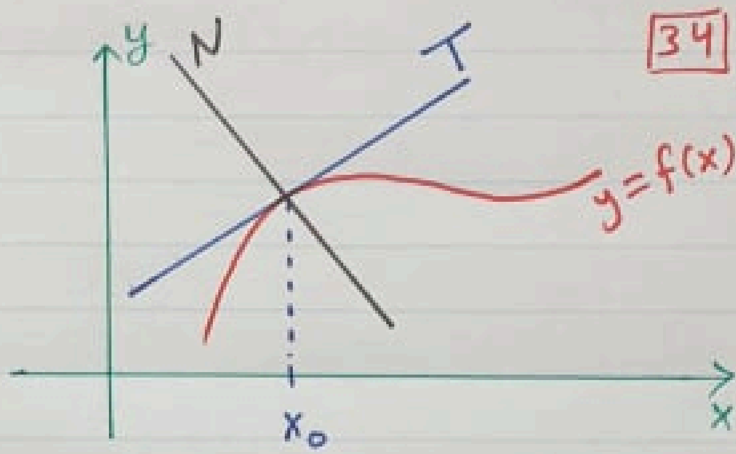
T : Tangent at x_0

$$y - y_0 = m_1 (x - x_0)$$

$$m_1 = f'(x_0)$$



N : Normal at x_0

$$y - y_0 = m_2 (x - x_0)$$

$$m_2 = \frac{-1}{m_1}$$

Exp Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

$$\frac{dy}{dx} = 6x^2 - 6x - 12 = 0 \qquad T \parallel x\text{-axis}$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2 \quad \text{or} \quad x = -1$$

when $x = 2 \Rightarrow y = 2(8) - 3(4) - 12(2) + 20 = 0$

$x = -1 \Rightarrow y = 2(-1) - 3(1) + 12 + 20 = 27$

The points are $(2, 0)$ and $(-1, 27)$

Exp Find slope at $x=0$ for $f(x) = \sec x \tan x$

slope at $x=0$ is $f'(0)$

$$f'(x) = \sec x \sec^2 x + \tan x \sec x \tan x$$

$$= \sec^3 x + \sec x \tan^2 x$$

$$f'(0) = (\sec 0)^3 + \sec(0)(\tan 0)^2 = (1)^3 + (1)(0) = 1 + 0 = 1$$

Implicit Differentiation

Exp Find $\frac{dy}{dx}$ for ① $x^2+y^2=5$

$$2x+2y\,y'=0$$

$$y'=\frac{dy}{dx}=-\frac{x}{y}$$

② $xy=\cos(xy)$

$$xy'+y=-\sin(xy)\left[xy'+y\right]$$

$$xy'+xy'\sin(xy)=-y-y\sin(xy)$$

$$xy'\left(1+\sin(xy)\right)=-y\left(1+\sin(xy)\right)$$

$$xy'=-y \quad\Rightarrow\quad y'=\frac{dy}{dx}=-\frac{y}{x}$$

Exp Find normal line to the curve $xy+2x-y=0$ that is parallel to the line $2x+y=0$

- The line $2x+y=0$ has slope $m_1=-2$ since $y=-2x$
- The normal line is parallel to $y=-2x$ so it hase slope m_1
- The curve $xy+2x-y=0$ has slope $m_2=\frac{-1}{m_1}=\frac{1}{2}$ at point of intersection of $y=-2x$ with curve.

