Electromagnetic Theory I

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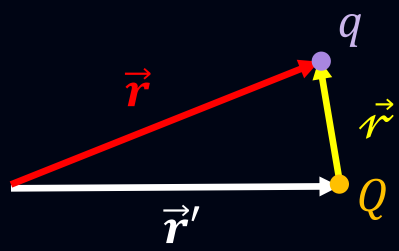
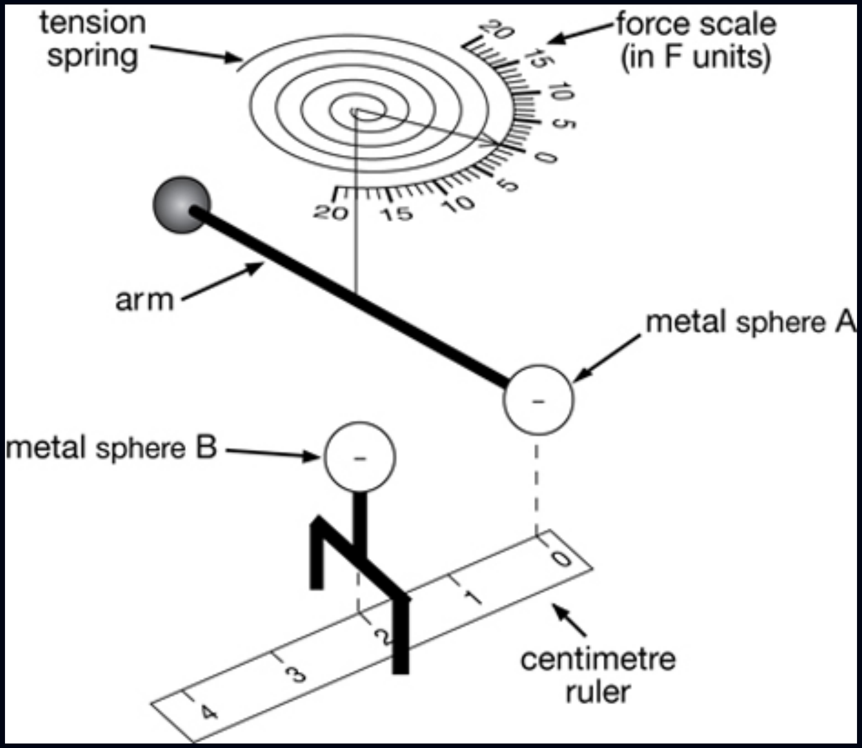
Chapter 2: Electrostatics

- ✓ The Electric Field
- ✓ Divergence and Curl of E
- ✓ Electric Potential
- ✓ Work and Energy in Electrostatics
- ✓ Conductors

Coulomb's Law

Experimental law (1785)





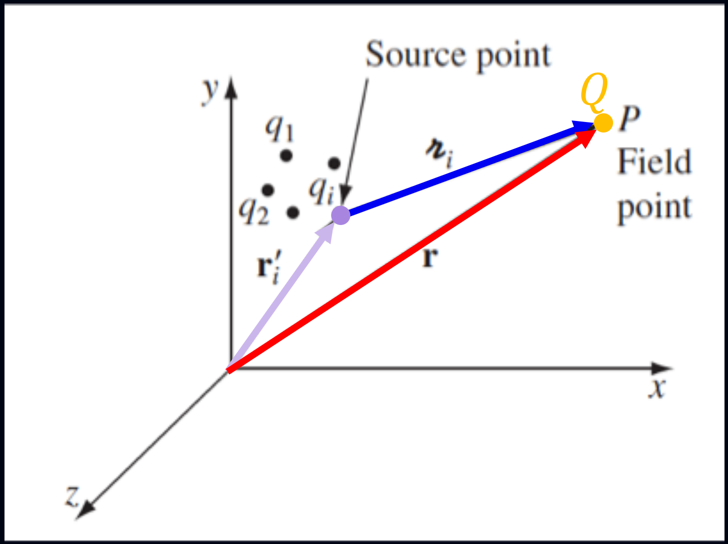
$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}^2} \hat{\mathscr{r}}$$

$\epsilon_0 = 8.85\times10^{-12} \frac{C^2}{N \cdot m^2}$: permittivity of free space

