M = 1 \Rightarrow |E_S| \le \frac{\pi-0}{180}\left(\frac{\pi}{4}\right)^4(1) \approx 0.00664$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	$\pi/4$	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_2	$\pi/2$	1	2	2
t_3	$3\pi/4$	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_4	π	0	1	0

(b) $\int_0^\pi \sin t\,dt = 2 \Rightarrow E_S = \int_0^\pi \sin t\,dt - S \approx 2 - 2.00456 = -0.00456 \Rightarrow |E_S| \approx 0.00456$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00456}{2} \times 100 \approx 0\%$

10. $\int_0^1 \sin \pi t\,dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;
$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.828$
$\Rightarrow T = \frac{1}{8}\left(2 + 2\sqrt{2}\right) \approx 0.60355$; $f(t) = \sin \pi t$
$\Rightarrow f'(t) = \pi \cos \pi t$
$\Rightarrow f''(t) = -\pi^2 \sin \pi t \Rightarrow M = \pi^2$
$\Rightarrow |E_T| \le \frac{1-0}{12}\left(\frac{1}{4}\right)^2(\pi^2) \approx 0.05140$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/4	$\sqrt{2}/2$	2	$\sqrt{2}$
t_2	1/2	1	2	2
t_3	3/4	$\sqrt{2}/2$	2	$\sqrt{2}$
t_4	1	0	1	0

(b) $\int_0^1 \sin \pi t\,dt = \left[-\frac{1}{\pi}\cos \pi t\right]_0^1 = \left(-\frac{1}{\pi}\cos \pi\right) - \left(-\frac{1}{\pi}\cos 0\right) = \frac{2}{\pi} \approx 0.63662 \Rightarrow |E_T| = \int_0^1 \sin \pi t\,dt - T$
$\approx \frac{2}{\pi} - 0.60355 = 0.03307$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03307}{\left(\frac{2}{\pi}\right)} \times 100 \approx 5\%$

II. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;
$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.65685$
$\Rightarrow S = \frac{1}{12}\left(2 + 4\sqrt{2}\right) \approx 0.63807$;
$f^{(3)}(t) = -\pi^3 \cos \pi t \Rightarrow f^{(4)}(t) = \pi^4 \sin \pi t$
$\Rightarrow M = \pi^4 \Rightarrow |E_S| \le \frac{1-0}{180}\left(\frac{1}{4}\right)^4(\pi^4) \approx 0.00211$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/4	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_2	1/2	1	2	2
t_3	3/4	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_4	1	0	1	0

(b) $\int_0^1 \sin \pi t\, dt = \frac{2}{\pi} \approx 0.63662 \Rightarrow E_S = \int_0^1 \sin \pi t\, dt - S \approx \frac{2}{\pi} - 0.63807 = -0.00145 \Rightarrow |E_S| \approx 0.00145$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00145}{\left(\frac{2}{\pi}\right)} \times 100 \approx 0\%$

11. (a) $M = 0$ (see Exercise 1): Then $n = 1 \Rightarrow \Delta x = 1 \Rightarrow |E_T| = \frac{1}{12}(1)^2(0) = 0 < 10^{-4}$
(b) $M = 0$ (see Exercise 1): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = \frac{1}{2} \Rightarrow |E_S| = \frac{1}{180}\left(\frac{1}{2}\right)^4(0) = 0 < 10^{-4}$

12. (a) $M = 0$ (see Exercise 2): Then $n = 1 \Rightarrow \Delta x = 2 \Rightarrow |E_T| = \frac{2}{12}(2)^2(0) = 0 < 10^{-4}$
(b) $M = 0$ (see Exercise 2): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

13. (a) $M = 2$ (see Exercise 3): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(2) = \frac{4}{3n^2} < 10^{-4} \Rightarrow n^2 > \frac{4}{3}(10^4) \Rightarrow n > \sqrt{\frac{4}{3}(10^4)}$
$\Rightarrow n > 115.4$, so let $n = 116$
(b) $M = 0$ (see Exercise 3): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

14. (a) $M = 2$ (see Exercise 4): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(2) = \frac{4}{3n^2} < 10^{-4} \Rightarrow n^2 > \frac{4}{3}(10^4) \Rightarrow n > \sqrt{\frac{4}{3}(10^4)}$
$\Rightarrow n > 115.4$, so let $n = 116$
(b) $M = 0$ (see Exercise 4): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

15. (a) $M = 12$ (see Exercise 5): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(12) = \frac{8}{n^2} < 10^{-4} \Rightarrow n^2 > 8(10^4) \Rightarrow n > \sqrt{8(10^4)}$
$\Rightarrow n > 282.8$, so let $n = 283$
(b) $M = 0$ (see Exercise 5): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

16. (a) $M = 6$ (see Exercise 6): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(6) = \frac{4}{n^2} < 10^{-4} \Rightarrow n^2 > 4(10^4) \Rightarrow n > \sqrt{4(10^4)}$
$= 200$, so let $n = 201$
(b) $M = 0$ (see Exercise 6): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

17. (a) $M = 6$ (see Exercise 7): Then $\Delta x = \frac{1}{n} \Rightarrow |E_T| \le \frac{1}{12}\left(\frac{1}{n}\right)^2(6) = \frac{1}{2n^2} < 10^{-4} \Rightarrow n^2 > \frac{1}{2}(10^4) \Rightarrow n > \sqrt{\frac{1}{2}(10^4)}$
$\Rightarrow n > 70.7$, so let $n = 71$
(b) $M = 120$ (see Exercise 7): Then $\Delta x = \frac{1}{n} \Rightarrow |E_S| = \frac{1}{180}\left(\frac{1}{n}\right)^4(120) = \frac{2}{3n^4} < 10^{-4} \Rightarrow n^4 > \frac{2}{3}(10^4)$
$\Rightarrow n > \sqrt[4]{\frac{2}{3}(10^4)} \Rightarrow n > 9.04$, so let $n = 10$ (n must be even)

18. (a) $M = 6$ (see Exercise 8): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(6) = \frac{4}{n^2} < 10^{-4} \Rightarrow n^2 > 4(10^4) \Rightarrow n > \sqrt{4(10^4)}$
$\Rightarrow n > 200$, so let $n = 201$
(b) $M = 120$ (see Exercise 8): Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \le \frac{2}{180}\left(\frac{2}{n}\right)^4(120) = \frac{64}{3n^4} < 10^{-4} \Rightarrow n^4 > \frac{64}{3}(10^4)$
$\Rightarrow n > \sqrt[4]{\frac{64}{3}(10^4)} \Rightarrow n > 21.5$, so let $n = 22$ (n must be even)

19. (a) $f(x) = \sqrt{x+1} \Rightarrow f'(x) = \frac{1}{2}(x+1)^{-1/2} \Rightarrow f''(x) = -\frac{1}{4}(x+1)^{-3/2} = -\frac{1}{4(\sqrt{x+1})^3} \Rightarrow M = \frac{1}{4(\sqrt{1})^3} = \frac{1}{4}$.

Then $\Delta x = \frac{3}{n} \Rightarrow |E_T| \le \frac{3}{12}\left(\frac{3}{n}\right)^2\left(\frac{1}{4}\right) = \frac{9}{16n^2} < 10^{-4} \Rightarrow n^2 > \frac{9}{16}(10^4) \Rightarrow n > \sqrt{\frac{9}{16}(10^4)} \Rightarrow n > 75$, so let $n = 76$

(b) $f^{(3)}(x) = \frac{3}{8}(x+1)^{-5/2} \Rightarrow f^{(4)}(x) = -\frac{15}{16}(x+1)^{-7/2} = -\frac{15}{16(\sqrt{x+1})^7} \Rightarrow M = \frac{15}{16(\sqrt{1})^7} = \frac{15}{16}$. Then $\Delta x = \frac{3}{n}$

$\Rightarrow |E_S| \le \frac{3}{180}\left(\frac{3}{n}\right)^4\left(\frac{15}{16}\right) = \frac{3^5(15)}{16(180)n^4} < 10^{-4} \Rightarrow n^4 > \frac{3^5(15)(10^4)}{16(180)} \Rightarrow n > \sqrt[4]{\frac{3^5(15)(10^4)}{16(180)}} \Rightarrow n > 10.6$, so let $n = 12$ (n must be even)

20. (a) $f(x) = \frac{1}{\sqrt{x+1}} \Rightarrow f'(x) = -\frac{1}{2}(x+1)^{-3/2} \Rightarrow f''(x) = \frac{3}{4}(x+1)^{-5/2} = \frac{3}{4(\sqrt{x+1})^5} \Rightarrow M = \frac{3}{4(\sqrt{1})^5} = \frac{3}{4}$.

Then $\Delta x = \frac{3}{n} \Rightarrow |E_T| \le \frac{3}{12}\left(\frac{3}{n}\right)^2\left(\frac{3}{4}\right) = \frac{3^4}{48n^2} < 10^{-4} \Rightarrow n^2 > \frac{3^4(10^4)}{48} \Rightarrow n > \sqrt{\frac{3^4(10^4)}{48}} \Rightarrow n > 129.9$, so let $n = 130$

(b) $f^{(3)}(x) = -\frac{15}{8}(x+1)^{-7/2} \Rightarrow f^{(4)}(x) = \frac{105}{16}(x+1)^{-9/2} = \frac{105}{16(\sqrt{x+1})^9} \Rightarrow M = \frac{105}{16(\sqrt{1})^9} = \frac{105}{16}$. Then $\Delta x = \frac{3}{n}$

$\Rightarrow |E_S| \le \frac{3}{180}\left(\frac{3}{n}\right)^4\left(\frac{105}{16}\right) = \frac{3^5(105)}{16(180)n^4} < 10^{-4} \Rightarrow n^4 > \frac{3^5(105)(10^4)}{16(180)} \Rightarrow n > \sqrt[4]{\frac{3^5(105)(10^4)}{16(180)}} \Rightarrow n > 17.25$, so let $n = 18$ (n must be even)

21. (a) $f(x) = \sin(x+1) \Rightarrow f'(x) = \cos(x+1) \Rightarrow f''(x) = -\sin(x+1) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(1)$ $= \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8(10^4)}{12} \Rightarrow n > \sqrt{\frac{8(10^4)}{12}} \Rightarrow n > 81.6$, so let $n = 82$

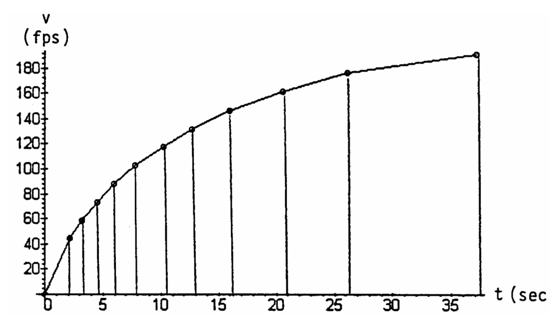
(b) $f^{(3)}(x) = -\cos(x+1) \Rightarrow f^{(4)}(x) = \sin(x+1) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \le \frac{2}{180}\left(\frac{2}{n}\right)^4(1) = \frac{32}{180n^4} < 10^{-4}$ $\Rightarrow n^4 > \frac{32(10^4)}{180} \Rightarrow n > \sqrt[4]{\frac{32(10^4)}{180}} \Rightarrow n > 6.49$, so let $n = 8$ (n must be even)

22. (a) $f(x) = \cos(x+\pi) \Rightarrow f'(x) = -\sin(x+\pi) \Rightarrow f''(x) = -\cos(x+\pi) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n}$ $\Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(1) = \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8(10^4)}{12} \Rightarrow n > \sqrt{\frac{8(10^4)}{12}} \Rightarrow n > 81.6$, so let $n = 82$

(b) $f^{(3)}(x) = \sin(x+\pi) \Rightarrow f^{(4)}(x) = \cos(x+\pi) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \le \frac{2}{180}\left(\frac{2}{n}\right)^4(1) = \frac{32}{180n^4} < 10^{-4}$ $\Rightarrow n^4 > \frac{32(10^4)}{180} \Rightarrow n > \sqrt[4]{\frac{32(10^4)}{180}} \Rightarrow n > 6.49$, so let $n = 8$ (n must be even)

23. $\frac{5}{2}(6.0 + 2(8.2) + 2(9.1) \ldots + 2(12.7) + 13.0)(30) = 15{,}990 \text{ ft}^3$.

24. Use the conversion 30 mph = 44 fps (ft per sec) since time is measured in seconds. The distance traveled as the car accelerates from, say, 40 mph = 58.67 fps to 50 mph = 73.33 fps in $(4.5 - 3.2) = 1.3$ sec is the area of the trapezoid (see figure) associated with that time interval: $\frac{1}{2}(58.67 + 73.33)(1.3) = 85.8$ ft. The total distance traveled by the Ford Mustang Cobra is the sum of all these eleven trapezoids (using $\frac{\Delta t}{2}$ and the table below):



v (mph)	0	30	40	50	60	70	80	90	100	110	120	130
v (fps)	0	44	58.67	73.33	88	102.67	117.33	132	146.67	161.33	176	190.67
t (sec)	0	2.2	3.2	4.5	5.9	7.8	10.2	12.7	16	20.6	26.2	37.1
Δt/2	0	1.1	0.5	0.65	0.7	0.95	1.2	1.25	1.65	2.3	2.8	5.45

$s = (44)(1.1) + (102.67)(0.5) + (132)(0.65) + (161.33)(0.7) + (190.67)(0.95) + (220)(1.2) + (249.33)(1.25) + (278.67)(1.65) + (308)(2.3) + (337.33)(2.8) + (366.67)(5.45) = 5166.346 \text{ ft} \approx 0.9785 \text{ mi}$

25. Using Simpson's Rule, $\Delta x = 1 \Rightarrow \frac{\Delta x}{3} = \frac{1}{3}$; $\sum my_i = 33.6 \Rightarrow$ Cross Section Area $\approx \frac{1}{3}(33.6) = 11.2 \text{ ft}^2$. Let x be the length of the tank. Then the Volume V = (Cross Sectional Area) x = 11.2x. Now 5000 lb of gasoline at 42 lb/ft^3 $\Rightarrow V = \frac{5000}{42} = 119.05 \text{ ft}^3$ $\Rightarrow 119.05 = 11.2x \Rightarrow x \approx 10.63$ ft

	x_i	y_i	m	my_i
x_0	0	1.5	1	1.5
x_1	1	1.6	4	6.4
x_2	2	1.8	2	3.6
x_3	3	1.9	4	7.6
x_4	4	2.0	2	4.0
x_5	5	2.1	4	8.4
x_6	6	2.1	1	2.1

26. $\frac{24}{2}[0.019 + 2(0.020) + 2(0.021) + \ldots + 2(0.031) + 0.035] = 4.2$ L

27. (a) $|E_S| \le \frac{b-a}{180}(\Delta x^4) M$; $n = 4 \Rightarrow \Delta x = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$; $|f^{(4)}| \le 1 \Rightarrow M = 1 \Rightarrow |E_S| \le \frac{(\frac{\pi}{2}-0)}{180}\left(\frac{\pi}{8}\right)^4(1) \approx 0.00021$

(b) $\Delta x = \frac{\pi}{8} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{24}$; $\sum mf(x_i) = 10.47208705$ $\Rightarrow S = \frac{\pi}{24}(10.47208705) \approx 1.37079$

	x_i	$f(x_i)$	m	$mf(x_{1i})$
x_0	0	1	1	1
x_1	$\pi/8$	0.974495358	4	3.897981432
x_2	$\pi/4$	0.900316316	2	1.800632632
x_3	$3\pi/8$	0.784213303	4	3.136853212
x_4	$\pi/2$	0.636619772	1	0.636619772

(c) $\approx \left(\frac{0.00021}{1.37079}\right) \times 100 \approx 0.015\%$

28. (a) $\Delta x = \frac{b-a}{n} = \frac{1-0}{10} = 0.1 \Rightarrow \text{erf}(1) = \frac{2}{\sqrt{3}}\left(\frac{0.1}{3}\right)(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 4y_9 + y_{10})$

$\frac{2}{30\sqrt{\pi}}(e^0 + 4e^{-0.01} + 2e^{-0.04} + 4e^{-0.09} + \ldots + 4e^{-0.81} + e^{-1}) \approx 0.843$

(b) $|E_S| \le \frac{1-0}{180}(0.1)^4(12) \approx 6.7 \times 10^{-6}$

29. $T = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{n-1} + y_n)$ where $\Delta x = \frac{b-a}{n}$ and f is continuous on [a, b]. So $T = \frac{b-a}{n}\frac{(y_0 + y_1 + y_1 + y_2 + y_2 + \ldots + y_{n-1} + y_{n-1} + y_n)}{2} = \frac{b-a}{n}\left(\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \ldots + \frac{f(x_{n-1})+f(x_n)}{2}\right)$.

Since f is continuous on each interval $[x_{k-1}, x_k]$, and $\frac{f(x_{k-1})+f(x_k)}{2}$ is always between $f(x_{k-1})$ and $f(x_k)$, there is a point c_k in $[x_{k-1}, x_k]$ with $f(c_k) = \frac{f(x_{k-1})+f(x_k)}{2}$; this is a consequence of the Intermediate Value Theorem. Thus our sum is $\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f(c_k)$ which has the form $\sum_{k=1}^{n}\Delta x_k f(c_k)$ with $\Delta x_k = \frac{b-a}{n}$ for all k. This is a Riemann Sum for f on [a, b].

30. $S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$ where n is even, $\Delta x = \frac{b-a}{n}$ and f is continuous on [a, b]. So $S = \frac{b-a}{n}\left(\frac{y_0 + 4y_1 + y_2}{3} + \frac{y_2 + 4y_3 + y_4}{3} + \frac{y_4 + 4y_5 + y_6}{3} + \ldots + \frac{y_{n-2} + 4y_{n-1} + y_n}{3}\right)$

$= \frac{b-a}{\frac{n}{2}}\left(\frac{f(x_0)+4f(x_1)+f(x_2)}{6} + \frac{f(x_2)+4f(x_3)+f(x_4)}{6} + \frac{f(x_4)+4f(x_5)+f(x_6)}{6} + \ldots + \frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right)$

$\frac{f(x_{2k})+4f(x_{2k+1})+f(x_{2k+2})}{6}$ is the average of the six values of the continuous function on the interval $[x_{2k}, x_{2k+2}]$, so it is between the minimum and maximum of f on this interval. By the Extreme Value Theorem for continuous functions, f takes on its maximum and minimum in this interval, so there are x_a and x_b with $x_{2k} \le x_a, x_b \le x_{2k+2}$ and $f(x_a) \le \frac{f(x_{2k})+4f(x_{2k+1})+f(x_{2k+2})}{6} \le f(x_b)$. By the Intermediate Value Theorem, there is c_k in $[x_{2k}, x_{2k+2}]$ with $f(c_k) = \frac{f(x_{2k})+4f(x_{2k+1})+f(x_{2k+2})}{6}$. So our sum has the form $\sum_{k=1}^{n/2}\Delta x_k f(c_k)$ with $\Delta x_k = \frac{b-a}{(n/2)}$, a Riemann sum for f on [a, b].

