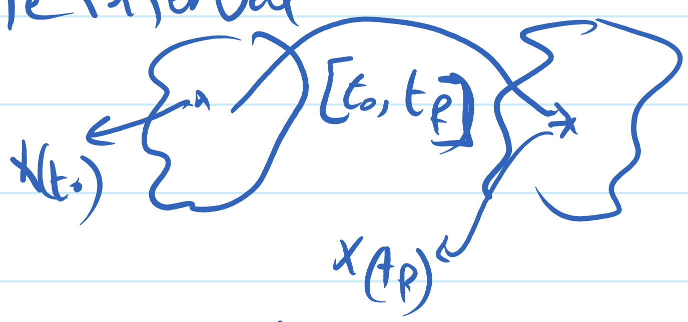
Controllability: A sys is said to be controllable at time t_0 if it is possible by means of unconstrained control vector $u(t)$ to transfer the sys. from initial state $x(t_0)$ to any other state $x(t_f)$ in finite interval



$x(t_0) = 0$

$x(t_f) = 0.1\ m$ ↗ state

* The sys is said completely controllable if and only if the vectors $B, AB, A^2B, \ldots, A^{n-1}B$ are linearly independent, or the controllability matrix (M) was full rank (n), where n # of states

$$M = [\,B \mid \underset{(n\times n)*(n\times r)\ \to\ n\times r}{AB} \mid \underset{(n\times n)(n\times n)\,*(n\times r)}{A^2B} \mid \cdots \mid A^{n-1}B\,]_{n\times n}$$

B vector: $n\times n$

B matrix: $n\times nr$

$4\times 4\times 2 \quad (4\times 8) \quad n=4,\ r=2$

$\det(M) \neq 0$ Fully state controllable

rank (M) , in matlab $M = \mathrm{ctrb}(A, B)$ ④

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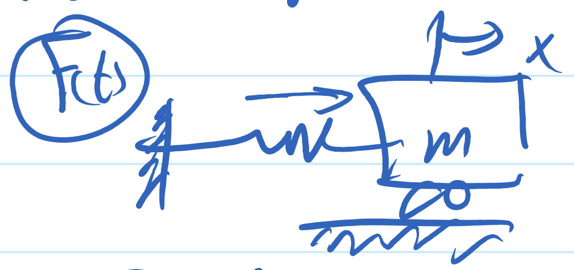
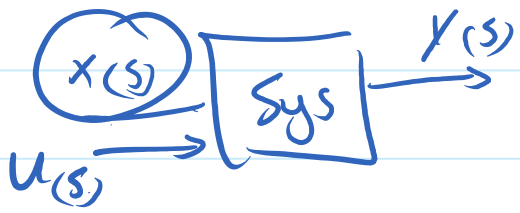
*** Observability :-** A Sys. is to be observable at time t_0 if with Sys in stable $X(t_0)$ it is possible to determine this state from the observation of the output over finite time





$$Z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \begin{array}{l} \leftarrow \text{measured} \\ \leftarrow (\times) \end{array}$$

* The Sys. is Completely state observable if and only if the row vectors $C, CA, CA^2, \ldots, CA^{n-1}$ are linearly independant or the rank of the observability matrix $(\mathcal{O})$ is full rank

$nP \times n$

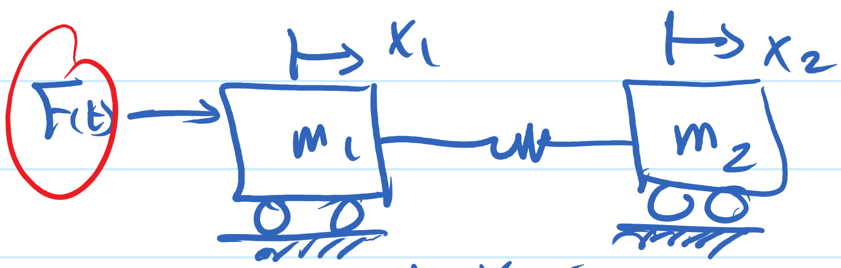
$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}_{nP \times n}$$

where n # of states

P # of outputs

$\det(\mathcal{O}) \neq 0$ fully state obsv.

