

3.5 Derivatives of Trigonometric Functions

62

1) If $f(x) = \sin x$, Then $f'(x) = \cos x$

Proof:
$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \left(\sin x \frac{\cos h - 1}{h} \right) + \lim_{h \rightarrow 0} \left(\cos x \frac{\sin h}{h} \right) \\ &= \sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= \sin x (0) + \cos x (1) = \cos x \end{aligned}$$

Example: Find y' for the following functions:

a) $y = 3x - \sin x$ $y' = 3 - \cos x$

b) $y = 3x \sin x$ $y' = 3x \cos x + 3 \sin x$

c) $y = \frac{\sin x}{3x}$ $y' = \frac{3x \cos x - 3 \sin x}{9x^2}$

2) If $f(x) = \cos x$, Then $f'(x) = -\sin x$

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Proof:
$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h} \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \left(\cos x \frac{\cosh - 1}{h} \right) - \lim_{h \rightarrow 0} \left(\sin x \frac{\sinh}{h} \right) \\
 &= \cos x \lim_{h \rightarrow 0} \frac{\cosh - 1}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sinh}{h} \\
 &= \cos x (0) - \sin x (1) = -\sin x
 \end{aligned}$$

Example: Find $\dot{y}$ for the following functions:

[a] $y = 1 - \sin x + \cos x \Rightarrow \dot{y} = -\cos x - \sin x$

[b] $y = (1 - \sin x) \cos x \Rightarrow \dot{y} = \sin x (\sin x - 1) - \cos x \cos x$
 $= \sin^2 x - \sin x - \cos^2 x$

[c] $y = \frac{\cos x}{1 - \sin x} \Rightarrow \dot{y} = \frac{-(1 - \sin x) \sin x + \cos x \cos x}{(1 - \sin x)^2}$
 $= \frac{\sin^2 x - \sin x + \cos^2 x}{(1 - \sin x)^2} = \frac{1 - \sin x}{(1 - \sin x)^2}$
 $= \frac{1}{1 - \sin x}$

Simple Harmonic Motion: الحركة التوافقية البسيطة

Example: A weight hanging from a spring is stretched down 3 units under its rest position and released at time $t = 0$. Its position at any time later on is

$$s = 3 \cos t$$

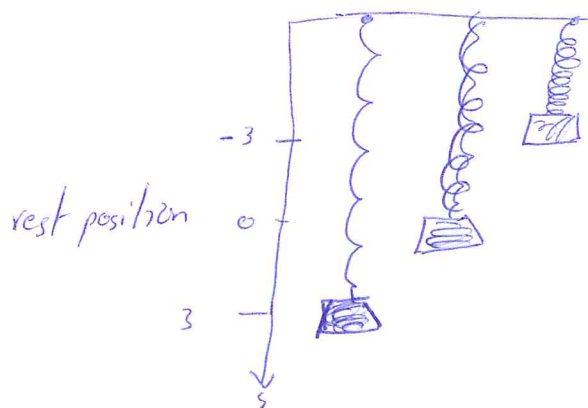
Find its velocity and acceleration at time t ?
 jerk

Position: $s = 3 \cos t$

Velocity $v = \frac{ds}{dt} = -3 \sin t$

Acceleration $a = \frac{dv}{dt} = -3 \cos t$

Jerk $j = \frac{da}{dt} = 3 \sin t$



⇒ As time passes, the weight moves down and up between $s = -3$ and $s = 3$ on s -axis.

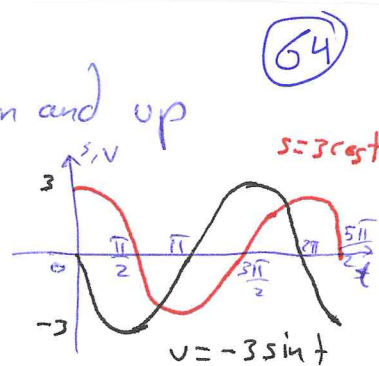
• The amplitude of the motion is 3

• The period of the motion is 2π

• $v = -3 \sin t$ gets greatest magnitude when $\cos t = 0$

• The speed $= |v| = 3 |\sin t|$

• The acceleration is always the exact opposite of the position value. When weight is above the rest position, gravity is pulling it back down, when the weight is below the rest position, the spring is pulling it back up.



[3] If $f(x) = \tan x$, then $f'(x) = \sec^2 x$.

Proof $f(x) = \tan x = \frac{\sin x}{\cos x}$

$$f'(x) = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

[4] If $f(x) = \cot x$, then $f'(x) = -\csc^2 x$

Proof $f(x) = \cot x = \frac{1}{\tan x}$

$$f'(x) = -\frac{\sec^2 x}{\tan^2 x} = -\frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

[5] If $f(x) = \sec x$, then $f'(x) = \sec x \tan x$

Proof $f(x) = \frac{1}{\cos x}$

$$f'(x) = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \frac{\sin x}{\cos x} = \sec x \tan x$$

(6) If $f(x) = \csc x$, then $f'(x) = -\csc x \cot x$ (65)

Proof: $f(x) = \frac{1}{\sin x}$

$$f'(x) = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x} \frac{\cos x}{\sin x} = -\csc x \cot x$$

Example: Let $y = \sec x \tan x$. Find y'

$$\begin{aligned} y' &= \sec x \sec^2 x + \tan x \sec x \tan x \\ &= \sec^3 x + \sec x \tan^2 x \end{aligned}$$

Example: Find $\lim_{x \rightarrow 0} \frac{\sqrt{2+\sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2+\sec 0}}{\cos(\pi - \tan 0)}$

$$= \frac{\sqrt{2+1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{\cos \pi} = -\sqrt{3}$$