Ch 6: 6.1 Eigenvalues and Eigenvectors

Def. Let A be $n\times n$ matrix.

- A scalar λ is an eigenvalue (or characterstic value) of the matrix A if $\exists$ a vector $x \neq 0$ s.t $Ax = \lambda x$.
- The vector x is the corresponding eigenvector (or characterstic vector) of λ.

Exp 1 Find the eigenvalues and the corresponding eigenvectors of the matrix $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$

- To find the eigenvalues we solve $Ax = \lambda x$. That is,

$\boxed{(A - \lambda I)x = 0}$ is homogenuous system with $x \neq 0$

Hence, $A - \lambda I$ must be singular $\Rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = 0 \Leftrightarrow (3-\lambda)(-2-\lambda) - 6 = 0 \Leftrightarrow \lambda^2 - \lambda - 12 = 0$$
$$\Leftrightarrow (\lambda - 4)(\lambda + 3) = 0 \Leftrightarrow \lambda_1 = 4 \text{ and } \lambda_2 = -3$$

- To find the eigenvector corresponding to the eigenvalue $\lambda_1 = 4$

we solve $(A - \lambda_1 I)x = 0 \Leftrightarrow \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $-x_1 + 2x_2 = 0$, $x_1 = 2x_2$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_2 = \alpha \text{ and } x_1 = 2\alpha$$

Hence $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is basis for $N(A - 4I)$

Thus, any nonzero vector multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector associated with $\lambda_1 = 4$

- " " " " " " " " $\lambda_2 = -3$

" " $(A - \lambda_2 I)x = 0 \Leftrightarrow \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $3x_1 + x_2 = 0$, $x_2 = -3x_1$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_2 = \alpha \Rightarrow x_1 = -\frac{1}{3}\alpha$$

Hence, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}\alpha \\ \alpha \end{pmatrix} = \frac{\alpha}{3}\begin{pmatrix} -1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ is basis for $N(A + 3I)$

Thus, any nonzero vector multiple of $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ is an eignvector associated with $\lambda = -3$.

Remark Let A be $n\times n$ matrix and λ be a scalar.
The following statements are equivalent:

1. λ is an eigenvalue of A.
2. $(A-\lambda I)x=0$ has a nontrivial solution.
3. $N(A-\lambda I)\neq\{0\}$
4. The matrix $A-\lambda I$ is singular
5. $|A-\lambda I|=0$

Def The sum of the diagonal elements of $A_{n\times n}$ is called trace of A denoted by $tr(A)$. That is, $tr(A)=\sum_{i=1}^{n} a_{ii}$

Result If $\lambda_1,\lambda_2,\dots,\lambda_n$ are eigenvalues of $A_{n\times n}$, then

1. $(\lambda_1)(\lambda_2)\cdots(\lambda_n)=|A|$
2. $\sum_{i=1}^{n}\lambda_i = tr(A)$

• Note that in Exp[1] $\Rightarrow A=\begin{bmatrix}3 & 2\\ 3 & -2\end{bmatrix} \Rightarrow |A|=-12$, $\lambda_1=4$ and $\lambda_2=-3$

$\Rightarrow$ Hence, $(\lambda_1)(\lambda_2)=|A|$ and $\lambda_1+\lambda_2=tr(A)$

• We denote $p(\lambda)=|A-\lambda I|$ the characteristic polynomial

and $0=|A-\lambda I|$ the characteristic equation

Exp[2] Compute the eigenvalues and basis for the corresponding eigenspaces of $A=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}$

• The characteristic equation is $|A-\lambda I|=0$

$\begin{vmatrix}1-\lambda & 2\\ -2 & 1-\lambda\end{vmatrix}=(1-\lambda)^2+4=0 \Leftrightarrow$ the roots "eigenvalues"

are $\lambda_1=1+2i$ and $\lambda_2=1-2i$

• To find the eigenspace corresponding to $\lambda_1=1+2i \Longrightarrow$

$(A-\lambda_1 I)x=0 \Longrightarrow \begin{bmatrix}-2i & 2\\ -2 & -2i\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ $x_2=\alpha$, $x_1=-i\alpha$

$\Rightarrow \left[\begin{array}{cc|c}1 & i & 0\\ 0 & 0 & 0\end{array}\right] \Rightarrow x=\begin{pmatrix}-i\alpha\\ \alpha\end{pmatrix}=i\alpha\begin{pmatrix}1\\ i\end{pmatrix}$

$-2i\,x_1 + 2x_2 = 0 \iff x_2 = i\,x_1$

$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha i \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ i \end{pmatrix}$ is the eigenvector associated with $\lambda_1 = 1+2i$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} \right\}$ is basis for the eigenspace for $\lambda_1 = 1+2i$

- Similarly $x = \alpha \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is the eigenvector associated with $\lambda_2 = 1-2i$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$ is a basis for $N(A - \lambda_2 I)$.

Note that in Exp[2] $\Rightarrow tr(A) = 2 = \lambda_1 + \lambda_2$

$\Rightarrow |A| = 5 = (\lambda_1)(\lambda_2)$

Def The matrix B is similar to matrix A if there exists a nonsingular matrix S s.t $B = S^{-1} A S$.

Exp* Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $S = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

- Find similar matrix B to A.

$B = S^{-1} A S = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 6 & 6 \end{bmatrix}$

Th Let A and B be $n\times n$ matrices. If B is similar to A, then the two matrices have the same characteristic polynomial and hence, the same eigenvalues.

In Exp* above: B similar to A ➡ The eigenvalues of the diagonal matrix A are $\lambda_1 = 2$, $\lambda_2 = 3$

➡ The eigenvalues of B are the same $\lambda_1 = 2$ and $\lambda_2 = 3$ since

$|B - \lambda I| = 0 \iff \begin{vmatrix} -1-\lambda & -2 \\ 6 & 6-\lambda \end{vmatrix} = 0 \iff (-1-\lambda)(6-\lambda) + 12 = 0$

$\iff \lambda^2 - 5\lambda + 6 = 0 \iff (\lambda - 2)(\lambda - 3) = 0 \iff \lambda_1 = 2,\ \lambda_2 = 3$