Electric Field

The principle of superposition

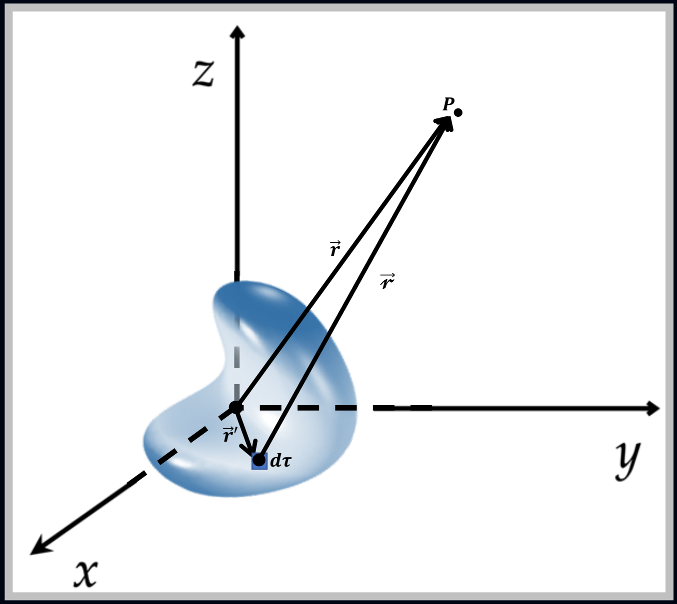
$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{1}{4\pi\epsilon_0}\frac{q_2 Q}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + \cdots + \frac{1}{4\pi\epsilon_0}\frac{q_N Q}{\mathscr{r}_N^2}\hat{\mathscr{r}}_N$$
$$= Q\left(\frac{1}{4\pi\epsilon_0}\frac{q_1}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{1}{4\pi\epsilon_0}\frac{q_2}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + \cdots + \frac{1}{4\pi\epsilon_0}\frac{q_N}{\mathscr{r}_N^2}\hat{\mathscr{r}}_N\right)$$
$$= Q\vec{E}$$



$$\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i$$

Continuous Charge Distribution

The electric field of a continuous charge distribution



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,dq$$

For a volume, surface, or line charge:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau' \,,\ \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,da' \,,\ \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,dl'$$

Continuous Charge Distribution

Example: find the electric field at distance z from the center of a spherical surface of radius R which carries a uniform charge distribution σ

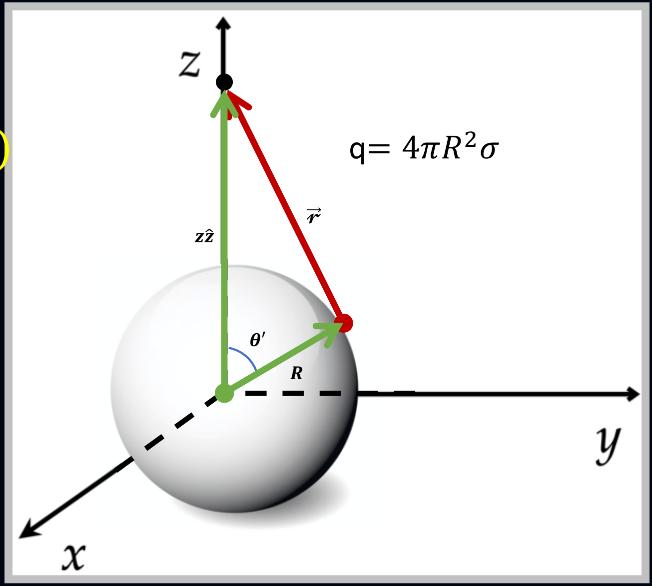
$$\vec{r} = z\hat{z}$$

$$\vec{r}\,' = R\hat{r}' = R(\sin\theta'\cos\phi'\ \hat{x} + \sin\theta'\sin\phi'\,\hat{y} + \cos\theta'\ \hat{z})$$

$$\vec{\mathscr{r}} = (-R\sin\theta'\cos\phi'\ \hat{x} - R\sin\theta'\sin\phi'\,\hat{y} + (z - R\cos\theta')\ \hat{z})$$

$$\mathscr{r} = \sqrt{R^2 + z^2 - 2Rz\cos\theta'}$$

$$da' = R^2\sin\theta'\,d\theta'\,d\phi';$$



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\,da'$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi} \frac{\sigma(-R\sin\theta'\cos\phi'\,\hat{x} - R\sin\theta'\sin\phi'\,\hat{y} + (z - R\cos\theta')\ \hat{z})}{(R^2 + z^2 - 2Rz\cos\theta')^{\frac{3}{2}}} R^2\sin\theta'\,d\theta'\,d\phi'$$

Continuous Charge Distribution

$$\vec{E}(\vec{r}) = \frac{\sigma R^2}{2\epsilon_0}\int_0^{\pi} \frac{(z - R\cos\theta')\,\hat{z}}{(R^2 + z^2 - 2Rz\cos\theta')^{\frac{3}{2}}} sin\theta' d\theta'$$

$$\vec{E}(\vec{r}) = \frac{\sigma R^2}{2\epsilon_0}\hat{z}\int_{-1}^{1} \frac{(z - Ru)}{(R^2 + z^2 - 2Rzu)^{\frac{3}{2}}} du$$

$$\vec{E}(\vec{r}) = \frac{\sigma R^2}{2\epsilon_0}\left(\frac{1}{z^2}\frac{(zu - R)}{\sqrt{(R^2 + z^2 - 2Rzu)}}\bigg|_{-1}^{1}\right)\hat{z}$$

$$\vec{E}(\vec{r}) = \frac{\sigma}{2\epsilon_0}\frac{R^2}{z^2}\left(\frac{(z - R)}{|z - R|} + \frac{(z + R)}{|z + R|}\right)\hat{z}$$