31. (a) $a = 1, e = \frac{1}{2} \Rightarrow \text{Length} = 4\int_0^{\pi/2}\sqrt{1 - \frac{1}{4}\cos^2 t}\,dt$
$= 2\int_0^{\pi/2}\sqrt{4 - \cos^2 t}\,dt = \int_0^{\pi/2} f(t)\,dt$; use the
Trapezoid Rule with $n = 10 \Rightarrow \Delta t = \frac{b-a}{n} = \frac{\left(\frac{\pi}{2}\right) - 0}{10}$
$= \frac{\pi}{20}$. $\int_0^{\pi/2}\sqrt{4 - \cos^2 t}\,dt \approx \sum_{n=0}^{10} mf(x_n) = 37.3686183$
$\Rightarrow T = \frac{\Delta t}{2}(37.3686183) = \frac{\pi}{40}(37.3686183)$
$= 2.934924419 \Rightarrow \text{Length} = 2(2.934924419)$
≈ 5.870
(b) $|f''(t)| < 1 \Rightarrow M = 1$
$\Rightarrow |E_T| \le \frac{b-a}{12}(\Delta t^2 M) \le \frac{\left(\frac{\pi}{2}\right) - 0}{12}\left(\frac{\pi}{20}\right)^2 1 \le 0.0032$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	1.732050808	1	1.732050808
x_1	$\pi/20$	1.739100843	2	3.478201686
x_2	$\pi/10$	1.759400893	2	3.518801786
x_3	$3\pi/20$	1.790560631	2	3.581121262
x_4	$\pi/5$	1.82906848	1	3.658136959
x_5	$\pi/4$	1.870828693	1	3.741657387
x_6	$3\pi/10$	1.911676881	2	3.823353762
x_7	$7\pi/20$	1.947791731	2	3.895583461
x_8	$2\pi/5$	1.975982919	2	3.951965839
x_9	$9\pi/20$	1.993872679	2	3.987745357
x_{10}	$\pi/2$	2	1	2

32. $\Delta x = \frac{\pi - 0}{8} = \frac{\pi}{8} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{24}; \sum mf(x_i) = 29.184807792$
$\Rightarrow S = \frac{\pi}{24}(29.18480779) \approx 3.82028$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	1.414213562	1	1.414213562
x_1	$\pi/8$	1.361452677	4	5.445810706
x_2	$\pi/4$	1.224744871	2	2.449489743
x_3	$3\pi/8$	1.070722471	4	4.282889883
x_4	$\pi/2$	1	2	2
x_5	$5\pi/8$	1.070722471	4	4.282889883
x_6	$3\pi/4$	1.224744871	2	2.449489743
x_7	$7\pi/8$	1.361452677	4	5.445810706
x_8	π	1.414213562	1	1.414213562

33. The length of the curve $y = \sin\left(\frac{3\pi}{20}x\right)$ from 0 to 20 is: $L = \int_0^{20}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$; $\frac{dy}{dx} = \frac{3\pi}{20}\cos\left(\frac{3\pi}{20}x\right) \Rightarrow \left(\frac{dy}{dx}\right)^2$
$= \frac{9\pi^2}{400}\cos^2\left(\frac{3\pi}{20}x\right) \Rightarrow L = \int_0^{20}\sqrt{1 + \frac{9\pi^2}{400}\cos^2\left(\frac{3\pi}{20}x\right)}\,dx$. Using numerical integration we find $L \approx 21.07$ in

34. First, we'll find the length of the cosine curve: $L = \int_{-25}^{25}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$; $\frac{dy}{dx} = -\frac{25\pi}{50}\sin\left(\frac{\pi x}{50}\right)$
$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{50}\right) \Rightarrow L = \int_{-25}^{25}\sqrt{1 + \frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{50}\right)}\,dx$. Using a numerical integrator we find
$L \approx 73.1848$ ft. Surface area is: $A = \text{length} \cdot \text{width} \approx (73.1848)(300) = 21{,}955.44$ ft.
Cost $= 1.75A = (1.75)(21{,}955.44) = \$38{,}422.02$. Answers may vary slightly, depending on the numerical integration used.

35. $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \cos^2 x \Rightarrow S = \int_0^{\pi} 2\pi(\sin x)\sqrt{1 + \cos^2 x}\,dx$; a numerical integration gives $S \approx 14.4$

36. $y = \frac{x^2}{4} \Rightarrow \frac{dy}{dx} = \frac{x}{2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4} \Rightarrow S = \int_0^2 2\pi\left(\frac{x^2}{4}\right)\sqrt{1 + \frac{x^2}{4}}\,dx$; a numerical integration gives $S \approx 5.28$

37. A calculator or computer numerical integrator yields $\sin^{-1} 0.6 \approx 0.643501109$.

38. A calculator or computer numerical integrator yields $\pi \approx 3.1415929$.

8.7 IMPROPER INTEGRALS

1. $\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b = \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

2. $\int_1^\infty \frac{dx}{x^{1.001}} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^{1.001}} = \lim_{b\to\infty} \left[-1000x^{-0.001}\right]_1^b = \lim_{b\to\infty} \left(\frac{-1000}{b^{0.001}} + 1000\right) = 1000$

3. $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} \int_b^1 x^{-1/2}\,dx = \lim_{b\to 0^+} \left[2x^{1/2}\right]_b^1 = \lim_{b\to 0^+} \left(2 - 2\sqrt{b}\right) = 2 - 0 = 2$

4. $\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-} \int_0^b (4-x)^{-1/2}dx = \lim_{b\to 4^-} \left[-2\sqrt{4-b} - \left(-2\sqrt{4}\right)\right] = 0 + 4 = 4$

5. $\int_{-1}^1 \frac{dx}{x^{2/3}} = \int_{-1}^0 \frac{dx}{x^{2/3}} + \int_0^1 \frac{dx}{x^{2/3}} = \lim_{b\to 0^-} \left[3x^{1/3}\right]_{-1}^b + \lim_{c\to 0^+} \left[3x^{1/3}\right]_c^1$
$= \lim_{b\to 0^-} \left[3b^{1/3} - 3(-1)^{1/3}\right] + \lim_{c\to 0^+} \left[3(1)^{1/3} - 3c^{1/3}\right] = (0+3) + (3-0) = 6$

6. $\int_{-8}^1 \frac{dx}{x^{1/3}} = \int_{-8}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}} = \lim_{b\to 0^-} \left[\frac{3}{2} x^{2/3}\right]_{-8}^b + \lim_{c\to 0^+} \left[\frac{3}{2} x^{2/3}\right]_c^1$
$= \lim_{b\to 0^-} \left[\frac{3}{2} b^{2/3} - \frac{3}{2}(-8)^{2/3}\right] + \lim_{c\to 0^+} \left[\frac{3}{2}(1)^{2/3} - \frac{3}{2} c^{2/3}\right] = \left[0 - \frac{3}{2}(4)\right] + \left(\frac{3}{2} - 0\right) = -\frac{9}{2}$

7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-} \left[\sin^{-1} x\right]_0^b = \lim_{b\to 1^-} \left(\sin^{-1} b - \sin^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

8. $\int_0^1 \frac{dr}{r^{0.999}} = \lim_{b\to 0^+} \left[1000r^{0.001}\right]_b^1 = \lim_{b\to 0^+} \left(1000 - 1000b^{0.001}\right) = 1000 - 0 = 1000$

9. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1} = \int_{-\infty}^{-2} \frac{dx}{x-1} - \int_{-\infty}^{-2} \frac{dx}{x+1} = \lim_{b\to -\infty} \left[\ln|x-1|\right]_b^{-2} - \lim_{b\to -\infty} \left[\ln|x+1|\right]_b^{-2} = \lim_{b\to -\infty} \left[\ln\left|\frac{x-1}{x+1}\right|\right]_b^{-2}$
$= \lim_{b\to -\infty} \left(\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right) = \ln 3 - \ln\left(\lim_{b\to -\infty} \frac{b-1}{b+1}\right) = \ln 3 - \ln 1 = \ln 3$

10. $\int_{-\infty}^2 \frac{2\,dx}{x^2+4} = \lim_{b\to -\infty} \left[\tan^{-1} \frac{x}{2}\right]_b^2 = \lim_{b\to -\infty} \left(\tan^{-1} 1 - \tan^{-1} \frac{b}{2}\right) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}$

11. $\int_2^\infty \frac{2\,dv}{v^2-v} = \lim_{b\to\infty} \left[2\ln\left|\frac{v-1}{v}\right|\right]_2^b = \lim_{b\to\infty} \left(2\ln\left|\frac{b-1}{b}\right| - 2\ln\left|\frac{2-1}{2}\right|\right) = 2\ln(1) - 2\ln\left(\frac{1}{2}\right) = 0 + 2\ln 2 = \ln 4$

12. $\int_2^\infty \frac{2\,dt}{t^2-1} = \lim_{b\to\infty} \left[\ln\left|\frac{t-1}{t+1}\right|\right]_2^b = \lim_{b\to\infty} \left(\ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right) = \ln(1) - \ln\left(\frac{1}{3}\right) = 0 + \ln 3 = \ln 3$

13. $\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2} = \int_{-\infty}^0 \frac{2x\,dx}{(x^2+1)^2} + \int_0^\infty \frac{2x\,dx}{(x^2+1)^2}$; $\begin{bmatrix} u = x^2+1 \\ du = 2x\,dx \end{bmatrix} \to \int_\infty^1 \frac{du}{u^2} + \int_1^\infty \frac{du}{u^2} = \lim_{b\to\infty} \left[-\frac{1}{u}\right]_b^1 + \lim_{c\to\infty} \left[-\frac{1}{u}\right]_1^c$
$= \lim_{b\to\infty} \left(-1 + \frac{1}{b}\right) + \lim_{c\to\infty} \left[-\frac{1}{c} - (-1)\right] = (-1+0) + (0+1) = 0$

14. $\int_{-\infty}^\infty \frac{x\,dx}{(x^2+4)^{3/2}} = \int_{-\infty}^0 \frac{x\,dx}{(x^2+4)^{3/2}} + \int_0^\infty \frac{x\,dx}{(x^2+4)^{3/2}}$; $\begin{bmatrix} u = x^2+4 \\ du = 2x\,dx \end{bmatrix} \to \int_\infty^4 \frac{du}{2u^{3/2}} + \int_4^\infty \frac{du}{2u^{3/2}}$
$= \lim_{b\to\infty} \left[-\frac{1}{\sqrt{u}}\right]_b^4 + \lim_{c\to\infty} \left[-\frac{1}{\sqrt{u}}\right]_4^c = \lim_{b\to\infty} \left(-\frac{1}{2} + \frac{1}{\sqrt{b}}\right) + \lim_{c\to\infty} \left(-\frac{1}{\sqrt{c}} + \frac{1}{2}\right) = \left(-\frac{1}{2} + 0\right) + \left(0 + \frac{1}{2}\right) = 0$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta;\ \begin{bmatrix} u=\theta^2+2\theta \\ du=2(\theta+1)\,d\theta \end{bmatrix} \to \int_0^3 \frac{du}{2\sqrt{u}} = \lim_{b\to 0^+} \int_b^3 \frac{du}{2\sqrt{u}} = \lim_{b\to 0^+} \left[\sqrt{u}\right]_b^3 = \lim_{b\to 0^+} \left(\sqrt{3}-\sqrt{b}\right) = \sqrt{3}-0$
$= \sqrt{3}$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds = \frac{1}{2}\int_0^2 \frac{2s\,ds}{\sqrt{4-s^2}} + \int_0^2 \frac{ds}{\sqrt{4-s^2}};\ \begin{bmatrix} u=4-s^2 \\ du=-2s\,ds \end{bmatrix} \to -\frac{1}{2}\int_4^0 \frac{du}{\sqrt{u}} + \lim_{c\to 2^-}\int_0^c \frac{ds}{\sqrt{4-s^2}}$
$= \lim_{b\to 0^+}\int_b^4 \frac{du}{2\sqrt{u}} + \lim_{c\to 2^-}\int_0^c \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to 0^+}\left[\sqrt{u}\right]_b^4 + \lim_{c\to 2^-}\left[\sin^{-1}\frac{s}{2}\right]_0^c$
$= \lim_{b\to 0^+}\left(2-\sqrt{b}\right) + \lim_{c\to 2^-}\left(\sin^{-1}\frac{c}{2} - \sin^{-1}0\right) = (2-0) + \left(\frac{\pi}{2}-0\right) = \frac{4+\pi}{2}$

17. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}};\ \begin{bmatrix} u=\sqrt{x} \\ du=\frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int_0^\infty \frac{2\,du}{u^2+1} = \lim_{b\to\infty}\int_0^b \frac{2\,du}{u^2+1} = \lim_{b\to\infty}\left[2\tan^{-1}u\right]_0^b$
$= \lim_{b\to\infty}\left(2\tan^{-1}b - 2\tan^{-1}0\right) = 2\left(\frac{\pi}{2}\right) - 2(0) = \pi$

18. $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{b\to 1^+}\int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty}\int_2^c \frac{dx}{x\sqrt{x^2-1}}$
$= \lim_{b\to 1^+}\left[\sec^{-1}|x|\right]_b^2 + \lim_{c\to\infty}\left[\sec^{-1}|x|\right]_2^c = \lim_{b\to 1^+}\left(\sec^{-1}2 - \sec^{-1}b\right) + \lim_{c\to\infty}\left(\sec^{-1}c - \sec^{-1}2\right)$
$= \left(\frac{\pi}{3}-0\right) + \left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{\pi}{2}$

19. $\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1}v)} = \lim_{b\to\infty}\left[\ln|1+\tan^{-1}v|\right]_0^b = \lim_{b\to\infty}\left[\ln|1+\tan^{-1}b|\right] - \ln|1+\tan^{-1}0|$
$= \ln\left(1+\frac{\pi}{2}\right) - \ln(1+0) = \ln\left(1+\frac{\pi}{2}\right)$

20. $\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx = \lim_{b\to\infty}\left[8\left(\tan^{-1}x\right)^2\right]_0^b = \lim_{b\to\infty}\left[8\left(\tan^{-1}b\right)^2\right] - 8\left(\tan^{-1}0\right)^2 = 8\left(\frac{\pi}{2}\right)^2 - 8(0) = 2\pi^2$

21. $\int_{-\infty}^0 \theta e^\theta\,d\theta = \lim_{b\to-\infty}\left[\theta e^\theta - e^\theta\right]_b^0 = \left(0\cdot e^0 - e^0\right) - \lim_{b\to-\infty}\left[be^b - e^b\right] = -1 - \lim_{b\to-\infty}\left(\frac{b-1}{e^{-b}}\right)$
$= -1 - \lim_{b\to-\infty}\left(\frac{1}{-e^{-b}}\right)$ (l'Hôpital's rule for $\frac{\infty}{\infty}$ form)
$= -1 - 0 = -1$

22. $\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta = \lim_{b\to\infty}\int_0^b 2e^{-\theta}\sin\theta\,d\theta$
$= \lim_{b\to\infty} 2\left[\frac{e^{-\theta}}{1+1}(-\sin\theta - \cos\theta)\right]_0^b$ (FORMULA 107 with $a=-1$, $b=1$)
$= \lim_{b\to\infty}\frac{-2(\sin b+\cos b)}{2e^b} + \frac{2(\sin 0+\cos 0)}{2e^0} = 0 + \frac{2(0+1)}{2} = 1$

23. $\int_{-\infty}^0 e^{-|x|}\,dx = \int_{-\infty}^0 e^x\,dx = \lim_{b\to-\infty}\left[e^x\right]_b^0 = \lim_{b\to-\infty}\left(1-e^b\right) = (1-0) = 1$

24. $\int_{-\infty}^\infty 2xe^{-x^2}\,dx = \int_{-\infty}^0 2xe^{-x^2}\,dx + \int_0^\infty 2xe^{-x^2}\,dx = \lim_{b\to-\infty}\left[-e^{-x^2}\right]_b^0 + \lim_{c\to\infty}\left[-e^{-x^2}\right]_0^c$
$= \lim_{b\to-\infty}\left[-1-\left(-e^{-b^2}\right)\right] + \lim_{c\to\infty}\left[-e^{-c^2}-(-1)\right] = (-1-0)+(0+1) = 0$

25. $\int_0^1 x\ln x\,dx = \lim_{b\to 0^+}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_b^1 = \left(\frac{1}{2}\ln 1 - \frac{1}{4}\right) - \lim_{b\to 0^+}\left(\frac{b^2}{2}\ln b - \frac{b^2}{4}\right) = -\frac{1}{4} - \lim_{b\to 0^+}\frac{\ln b}{\left(\frac{2}{b^2}\right)} + 0$
$= -\frac{1}{4} - \lim_{b\to 0^+}\frac{\left(\frac{1}{b}\right)}{\left(-\frac{4}{b^3}\right)} = -\frac{1}{4} + \lim_{b\to 0^+}\left(\frac{b^2}{4}\right) = -\frac{1}{4} + 0 = -\frac{1}{4}$

26. $\int_0^1 (-\ln x)\,dx = \lim_{b\to 0^+} [x - x\ln x]_b^1 = [1 - 1\ln 1] - \lim_{b\to 0^+} [b - b\ln b] = 1 - 0 + \lim_{b\to 0^+} \frac{\ln b}{\left(\frac{1}{b}\right)} = 1 + \lim_{b\to 0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$
$= 1 - \lim_{b\to 0^+} b = 1 - 0 = 1$

27. $\int_0^2 \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to 2^-} \left[\sin^{-1} \frac{s}{2}\right]_0^b = \lim_{b\to 2^-} \left(\sin^{-1} \frac{b}{2}\right) - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

28. $\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}} = \lim_{b\to 1^-} \left[2\sin^{-1}(r^2)\right]_0^b = \lim_{b\to 1^-} \left[2\sin^{-1}(b^2)\right] - 2\sin^{-1} 0 = 2\cdot\frac{\pi}{2} - 0 = \pi$

29. $\int_1^2 \frac{ds}{s\sqrt{s^2-1}} = \lim_{b\to 1^+} \left[\sec^{-1} s\right]_b^2 = \sec^{-1} 2 - \lim_{b\to 1^+} \sec^{-1} b = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

30. $\int_2^4 \frac{dt}{t\sqrt{t^2-4}} = \lim_{b\to 2^+} \left[\frac{1}{2}\sec^{-1}\frac{t}{2}\right]_b^4 = \lim_{b\to 2^+} \left[\left(\frac{1}{2}\sec^{-1}\frac{4}{2}\right) - \frac{1}{2}\sec^{-1}\left(\frac{b}{2}\right)\right] = \frac{1}{2}\left(\frac{\pi}{3}\right) - \frac{1}{2}\cdot 0 = \frac{\pi}{6}$

31. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}} = \lim_{b\to 0^-} \int_{-1}^b \frac{dx}{\sqrt{-x}} + \lim_{c\to 0^+} \int_c^4 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^-} \left[-2\sqrt{-x}\right]_{-1}^b + \lim_{c\to 0^+} \left[2\sqrt{x}\right]_c^4$
$= \lim_{b\to 0^-} \left(-2\sqrt{-b}\right) - \left(-2\sqrt{-(-1)}\right) + 2\sqrt{4} - \lim_{c\to 0^+} 2\sqrt{c} = 0 + 2 + 2\cdot 2 - 0 = 6$

32. $\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{b\to 1^-} \left[-2\sqrt{1-x}\right]_0^b + \lim_{c\to 1^+} \left[2\sqrt{x-1}\right]_c^2$
$= \lim_{b\to 1^-} \left(-2\sqrt{1-b}\right) - \left(-2\sqrt{1-0}\right) + 2\sqrt{2-1} - \lim_{c\to 1^+} \left(2\sqrt{c-1}\right) = 0 + 2 + 2 - 0 = 4$

33. $\int_{-1}^\infty \frac{d\theta}{\theta^2+5\theta+6} = \lim_{b\to\infty} \left[\ln\left|\frac{\theta+2}{\theta+3}\right|\right]_{-1}^b = \lim_{b\to\infty} \left[\ln\left|\frac{b+2}{b+3}\right|\right] - \ln\left|\frac{-1+2}{-1+3}\right| = 0 - \ln\left(\frac{1}{2}\right) = \ln 2$

34. $\int_0^\infty \frac{dx}{(x+1)(x^2+1)} = \lim_{b\to\infty} \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1} x\right]_0^b = \lim_{b\to\infty} \left[\frac{1}{2}\ln\left(\frac{x+1}{\sqrt{x^2+1}}\right) + \frac{1}{2}\tan^{-1} x\right]_0^b$
$= \lim_{b\to\infty} \left[\frac{1}{2}\ln\left(\frac{b+1}{\sqrt{b^2+1}}\right) + \frac{1}{2}\tan^{-1} b\right] - \left[\frac{1}{2}\ln\frac{1}{\sqrt{1}} + \frac{1}{2}\tan^{-1} 0\right] = \frac{1}{2}\ln 1 + \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\ln 1 - \frac{1}{2}\cdot 0 = \frac{\pi}{4}$

35. $\int_0^{\pi/2} \tan\theta\,d\theta = \lim_{b\to\frac{\pi}{2}^-} \left[-\ln|\cos\theta|\right]_0^b = \lim_{b\to\frac{\pi}{2}^-} \left[-\ln|\cos b|\right] + \ln 1 = \lim_{b\to\frac{\pi}{2}^-} \left[-\ln|\cos b|\right] = +\infty$, the integral diverges

36. $\int_0^{\pi/2} \cot\theta\,d\theta = \lim_{b\to 0^+} \left[\ln|\sin\theta|\right]_b^{\pi/2} = \ln 1 - \lim_{b\to 0^+} \left[\ln|\sin b|\right] = -\lim_{b\to 0^+} \left[\ln|\sin b|\right] = +\infty$, the integral diverges

37. $\int_0^\pi \frac{\sin\theta\,d\theta}{\sqrt{\pi-\theta}}$; $[\pi - \theta = x] \rightarrow -\int_\pi^0 \frac{\sin x\,dx}{\sqrt{x}} = \int_0^\pi \frac{\sin x\,dx}{\sqrt{x}}$. Since $0 \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ for all $0 \le x \le \pi$ and $\int_0^\pi \frac{dx}{\sqrt{x}}$ converges, then $\int_0^\pi \frac{\sin x}{\sqrt{x}}\,dx$ converges by the Direct Comparison Test.