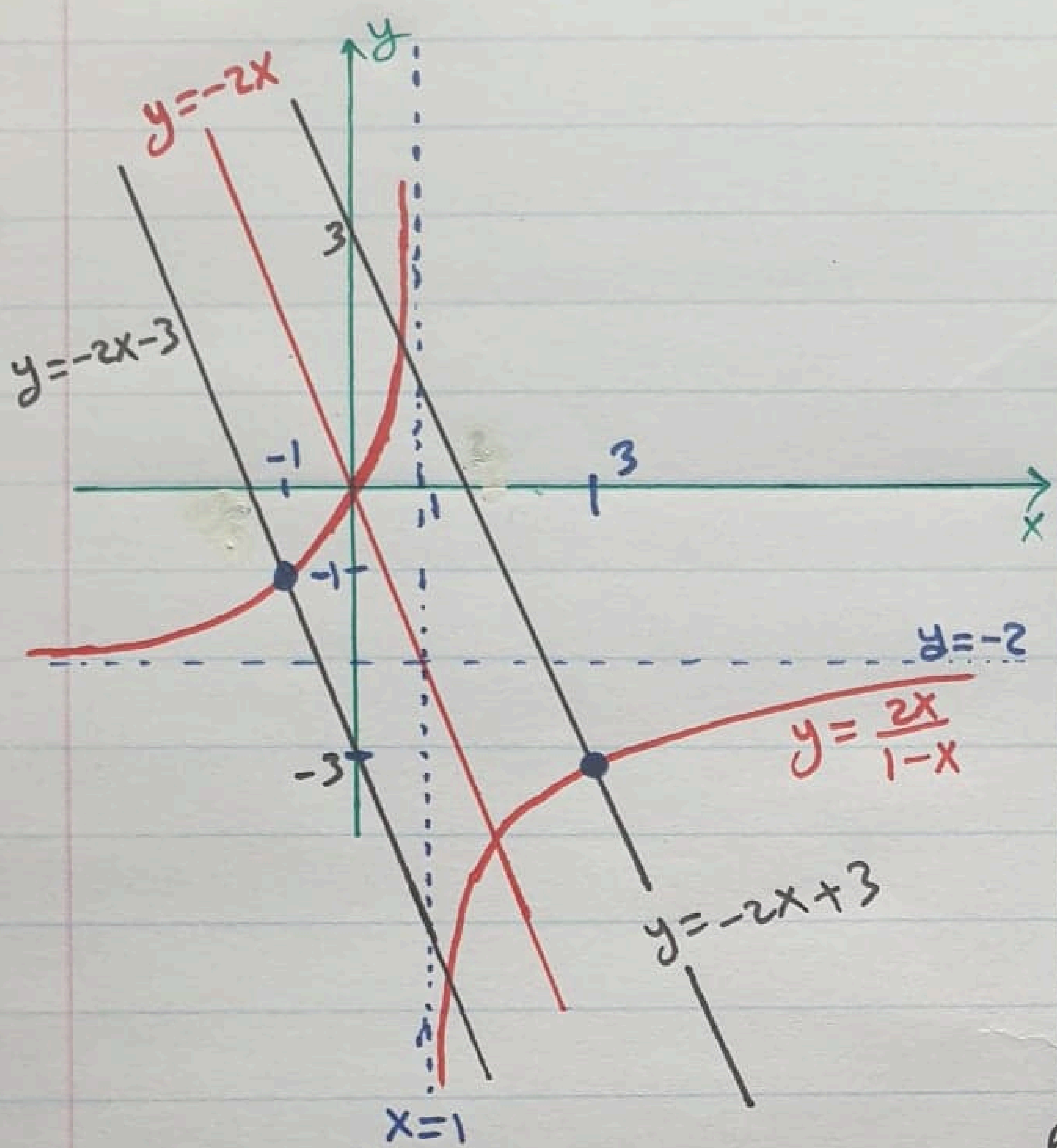
$$xy'+y+2-y'=0 \quad\Rightarrow\quad y'(x-1)=-y-2$$

$$y'=\frac{dy}{dx}=\frac{-y-2}{x-1}=\frac{-(y+2)}{-(1-x)}=\frac{y+2}{1-x}=\frac{1}{2}$$

$$2(y+2)=1-x \quad\Rightarrow 2y+4=1-x \quad\Rightarrow x=-2y-3 \quad ①$$

substitue ① in the curve to find the points of intersection the line with curve:

$xy + 2x - y = 0 \Rightarrow (-2y-3)y + 2(-2y-3) - y = 0$



$-2y^2 - 3y - 4y - 6 - y = 0$

$-2y^2 - 8y - 6 = 0$

$y^2 + 4y + 3 = 0$

$(y+1)(y+3) = 0$

$y = -1$ or $y = -3$

$\Downarrow$ from ① $\Downarrow$

$x = -2(-1) - 3 = -1$

$\Downarrow$ from ① $\Downarrow$

$x = -2(-3) - 3 = 3$

Points $(-1,-1)$, $(3,-3)$

Point $(-1,-1)$

Normal line $y - y_0 = m_1 (x - x_0)$

$y - -1 = -2(x - -1)$

$y + 1 = -2(x+1)$

$y = -2x - 3$

Point $(3,-3)$

Normal line $y - y_0 = m_1 (x - x_0)$

$y - -3 = -2(x-3)$

$y + 3 = -2x + 6$

$y = -2x + 3$

Curve: $xy + 2x - y = 0$

$y(x-1) = -2x$

$y = \frac{-2x}{x-1}$

$y = \frac{2x}{1-x}$

$x=1$ is V. Asy.