$\text{rank}(\mathcal{O})$ ✓



$\rightarrow y = x_2 \quad (P = 1 \quad)$

$$Z = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} \begin{array}{l} \leftarrow \text{measured} \\ \leftarrow \times \\ \leftarrow \times \\ \leftarrow \times \end{array}$$

4 states

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Ex:- $\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ check the Controllability

2×2 $(n\times n)$ $n\times r$ $n=2$ $r=1$

$M = [B \; \cdots\cdots \; A^{n-1}B] = [B \quad AB]$

$= \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \; \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]$

2×1 2×1

$M = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \det(M) = 0$

The Sys. is not fully state Controllable.

Ex: $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$

$y = [4 \quad 5 \quad 1] x$ $n=3$

$p \times n$ $p=1$

$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \equiv \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$ for this example

$$O = \begin{array}{c} \left[\begin{array}{ccc} 4 & 5 & 1 \\ \hline \multicolumn{3}{c}{\begin{bmatrix} 4 & 5 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}} \\ \hline \multicolumn{3}{c}{CA * \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}} \end{array}\right] \end{array} \begin{array}{l} C \\ CA \\ CA^2 \end{array}$$

$$O = \begin{pmatrix} 4 & 5 & 1 \\ -6 & -7 & -1 \\ 6 & 5 & -1 \end{pmatrix}$$

in matlab

O = obsv(A, C)

rank(O) = 2

not fully state observable

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* Diagonalized Method:

$A \xrightarrow[\text{Transformation}]{\text{Similarity}} \bar{A}$ (diagonalized)

real number + no repeated

There are two Cases:

⊗ Case 1: all the eigenvalues are distinct or Complex eigenvalues

4 states

$\lambda_1 = -2+8j$
$\lambda_2 = -2-8j$
$\lambda_3 = -5$
$\lambda_4 = -10$

$$Q = [\vec{q}_1 \quad \vec{q}_2 \; ---- \; \vec{q}_n] \quad \text{or} \quad [\hat{\vec{q}}_1 \; ---- \; \hat{\vec{q}}_n]$$

eigenvectors — normalized eigenvectors

Q :- Similarity Transformation matrix

$$\bar{A} = Q^{-1} A Q = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{bmatrix}$$

diagonal form

The eigenvalues for A and $\bar{A}$ are similar

$(\lambda I - A) = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Ex: $\dot{x} = \begin{bmatrix} 6 & 2 \\ 4 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ find the diagonal form

$$|(\lambda I - A)| = 0 \Rightarrow \left| \begin{bmatrix} \lambda-6 & -2 \\ -4 & \lambda-1 \end{bmatrix} \right| = 0$$

$$\lambda^2 - \lambda - 6\lambda + 6 - 8 = 0 \Rightarrow \lambda^2 - 7\lambda - 2 = 0$$

$\lambda_1 = 7.2749$
$\lambda_2 = -0.2749$

real + no repeated (distinct)

$$Q = \begin{bmatrix} 0.8432 & -0.3037 \\ 0.5375 & 0.9528 \end{bmatrix}$$

$\hat{\vec{q}}_1 \qquad \hat{\vec{q}}_2$

Case 2:- if matrix involves multiple repeated eigen values. There for the diagonalized form is impossible in such case. but we will get Jordan matrix

Jordan Canonical matrix

states (n) 6: $\underbrace{\overset{-6}{\lambda_1}, \overset{-6}{\lambda_1}, \overset{-6}{\lambda_1}}_{\text{repeated}}, \underbrace{\overset{-5}{\lambda_4}, \overset{-5}{\lambda_4}}_{\text{repeated}}, \underbrace{\overset{-10}{\lambda_6} \cdots \overset{-12}{\lambda_n}}_{\text{distinct}}$

$Q = [q_1 \; q_2 \cdots q_n]$ eigenvectors

$$J = Q^{-1} A Q = \begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{matrix}} & & & & 0 \\ & \boxed{\begin{matrix} \lambda_4 & 1 \\ 0 & \lambda_4 \end{matrix}} & & & \\ & & \lambda_6 & & \\ & & & \ddots & \\ 0 & & & & \lambda_n \end{bmatrix}$$

3x3

Jordan block (J_1)

Jordan block (J_2)

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$$J = \begin{bmatrix} \boxed{\begin{matrix} -6 & 1 & 0 \\ 0 & -6 & 1 \\ 0 & 0 & -6 \end{matrix}} & & & & 0 \\ & \boxed{\begin{matrix} -5 & 1 \\ 0 & -5 \end{matrix}} & & & \\ & & -10 & & \\ & & & \ddots & \\ 0 & & & & -12 \end{bmatrix}$$