38. $\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\,d\theta}{(\pi-2\theta)^{1/3}}$; $\begin{bmatrix} x = \pi - 2\theta \\ \theta = \frac{\pi}{2} - \frac{x}{2} \\ d\theta = -\frac{dx}{2} \end{bmatrix} \rightarrow \int_{2\pi}^0 \frac{-\cos\left(\frac{\pi}{2}-\frac{x}{2}\right)dx}{2x^{1/3}} = \int_0^{2\pi} \frac{\sin\left(\frac{x}{2}\right)dx}{2x^{1/3}}$. Since $0 \le \frac{\sin\frac{x}{2}}{2x^{1/3}} \le \frac{1}{2x^{1/3}}$ for all $0 \le x \le 2\pi$ and $\int_0^{2\pi} \frac{dx}{2x^{1/3}}$ converges, then $\int_0^{2\pi} \frac{\sin\frac{x}{2}\,dx}{2x^{1/3}}$ converges by the Direct Comparison Test.

39. $\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$; $\left[\frac{1}{x} = y\right] \rightarrow \int_\infty^{1/\ln 2} \frac{y^2e^{-y}\,dy}{-y^2} = \int_{1/\ln 2}^\infty e^{-y}\,dy = \lim_{b\to\infty} \left[-e^{-y}\right]_{1/\ln 2}^b = \lim_{b\to\infty} \left[-e^{-b}\right] - \left[-e^{-1/\ln 2}\right]$
$= 0 + e^{-1/\ln 2} = e^{-1/\ln 2}$, so the integral converges.

40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$; $[y = \sqrt{x}] \rightarrow 2\int_0^1 e^{-y}\,dy = 2 - \frac{2}{e}$, so the integral converges.

41. $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$. Since for $0 \le t \le \pi$, $0 \le \frac{1}{\sqrt{t} + \sin t} \le \frac{1}{\sqrt{t}}$ and $\int_0^{\pi} \frac{dt}{\sqrt{t}}$ converges, then the original integral converges as well by the Direct Comparison Test.

42. $\int_0^1 \frac{dt}{t - \sin t}$; let $f(t) = \frac{1}{t - \sin t}$ and $g(t) = \frac{1}{t^3}$, then $\lim\limits_{t \to 0} \frac{f(t)}{g(t)} = \lim\limits_{t \to 0} \frac{t^3}{t - \sin t} = \lim\limits_{t \to 0} \frac{3t^2}{1 - \cos t} = \lim\limits_{t \to 0} \frac{6t}{\sin t}$ $= \lim\limits_{t \to 0} \frac{6}{\cos t} = 6$. Now, $\int_0^1 \frac{dt}{t^3} = \lim\limits_{b \to 0^+} \left[-\frac{1}{2t^2}\right]_b^1 = -\frac{1}{2} - \lim\limits_{b \to 0^+} \left[-\frac{1}{2b^2}\right] = +\infty$, which diverges $\Rightarrow \int_0^1 \frac{dt}{t - \sin t}$ diverges by the Limit Comparison Test.

43. $\int_0^2 \frac{dx}{1 - x^2} = \int_0^1 \frac{dx}{1 - x^2} + \int_1^2 \frac{dx}{1 - x^2}$ and $\int_0^1 \frac{dx}{1 - x^2} = \lim\limits_{b \to 1^-} \left[\frac{1}{2} \ln \left|\frac{1 + x}{1 - x}\right|\right]_0^b = \lim\limits_{b \to 1^-} \left[\frac{1}{2} \ln \left|\frac{1 + b}{1 - b}\right|\right] - 0 = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1 - x^2}$ diverges as well.

44. $\int_0^2 \frac{dx}{1 - x} = \int_0^1 \frac{dx}{1 - x} + \int_1^2 \frac{dx}{1 - x}$ and $\int_0^1 \frac{dx}{1 - x} = \lim\limits_{b \to 1^-} [-\ln(1 - x)]_0^b = \lim\limits_{b \to 1^-} [-\ln(1 - b)] - 0 = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1 - x}$ diverges as well.

45. $\int_{-1}^1 \ln|x|\,dx = \int_{-1}^0 \ln(-x)\,dx + \int_0^1 \ln x\,dx$; $\int_0^1 \ln x\,dx = \lim\limits_{b \to 0^+} [x \ln x - x]_b^1 = [1 \cdot 0 - 1] - \lim\limits_{b \to 0^+} [b \ln b - b]$ $= -1 - 0 = -1$; $\int_{-1}^0 \ln(-x)\,dx = -1 \Rightarrow \int_{-1}^1 \ln|x|\,dx = -2$ converges.

46. $\int_{-1}^1 (-x \ln|x|)\,dx = \int_{-1}^0 [-x \ln(-x)]\,dx + \int_0^1 (-x \ln x)\,dx = \lim\limits_{b \to 0^+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_b^1 - \lim\limits_{c \to 0^+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_c^1$ $= \left[\frac{1}{2} \ln 1 - \frac{1}{4}\right] - \lim\limits_{b \to 0^+} \left[\frac{b^2}{2} \ln b - \frac{b^2}{4}\right] - \left[\frac{1}{2} \ln 1 - \frac{1}{4}\right] + \lim\limits_{c \to 0^+} \left[\frac{c^2}{2} \ln c - \frac{c^2}{4}\right] = -\frac{1}{4} - 0 + \frac{1}{4} + 0 = 0 \Rightarrow$ the integral converges (see Exercise 25 for the limit calculations).

47. $\int_1^{\infty} \frac{dx}{1 + x^3}$; $0 \le \frac{1}{x^3 + 1} \le \frac{1}{x^3}$ for $1 \le x < \infty$ and $\int_1^{\infty} \frac{dx}{x^3}$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{1 + x^3}$ converges by the Direct Comparison Test.

48. $\int_4^{\infty} \frac{dx}{\sqrt{x} - 1}$; $\lim\limits_{x \to \infty} \frac{\left(\frac{1}{\sqrt{x} - 1}\right)}{\left(\frac{1}{\sqrt{x}}\right)} = \lim\limits_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x} - 1} = \lim\limits_{x \to \infty} \frac{1}{1 - \frac{1}{\sqrt{x}}} = \frac{1}{1 - 0} = 1$ and $\int_4^{\infty} \frac{dx}{\sqrt{x}} = \lim\limits_{b \to \infty} \left[2\sqrt{x}\right]_4^b = \infty$, which diverges $\Rightarrow \int_4^{\infty} \frac{dx}{\sqrt{x} - 1}$ diverges by the Limit Comparison Test.

49. $\int_2^{\infty} \frac{dv}{\sqrt{v - 1}}$; $\lim\limits_{v \to \infty} \frac{\left(\frac{1}{\sqrt{v - 1}}\right)}{\left(\frac{1}{\sqrt{v}}\right)} = \lim\limits_{v \to \infty} \frac{\sqrt{v}}{\sqrt{v - 1}} = \lim\limits_{v \to \infty} \frac{1}{\sqrt{1 - \frac{1}{v}}} = \frac{1}{\sqrt{1 - 0}} = 1$ and $\int_2^{\infty} \frac{dv}{\sqrt{v}} = \lim\limits_{b \to \infty} \left[2\sqrt{v}\right]_2^b = \infty$, which diverges $\Rightarrow \int_2^{\infty} \frac{dv}{\sqrt{v - 1}}$ diverges by the Limit Comparison Test.

50. $\int_0^{\infty} \frac{d\theta}{1 + e^{\theta}}$; $0 \le \frac{1}{1 + e^{\theta}} \le \frac{1}{e^{\theta}}$ for $0 \le \theta < \infty$ and $\int_0^{\infty} \frac{d\theta}{e^{\theta}} = \lim\limits_{b \to \infty} \left[-e^{-\theta}\right]_0^b = \lim\limits_{b \to \infty} (-e^{-b} + 1) = 1 \Rightarrow \int_0^{\infty} \frac{d\theta}{e^{\theta}}$ converges $\Rightarrow \int_0^{\infty} \frac{d\theta}{1 + e^{\theta}}$ converges by the Direct Comparison Test.

51. $\int_0^{\infty} \frac{dx}{\sqrt{x^6 + 1}} = \int_0^1 \frac{dx}{\sqrt{x^6 + 1}} + \int_1^{\infty} \frac{dx}{\sqrt{x^6 + 1}} < \int_0^1 \frac{dx}{\sqrt{x^6 + 1}} + \int_1^{\infty} \frac{dx}{x^3}$ and $\int_1^{\infty} \frac{dx}{x^3} = \lim\limits_{b \to \infty} \left[-\frac{1}{2x^2}\right]_1^b$ $= \lim\limits_{b \to \infty} \left(-\frac{1}{2b^2} + \frac{1}{2}\right) = \frac{1}{2} \Rightarrow \int_0^{\infty} \frac{dx}{\sqrt{x^6 + 1}}$ converges by the Direct Comparison Test.

52. $\int_2^{\infty} \frac{dx}{\sqrt{x^2-1}}$; $\lim_{x \to \infty} \frac{\left(\frac{1}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to \infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$; $\int_2^{\infty} \frac{1}{x}\, dx = \lim_{b \to \infty} [\ln b]_2^b = \infty$, which diverges $\Rightarrow \int_2^{\infty} \frac{dx}{\sqrt{x^2-1}}$ diverges by the Limit Comparison Test.

53. $\int_1^{\infty} \frac{\sqrt{x+1}}{x^2}\, dx$; $\lim_{x \to \infty} \frac{\left(\frac{\sqrt{x}}{x^2}\right)}{\left(\frac{\sqrt{x+1}}{x^2}\right)} = \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x+1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{1}{x}}} = 1$; $\int_1^{\infty} \frac{\sqrt{x}}{x^2}\, dx = \int_1^{\infty} \frac{dx}{x^{3/2}}$ $= \lim_{b \to \infty} \left[-2x^{-1/2}\right]_1^b = \lim_{b \to \infty} \left(\frac{-2}{\sqrt{b}} + 2\right) = 2 \Rightarrow \int_1^{\infty} \frac{\sqrt{x+1}}{x^2}\, dx$ converges by the Limit Comparison Test.

54. $\int_2^{\infty} \frac{x\, dx}{\sqrt{x^4-1}}$; $\lim_{x \to \infty} \frac{\left(\frac{x}{\sqrt{x^4-1}}\right)}{\left(\frac{x}{\sqrt{x^4}}\right)} = \lim_{x \to \infty} \frac{\sqrt{x^4}}{\sqrt{x^4-1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1-\frac{1}{x^4}}} = 1$; $\int_2^{\infty} \frac{x\, dx}{\sqrt{x^4}} = \int_2^{\infty} \frac{dx}{x} = \lim_{b \to \infty} [\ln x]_2^b = \infty$, which diverges $\Rightarrow \int_2^{\infty} \frac{x\, dx}{\sqrt{x^4-1}}$ diverges by the Limit Comparison Test.

55. $\int_{\pi}^{\infty} \frac{2+\cos x}{x}\, dx$; $0 < \frac{1}{x} \le \frac{2+\cos x}{x}$ for $x \ge \pi$ and $\int_{\pi}^{\infty} \frac{dx}{x} = \lim_{b \to \infty} [\ln x]_{\pi}^b = \infty$, which diverges $\Rightarrow \int_{\pi}^{\infty} \frac{2+\cos x}{x}\, dx$ diverges by the Direct Comparison Test.

56. $\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\, dx$; $0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$ for $x \ge \pi$ and $\int_{\pi}^{\infty} \frac{2}{x^2}\, dx = \lim_{b \to \infty} \left[-\frac{2}{x}\right]_{\pi}^b = \lim_{b \to \infty} \left(-\frac{2}{b} + \frac{2}{\pi}\right) = \frac{2}{\pi}$ $\Rightarrow \int_{\pi}^{\infty} \frac{2\, dx}{x^2}$ converges $\Rightarrow \int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\, dx$ converges by the Direct Comparison Test.

57. $\int_4^{\infty} \frac{2\, dt}{t^{3/2}-1}$; $\lim_{t \to \infty} \frac{t^{3/2}}{t^{3/2}-1} = 1$ and $\int_4^{\infty} \frac{2\, dt}{t^{3/2}} = \lim_{b \to \infty} \left[-4t^{-1/2}\right]_4^b = \lim_{b \to \infty} \left(\frac{-4}{\sqrt{b}} + 2\right) = 2 \Rightarrow \int_4^{\infty} \frac{2\, dt}{t^{3/2}}$ converges $\Rightarrow \int_4^{\infty} \frac{2\, dt}{t^{3/2}+1}$ converges by the Limit Comparison Test.

58. $\int_2^{\infty} \frac{dx}{\ln x}$; $0 < \frac{1}{x} < \frac{1}{\ln x}$ for $x > 2$ and $\int_2^{\infty} \frac{dx}{x}$ diverges $\Rightarrow \int_2^{\infty} \frac{dx}{\ln x}$ diverges by the Direct Comparison Test.

59. $\int_1^{\infty} \frac{e^x}{x}\, dx$; $0 < \frac{1}{x} < \frac{e^x}{x}$ for $x > 1$ and $\int_1^{\infty} \frac{dx}{x}$ diverges $\Rightarrow \int_1^{\infty} \frac{e^x\, dx}{x}$ diverges by the Direct Comparison Test.

60. $\int_{e^e}^{\infty} \ln(\ln x)\, dx$; $[x = e^y] \rightarrow \int_e^{\infty} (\ln y)\, e^y\, dy$; $0 < \ln y < (\ln y)\, e^y$ for $y \ge e$ and $\int_e^{\infty} \ln y\, dy = \lim_{b \to \infty} [y \ln y - y]_e^b = \infty$, which diverges $\Rightarrow \int_e^{\infty} \ln e^y\, dy$ diverges $\Rightarrow \int_{e^e}^{\infty} \ln(\ln x)\, dx$ diverges by the Direct Comparison Test.

61. $\int_1^{\infty} \frac{dx}{\sqrt{e^x - x}}$; $\lim_{x \to \infty} \frac{\left(\frac{1}{\sqrt{e^x-x}}\right)}{\left(\frac{1}{\sqrt{e^x}}\right)} = \lim_{x \to \infty} \frac{\sqrt{e^x}}{\sqrt{e^x-x}} = \lim_{x \to \infty} \frac{1}{\sqrt{1-\frac{x}{e^x}}} = \frac{1}{\sqrt{1-0}} = 1$; $\int_1^{\infty} \frac{dx}{\sqrt{e^x}} = \int_1^{\infty} e^{-x/2}\, dx$ $= \lim_{b \to \infty} \left[-2e^{-x/2}\right]_1^b = \lim_{b \to \infty} \left(-2e^{-b/2} + 2e^{-1/2}\right) = \frac{2}{\sqrt{e}} \Rightarrow \int_1^{\infty} e^{-x/2}\, dx$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{\sqrt{e^x-x}}$ converges by the Limit Comparison Test.

62. $\int_1^{\infty} \frac{dx}{e^x - 2^x}$; $\lim_{x \to \infty} \frac{\left(\frac{1}{e^x-2^x}\right)}{\left(\frac{1}{e^x}\right)} = \lim_{x \to \infty} \frac{e^x}{e^x-2^x} = \lim_{x \to \infty} \frac{1}{1-\left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = 1$ and $\int_1^{\infty} \frac{dx}{e^x} = \lim_{b \to \infty} \left[-e^{-x}\right]_1^b$ $= \lim_{b \to \infty} \left(-e^{-b} + e^{-1}\right) = \frac{1}{e} \Rightarrow \int_1^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow \int_1^{\infty} \frac{dx}{e^x-2^x}$ converges by the Limit Comparison Test.

63. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}} = 2\int_0^{\infty} \frac{dx}{\sqrt{x^4+1}}$; $\int_0^{\infty} \frac{dx}{\sqrt{x^4+1}} = \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^{\infty} \frac{dx}{\sqrt{x^4+1}} < \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^{\infty} \frac{dx}{x^2}$ and $\int_1^{\infty} \frac{dx}{x^2} = \lim_{b \to \infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b \to \infty} \left(-\frac{1}{b} + 1\right) = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$ converges by the Direct Comparison Test.

64. $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = 2\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$; $0 < \frac{1}{e^x + e^{-x}} < \frac{1}{e^x}$ for $x > 0$; $\int_0^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow$ $2\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$ converges by the Direct Comparison Test.

65. (a) $\int_1^2 \frac{dx}{x(\ln x)^p}$; $[t = \ln x]$ $\rightarrow \int_0^{\ln 2} \frac{dt}{t^p} = \lim_{b \to 0^+} \left[\frac{1}{-p+1} t^{1-p}\right]_b^{\ln 2} = \lim_{b \to 0^+} \frac{b^{1-p}}{p-1} + \frac{1}{1-p} (\ln 2)^{1-p}$

$\Rightarrow$ the integral converges for $p < 1$ and diverges for $p \ge 1$

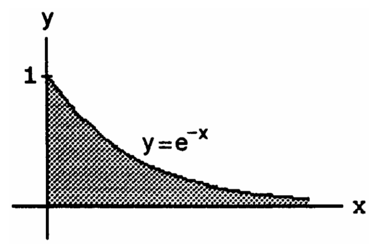
(b) $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$; $[t = \ln x]$ $\rightarrow \int_{\ln 2}^{\infty} \frac{dt}{t^p}$ and this integral is essentially the same as in Exercise 65(a): it converges for $p > 1$ and diverges for $p \le 1$

66. $\int_0^{\infty} \frac{2x\,dx}{x^2+1} = \lim_{b \to \infty} \left[\ln(x^2+1)\right]_0^b = \lim_{b \to \infty} \left[\ln(b^2+1)\right] - 0 = \lim_{b \to \infty} \ln(b^2+1) = \infty \Rightarrow$ the integral $\int_{-\infty}^{\infty} \frac{2x}{x^2+1}\,dx$ diverges. But $\lim_{b \to \infty} \int_{-\infty}^{b} \frac{2x\,dx}{x^2+1} = \lim_{b \to \infty} \left[\ln(x^2+1)\right]_{-b}^{b} = \lim_{b \to \infty} \left[\ln(b^2+1) - \ln(b^2+1)\right] = \lim_{b \to \infty} \ln\left(\frac{b^2+1}{b^2+1}\right)$

$= \lim_{b \to \infty} (\ln 1) = 0$

67. $A = \int_0^{\infty} e^{-x}\,dx = \lim_{b \to \infty} \left[-e^{-x}\right]_0^b = \lim_{b \to \infty} (-e^{-b}) - (-e^{-0})$

$= 0 + 1 = 1$



68. $\bar{x} = \frac{1}{A}\int_0^{\infty} xe^{-x}\,dx = \lim_{b \to \infty} \left[-xe^{-x} - e^{-x}\right]_0^b = \lim_{b \to \infty} (-be^{-b} - e^{-b}) - (-0 \cdot e^{-0} - e^{-0}) = 0 + 1 = 1$;

$\bar{y} = \frac{1}{2A}\int_0^{\infty} (e^{-x})^2\,dx = \frac{1}{2}\int_0^{\infty} e^{-2x}\,dx = \lim_{b \to \infty} \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_0^b = \lim_{b \to \infty} \frac{1}{2}\left(-\frac{1}{2}e^{-2b}\right) - \frac{1}{2}\left(-\frac{1}{2}e^{-2 \cdot 0}\right) = 0 + \frac{1}{4} = \frac{1}{4}$

69. $V = \int_0^{\infty} 2\pi xe^{-x}\,dx = 2\pi\int_0^{\infty} xe^{-x}\,dx = 2\pi \lim_{b \to \infty} \left[-xe^{-x} - e^{-x}\right]_0^b = 2\pi\left[\lim_{b \to \infty} (-be^{-b} - e^{-b}) - 1\right] = 2\pi$

70. $V = \int_0^{\infty} \pi (e^{-x})^2\,dx = \pi\int_0^{\infty} e^{-2x}\,dx = \pi \lim_{b \to \infty} \left[-\frac{1}{2}e^{-2x}\right]_0^b = \pi \lim_{b \to \infty} \left(-\frac{1}{2}e^{-2b} + \frac{1}{2}\right) = \frac{\pi}{2}$

71. $A = \int_0^{\pi/2} (\sec x - \tan x)\,dx = \lim_{b \to \frac{\pi}{2}^-} \left[\ln|\sec x + \tan x| - \ln|\sec x|\right]_0^b = \lim_{b \to \frac{\pi}{2}^-} \left(\ln\left|1 + \frac{\tan b}{\sec b}\right| - \ln|1+0|\right)$

$= \lim_{b \to \frac{\pi}{2}^-} \ln|1 + \sin b| = \ln 2$

72. (a) $V = \int_0^{\pi/2} \pi \sec^2 x\,dx - \int_0^{\pi/2} \pi \tan^2 x\,dx = \pi\int_0^{\pi/2} (\sec^2 x - \tan^2 x)\,dx = \int_0^{\pi/2} \pi\left[\sec^2 x - (\sec^2 x - 1)\right]dx$

$= \pi\int_0^{\pi/2} dx = \frac{\pi^2}{2}$

(b) $S_{\text{outer}} = \int_0^{\pi/2} 2\pi \sec x\sqrt{1 + \sec^2 x \tan^2 x}\,dx \ge \int_0^{\pi/2} 2\pi \sec x(\sec x \tan x)\,dx = \pi \lim_{b \to \frac{\pi}{2}^-} \left[\tan^2 x\right]_0^b$

$= \pi\left[\lim_{b \to \frac{\pi}{2}^-} [\tan^2 b] - 0\right] = \pi \lim_{b \to \frac{\pi}{2}^-} (\tan^2 b) = \infty \Rightarrow S_{\text{outer}}$ diverges; $S_{\text{inner}} = \int_0^{\pi/2} 2\pi \tan x\sqrt{1 + \sec^4 x}\,dx$

$\ge \int_0^{\pi/2} 2\pi \tan x \sec^2 x\,dx = \pi \lim_{b \to \frac{\pi}{2}^-} \left[\tan^2 x\right]_0^b = \pi\left[\lim_{b \to \frac{\pi}{2}^-} [\tan^2 b] - 0\right] = \pi \lim_{b \to \frac{\pi}{2}^-} (\tan^2 b) = \infty$

$\Rightarrow S_{\text{inner}}$ diverges

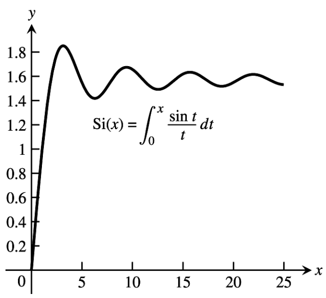
73. (a) $\int_3^\infty e^{-3x}\,dx = \lim\limits_{b\to\infty} \left[-\frac{1}{3}e^{-3x}\right]_3^b = \lim\limits_{b\to\infty} \left(-\frac{1}{3}e^{-3b}\right) - \left(-\frac{1}{3}e^{-3\cdot 3}\right) = 0 + \frac{1}{3}\cdot e^{-9} = \frac{1}{3}e^{-9}$
$\approx 0.0000411 < 0.000042$. Since $e^{-x^2} \le e^{-3x}$ for $x > 3$, then $\int_3^\infty e^{-x^2}\,dx < 0.000042$ and therefore
$\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error greater than 0.000042.