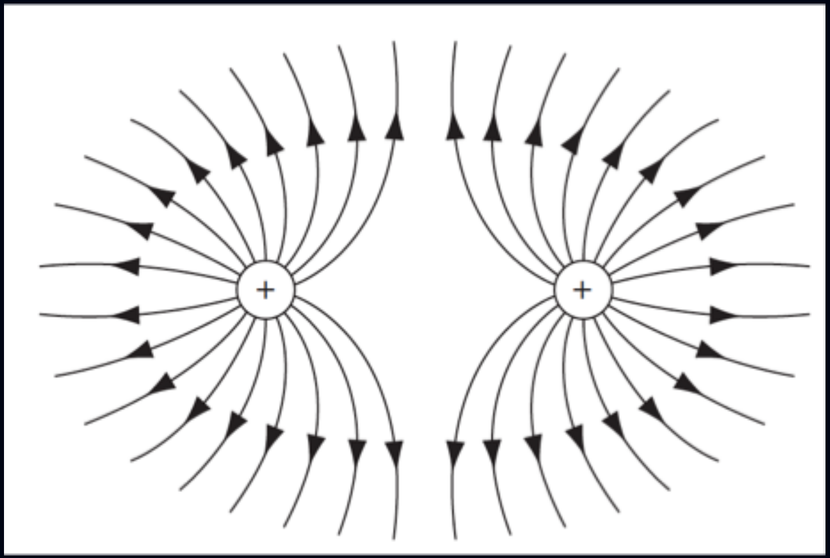
$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{\sigma}{\epsilon_0}\dfrac{R^2}{z^2}\hat{z} = \dfrac{q}{4\pi\epsilon_0 z^2}\hat{z} & z > R \\ 0 & z < R \end{cases}$$

Field Lines, Flux and Gauss's Law

Field lines:

The magnitude of the field is indicated by the density of the field lines:

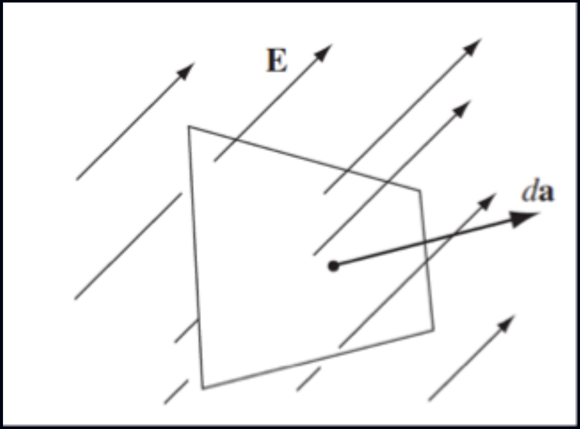
- it's strong near the center where the field lines are close together,
- it's weak farther out, where they are relatively far apart.
- The filed lines begin on positive charges and end on negative ones;
- they cannot simply terminate in midair, though they may extend out to infinity;
- they can never cross.



Field Lines, Flux and Gauss's Law

Field flux: Number of field lines passing through surface

$$\Phi_E = \int \vec{E} \cdot d\vec{a}$$



- $(\vec{E} \cdot d\vec{a})$ is proportional to the number of lines passing through the infinitesimal area $d\vec{a}$.
- The dot product picks out the component of $d\vec{a}$ along the direction of $\vec{E}$,
- It is only the area in the plane perpendicular to $\vec{E}$.
- The number of field lines is proportional to the magnitude of a charge.
- Therefore, "the flux through any closed surface is a measure of the total charge inside". This is the essence of Gauss's Law.

Flux and Gauss's Law

In the case of a point charge q at the origin, the flux of $\vec{E}$ through a sphere of radius r is

$$\Phi_E = \oint \vec{E} \cdot d\vec{a} = \oint \frac{q}{4\pi\epsilon_0 r^2}\hat{r} \cdot r^2 \sin\theta \, d\theta d\phi \hat{r} = \frac{q}{\epsilon_0}$$

Now suppose a collection of charges. According to the principle of superposition, the total field is the (vector) sum of all the individual fields:

$$\Phi_E = \oint \vec{E} \cdot d\vec{a} = \oint \left(\vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_N\right) \cdot d\vec{a} = \frac{q_1}{\epsilon_0} + \frac{q_2}{\epsilon_0} + \cdots + \frac{q_N}{\epsilon_0} = \frac{Q_{enc}}{\epsilon_0}$$

Gauss's Law $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

Flux and Gauss's Law

$$\int_V \vec{\nabla}\cdot\vec{E}\,d\tau = \oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_V \rho(\vec{r})\,d\tau$$

Gauss's Law in differential form $\vec{\nabla}\cdot\vec{E} = \rho(\vec{r})/\epsilon_0$

In general, we can calculate the divergence of $\vec{E}$ directly from the Coulomb' s Law of

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau' \rightarrow \vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau' = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{r}\,')\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

$$\rightarrow \vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{r}\,')4\pi\delta(\vec{r}-\vec{r}\,')d\tau' = \rho(\vec{r})/\epsilon_0$$