(b) $\int_0^3 e^{-x^2}\,dx \cong 0.88621$

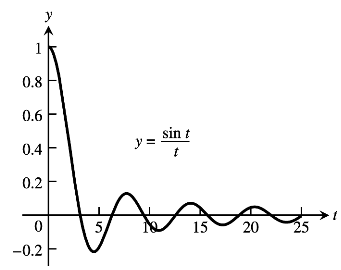
74. (a) $V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi \lim\limits_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \pi\left[\lim\limits_{b\to\infty}\left(-\frac{1}{b}\right) - \left(-\frac{1}{1}\right)\right] = \pi(0+1) = \pi$

(b) When you take the limit to ∞, you are no longer modeling the real world which is finite. The comparison step in the modeling process discussed in Section 4.2 relating the mathematical world to the real world fails to hold.

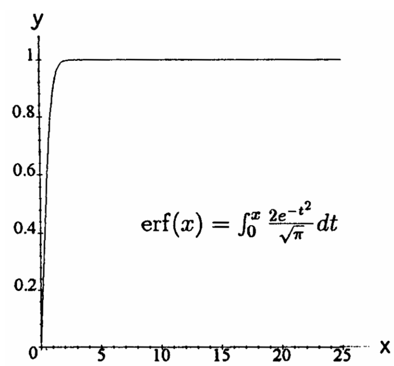
75. (a)



(b) > int((sin(t))/t, t=0..infinity); (answer is $\frac{\pi}{2}$)

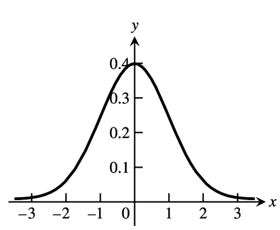


76. (a)



(b) > f:= 2*exp(−t^2)/sqrt(Pi);
> int(f, t=0..infinity); (answer is 1)

77. (a) $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$
f is increasing on $(-\infty, 0]$. f is decreasing on $[0, \infty)$.
f has a local maximum at $(0, f(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right)$



(b) Maple commands:
>f: = exp(−x^2/2)(sqrt(2*pi);
>int(f, x = −1..1); ≈ 0.683
>int(f, x = −2..2); ≈ 0.954
>int(f, x = −3..3); ≈ 0.997

(c) Part (b) suggests that as n increases, the integral approaches 1. We can take $\int_{-n}^{n} f(x)\,dx$ as close to 1 as we want by choosing $n > 1$ large enough. Also, we can make $\int_{n}^{\infty} f(x)\,dx$ and $\int_{-\infty}^{-n} f(x)\,dx$ as small as we want by choosing n large enough. This is because $0 < f(x) < e^{-x/2}$ for $x > 1$. (Likewise, $0 < f(x) < e^{x/2}$ for $x < -1$.)
Thus, $\int_{n}^{\infty} f(x)\,dx < \int_{n}^{\infty} e^{-x/2}dx$.
$\int_{n}^{\infty} e^{-x/2}dx = \lim_{c\to\infty}\int_{n}^{c} e^{-x/2}dx = \lim_{c\to\infty}\left[-2e^{-x/2}\right]_{n}^{c} = \lim_{c\to\infty}\left[-2e^{-c/2} + 2e^{-n/2}\right] = 2e^{-n/2}$
As $n \to \infty$, $2e^{-n/2} \to 0$, for large enough n, $\int_{n}^{\infty} f(x)\,dx$ is as small as we want. Likewise for large enough n, $\int_{-\infty}^{-n} f(x)\,dx$ is as small as we want.

78. (a) The statement is true since $\int_{-\infty}^{b} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx$, $\int_{b}^{\infty} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx - \int_{a}^{b} f(x)\,dx$ and $\int_{a}^{b} f(x)\,dx$ exists since f(x) is integrable on every interval [a, b].

(b) $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx - \int_{a}^{b} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$
$= \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$

79. Example CAS commands:
Maple:
```
f := (x,p) -> x^p*ln(x);
domain := 0..exp(1);
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=domain, y=-50..10, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9], thickness=[3,4,1,2,0],
      legend=["p= -2","p = -1","p = 0","p = 1","p = 2"], title="#79 (Section 8.7)" );
q1 := Int( f(x,p), x=domain );
q2 := value( q1 );
q3 := simplify( q2 ) assuming p>-1;
q4 := simplify( q2 ) assuming p<-1;
q5 := value( eval( q1, p=-1 ) );
i1 := q1 = piecewise( p<-1, q4, p=-1, q5, p>-1, q3 );
```

80. Example CAS commands:
Maple:
```
f := (x,p) -> x^p*ln(x);
domain := exp(1)..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=exp(1)..10, y=0..100, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9], thickness=[3,4,1,2,0],
      legend=["p = -2","p = -1","p = 0","p = 1","p = 2"], title="#80 (Section 8.7)" );
q6 := Int( f(x,p), x=domain );
q7 := value( q6 );
q8 := simplify( q7 ) assuming p>-1;
q9 := simplify( q7 ) assuming p<-1;
```

```
q10 := value( eval( q6, p=-1 ) );
i2 := q6 = piecewise( p<-1, q9, p=-1, q10, p>-1, q8 );
```

81. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(x);
domain := 0..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=0..10, y=-50..50, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9], thickness=[3,4,1,2,0],
    legend=["p = -2","p = -1","p = 0","p = 1","p = 2"], title="#81 (Section 8.7)" );
q11 := Int( f(x,p), x=domain ):
q11 = lhs(i1+i2);
`` = rhs(i1+i2);
`` = piecewise( p<-1, q4+q9, p=-1, q5+q10, p>-1, q3+q8 );
`` = piecewise( p<-1, -infinity, p=-1, undefined, p>-1, infinity );
```

82. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(abs(x));
domain := -infinity..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=-4..4, y=-20..10, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9],
    legend=["p = -2","p = -1","p = 0","p = 1","p = 2"], title="#82 (Section 8.7)" );
q12 := Int( f(x,p), x=domain );
q12p := Int( f(x,p), x=0..infinity );
q12n := Int( f(x,p), x=-infinity..0 );
q12 = q12p + q12n;
`` = simplify( q12p+q12n );
```

79-82. Example CAS commands:

Mathematica: (functions and domains may vary)

```
Clear[x, f, p]
f[x_]:= x^p Log[Abs[x]]
int = Integrate[f[x], {x, e, 100)]
int /. p → 2.5
```

In order to plot the function, a value for p must be selected.

```
p = 3;
Plot[f[x], {x, 2.72, 10}]
```

CHAPTER 8 PRACTICE EXERCISES

1. $u = \ln(x+1)$, $du = \frac{dx}{x+1}$; $dv = dx$, $v = x$;

$\int \ln(x+1)\,dx = x\ln(x+1) - \int \frac{x}{x+1}\,dx = x\ln(x+1) - \int dx + \int \frac{dx}{x+1} = x\ln(x+1) - x + \ln(x+1) + C_1$
$= (x+1)\ln(x+1) - x + C_1 = (x+1)\ln(x+1) - (x+1) + C$, where $C = C_1 + 1$

2. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x^2\,dx$, $v = \frac{1}{3}x^3$;

$\int x^2 \ln x\,dx = \frac{1}{3}x^3\ln x - \int \frac{1}{3}x^3\left(\frac{1}{x}\right)dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

3. $u = \tan^{-1} 3x$, $du = \frac{3\,dx}{1+9x^2}$; $dv = dx$, $v = x$;

$$\int \tan^{-1} 3x\,dx = x\tan^{-1} 3x - \int \frac{3x\,dx}{1+9x^2}\,;\ \begin{bmatrix} y = 1+9x^2 \\ dy = 18x\,dx \end{bmatrix} \to\ x\tan^{-1} 3x - \frac{1}{6}\int \frac{dy}{y}$$
$$= x\tan^{-1}(3x) - \frac{1}{6}\ln\left(1+9x^2\right) + C$$

4. $u = \cos^{-1}\left(\frac{x}{2}\right)$, $du = \frac{-\,dx}{\sqrt{4-x^2}}$; $dv = dx$, $v = x$;

$$\int \cos^{-1}\left(\tfrac{x}{2}\right) dx = x\cos^{-1}\left(\tfrac{x}{2}\right) + \int \frac{x\,dx}{\sqrt{4-x^2}}\,;\ \begin{bmatrix} y = 4-x^2 \\ dy = -2x\,dx \end{bmatrix} \to\ x\cos^{-1}\left(\tfrac{x}{2}\right) - \frac{1}{2}\int \frac{dy}{\sqrt{y}}$$
$$= x\cos^{-1}\left(\tfrac{x}{2}\right) - \sqrt{4-x^2} + C = x\cos^{-1}\left(\tfrac{x}{2}\right) - 2\sqrt{1-\left(\tfrac{x}{2}\right)^2} + C$$

5.

		e^x
$(x+1)^2$	$\xrightarrow{(+)}$	e^x
$2(x+1)$	$\xrightarrow{(-)}$	e^x
2	$\xrightarrow{(+)}$	e^x
0		

$$\Rightarrow\ \int (x+1)^2 e^x\,dx = \left[(x+1)^2 - 2(x+1) + 2\right] e^x + C$$

6.

		$\sin(1-x)$
x^2	$\xrightarrow{(+)}$	$\cos(1-x)$
$2x$	$\xrightarrow{(-)}$	$-\sin(1-x)$
2	$\xrightarrow{(+)}$	$-\cos(1-x)$
0		

$$\Rightarrow\ \int x^2\sin(1-x)\,dx = x^2\cos(1-x) + 2x\sin(1-x) - 2\cos(1-x) + C$$

7. $u = \cos 2x$, $du = -2\sin 2x\,dx$; $dv = e^x\,dx$, $v = e^x$;

$$I = \int e^x\cos 2x\,dx = e^x\cos 2x + 2\int e^x\sin 2x\,dx;$$

$u = \sin 2x$, $du = 2\cos 2x\,dx$; $dv = e^x\,dx$, $v = e^x$;

$$I = e^x\cos 2x + 2\left[e^x\sin 2x - 2\int e^x\cos 2x\,dx\right] = e^x\cos 2x + 2e^x\sin 2x - 4I\ \Rightarrow\ I = \frac{e^x\cos 2x}{5} + \frac{2e^x\sin 2x}{5} + C$$

8. $u = \sin 3x$, $du = 3\cos 3x\,dx$; $dv = e^{-2x}\,dx$, $v = -\frac{1}{2}e^{-2x}$;

$$I = \int e^{-2x}\sin 3x\,dx = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\int e^{-2x}\cos 3x\,dx;$$

$u = \cos 3x$, $du = -3\sin 3x\,dx$; $dv = e^{-2x}\,dx$, $v = -\frac{1}{2}e^{-2x}$;

$$I = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\left[-\frac{1}{2}e^{-2x}\cos 3x - \frac{3}{2}\int e^{-2x}\sin 3x\,dx\right] = -\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x - \frac{9}{4}I$$
$$\Rightarrow\ I = \frac{4}{13}\left(-\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x\right) + C = -\frac{2}{13}e^{-2x}\sin 3x - \frac{3}{13}e^{-2x}\cos 3x + C$$

9. $\int \frac{x\,dx}{x^2-3x+2} = \int \frac{2\,dx}{x-2} - \int \frac{dx}{x-1} = 2\ln|x-2| - \ln|x-1| + C$

10. $\int \frac{x\,dx}{x^2+4x+3} = \frac{3}{2}\int \frac{dx}{x+3} - \frac{1}{2}\int \frac{dx}{x+1} = \frac{3}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + C$

11. $\int \frac{dx}{x(x+1)^2} = \int \left(\frac{1}{x} - \frac{1}{x+1} + \frac{-1}{(x+1)^2}\right) dx = \ln|x| - \ln|x+1| + \frac{1}{x+1} + C$

12. $\int \frac{x+1}{x^2(x-1)}\,dx = \int \left(\frac{2}{x-1} - \frac{2}{x} - \frac{1}{x^2}\right) dx = 2\ln\left|\frac{x-1}{x}\right| + \frac{1}{x} + C = -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C$

13. $\int \frac{\sin\theta\, d\theta}{\cos^2\theta + \cos\theta - 2}$; $[\cos\theta = y]$ $\rightarrow$ $-\int \frac{dy}{y^2+y-2} = -\frac{1}{3}\int \frac{dy}{y-1} + \frac{1}{3}\int \frac{dy}{y+2} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C$
$= \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

14. $\int \frac{\cos\theta\, d\theta}{\sin^2\theta + \sin\theta - 6}$; $[\sin\theta = x]$ $\rightarrow$ $\int \frac{dx}{x^2+x-6} = \frac{1}{5}\int \frac{dx}{x-2} - \frac{1}{5}\int \frac{dx}{x+3} = \frac{1}{5}\ln\left|\frac{\sin\theta-2}{\sin\theta+3}\right| + C$

15. $\int \frac{3x^2+4x+4}{x^3+x}\,dx = \int \frac{4}{x}\,dx - \int \frac{x-4}{x^2+1}\,dx = 4\ln|x| - \frac{1}{2}\ln(x^2+1) + 4\tan^{-1}x + C$

16. $\int \frac{4x\,dx}{x^3+4x} = \int \frac{4\,dx}{x^2+4} = 2\tan^{-1}\left(\frac{x}{2}\right) + C$

17. $\int \frac{(v+3)\,dv}{2v^3-8v} = \frac{1}{2}\int \left(-\frac{3}{4v} + \frac{5}{8(v-2)} + \frac{1}{8(v+2)}\right) dv = -\frac{3}{8}\ln|v| + \frac{5}{16}\ln|v-2| + \frac{1}{16}\ln|v+2| + C$
$= \frac{1}{16}\ln\left|\frac{(v-2)^5(v+2)}{v^6}\right| + C$

18. $\int \frac{(3v-7)\,dv}{(v-1)(v-2)(v-3)} = \int \frac{(-2)\,dv}{v-1} + \int \frac{dv}{v-2} + \int \frac{dv}{v-3} = \ln\left|\frac{(v-2)(v-3)}{(v-1)^2}\right| + C$

19. $\int \frac{dt}{t^4+4t^2+3} = \frac{1}{2}\int \frac{dt}{t^2+1} - \frac{1}{2}\int \frac{dt}{t^2+3} = \frac{1}{2}\tan^{-1}t - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C = \frac{1}{2}\tan^{-1}t - \frac{\sqrt{3}}{6}\tan^{-1}\frac{t}{\sqrt{3}} + C$

20. $\int \frac{t\,dt}{t^4-t^2-2} = \frac{1}{3}\int \frac{t\,dt}{t^2-2} - \frac{1}{3}\int \frac{t\,dt}{t^2+1} = \frac{1}{6}\ln|t^2-2| - \frac{1}{6}\ln(t^2+1) + C$

21. $\int \frac{x^3+x^2}{x^2+x-2}\,dx = \int \left(x + \frac{2x}{x^2+x-2}\right) dx = \int x\,dx + \frac{2}{3}\int \frac{dx}{x-1} + \frac{4}{3}\int \frac{dx}{x+2} = \frac{x^2}{2} + \frac{4}{3}\ln|x+2| + \frac{2}{3}\ln|x-1| + C$

22. $\int \frac{x^3+1}{x^3-x}\,dx = \int \left(1 + \frac{x+1}{x^3-x}\right) dx = \int \left[1 + \frac{1}{x(x-1)}\right] dx = \int dx + \int \frac{dx}{x-1} - \int \frac{dx}{x} = x + \ln|x-1| - \ln|x| + C$

23. $\int \frac{x^3+4x^2}{x^2+4x+3}\,dx = \int \left(x - \frac{3x}{x^2+4x+3}\right) dx = \int x\,dx + \frac{3}{2}\int \frac{dx}{x+1} - \frac{9}{2}\int \frac{dx}{x+3} = \frac{x^2}{2} - \frac{9}{2}\ln|x+3| + \frac{3}{2}\ln|x+1| + C$

24. $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx = \int \left[(2x-3) + \frac{x}{x^2+2x-8}\right] dx = \int (2x-3)\,dx + \frac{1}{3}\int \frac{dx}{x-2} + \frac{2}{3}\int \frac{dx}{x+4}$
$= x^2 - 3x + \frac{2}{3}\ln|x+4| + \frac{1}{3}\ln|x-2| + C$

25. $\int \frac{dx}{x\left(3\sqrt{x+1}\right)}$; $\begin{bmatrix} u = \sqrt{x+1} \\ du = \frac{dx}{2\sqrt{x+1}} \\ dx = 2u\,du \end{bmatrix}$ $\rightarrow$ $\frac{2}{3}\int \frac{u\,du}{(u^2-1)\,u} = \frac{1}{3}\int \frac{du}{u-1} - \frac{1}{3}\int \frac{du}{u+1} = \frac{1}{3}\ln|u-1| - \frac{1}{3}\ln|u+1| + C$
$= \frac{1}{3}\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

26. $\int \frac{dx}{x\left(1+\sqrt[3]{x}\right)}$; $\begin{bmatrix} u = \sqrt[3]{x} \\ du = \frac{dx}{3x^{2/3}} \\ dx = 3u^2\,du \end{bmatrix}$ $\rightarrow \int \frac{3u^2\,du}{u^3(1+u)} = 3\int \frac{du}{u(1+u)} = 3\ln\left|\frac{u}{u+1}\right| + C = 3\ln\left|\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}\right| + C$

27. $\int \frac{ds}{e^s-1}$; $\begin{bmatrix} u = e^s - 1 \\ du = e^s\,ds \\ ds = \frac{du}{u+1} \end{bmatrix}$ $\rightarrow$ $\int \frac{du}{u(u+1)} = -\int \frac{du}{u+1} + \int \frac{du}{u} = \ln\left|\frac{u}{u+1}\right| + C = \ln\left|\frac{e^s-1}{e^s}\right| + C = \ln|1-e^{-s}| + C$

28. $\int \frac{ds}{\sqrt{e^s+1}}$; $\begin{bmatrix} u = \sqrt{e^s+1} \\ du = \frac{e^s\,ds}{2\sqrt{e^s+1}} \\ ds = \frac{2u\,du}{u^2-1} \end{bmatrix} \rightarrow \int \frac{2u\,du}{u(u^2-1)} = 2\int \frac{du}{(u+1)(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u+1} = \ln\left|\frac{u-1}{u+1}\right| + C$
$= \ln\left|\frac{\sqrt{e^s+1}-1}{\sqrt{e^s+1}+1}\right| + C$

29. (a) $\int \frac{y\,dy}{\sqrt{16-y^2}} = -\frac{1}{2}\int \frac{d(16-y^2)}{\sqrt{16-y^2}} = -\sqrt{16-y^2} + C$
(b) $\int \frac{y\,dy}{\sqrt{16-y^2}}$; $[y = 4\sin x] \rightarrow 4\int \frac{\sin x\cos x\,dx}{\cos x} = -4\cos x + C = -\frac{4\sqrt{16-y^2}}{4} + C = -\sqrt{16-y^2} + C$

30. (a) $\int \frac{x\,dx}{\sqrt{4+x^2}} = \frac{1}{2}\int \frac{d(4+x^2)}{\sqrt{4+x^2}} = \sqrt{4+x^2} + C$
(b) $\int \frac{x\,dx}{\sqrt{4+x^2}}$; $[x = 2\tan y] \rightarrow \int \frac{2\tan y \cdot 2\sec^2 y\,dy}{2\sec y} = 2\int \sec y\tan y\,dy = 2\sec y + C = \sqrt{4+x^2} + C$

31. (a) $\int \frac{x\,dx}{4-x^2} = -\frac{1}{2}\int \frac{d(4-x^2)}{4-x^2} = -\frac{1}{2}\ln|4-x^2| + C$
(b) $\int \frac{x\,dx}{4-x^2}$; $[x = 2\sin\theta] \rightarrow \int \frac{2\sin\theta \cdot 2\cos\theta\,d\theta}{4\cos^2\theta} = \int \tan\theta\,d\theta = -\ln|\cos\theta| + C = -\ln\left(\frac{\sqrt{4-x^2}}{2}\right) + C$
$= -\frac{1}{2}\ln|4-x^2| + C$

32. (a) $\int \frac{t\,dt}{\sqrt{4t^2-1}} = \frac{1}{8}\int \frac{d(4t^2-1)}{\sqrt{4t^2-1}} = \frac{1}{4}\sqrt{4t^2-1} + C$
(b) $\int \frac{t\,dt}{\sqrt{4t^2-1}}$; $\left[t = \frac{1}{2}\sec\theta\right] \rightarrow \int \frac{\frac{1}{2}\sec\theta\tan\theta \cdot \frac{1}{2}\sec\theta\,d\theta}{\tan\theta} = \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{\tan\theta}{4} + C = \frac{\sqrt{4t^2-1}}{4} + C$

33. $\int \frac{x\,dx}{9-x^2}$; $\begin{bmatrix} u = 9-x^2 \\ du = -2x\,dx \end{bmatrix} \rightarrow -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C = \ln\frac{1}{\sqrt{u}} + C = \ln\frac{1}{\sqrt{9-x^2}} + C$

34. $\int \frac{dx}{x(9-x^2)} = \frac{1}{9}\int \frac{dx}{x} + \frac{1}{18}\int \frac{dx}{3-x} - \frac{1}{18}\int \frac{dx}{3+x} = \frac{1}{9}\ln|x| - \frac{1}{18}\ln|3-x| - \frac{1}{18}\ln|3+x| + C$
$= \frac{1}{9}\ln|x| - \frac{1}{18}\ln|9-x^2| + C$

35. $\int \frac{dx}{9-x^2} = \frac{1}{6}\int \frac{dx}{3-x} + \frac{1}{6}\int \frac{dx}{3+x} = -\frac{1}{6}\ln|3-x| + \frac{1}{6}\ln|3+x| + C = \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + C$

36. $\int \frac{dx}{\sqrt{9-x^2}}$; $\begin{bmatrix} x = 3\sin\theta \\ dx = 3\cos\theta\,d\theta \end{bmatrix} \rightarrow \int \frac{3\cos\theta}{3\cos\theta}\,d\theta = \int d\theta = \theta + C = \sin^{-1}\frac{x}{3} + C$

37. $\int \sin^3 x\cos^4 x\,dx = \int \cos^4 x(1-\cos^2 x)\sin x\,dx = \int \cos^4 x\sin x\,dx - \int \cos^6 x\sin x\,dx = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$

38. $\int \cos^5 x\sin^5 x\,dx = \int \sin^5 x\cos^4 x\cos x\,dx = \int \sin^5 x\,(1-\sin^2 x)^2\cos x\,dx$
$= \int \sin^5 x\cos x\,dx - 2\int \sin^7 x\cos x\,dx + \int \sin^9 x\cos x\,dx = \frac{\sin^6 x}{6} - \frac{2\sin^8 x}{8} + \frac{\sin^{10} x}{10} + C$

39. $\int \tan^4 x\sec^2 x\,dx = \frac{\tan^5 x}{5} + C$

40. $\int \tan^3 x\sec^3 x\,dx = \int (\sec^2 x - 1)\sec^2 x \cdot \sec x \cdot \tan x\,dx = \int \sec^4 x \cdot \sec x \cdot \tan x\,dx - \int \sec^2 x \cdot \sec x \cdot \tan x\,dx$
$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

41. $\int \sin 5\theta \cos 6\theta \, d\theta = \frac{1}{2}\int (\sin(-\theta) + \sin(11\theta))\, d\theta = \frac{1}{2}\int \sin(-\theta)\, d\theta + \frac{1}{2}\int \sin(11\theta)\, d\theta = \frac{1}{2}\cos(-\theta) - \frac{1}{22}\cos 11\theta + C$
$= \frac{1}{2}\cos\theta - \frac{1}{22}\cos 11\theta + C$

42. $\int \cos 3\theta \cos 3\theta \, d\theta = \frac{1}{2}\int \left(\cos 0 + \cos 6\theta\right) d\theta = \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 6\theta \, d\theta = \frac{1}{2}\theta + \frac{1}{12}\sin 6\theta + C$

43. $\int \sqrt{1+\cos\left(\frac{t}{2}\right)}\, dt = \int \sqrt{2}\left|\cos\ \frac{t}{4}\right| dt = 4\sqrt{2}\left|\sin\ \frac{t}{4}\right| + C$

44. $\int e^t\sqrt{\tan^2 e^t + 1}\, dt = \int |\sec e^t|\, e^t\, dt = \ln|\sec e^t + \tan e^t| + C$

45. $|E_s| \le \frac{3-1}{180}(\triangle x)^4 M$ where $\triangle x = \frac{3-1}{n} = \frac{2}{n}$; $f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f'''(x) = -6x^{-4}$
$\Rightarrow f^{(4)}(x) = 24x^{-5}$ which is decreasing on $[1,3]$ $\Rightarrow$ maximum of $f^{(4)}(x)$ on $[1,3]$ is $f^{(4)}(1) = 24 \Rightarrow M = 24$. Then
$|E_s| \le 0.0001 \Rightarrow \left(\frac{3-1}{180}\right)\left(\frac{2}{n}\right)^4(24) \le 0.0001 \Rightarrow \left(\frac{768}{180}\right)\left(\frac{1}{n^4}\right) \le 0.0001 \Rightarrow \frac{1}{n^4} \le (0.0001)\left(\frac{180}{768}\right) \Rightarrow n^4 \ge 10{,}000\left(\frac{768}{180}\right)$
$\Rightarrow n \ge 14.37 \Rightarrow n \ge 16$ (n must be even)

46. $|E_T| \le \frac{1-0}{12}(\triangle x)^2 M$ where $\triangle x = \frac{1-0}{n} = \frac{1}{n}$; $0 \le f''(x) \le 8 \Rightarrow M = 8$. Then $|E_T| \le 10^{-3} \Rightarrow \frac{1}{12}\left(\frac{1}{n}\right)^2(8) \le 10^{-3}$
$\Rightarrow \frac{2}{3n^2} \le 10^{-3} \Rightarrow \frac{3n^2}{2} \ge 1000 \Rightarrow n^2 \ge \frac{2000}{3} \Rightarrow n \ge 25.82 \Rightarrow n \ge 26$

47. $\triangle x = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6} \Rightarrow \frac{\triangle x}{2} = \frac{\pi}{12}$;
$\sum_{i=0}^{6} mf(x_i) = 12 \Rightarrow T = \left(\frac{\pi}{12}\right)(12) = \pi$;

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	$\pi/6$	1/2	2	1
x_2	$\pi/3$	3/2	2	3
x_3	$\pi/2$	2	2	4
x_4	$2\pi/3$	3/2	2	3
x_5	$5\pi/6$	1/2	2	1
x_6	π	0	1	0

$\sum_{i=0}^{6} mf(x_i) = 18$ and $\frac{\triangle x}{3} = \frac{\pi}{18} \Rightarrow$
$S = \left(\frac{\pi}{18}\right)(18) = \pi$.

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	$\pi/6$	1/2	4	2
x_2	$\pi/3$	3/2	2	3
x_3	$\pi/2$	2	4	8
x_4	$2\pi/3$	3/2	2	3
x_5	$5\pi/6$	1/2	4	2
x_6	π	0	1	0

48. $\left|f^{(4)}(x)\right| \le 3 \Rightarrow M = 3$; $\triangle x = \frac{2-1}{n} = \frac{1}{n}$. Hence $|E_s| \le 10^{-5} \Rightarrow \left(\frac{2-1}{180}\right)\left(\frac{1}{n}\right)^4(3) \le 10^{-5} \Rightarrow \frac{1}{60n^4} \le 10^{-5} \Rightarrow n^4 \ge \frac{10^5}{60}$
$\Rightarrow n \ge 6.38 \Rightarrow n \ge 8$ (n must be even)

49. $y_{av} = \frac{1}{365-0}\int_0^{365}\left[37\sin\left(\frac{2\pi}{365}(x-101)\right) + 25\right] dx = \frac{1}{365}\left[-37\left(\frac{365}{2\pi}\cos\left(\frac{2\pi}{365}(x-101)\right) + 25x\right)\right]_0^{365}$
$= \frac{1}{365}\left[\left(-37\left(\frac{365}{2\pi}\right)\cos\left[\frac{2\pi}{365}(365-101)\right] + 25(365)\right) - \left(-37\left(\frac{365}{2\pi}\right)\cos\left[\frac{2\pi}{365}(0-101)\right] + 25(0)\right)\right]$
$= -\frac{37}{2\pi}\cos\left(\frac{2\pi}{365}(264)\right) + 25 + \frac{37}{2\pi}\cos\left(\frac{2\pi}{365}(-101)\right) = -\frac{37}{2\pi}\left(\cos\left(\frac{2\pi}{365}(264)\right) - \cos\left(\frac{2\pi}{365}(-101)\right)\right) + 25$
$\approx -\frac{37}{2\pi}(0.16705 - 0.16705) + 25 = 25°$ F

50. $av(C_v) = \frac{1}{675-20}\int_{20}^{675}\left[8.27 + 10^{-5}(26T - 1.87T^2)\right] dT = \frac{1}{655}\left[8.27T + \frac{13}{10^5}T^2 - \frac{0.62333}{10^5}T^3\right]_{20}^{675}$
$\approx \frac{1}{655}\left[(5582.25 + 59.23125 - 1917.03194) - (165.4 + 0.052 - 0.04987)\right] \approx 5.434$;
$8.27 + 10^{-5}(26T - 1.87T^2) = 5.434 \Rightarrow 1.87T^2 - 26T - 283{,}600 = 0 \Rightarrow T \approx \frac{26 + \sqrt{676 + 4(1.87)(283{,}600)}}{2(1.87)} \approx 396.45°$ C

51. (a) Each interval is 5 min $= \frac{1}{12}$ hour.

$\frac{1}{24}[2.5 + 2(2.4) + 2(2.3) + \ldots + 2(2.4) + 2.3] = \frac{29}{12} \approx 2.42$ gal

(b) $(60 \text{ mph})\left(\frac{12}{29} \text{ hours/gal}\right) \approx 24.83$ mi/gal

52. Using the Simpson's rule, $\triangle x = 15 \Rightarrow \frac{\triangle x}{3} = 5$;
$\sum mf(x_i) = 1211.8 \Rightarrow$ Area $\approx (1211.8)(5) = 6059$ ft^2;
The cost is Area $\cdot$ (\$2.10/ft^2) $\approx$ (6059 ft^2)(\$2.10/ft^2)
$=$ \$12,723.90 $\Rightarrow$ the job cannot be done for \$11,000.

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	15	36	4	144
x_2	30	54	2	108
x_3	45	51	4	204
x_4	60	49.5	2	99
x_5	75	54	4	216
x_6	90	64.4	2	128.8
x_7	105	67.5	4	270
x_8	120	42	1	42

53. $\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{b \to 3^-} \int_0^b \frac{dx}{\sqrt{9-x^2}} = \lim_{b \to 3^-} \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^b = \lim_{b \to 3^-} \sin^{-1}\left(\frac{b}{3}\right) - \sin^{-1}\left(\frac{0}{3}\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

54. $\int_0^1 \ln x\, dx = \lim_{b \to 0^+} [x \ln x - x]_b^1 = (1 \cdot \ln 1 - 1) - \lim_{b \to 0^+} [b \ln b - b] = -1 - \lim_{b \to 0^+} \frac{\ln b}{\left(\frac{1}{b}\right)} = -1 - \lim_{b \to 0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$
$= -1 + 0 = -1$

55. $\int_{-1}^1 \frac{dy}{y^{2/3}} = \int_{-1}^0 \frac{dy}{y^{2/3}} + \int_0^1 \frac{dy}{y^{2/3}} = 2\int_0^1 \frac{dy}{y^{2/3}} = 2 \cdot 3 \lim_{b \to 0^+} \left[y^{1/3}\right]_b^1 = 6\left(1 - \lim_{b \to 0^+} b^{1/3}\right) = 6$

56. $\int_{-2}^{\infty} \frac{d\theta}{(\theta+1)^{3/5}} = \int_{-2}^{-1} \frac{d\theta}{(\theta+1)^{3/5}} + \int_{-1}^{2} \frac{d\theta}{(\theta+1)^{3/5}} + \int_2^{\infty} \frac{d\theta}{(\theta+1)^{3/5}}$ converges if each integral converges, but
$\lim_{\theta \to \infty} \frac{\theta^{3/5}}{(\theta+1)^{3/5}} = 1$ and $\int_2^{\infty} \frac{d\theta}{\theta^{3/5}}$ diverges $\Rightarrow \int_{-2}^{\infty} \frac{d\theta}{(\theta+1)^{3/5}}$ diverges

57. $\int_3^{\infty} \frac{2\,du}{u^2-2u} = \int_3^{\infty} \frac{du}{u-2} - \int_3^{\infty} \frac{du}{u} = \lim_{b \to \infty} \left[\ln\left|\frac{u-2}{u}\right|\right]_3^b = \lim_{b \to \infty} \left[\ln\left|\frac{b-2}{b}\right|\right] - \ln\left|\frac{3-2}{3}\right| = 0 - \ln\left(\frac{1}{3}\right) = \ln 3$

58. $\int_1^{\infty} \frac{3v-1}{4v^3-v^2}\, dv = \int_1^{\infty} \left(\frac{1}{v} + \frac{1}{v^2} - \frac{4}{4v-1}\right) dv = \lim_{b \to \infty} \left[\ln v - \frac{1}{v} - \ln(4v-1)\right]_1^b$
$= \lim_{b \to \infty} \left[\ln\left(\frac{b}{4b-1}\right) - \frac{1}{b}\right] - (\ln 1 - 1 - \ln 3) = \ln\frac{1}{4} + 1 + \ln 3 = 1 + \ln\frac{3}{4}$

59. $\int_0^{\infty} x^2 e^{-x}\, dx = \lim_{b \to \infty} \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^b = \lim_{b \to \infty} \left(-b^2e^{-b} - 2be^{-b} - 2e^{-b}\right) - (-2) = 0 + 2 = 2$

60. $\int_{-\infty}^0 xe^{3x}\, dx = \lim_{b \to -\infty} \left[\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right]_b^0 = -\frac{1}{9} - \lim_{b \to -\infty} \left(\frac{b}{3}e^{3b} - \frac{1}{9}e^{3b}\right) = -\frac{1}{9} - 0 = -\frac{1}{9}$

61. $\int_{-\infty}^{\infty} \frac{dx}{4x^2+9} = 2\int_0^{\infty} \frac{dx}{4x^2+9} = \frac{1}{2}\int_0^{\infty} \frac{dx}{x^2+\frac{9}{4}} = \frac{1}{2} \lim_{b \to \infty} \left[\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)\right]_0^b = \frac{1}{2} \lim_{b \to \infty} \left[\frac{2}{3}\tan^{-1}\left(\frac{2b}{3}\right)\right] - \frac{1}{3}\tan^{-1}(0)$
$= \frac{1}{2}\left(\frac{2}{3} \cdot \frac{\pi}{2}\right) - 0 = \frac{\pi}{6}$

62. $\int_{-\infty}^{\infty} \frac{4\,dx}{x^2+16} = 2\int_0^{\infty} \frac{4\,dx}{x^2+16} = 2 \lim_{b \to \infty} \left[\tan^{-1}\left(\frac{x}{4}\right)\right]_0^b = 2\left(\lim_{b \to \infty} \left[\tan^{-1}\left(\frac{b}{4}\right)\right] - \tan^{-1}(0)\right) = 2\left(\frac{\pi}{2}\right) - 0 = \pi$

63. $\lim_{\theta \to \infty} \frac{\theta}{\sqrt{\theta^2+1}} = 1$ and $\int_6^{\infty} \frac{d\theta}{\theta}$ diverges $\Rightarrow \int_6^{\infty} \frac{d\theta}{\sqrt{\theta^2+1}}$ diverges

64. $I = \int_0^\infty e^{-u} \cos u \, du = \lim_{b \to \infty} \left[-e^{-u} \cos u\right]_0^b - \int_0^\infty e^{-u} \sin u \, du = 1 + \lim_{b \to \infty} \left[e^{-u} \sin u\right]_0^b - \int_0^\infty (e^{-u}) \cos u \, du$
$\Rightarrow I = 1 + 0 - I \Rightarrow 2I = 1 \Rightarrow I = \frac{1}{2}$ converges

65. $\int_1^\infty \frac{\ln z}{z} \, dz = \int_1^e \frac{\ln z}{z} \, dz + \int_e^\infty \frac{\ln z}{z} \, dz = \left[\frac{(\ln z)^2}{2}\right]_1^e + \lim_{b \to \infty} \left[\frac{(\ln z)^2}{2}\right]_e^b = \left(\frac{1^2}{2} - 0\right) + \lim_{b \to \infty} \left[\frac{(\ln b)^2}{2} - \frac{1}{2}\right] = \infty$
$\Rightarrow$ diverges

66. $0 < \frac{e^{-t}}{\sqrt{t}} \le e^{-t}$ for $t \ge 1$ and $\int_1^\infty e^{-t} \, dt$ converges $\Rightarrow \int_1^\infty \frac{e^{-t}}{\sqrt{t}} \, dt$ converges

67. $\int_{-\infty}^\infty \frac{2 \, dx}{e^x + e^{-x}} = 2\int_0^\infty \frac{2 \, dx}{e^x + e^{-x}} < \int_0^\infty \frac{4 \, dx}{e^x}$ converges $\Rightarrow \int_{-\infty}^\infty \frac{2 \, dx}{e^x + e^{-x}}$ converges

68. $\int_{-\infty}^\infty \frac{dx}{x^2(1+e^x)} = \int_{-\infty}^{-1} \frac{dx}{x^2(1+e^x)} + \int_{-1}^0 \frac{dx}{x^2(1+e^x)} + \int_0^1 \frac{dx}{x^2(1+e^x)} + \int_1^\infty \frac{dx}{x^2(1+e^x)}$;
$\lim_{x \to 0} \frac{\left(\frac{1}{x^2}\right)}{\left[\frac{1}{x^2(1+e^x)}\right]} = \lim_{x \to 0} \frac{x^2(1+e^x)}{x^2} = \lim_{x \to 0} (1+e^x) = 2$ and $\int_0^1 \frac{dx}{x^2}$ diverges $\Rightarrow \int_0^1 \frac{dx}{x^2(1+e^x)}$ diverges
$\Rightarrow \int_{-\infty}^\infty \frac{dx}{x^2(1+e^x)}$ diverges

69. $\int \frac{x \, dx}{1+\sqrt{x}}$; $\begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int \frac{u^2 \cdot 2u \, du}{1+u} = \int \left(2u^2 - 2u + 2 - \frac{2}{1+u}\right) du = \frac{2}{3}u^3 - u^2 + 2u - 2\ln|1+u| + C$
$= \frac{2x^{3/2}}{3} - x + 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$

70. $\int \frac{x^3+2}{4-x^2} \, dx = -\int \left(x + \frac{4x+2}{x^2-4}\right) dx = -\int x \, dx - \frac{3}{2}\int \frac{dx}{x+2} - \frac{5}{2}\int \frac{dx}{x-2} = -\frac{x^2}{2} - \frac{3}{2}\ln|x+2| - \frac{5}{2}\ln|x-2| + C$

71. $\int \frac{dx}{x(x^2+1)^2}$; $\begin{bmatrix} x = \tan\theta \\ dx = \sec^2\theta \, d\theta \end{bmatrix} \to \int \frac{\sec^2\theta \, d\theta}{\tan\theta \sec^4\theta} = \int \frac{\cos^3\theta \, d\theta}{\sin\theta} = \int \left(\frac{1-\sin^2\theta}{\sin\theta}\right) d(\sin\theta)$
$= \ln|\sin\theta| - \frac{1}{2}\sin^2\theta + C = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| - \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)^2 + C$

72. $\int \frac{dx}{\sqrt{-2x-x^2}} = \int \frac{d(x+1)}{\sqrt{1-(x+1)^2}} = \sin^{-1}(x+1) + C$

73. $\int \frac{2-\cos x+\sin x}{\sin^2 x} \, dx = \int 2\csc^2 x \, dx - \int \frac{\cos x \, dx}{\sin^2 x} + \int \csc x \, dx = -2\cot x + \frac{1}{\sin x} - \ln|\csc x + \cot x| + C$
$= -2\cot x + \csc x - \ln|\csc x + \cot x| + C$

74. $\int \frac{\sin^2\theta}{\cos^2\theta} \, d\theta = \int \frac{1-\cos^2\theta}{\cos^2\theta} \, d\theta = \int \sec^2\theta \, d\theta - \int d\theta = \tan\theta - \theta + C$

75. $\int \frac{9 \, dv}{81-v^4} = \frac{1}{2}\int \frac{dv}{v^2+9} + \frac{1}{12}\int \frac{dv}{3-v} + \frac{1}{12}\int \frac{dv}{3+v} = \frac{1}{12}\ln\left|\frac{3+v}{3-v}\right| + \frac{1}{6}\tan^{-1}\frac{v}{3} + C$

76. $\int_2^\infty \frac{dx}{(x-1)^2} = \lim_{b \to \infty} \left[\frac{1}{1-x}\right]_2^b = \lim_{b \to \infty} \left[\frac{1}{1-b} - (-1)\right] = 0 + 1 = 1$

77.

		$\cos(2\theta+1)$
θ	$\xrightarrow{(+)}$	$\frac{1}{2}\sin(2\theta+1)$
1	$\xrightarrow{(-)}$	$-\frac{1}{4}\cos(2\theta+1)$
0		

$\Rightarrow \int \theta\cos(2\theta+1) \, d\theta = \frac{\theta}{2}\sin(2\theta+1) + \frac{1}{4}\cos(2\theta+1) + C$

78. $\int \frac{x^3\,dx}{x^2-2x+1} = \int \left(x+2+\frac{3x-2}{x^2-2x+1}\right) dx = \int (x+2)\,dx + 3\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$
$= \frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C$

79. $\int \frac{\sin 2\theta\,d\theta}{(1+\cos 2\theta)^2} = -\frac{1}{2}\int \frac{d(1+\cos 2\theta)}{(1+\cos 2\theta)^2} = \frac{1}{2(1+\cos 2\theta)} + C = \frac{1}{4}\sec^2\theta + C$

80. $\int_{\pi/4}^{\pi/2} \sqrt{1+\cos 4x}\,dx = -\sqrt{2}\int_{\pi/4}^{\pi/2} \cos 2x\,dx = \left[-\frac{\sqrt{2}}{2}\sin 2x\right]_{\pi/4}^{\pi/2} = \frac{\sqrt{2}}{2}$

81. $\int \frac{x\,dx}{\sqrt{2-x}}$; $\begin{bmatrix} y = 2-x \\ dy = -dx \end{bmatrix} \rightarrow -\int \frac{(2-y)\,dy}{\sqrt{y}} = \frac{2}{3}y^{3/2} - 4y^{1/2} + C = \frac{2}{3}(2-x)^{3/2} - 4(2-x)^{1/2} + C$
$= 2\left[\frac{\left(\sqrt{2-x}\right)^3}{3} - 2\sqrt{2-x}\right] + C$

82. $\int \frac{\sqrt{1-v^2}}{v^2}\,dv$; $[v = \sin\theta] \rightarrow \int \frac{\cos\theta\cdot\cos\theta\,d\theta}{\sin^2\theta} = \int \frac{(1-\sin^2\theta)\,d\theta}{\sin^2\theta} = \int \csc^2\theta\,d\theta - \int d\theta = \cot\theta - \theta + C$
$= -\sin^{-1} v - \frac{\sqrt{1-v^2}}{v} + C$

83. $\int \frac{dy}{y^2-2y+2} = \int \frac{d(y-1)}{(y-1)^2+1} = \tan^{-1}(y-1) + C$

84. $\int \frac{x\,dx}{\sqrt{8-2x^2-x^4}} = \frac{1}{2}\int \frac{d(x^2+1)}{\sqrt{9-(x^2+1)^2}} = \frac{1}{2}\sin^{-1}\left(\frac{x^2+1}{3}\right) + C$

85. $\int \frac{z+1}{z^2(z^2+4)}\,dz = \frac{1}{4}\int \left(\frac{1}{z} + \frac{1}{z^2} - \frac{z+1}{z^2+4}\right) dz = \frac{1}{4}\ln|z| - \frac{1}{4z} - \frac{1}{8}\ln(z^2+4) - \frac{1}{8}\tan^{-1}\frac{z}{2} + C$

86. $\int x^3 e^{x^2}\,dx = \frac{1}{2}\int x^2 e^{x^2}\,d(x^2) = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C = \frac{(x^2-1)\,e^{x^2}}{2} + C$

87. $\int \frac{t\,dt}{\sqrt{9-4t^2}} = -\frac{1}{8}\int \frac{d(9-4t^2)}{\sqrt{9-4t^2}} = -\frac{1}{4}\sqrt{9-4t^2} + C$

88. $u = \tan^{-1} x$, $du = \frac{dx}{1+x^2}$; $dv = \frac{dx}{x^2}$, $v = -\frac{1}{x}$;
$\int \frac{\tan^{-1} x\,dx}{x^2} = -\frac{1}{x}\tan^{-1} x + \int \frac{dx}{x(1+x^2)} = -\frac{1}{x}\tan^{-1} x + \int \frac{dx}{x} - \int \frac{x\,dx}{1+x^2}$
$= -\frac{1}{x}\tan^{-1} x + \ln|x| - \frac{1}{2}\ln(1+x^2) + C = -\frac{\tan^{-1} x}{x} + \ln|x| - \ln\sqrt{1+x^2} + C$

89. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2}$; $[e^t = x] \rightarrow \int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \ln|x+1| - \ln|x+2| + C = \ln\left|\frac{x+1}{x+2}\right| + C$
$= \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

90. $\int \tan^3 t\,dt = \int (\tan t)(\sec^2 t - 1)\,dt = \frac{\tan^2 t}{2} - \int \tan t\,dt = \frac{\tan^2 t}{2} - \ln|\sec t| + C$

91. $\int_1^\infty \frac{\ln y\,dy}{y^3}$; $\begin{bmatrix} x = \ln y \\ dx = \frac{dy}{y} \\ dy = e^x\,dx \end{bmatrix} \rightarrow \int_0^\infty \frac{x\cdot e^x}{e^{3x}}\,dx = \int_0^\infty xe^{-2x}\,dx = \lim_{b\to\infty}\left[-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^b$
$= \lim_{b\to\infty}\left(\frac{-b}{2e^{2b}} - \frac{1}{4e^{2b}}\right) - \left(0 - \frac{1}{4}\right) = \frac{1}{4}$

92. $\int \frac{\cot v\,dv}{\ln(\sin v)} = \int \frac{\cos v\,dv}{(\sin v)\ln(\sin v)}$; $\begin{bmatrix} u = \ln(\sin v) \\ du = \frac{\cos v\,dv}{\sin v} \end{bmatrix} \rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|\ln(\sin v)| + C$

93. $\int e^{\ln \sqrt{x}}\,dx = \int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$

94. $\int e^{\theta}\sqrt{3+4e^{\theta}}\,d\theta;\ \begin{bmatrix} u = 4e^{\theta} \\ du = 4e^{\theta}\,d\theta \end{bmatrix} \rightarrow \frac{1}{4}\int \sqrt{3+u}\,du = \frac{1}{4}\cdot\frac{2}{3}(3+u)^{3/2} + C = \frac{1}{6}\left(3+4e^{\theta}\right)^{3/2} + C$

95. $\int \frac{\sin 5t\,dt}{1+(\cos 5t)^2};\ \begin{bmatrix} u = \cos 5t \\ du = -5\sin 5t\,dt \end{bmatrix} \rightarrow -\frac{1}{5}\int \frac{du}{1+u^2} = -\frac{1}{5}\tan^{-1} u + C = -\frac{1}{5}\tan^{-1}(\cos 5t) + C$

96. $\int \frac{dv}{\sqrt{e^{2v}-1}};\ \begin{bmatrix} x = e^{v} \\ dx = e^{v}\,dv \end{bmatrix} \rightarrow \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C = \sec^{-1}(e^{v}) + C$

97. $\int \frac{dr}{1+\sqrt{r}};\ \begin{bmatrix} u = \sqrt{r} \\ du = \frac{dr}{2\sqrt{r}} \end{bmatrix} \rightarrow \int \frac{2u\,du}{1+u} = \int \left(2 - \frac{2}{1+u}\right)du = 2u - 2\ln|1+u| + C = 2\sqrt{r} - 2\ln\left(1+\sqrt{r}\right) + C$

98. $\int \frac{4x^3-20x}{x^4-10x^2+9}\,dx = \int \frac{d\left(x^4-10x^2+9\right)}{x^4-10x^2+9} = \ln|x^4-10x^2+9| + C$

99. $\int \frac{x^3}{1+x^2}\,dx = \int \left(x - \frac{x}{1+x^2}\right)dx = \int x\,dx - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(1+x^2) + C$

100. $\int \frac{x^2}{1+x^3}\,dx = 3\int \frac{3x^2}{1+x^3}\,dx = 3\ln|1+x^3| + C$

101. $\int \frac{1+x^2}{1+x^3}\,dx;\ \frac{1+x^2}{1+x^3} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2} \Rightarrow 1+x^2 = A(1-x+x^2) + (Bx+C)(1+x)$
$= (A+B)x^2 + (-A+B+C)x + (A+C) \Rightarrow A+B = 1,\ -A+B+C = 0,\ A+C = 1 \Rightarrow A = \frac{2}{3}, B = \frac{1}{3}, C = \frac{1}{3};$
$\int \frac{1+x^2}{1+x^3}\,dx = \int \left(\frac{2/3}{1+x} + \frac{(1/3)x+1/3}{1-x+x^2}\right)dx = \frac{2}{3}\int \frac{1}{1+x}\,dx + \frac{1}{3}\int \frac{x+1}{1-x+x^2}\,dx = \frac{2}{3}\int \frac{1}{1+x}\,dx + \frac{1}{3}\int \frac{x+1}{\frac{3}{4}+\left(x-\frac{1}{2}\right)^2}\,dx;$
$\begin{bmatrix} u = x - \frac{1}{2} \\ du = dx \end{bmatrix} \rightarrow \frac{1}{3}\int \frac{u+\frac{3}{2}}{\frac{3}{4}+u^2}\,du = \frac{1}{3}\int \frac{u}{\frac{3}{4}+u^2}\,du + \frac{1}{2}\int \frac{1}{\frac{3}{4}+u^2}\,du = \frac{1}{6}\ln\left|\frac{3}{4} + u^2\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}/2}\right)$
$= \frac{1}{6}\ln\left|\frac{3}{4} + \left(x-\frac{1}{2}\right)^2\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right) = \frac{1}{6}\ln|1 - x + x^2| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$
$\Rightarrow \frac{2}{3}\int \frac{1}{1+x}\,dx + \frac{1}{3}\int \frac{x+1}{1-x+x^2}\,dx = \frac{2}{3}\ln|1+x| + \frac{1}{6}\ln|1 - x + x^2| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

102. $\int \frac{1+x^2}{(1+x)^3}\,dx;\ \begin{bmatrix} u = 1+x \\ du = dx \end{bmatrix} \rightarrow \int \frac{1+(u-1)^2}{u^3}\,du = \int \frac{u^2-2u+2}{u^3}\,du = \int \frac{1}{u}\,du - \int \frac{2}{u^2}\,du + \int \frac{2}{u^3}\,du = \ln|u| + \frac{2}{u} - \frac{1}{u^2} + C$
$= \ln|1+x| + \frac{2}{1+x} - \frac{1}{(1+x)^2} + C$

103. $\int \sqrt{x}\sqrt{1+\sqrt{x}}\,dx;\ \begin{bmatrix} w = \sqrt{x} \Rightarrow w^2 = x \\ 2w\,dw = dx \end{bmatrix} \rightarrow \int 2w^2\sqrt{1+w}\,dw$

		$\sqrt{1+w}$
$2w^2$	$\xrightarrow{(+)}$	$\frac{2}{3}(1+w)^{3/2}$
$4w$	$\xrightarrow{(-)}$	$\frac{4}{15}(1+w)^{5/2}$
4	$\xrightarrow{(+)}$	$\frac{8}{105}(1+w)^{7/2}$
0		

$\Rightarrow \int 2w^2\sqrt{1+w}\,dw = \frac{4}{3}w^2(1+w)^{3/2} - \frac{16}{15}w(1+w)^{5/2} + \frac{32}{105}(1+w)^{7/2} + C$
$= \frac{4}{3}x\left(1+\sqrt{x}\right)^{3/2} - \frac{16}{15}\sqrt{x}\left(1+\sqrt{x}\right)^{5/2} + \frac{32}{105}\left(1+\sqrt{x}\right)^{7/2} + C$

104. $\int \sqrt{1+\sqrt{1+x}}\,dx$; $\left[\begin{matrix} w=\sqrt{1+x} \Rightarrow w^2 = 1+x \\ 2w\,dw = dx \end{matrix}\right] \rightarrow \int 2w\sqrt{1+w}\,dw$;

$\left[u = 2w,\ du = 2\,dw,\ dv = \sqrt{1+w}\,dw,\ v = \frac{2}{3}(1+w)^{3/2}\right]$

$\int 2w\sqrt{1+w}\,dw = \frac{4}{3}w(1+w)^{3/2} - \int \frac{4}{3}(1+w)^{3/2}\,dw = \frac{4}{3}w(1+w)^{3/2} - \frac{8}{15}(1+w)^{5/2} + C$

$= \frac{4}{3}\sqrt{1+x}\left(1+\sqrt{1+x}\right)^{3/2} - \frac{8}{15}\left(1+\sqrt{1+x}\right)^{5/2} + C$

105. $\int \frac{1}{\sqrt{x}\sqrt{1+x}}\,dx$; $\left[\begin{matrix} u=\sqrt{x} \Rightarrow u^2 = x \\ 2u\,du = dx \end{matrix}\right] \rightarrow \int \frac{2}{\sqrt{1+u^2}}\,du$; $\left[u = \tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ du = \sec^2\theta\,d\theta,\ \sqrt{1+u^2} = \sec\theta\right]$

$\int \frac{2}{\sqrt{1+u^2}}\,du = \int \frac{2\sec^2\theta}{\sec\theta}\,d\theta = \int 2\sec\theta\,d\theta = 2\ln|\sec\theta + \tan\theta| + C = 2\ln\left|\sqrt{1+u^2} + u\right| + C$

$= 2\ln\left|\sqrt{1+x} + \sqrt{x}\right| + C$

106. $\int_0^{1/2} \sqrt{1+\sqrt{1-x^2}}\,dx$;

$\left[x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ \sqrt{1-x^2} = \cos\theta,\ x = 0 = \sin\theta \Rightarrow \theta = 0,\ x = \frac{1}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{6}\right]$

$\rightarrow \int_0^{\pi/6} \sqrt{1+\cos\theta}\cos\theta\,d\theta = \int_0^{\pi/6} \frac{\sqrt{1-\cos^2\theta}}{\sqrt{1-\cos\theta}}\cos\theta\,d\theta = \int_0^{\pi/6} \frac{\sin\theta\cos\theta}{\sqrt{1-\cos\theta}}\,d\theta = \lim_{c\to 0^+} \int_c^{\pi/6} \frac{\sin\theta\cos\theta}{\sqrt{1-\cos\theta}}\,d\theta$;

$\left[u = \cos\theta,\ du = -\sin\theta\,d\theta,\ dv = \frac{\sin\theta}{\sqrt{1-\cos\theta}}\,d\theta,\ v = 2(1-\cos\theta)^{1/2}\right]$

$= \lim_{c\to 0^+}\left[\left[2\cos\theta\,(1-\cos\theta)^{1/2}\right]_c^{\pi/6} + \int_c^{\pi/6} 2(1-\cos\theta)^{1/2}\sin\theta\,d\theta\right]$

$= \lim_{c\to 0^+}\left[\left(2\cos\left(\frac{\pi}{6}\right)\left(1-\cos\left(\frac{\pi}{6}\right)\right)^{1/2} - 2\cos c\,(1-\cos c)^{1/2}\right) + \left[\frac{4}{3}(1-\cos\theta)^{3/2}\right]_c^{\pi/6}\right]$

$= \lim_{c\to 0^+}\left[\sqrt{3}\left(1-\frac{\sqrt{3}}{2}\right)^{1/2} - 2\cos c\,(1-\cos c)^{1/2} + \left(\frac{4}{3}\left(1-\cos\left(\frac{\pi}{6}\right)\right)^{3/2} - \frac{4}{3}(1-\cos c)^{3/2}\right)\right]$

$= \lim_{c\to 0^+}\left[\sqrt{3}\left(1-\frac{\sqrt{3}}{2}\right)^{1/2} - 2\cos c\,(1-\cos c)^{1/2} + \frac{4}{3}\left(1-\frac{\sqrt{3}}{2}\right)^{3/2} - \frac{4}{3}(1-\cos c)^{3/2}\right]$

$= \sqrt{3}\left(1-\frac{\sqrt{3}}{2}\right)^{1/2} + \frac{4}{3}\left(1-\frac{\sqrt{3}}{2}\right)^{3/2} = \left(1-\frac{\sqrt{3}}{2}\right)^{1/2}\left(\frac{4+\sqrt{3}}{3}\right) = \frac{\left(4+\sqrt{3}\right)\sqrt{2-\sqrt{3}}}{3\sqrt{2}}$

107. $\int \frac{\ln x}{x + x\ln x}\,dx = \int \frac{\ln x}{x(1+\ln x)}\,dx$; $\left[\begin{matrix} u = 1+\ln x \\ du = \frac{1}{x}dx \end{matrix}\right] \rightarrow \int \frac{u-1}{u}\,du = \int du - \int \frac{1}{u}\,du = u - \ln|u| + C$

$= (1+\ln x) - \ln|1+\ln x| + C = \ln x - \ln|1+\ln x| + C$

108. $\int \frac{1}{x\ln x\cdot\ln(\ln x)}\,dx$; $\left[\begin{matrix} u = \ln(\ln x) \\ du = \frac{1}{x\ln x}dx \end{matrix}\right] \rightarrow \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln(\ln x)| + C$

109. $\int \frac{x^{\ln x}\ln x}{x}dx$; $\left[u = x^{\ln x} \Rightarrow \ln u = \ln x^{\ln x} = (\ln x)^2 \Rightarrow \frac{1}{u}du = \frac{2\ln x}{x}\,dx \Rightarrow du = \frac{2u\ln x}{x}\,dx = \frac{2x^{\ln x}\ln x}{x}\,dx\right] \rightarrow \frac{1}{2}\int du$

$= \frac{1}{2}u + C = \frac{1}{2}x^{\ln x} + C$

110. $\int (\ln x)^{\ln x}\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]dx$; $\left[u = (\ln x)^{\ln x} \Rightarrow \ln u = \ln(\ln x)^{\ln x} = (\ln x)\ln(\ln x) \Rightarrow \frac{1}{u}du = \left(\frac{(\ln x)}{x\ln x} + \frac{\ln(\ln x)}{x}\right)dx\right.$

$\left.\Rightarrow du = u\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]dx = (\ln x)^{\ln x}\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]dx\right] \rightarrow \int du = u + C = (\ln x)^{\ln x} + C$

111. $\int \frac{1}{x\sqrt{1-x^4}}\,dx = \int \frac{x}{x^2\sqrt{1-x^4}}\,dx;\ \left[x^2 = \sin\theta,\ 0 \le \theta < \frac{\pi}{2},\ 2x\,dx = \cos\theta\,d\theta,\ \sqrt{1-x^4} = \cos\theta\right] \to \frac{1}{2}\int \frac{\cos\theta}{\sin\theta\cos\theta}\,d\theta$
$= \frac{1}{2}\int \csc\theta\,d\theta = -\frac{1}{2}\ln|\csc\theta + \cot\theta| + C = -\frac{1}{2}\ln\left|\frac{1}{x^2} + \frac{\sqrt{1-x^4}}{x^2}\right| + C = -\frac{1}{2}\ln\left|\frac{1+\sqrt{1-x^4}}{x^2}\right| + C$

112. $\int \frac{\sqrt{1-x}}{x}\,dx;\ \left[u = \sqrt{1-x} \Rightarrow u^2 = 1-x \Rightarrow 2u\,du = -dx\right] \to \int \frac{-2u^2}{1-u^2}\,du = \int \frac{2u^2}{u^2-1}\,du = \int \left(2 + \frac{2}{u^2-1}\right)du;$
$\frac{2}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1} \Rightarrow 2 = A(u+1) + B(u-1) = (A+B)u + A - B \Rightarrow A + B = 0,\ A - B = 2$
$\Rightarrow A = 1 \Rightarrow B = -1;\ \int \left(2 + \frac{2}{u^2-1}\right)du = \int 2\,du + \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du$
$= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1-x} + \frac{1}{2}\ln\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right| + C$

113. (a) $\int_0^a f(a-x)\,dx;\ \left[u = a - x \Rightarrow du = -dx,\ x = 0 \Rightarrow u = a,\ x = a \Rightarrow u = 0\right] \to -\int_a^0 f(u)\,du = \int_0^a f(u)\,du$, which is the same integral as $\int_0^a f(x)\,dx$.

(b) $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)}\,dx = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x}{\sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x + \cos\left(\frac{\pi}{2}\right)\cos x + \sin\left(\frac{\pi}{2}\right)\sin x}\,dx$
$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx \Rightarrow 2\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} dx$
$= \left[x\right]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow 2\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{2} \Rightarrow \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{4}$

114. $\int \frac{\sin x}{\sin x + \cos x}\,dx = \int \frac{\sin x + \cos x - \cos x + \sin x - \sin x}{\sin x + \cos x}\,dx = \int \frac{\sin x + \cos x}{\sin x + \cos x}\,dx + \int \frac{-\cos x + \sin x}{\sin x + \cos x}\,dx + \int \frac{-\sin x}{\sin x + \cos x}\,dx$
$= \int dx - \int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx - \int \frac{\sin x}{\sin x + \cos x}\,dx = x - \ln|\sin x + \cos x| - \int \frac{\sin x}{\sin x + \cos x}\,dx$
$\Rightarrow 2\int \frac{\sin x}{\sin x + \cos x}\,dx = x - \ln|\sin x + \cos x| \Rightarrow \int \frac{\sin x}{\sin x + \cos x}\,dx = \frac{x}{2} - \frac{1}{2}\ln|\sin x + \cos x| + C$

115. $\int \frac{\sin^2 x}{1 + \sin^2 x}\,dx = \int \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}}\,dx = \int \frac{\tan^2 x}{\sec^2 x + \tan^2 x}\,dx = \int \frac{\tan^2 x + \sec^2 x - \sec^2 x}{\sec^2 x + \tan^2 x}\,dx = \int \frac{\tan^2 x + \sec^2 x}{\sec^2 x + \tan^2 x}\,dx - \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\,dx$
$= \int dx - \int \frac{\sec^2 x}{1 + 2\tan^2 x}\,dx = x - \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right) + C$

116. $\int \frac{1 - \cos x}{1 + \cos x}\,dx = \int \frac{(1-\cos x)^2}{1 - \cos^2 x}\,dx = \int \frac{1 - 2\cos x + \cos^2 x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x}\,dx - \int \frac{2\cos x}{\sin^2 x}\,dx + \int \frac{\cos^2 x}{\sin^2 x}\,dx$
$= \int \csc^2 x\,dx - 2\int \csc x\cot x\,dx + \int \cot^2 x\,dx = -\cot x + 2\csc x + \int \left(\csc^2 x - 1\right)dx = = -2\cot x + 2\csc x - x + C$

CHAPTER 8 ADDITIONAL AND ADVANCED EXERCISES

1. $u = \left(\sin^{-1} x\right)^2,\ du = \frac{2\sin^{-1} x\,dx}{\sqrt{1-x^2}};\ dv = dx,\ v = x;$
$\int \left(\sin^{-1} x\right)^2 dx = x\left(\sin^{-1} x\right)^2 - \int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}};$
$u = \sin^{-1} x,\ du = \frac{dx}{\sqrt{1-x^2}};\ dv = -\frac{2x\,dx}{\sqrt{1-x^2}},\ v = 2\sqrt{1-x^2};$
$-\int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}} = 2\left(\sin^{-1} x\right)\sqrt{1-x^2} - \int 2\,dx = 2\left(\sin^{-1} x\right)\sqrt{1-x^2} - 2x + C$; therefore
$\int \left(\sin^{-1} x\right)^2 dx = x\left(\sin^{-1} x\right)^2 + 2\left(\sin^{-1} x\right)\sqrt{1-x^2} - 2x + C$

2. $\frac{1}{x} = \frac{1}{x},$
$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1},$
$\frac{1}{x(x+1)(x+2)} = \frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)},$

$\frac{1}{x(x+1)(x+2)(x+3)} = \frac{1}{6x} - \frac{1}{2(x+1)} + \frac{1}{2(x+2)} - \frac{1}{6(x+3)}$,

$\frac{1}{x(x+1)(x+2)(x+3)(x+4)} = \frac{1}{24x} - \frac{1}{6(x+1)} + \frac{1}{4(x+2)} - \frac{1}{6(x+3)} + \frac{1}{24(x+4)} \Rightarrow$ the following pattern:

$\frac{1}{x(x+1)(x+2)\cdots(x+m)} = \sum_{k=0}^{m} \frac{(-1)^k}{(k!)(m-k)!(x+k)}$; therefore $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

$= \sum_{k=0}^{m} \left[\frac{(-1)^k}{(k!)(m-k)!} \ln|x+k|\right] + C$

3. $u = \sin^{-1} x,\ du = \frac{dx}{\sqrt{1-x^2}}$; $dv = x\,dx,\ v = \frac{x^2}{2}$;

$\int x \sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \int \frac{x^2\,dx}{2\sqrt{1-x^2}}$; $\begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{bmatrix} \rightarrow \int x \sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \int \frac{\sin^2\theta\cos\theta\,d\theta}{2\cos\theta}$

$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \sin^2\theta\,d\theta = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C = \frac{x^2}{2}\sin^{-1} x + \frac{\sin\theta\cos\theta - \theta}{4} + C$

$= \frac{x^2}{2}\sin^{-1} x + \frac{x\sqrt{1-x^2} - \sin^{-1} x}{4} + C$

4. $\int \sin^{-1}\sqrt{y}\,dy$; $\begin{bmatrix} z = \sqrt{y} \\ dz = \frac{dy}{2\sqrt{y}} \end{bmatrix} \rightarrow \int 2z \sin^{-1} z\,dz$; from Exercise 3, $\int z \sin^{-1} z\,dz$

$= \frac{z^2 \sin^{-1} z}{2} + \frac{z\sqrt{1-z^2} - \sin^{-1} z}{4} + C \Rightarrow \int \sin^{-1}\sqrt{y}\,dy = y \sin^{-1}\sqrt{y} + \frac{\sqrt{y}\sqrt{1-y} - \sin^{-1}\sqrt{y}}{2} + C$

$= y \sin^{-1}\sqrt{y} + \frac{\sqrt{y-y^2}}{2} - \frac{\sin^{-1}\sqrt{y}}{2} + C$

5. $\int \frac{dt}{t-\sqrt{1-t^2}}$; $\begin{bmatrix} t = \sin\theta \\ dt = \cos\theta\,d\theta \end{bmatrix} \rightarrow \int \frac{\cos\theta\,d\theta}{\sin\theta - \cos\theta} = \int \frac{d\theta}{\tan\theta - 1}$; $\begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\,d\theta \\ d\theta = \frac{du}{u^2+1} \end{bmatrix} \rightarrow \int \frac{du}{(u-1)(u^2+1)}$

$= \frac{1}{2}\int \frac{du}{u-1} - \frac{1}{2}\int \frac{du}{u^2+1} - \frac{1}{2}\int \frac{u\,du}{u^2+1} = \frac{1}{2}\ln\left|\frac{u-1}{\sqrt{u^2+1}}\right| - \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\ln\left|\frac{\tan\theta - 1}{\sec\theta}\right| - \frac{1}{2}\theta + C$

$= \frac{1}{2}\ln\left(t - \sqrt{1-t^2}\right) - \frac{1}{2}\sin^{-1} t + C$

6. $\int \frac{1}{x^4+4}\,dx = \int \frac{1}{(x^2+2)^2 - 4x^2}\,dx = \int \frac{1}{(x^2+2x+2)(x^2-2x+2)}\,dx$

$= \frac{1}{16}\int \left[\frac{2x+2}{x^2+2x+2} + \frac{2}{(x+1)^2+1} - \frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1}\right] dx$

$= \frac{1}{16}\ln\left|\frac{x^2+2x+2}{x^2-2x+2}\right| + \frac{1}{8}\left[\tan^{-1}(x+1) + \tan^{-1}(x-1)\right] + C$

7. $\lim_{x\to\infty} \int_{-x}^{x} \sin t\,dt = \lim_{x\to\infty} [-\cos t]_{-x}^{x} = \lim_{x\to\infty} [-\cos x + \cos(-x)] = \lim_{x\to\infty} (-\cos x + \cos x) = \lim_{x\to\infty} 0 = 0$

8. $\lim_{x\to 0^+} \int_x^1 \frac{\cos t}{t^2}\,dt$; $\lim_{t\to 0^+} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{\cos t}{t^2}\right)} = \lim_{t\to 0^+} \frac{1}{\cos t} = 1 \Rightarrow \lim_{x\to 0^+} \int_x^1 \frac{\cos t}{t^2}\,dt$ diverges since $\int_0^1 \frac{dt}{t^2}$ diverges; thus

$\lim_{x\to 0^+} x \int_x^1 \frac{\cos t}{t^2}\,dt$ is an indeterminate $0 \cdot \infty$ form and we apply l'Hôpital's rule:

$\lim_{x\to 0^+} x \int_x^1 \frac{\cos t}{t^2}\,dt = \lim_{x\to 0^+} \frac{-\int_1^x \frac{\cos t}{t^2}\,dt}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{-\left(\frac{\cos x}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} \cos x = 1$

9. $\lim_{n\to\infty} \sum_{k=1}^{n} \ln \sqrt[n]{1+\frac{k}{n}} = \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 + k\left(\frac{1}{n}\right)\right)\left(\frac{1}{n}\right) = \int_0^1 \ln(1+x)\,dx$; $\begin{bmatrix} u = 1+x,\ du = dx \\ x = 0 \Rightarrow u = 1,\ x = 1 \Rightarrow u = 2 \end{bmatrix}$

$\rightarrow \int_1^2 \ln u\,du = [u \ln u - u]_1^2 = (2\ln 2 - 2) - (\ln 1 - 1) = 2\ln 2 - 1 = \ln 4 - 1$

10. $\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2-k^2}} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(\frac{n}{\sqrt{n^2-k^2}}\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(\frac{1}{\sqrt{1-\left[k\left(\frac{1}{n}\right)\right]^2}}\right)\left(\frac{1}{n}\right)$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1} x\right]_0^1 = \frac{\pi}{2}$

11. $\frac{dy}{dx} = \sqrt{\cos 2x} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cos 2x = 2\cos^2 x$; $L = \int_0^{\pi/4} \sqrt{1 + \left(\sqrt{\cos 2t}\right)^2}\,dt = \sqrt{2}\int_0^{\pi/4} \sqrt{\cos^2 t}\,dt$

$= \sqrt{2}\,\left[\sin t\right]_0^{\pi/4} = 1$

12. $\frac{dy}{dx} = \frac{-2x}{1-x^2} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2} = \frac{1+2x^2+x^4}{(1-x^2)^2} = \left(\frac{1+x^2}{1-x^2}\right)^2$; $L = \int_0^{1/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_0^{1/2} \left(\frac{1+x^2}{1-x^2}\right) dx = \int_0^{1/2} \left(-1 + \frac{2}{1-x^2}\right) dx = \int_0^{1/2} \left(-1 + \frac{1}{1+x} + \frac{1}{1-x}\right) dx = \left[-x + \ln\left|\frac{1+x}{1-x}\right|\right]_0^{1/2}$

$= \left(-\frac{1}{2} + \ln 3\right) - (0 + \ln 1) = \ln 3 - \frac{1}{2}$

13. $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi xy\,dx$

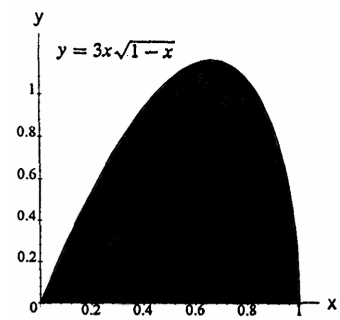
$= 6\pi \int_0^1 x^2\sqrt{1-x}\,dx$; $\begin{bmatrix} u = 1-x \\ du = -\,dx \\ x^2 = (1-u)^2 \end{bmatrix}$

$\rightarrow\ -6\pi \int_1^0 (1-u)^2\sqrt{u}\,du$

$= -6\pi \int_1^0 \left(u^{1/2} - 2u^{3/2} + u^{5/2}\right) du$

$= -6\pi \left[\frac{2}{3}u^{3/2} - \frac{4}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right]_1^0 = 6\pi\left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7}\right)$

$= 6\pi\left(\frac{70-84+30}{105}\right) = 6\pi\left(\frac{16}{105}\right) = \frac{32\pi}{35}$

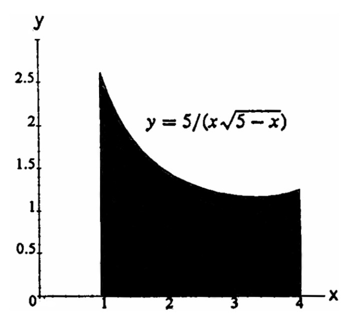


14. $V = \int_a^b \pi y^2\,dx = \pi \int_1^4 \frac{25\,dx}{x^2(5-x)}$

$= \pi \int_1^4 \left(\frac{dx}{x} + \frac{5\,dx}{x^2} + \frac{dx}{5-x}\right)$

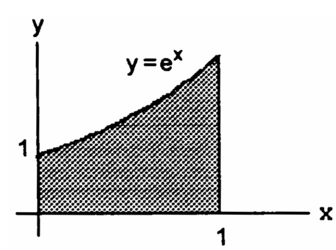
$= \pi\left[\ln\left|\frac{x}{5-x}\right| - \frac{5}{x}\right]_1^4 = \pi\left(\ln 4 - \frac{5}{4}\right) - \pi\left(\ln\frac{1}{4} - 5\right)$

$= \frac{15\pi}{4} + 2\pi \ln 4$

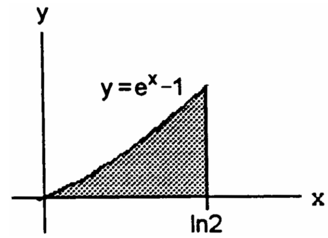


15. $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x e^x\,dx$

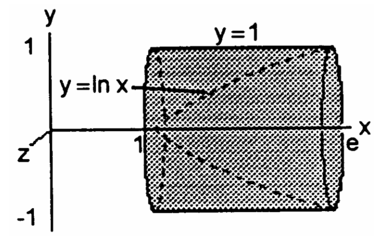
$= 2\pi\left[xe^x - e^x\right]_0^1 = 2\pi$



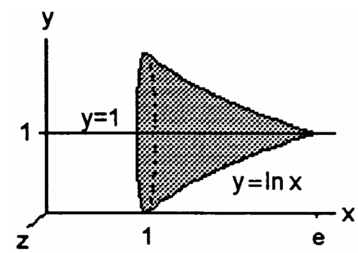
16. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x)(e^x - 1)\,dx$
$= 2\pi \int_0^{\ln 2} [(\ln 2)\,e^x - \ln 2 - xe^x + x]\,dx$
$= 2\pi \left[(\ln 2)\,e^x - (\ln 2)x - xe^x + e^x + \frac{x^2}{2}\right]_0^{\ln 2}$
$= 2\pi \left[2 \ln 2 - (\ln 2)^2 - 2 \ln 2 + 2 + \frac{(\ln 2)^2}{2}\right] - 2\pi(\ln 2 + 1)$
$= 2\pi \left[-\frac{(\ln 2)^2}{2} - \ln 2 + 1\right]$



17. (a) $V = \int_1^e \pi\,[1 - (\ln x)^2]\,dx$
$= \pi\,[x - x(\ln x)^2]_1^e + 2\pi \int_1^e \ln x\,dx$
(FORMULA 110)
$= \pi\,[x - x(\ln x)^2 + 2(x \ln x - x)]_1^e$
$= \pi\,[-x - x(\ln x)^2 + 2x \ln x]_1^e$
$= \pi\,[-e - e + 2e - (-1)] = \pi$



(b) $V = \int_1^e \pi(1 - \ln x)^2\,dx = \pi \int_1^e [1 - 2 \ln x + (\ln x)^2]\,dx$
$= \pi\,[x - 2(x \ln x - x) + x(\ln x)^2]_1^e - 2\pi \int_1^e \ln x\,dx$
$= \pi\,[x - 2(x \ln x - x) + x(\ln x)^2 - 2(x \ln x - x)]_1^e$
$= \pi\,[5x - 4x \ln x + x(\ln x)^2]_1^e$
$= \pi\,[(5e - 4e + e) - (5)] = \pi(2e - 5)$



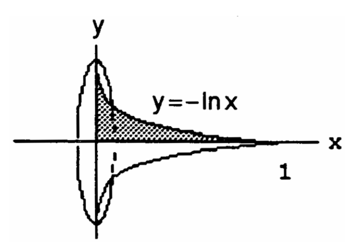
18. (a) $V = \pi \int_0^1 \left[(e^y)^2 - 1\right] dy = \pi \int_0^1 (e^{2y} - 1)\,dy = \pi \left[\frac{e^{2y}}{2} - y\right]_0^1 = \pi \left[\frac{e^2}{2} - 1 - \left(\frac{1}{2}\right)\right] = \frac{\pi\,(e^2 - 3)}{2}$

(b) $V = \pi \int_0^1 (e^y - 1)^2\,dy = \pi \int_0^1 (e^{2y} - 2e^y + 1)\,dy = \pi \left[\frac{e^{2y}}{2} - 2e^y + y\right]_0^1 = \pi \left[\left(\frac{e^2}{2} - 2e + 1\right) - \left(\frac{1}{2} - 2\right)\right]$
$= \pi \left(\frac{e^2}{2} - 2e + \frac{5}{2}\right) = \frac{\pi\,(e^2 - 4e + 5)}{2}$

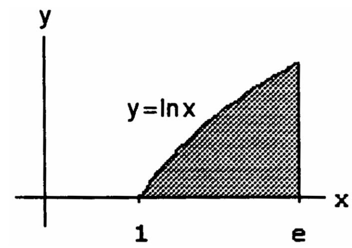
19. (a) $\lim_{x \to 0^+} x \ln x = 0 \Rightarrow \lim_{x \to 0^+} f(x) = 0 = f(0) \Rightarrow$ f is continuous

(b) $V = \int_0^2 \pi x^2 (\ln x)^2\,dx;\ \begin{bmatrix} u = (\ln x)^2 \\ du = (2 \ln x)\,\frac{dx}{x} \\ dv = x^2 dx \\ v = \frac{x^3}{3} \end{bmatrix} \rightarrow \pi \left(\lim_{b \to 0^+} \left[\frac{x^3}{3} (\ln x)^2\right]_b^2 - \int_0^2 \left(\frac{x^3}{3}\right)(2 \ln x)\,\frac{dx}{x}\right)$
$= \pi \left[\left(\frac{8}{3}\right)(\ln 2)^2 - \left(\frac{2}{3}\right) \lim_{b \to 0^+} \left[\frac{x^3}{3} \ln x - \frac{x^3}{9}\right]_b^2\right] = \pi \left[\frac{8(\ln 2)^2}{3} - \frac{16(\ln 2)}{9} + \frac{16}{27}\right]$

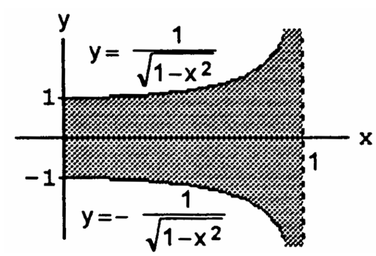
20. $V = \int_0^1 \pi(-\ln x)^2\,dx$
$= \pi \left(\lim_{b \to 0^+} [x(\ln x)^2]_b^1 - 2 \int_0^1 \ln x\,dx\right)$
$= -2\pi \lim_{b \to 0^+} [x \ln x - x]_b^1 = 2\pi$



21. $M = \int_1^e \ln x\, dx = [x \ln x - x]_1^e = (e - e) - (0 - 1) = 1;$
$M_x = \int_1^e (\ln x)\left(\frac{\ln x}{2}\right) dx = \frac{1}{2}\int_1^e (\ln x)^2\, dx$
$= \frac{1}{2}\left([x(\ln x)^2]_1^e - 2\int_1^e \ln x\, dx\right) = \frac{1}{2}(e - 2);$
$M_y = \int_1^e x \ln x\, dx = \left[\frac{x^2 \ln x}{2}\right]_1^e - \frac{1}{2}\int_1^e x\, dx$
$= \frac{1}{2}\left[x^2 \ln x - \frac{x^2}{2}\right]_1^e = \frac{1}{2}\left[\left(e^2 - \frac{e^2}{2}\right) + \frac{1}{2}\right] = \frac{1}{4}(e^2 + 1);$
therefore, $\overline{x} = \frac{M_y}{M} = \frac{e^2+1}{4}$ and $\overline{y} = \frac{M_x}{M} = \frac{e-2}{2}$



22. $M = \int_0^1 \frac{2\, dx}{\sqrt{1-x^2}} = 2\left[\sin^{-1} x\right]_0^1 = \pi;$
$M_y = \int_0^1 \frac{2x\, dx}{\sqrt{1-x^2}} = 2\left[-\sqrt{1-x^2}\right]_0^1 = 2;$
therefore, $\overline{x} = \frac{M_y}{M} = \frac{2}{\pi}$ and $\overline{y} = 0$ by symmetry



23. $L = \int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx = \int_1^e \frac{\sqrt{x^2+1}}{x}\, dx; \begin{bmatrix} x = \tan\theta \\ dx = \sec^2\theta\, d\theta \end{bmatrix} \to L = \int_{\pi/4}^{\tan^{-1} e} \frac{\sec\theta \cdot \sec^2\theta\, d\theta}{\tan\theta}$
$= \int_{\pi/4}^{\tan^{-1} e} \frac{(\sec\theta)(\tan^2\theta + 1)}{\tan\theta}\, d\theta = \int_{\pi/4}^{\tan^{-1} e} (\tan\theta \sec\theta + \csc\theta)\, d\theta = [\sec\theta - \ln|\csc\theta + \cot\theta|]_{\pi/4}^{\tan^{-1} e}$
$= \left(\sqrt{1+e^2} - \ln\left|\frac{\sqrt{1+e^2}}{e} + \frac{1}{e}\right|\right) - \left[\sqrt{2} - \ln\left(1 + \sqrt{2}\right)\right] = \sqrt{1+e^2} - \ln\left(\frac{\sqrt{1+e^2}}{e} + \frac{1}{e}\right) - \sqrt{2} + \ln\left(1 + \sqrt{2}\right)$

24. $y = \ln x \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + x^2 \Rightarrow S = 2\pi\int_c^d x\sqrt{1+x^2}\, dy \Rightarrow S = 2\pi\int_0^1 e^y\sqrt{1+e^{2y}}\, dy; \begin{bmatrix} u = e^y \\ du = e^y\, dy \end{bmatrix}$
$\to S = 2\pi\int_1^e \sqrt{1+u^2}\, du; \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{bmatrix} \to 2\pi\int_{\pi/4}^{\tan^{-1} e} \sec\theta \cdot \sec^2\theta\, d\theta$
$= 2\pi\left(\frac{1}{2}\right)[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]_{\pi/4}^{\tan^{-1} e} = \pi\left[\left(\sqrt{1+e^2}\right)e + \ln\left|\sqrt{1+e^2} + e\right|\right] - \pi\left[\sqrt{2}\cdot 1 + \ln\left(\sqrt{2} + 1\right)\right]$
$= \pi\left[e\sqrt{1+e^2} + \ln\left(\frac{\sqrt{1+e^2}+e}{\sqrt{2}+1}\right) - \sqrt{2}\right]$

25. $S = 2\pi\int_{-1}^1 f(x)\sqrt{1 + [f'(x)]^2}\, dx; f(x) = \left(1 - x^{2/3}\right)^{3/2} \Rightarrow [f'(x)]^2 + 1 = \frac{1}{x^{2/3}} \Rightarrow S = 2\pi\int_{-1}^1 \left(1 - x^{2/3}\right)^{3/2} \cdot \frac{dx}{\sqrt{x^{2/3}}}$
$= 4\pi\int_0^1 \left(1 - x^{2/3}\right)^{3/2}\left(\frac{1}{x^{1/3}}\right) dx; \begin{bmatrix} u = x^{2/3} \\ du = \frac{2}{3}\frac{dx}{x^{1/3}} \end{bmatrix} \to 4 \cdot \frac{3}{2}\pi\int_0^1 (1-u)^{3/2}\, du = -6\pi\int_0^1 (1-u)^{3/2}\, d(1-u)$
$= -6\pi \cdot \frac{2}{5}\left[(1-u)^{5/2}\right]_0^1 = \frac{12\pi}{5}$

26. $y = \int_1^x \sqrt{\sqrt{t} - 1}\, dt \Rightarrow \frac{dy}{dx} = \sqrt{\sqrt{x} - 1} \Rightarrow L = \int_1^{16}\sqrt{1 + \left(\sqrt{\sqrt{x} - 1}\right)^2}\, dx = \int_1^{16}\sqrt{1 + \sqrt{x} - 1}\, dx$
$= \int_1^{16} \sqrt[4]{x}\, dx = \left[\frac{4}{5}x^{5/4}\right]_1^{16} = \frac{4}{5}(16)^{5/4} - \frac{4}{5}(1)^{5/4} = \frac{124}{5}$

27. $\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right) dx = \lim_{b\to\infty}\int_1^b \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right) dx = \lim_{b\to\infty}\left[\frac{a}{2}\ln(x^2+1) - \frac{1}{2}\ln x\right]_1^b = \lim_{b\to\infty}\left[\frac{1}{2}\ln\frac{(x^2+1)^a}{x}\right]_1^b$
$= \lim_{b\to\infty}\frac{1}{2}\left[\ln\frac{(b^2+1)^a}{b} - \ln 2^a\right]; \lim_{b\to\infty}\frac{(b^2+1)^a}{b} > \lim_{b\to\infty}\frac{b^{2a}}{b} = \lim_{b\to\infty} b^{2\left(a - \frac{1}{2}\right)} = \infty$ if $a > \frac{1}{2} \Rightarrow$ the improper integral diverges if $a > \frac{1}{2}$; for $a = \frac{1}{2}$: $\lim_{b\to\infty}\frac{\sqrt{b^2+1}}{b} = \lim_{b\to\infty}\sqrt{1 + \frac{1}{b^2}} = 1 \Rightarrow \lim_{b\to\infty}\frac{1}{2}\left[\ln\frac{(b^2+1)^{1/2}}{b} - \ln 2^{1/2}\right]$

$= \frac{1}{2}\left(\ln 1 - \frac{1}{2}\ln 2\right) = -\frac{\ln 2}{4}$; if $a < \frac{1}{2}$: $0 \le \lim_{b\to\infty} \frac{(b^2+1)^a}{b} < \lim_{b\to\infty} \frac{(b+1)^{2a}}{b+1} = \lim_{b\to\infty} (b+1)^{2a-1} = 0$

$\Rightarrow \lim_{b\to\infty} \ln \frac{(b^2+1)^a}{b} = -\infty \Rightarrow$ the improper integral diverges if $a < \frac{1}{2}$; in summary, the improper integral $\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right) dx$ converges only when $a = \frac{1}{2}$ and has the value $-\frac{\ln 2}{4}$

28. $G(x) = \lim_{b\to\infty} \int_0^b e^{-xt}\, dt = \lim_{b\to\infty} \left[-\frac{1}{x} e^{-xt}\right]_0^b = \lim_{b\to\infty} \left(\frac{1-e^{-xb}}{x}\right) = \frac{1-0}{x} = \frac{1}{x}$ if $x > 0 \Rightarrow xG(x) = x\left(\frac{1}{x}\right) = 1$ if $x > 0$

29. $A = \int_1^\infty \frac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \le 1$. Thus, $p \le 1$ for infinite area. The volume of the solid of revolution about the x-axis is $V = \int_1^\infty \pi\left(\frac{1}{x^p}\right)^2 dx = \pi \int_1^\infty \frac{dx}{x^{2p}}$ which converges if $2p > 1$ and diverges if $2p \le 1$. Thus we want $p > \frac{1}{2}$ for finite volume. In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $\frac{1}{2} < p \le 1$.

30. The area is given by the integral $A = \int_0^1 \frac{dx}{x^p}$;

$p = 1$: $A = \lim_{b\to 0^+} [\ln x]_b^1 = -\lim_{b\to 0^+} \ln b = \infty$, diverges;

$p > 1$: $A = \lim_{b\to 0^+} [x^{1-p}]_b^1 = 1 - \lim_{b\to 0^+} b^{1-p} = -\infty$, diverges;

$p < 1$: $A = \lim_{b\to 0^+} [x^{1-p}]_b^1 = 1 - \lim_{b\to 0^+} b^{1-p} = 1 - 0$, converges; thus, $p \ge 1$ for infinite area.

The volume of the solid of revolution about the x-axis is $V_x = \pi\int_0^1 \frac{dx}{x^{2p}}$ which converges if $2p < 1$ or $p < \frac{1}{2}$, and diverges if $p \ge \frac{1}{2}$. Thus, V_x is infinite whenever the area is infinite $(p \ge 1)$.

The volume of the solid of revolution about the y-axis is $V_y = \pi \int_1^\infty [R(y)]^2\, dy = \pi \int_1^\infty \frac{dy}{y^{2/p}}$ which converges if $\frac{2}{p} > 1 \Leftrightarrow p < 2$ (see Exercise 29). In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $1 \le p < 2$, as described above.

31. (a) $\Gamma(1) = \int_0^\infty e^{-t}\, dt = \lim_{b\to\infty} \int_0^b e^{-t}\, dt = \lim_{b\to\infty} \left[-e^{-t}\right]_0^b = \lim_{b\to\infty} \left[-\frac{1}{e^b} - (-1)\right] = 0 + 1 = 1$

(b) $u = t^x$, $du = xt^{x-1}\, dt$; $dv = e^{-t}\, dt$, $v = -e^{-t}$; x = fixed positive real

$\Rightarrow \Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt = \lim_{b\to\infty} \left[-t^x e^{-t}\right]_0^b + x\int_0^\infty t^{x-1} e^{-t}\, dt = \lim_{b\to\infty} \left(-\frac{b^x}{e^b} + 0^x e^0\right) + x\Gamma(x) = x\Gamma(x)$

(c) $\Gamma(n+1) = n\Gamma(n) = n!$:

$n = 0$: $\Gamma(0+1) = \Gamma(1) = 0!$;

$n = k$: Assume $\Gamma(k+1) = k!$		for some $k > 0$;
$n = k+1$: $\Gamma(k+1+1)$	$= (k+1)\,\Gamma(k+1)$	from part (b)
	$= (k+1)k!$	induction hypothesis
	$= (k+1)!$	definition of factorial

Thus, $\Gamma(n+1) = n\Gamma(n) = n!$ for every positive integer n.

32. (a) $\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}$ and $n\Gamma(n) = n! \Rightarrow n! \approx n\left(\frac{n}{e}\right)^n \sqrt{\frac{2\pi}{n}} = \left(\frac{n}{e}\right)^n \sqrt{2n\pi}$

(b)

n	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}$	calculator
10	3598695.619	3628800
20	2.4227868×10^{18}	2.432902×10^{18}
30	2.6451710×10^{32}	2.652528×10^{32}
40	8.1421726×10^{47}	8.1591528×10^{47}
50	3.0363446×10^{64}	3.0414093×10^{64}
60	8.3094383×10^{81}	8.3209871×10^{81}

(c)

n	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}$	$\left(\frac{n}{e}\right)^n \sqrt{2n\pi}\, e^{1/12n}$	calculator
10	3598695.619	3628810.051	3628800

33.

e^{2x}	$(+)$	$\cos 3x$
$2e^{2x}$	$(-)$	$\frac{1}{3}\sin 3x$
$4e^{2x}$	$(+)$	$-\frac{1}{9}\cos 3x$

$I = \frac{e^{2x}}{3}\sin 3x + \frac{2e^{2x}}{9}\cos 3x - \frac{4}{9} I \Rightarrow \frac{13}{9} I = \frac{e^{2x}}{9}(3\sin 3x + 2\cos 3x) \Rightarrow I = \frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$

34.

e^{3x}	$(+)$	$\sin 4x$
$3e^{3x}$	$(-)$	$-\frac{1}{4}\cos 4x$
$9e^{3x}$	$(+)$	$-\frac{1}{16}\sin 4x$

$I = -\frac{e^{3x}}{4}\cos 4x + \frac{3e^{3x}}{16}\sin 4x - \frac{9}{16} I \Rightarrow \frac{25}{16} I = \frac{e^{3x}}{16}(3\sin 4x - 4\cos 4x) \Rightarrow I = \frac{e^{3x}}{25}(3\sin 4x - 4\cos 4x) + C$

35.

$\sin 3x$	$(+)$	$\sin x$
$3\cos 3x$	$(-)$	$-\cos x$
$-9\sin 3x$	$(+)$	$-\sin x$

$I = -\sin 3x \cos x + 3\cos 3x \sin x + 9I \Rightarrow -8I = -\sin 3x \cos x + 3\cos 3x \sin x$

$\Rightarrow I = \frac{\sin 3x \cos x - 3\cos 3x \sin x}{8} + C$

36.

$\cos 5x$	$(+)$	$\sin 4x$
$-\sin 5x$	$(-)$	$-\frac{1}{4}\cos 4x$
$-25\cos 5x$	$(+)$	$-\frac{1}{16}\sin 4$

$I = -\frac{1}{4}\cos 5x \cos 4x - \frac{5}{16}\sin 5x \sin 4x + \frac{25}{16} I \Rightarrow -\frac{9}{16} I = -\frac{1}{4}\cos 5x \cos 4x - \frac{5}{16}\sin 5x \sin 4x$

$\Rightarrow I = \frac{1}{9}(4\cos 5x \cos 4x + 5\sin 5x \sin 4x) + C$

37.

e^{ax}	$(+)$	$\sin bx$
ae^{ax}	$(-)$	$-\frac{1}{b}\cos bx$
a^2e^{ax}	$(+)$	$-\frac{1}{b^2}\sin bx$

$I = -\frac{e^{ax}}{b}\cos bx + \frac{ae^{ax}}{b^2}\sin bx - \frac{a^2}{b^2} I \Rightarrow \left(\frac{a^2+b^2}{b^2}\right) I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx)$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

38.
$$\begin{array}{lcl} e^{ax} & (+) & \cos bx \\ ae^{ax} & (-) & \frac{1}{b}\sin bx \\ a^2e^{ax} & (+) & -\frac{1}{b^2}\cos bx \end{array}$$

$I = \frac{e^{ax}}{b}\sin bx + \frac{ae^{ax}}{b^2}\cos bx - \frac{a^2}{b^2}I \Rightarrow \left(\frac{a^2+b^2}{b^2}\right)I = \frac{e^{ax}}{b^2}(a\cos bx + b\sin bx)$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

39.
$$\begin{array}{lcl} \ln(ax) & (+) & 1 \\ \frac{1}{x} & (-) & x \end{array}$$

$I = x\ln(ax) - \int \left(\frac{1}{x}\right) x\,dx = x\ln(ax) - x + C$

40.
$$\begin{array}{lcl} \ln(ax) & (+) & x^2 \\ \frac{1}{x} & (-) & \frac{1}{3}x^3 \end{array}$$

$I = \frac{1}{3}x^3\ln(ax) - \int\left(\frac{1}{x}\right)\left(\frac{x^3}{3}\right)dx = \frac{1}{3}x^3\ln(ax) - \frac{1}{9}x^3 + C$

41. $\int \frac{dx}{1-\sin x} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1-\left(\frac{2z}{1+z^2}\right)} = \int \frac{2\,dz}{(1-z)^2} = \frac{2}{1-z} + C = \frac{2}{1-\tan\left(\frac{x}{2}\right)} + C$

42. $\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1+\left(\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{1+z^2+2z+1-z^2} = \int \frac{dz}{1+z} = \ln|1+z| + C$

$= \ln\left|\tan\left(\frac{x}{2}\right)+1\right| + C$

43. $\int_0^{\pi/2} \frac{dx}{1+\sin x} = \int_0^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1+\left(\frac{2z}{1+z^2}\right)} = \int_0^1 \frac{2\,dz}{(1+z)^2} = -\left[\frac{2}{1+z}\right]_0^1 = -(1-2) = 1$

44. $\int_{\pi/3}^{\pi/2} \frac{dx}{1-\cos x} = \int_{1/\sqrt{3}}^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1-\left(\frac{1-z^2}{1+z^2}\right)} = \int_{1/\sqrt{3}}^1 \frac{dz}{z^2} = \left[-\frac{1}{z}\right]_{1/\sqrt{3}}^1 = \sqrt{3}-1$

45. $\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta} = \int_0^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{2+\left(\frac{1-z^2}{1+z^2}\right)} = \int_0^1 \frac{2\,dz}{2+2z^2+1-z^2} = \int_0^1 \frac{2\,dz}{z^2+3} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{z}{\sqrt{3}}\right]_0^1 = \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}$

$= \frac{\pi}{3\sqrt{3}} = \frac{\sqrt{3}\pi}{9}$

46. $\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta+\sin\theta} = \int_1^{\sqrt{3}} \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\,dz}{1+z^2}\right)}{\left[\frac{2z(1-z^2)}{(1+z^2)^2}+\left(\frac{2z}{1+z^2}\right)\right]} = \int_1^{\sqrt{3}} \frac{2(1-z^2)\,dz}{2z-2z^3+2z+2z^3} = \int_1^{\sqrt{3}} \frac{1-z^2}{2z}\,dz$

$= \left[\frac{1}{2}\ln z - \frac{z^2}{4}\right]_1^{\sqrt{3}} = \left(\frac{1}{2}\ln\sqrt{3}-\frac{3}{4}\right) - \left(0-\frac{1}{4}\right) = \frac{\ln 3}{4} - \frac{1}{2} = \frac{1}{4}(\ln 3 - 2) = \frac{1}{2}\left(\ln\sqrt{3}-1\right)$

47. $\int \frac{dt}{\sin t-\cos t} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2}-\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{2z-1+z^2} = \int \frac{2\,dz}{(z+1)^2-2} = \frac{1}{\sqrt{2}}\ln\left|\frac{z+1-\sqrt{2}}{z+1+\sqrt{2}}\right| + C$

$= \frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{t}{2}\right)+1-\sqrt{2}}{\tan\left(\frac{t}{2}\right)+1+\sqrt{2}}\right| + C$

48. $\int \frac{\cos t\, dt}{1-\cos t} = \int \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\, dz}{1+z^2}\right)}{1-\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\left(1-z^2\right) dz}{\left(1+z^2\right)^2-\left(1+z^2\right)\left(1-z^2\right)} = \int \frac{2\left(1-z^2\right) dz}{\left(1+z^2\right)\left(1+z^2-1+z^2\right)}$

$= \int \frac{\left(1-z^2\right) dz}{\left(1+z^2\right) z^2} = \int \frac{dz}{z^2\left(1+z^2\right)} - \int \frac{dz}{1+z^2} = \int \frac{dz}{z^2} - 2\int \frac{dz}{z^2+1} = -\frac{1}{z} - 2\tan^{-1} z + C = -\cot\left(\frac{t}{2}\right) - t + C$

49. $\int \sec\theta\, d\theta = \int \frac{d\theta}{\cos\theta} = \int \frac{\left(\frac{2\, dz}{1+z^2}\right)}{\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\, dz}{1-z^2} = \int \frac{2\, dz}{(1+z)(1-z)} = \int \frac{dz}{1+z} + \int \frac{dz}{1-z}$

$= \ln|1+z| - \ln|1-z| + C = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$

50. $\int \csc\theta\, d\theta = \int \frac{d\theta}{\sin\theta} = \int \frac{\left(\frac{2\, dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2}\right)} = \int \frac{dz}{z} = \ln|z| + C = \ln\left|\tan\frac{\theta}{2}\right| + C$