$y=-2$ is H. Asy.

Linearization

- The best linear function that approximates the diff curve $f(x)$ near $x=a$ is its tangent line given by

$$L(x) = f(a) + f'(a)(x-a)$$

$y - y_0 = m(x-x_0)$
$y - f(a) = f'(a)(x-a)$
$y = f(a) + f'(a)(x-a)$

- $L(x)$ is called Linearization of $f(x)$ at $x=a$
- The approximation $f(x) \approx L(x)$ is called the standard linear approximation of f at $x=a$

Exp Consider the function $f(x) = \sqrt{1+x}$

1. Find linearization of f at $x=0$

$L(x) = f(0) + f'(0)(x-0)$ $\qquad f'(x) = \frac{1}{2}\frac{1}{\sqrt{1+x}}$

$\boxed{L(x) = 1 + \frac{1}{2}x}$ $\qquad f'(0) = \frac{1}{2}$

2. Use linearization to estimate $f(0.2)$ and $f(0.04)$

$$f(0.2) \approx L(0.2) = 1 + \frac{0.2}{2} = 1 + 0.1 = 1.1$$

$$f(0.04) \approx L(0.04) = 1 + \frac{0.04}{2} = 1 + 0.02 = 1.02$$

3. Find the true values of $f(0.2)$ and $f(0.04)$

$$f(0.2) = \sqrt{1+0.2} = \sqrt{1.2} = 1.095445115$$

$$f(0.04) = \sqrt{1+0.04} = \sqrt{1.04} = 1.019803902$$

4. Find Error and compare

Error = |True value − Estimated Value|

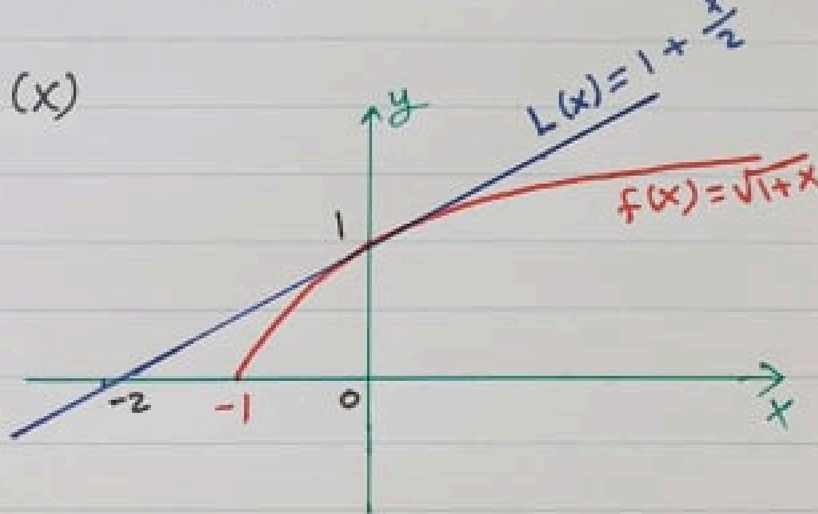
$E_1 = |f(0.2) - L(0.2)| \approx 0.00455 < 0.005 = 5 \times 10^{-3}$

$E_2 = |f(0.04) - L(0.04)| \approx 0.000196 < 0.0002 = 2 \times 10^{-4}$

As $x \longrightarrow 0 \Rightarrow$ Error decreases

This means $L(x)$ gives better approximation

(5) sketch $f(x)$ and $L(x)$



Exp Find linearization of $f(x) = \cos x + 1$ at $x = \frac{\pi}{3}$

$L(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right)$

$= \left(\cos\frac{\pi}{3} + 1\right) + \left(\frac{-\sqrt{3}}{2}\right)\left(x - \frac{\pi}{3}\right)$

$= \left(\frac{1}{2} + 1\right) - \frac{\sqrt{3}}{2}x + \frac{\sqrt{3}\,\pi}{6}$

$= \frac{3}{2} - \frac{\sqrt{3}}{2}x + \frac{\sqrt{3}\,\pi}{6}$

$f'(x) = -\sin x$

$f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3}$

$= -\frac{\sqrt{3}}{2}$

$L(x) = \left(\frac{3}{2} + \frac{\sqrt{3}\,\pi}{6}\right) - \frac{\sqrt{3}}{2}x$

Remark: $f(x) = (1+x)^k \Rightarrow L(x) = 1 + kx$ only near $x = 0$

where $k \in \mathbb{R}$

Differential

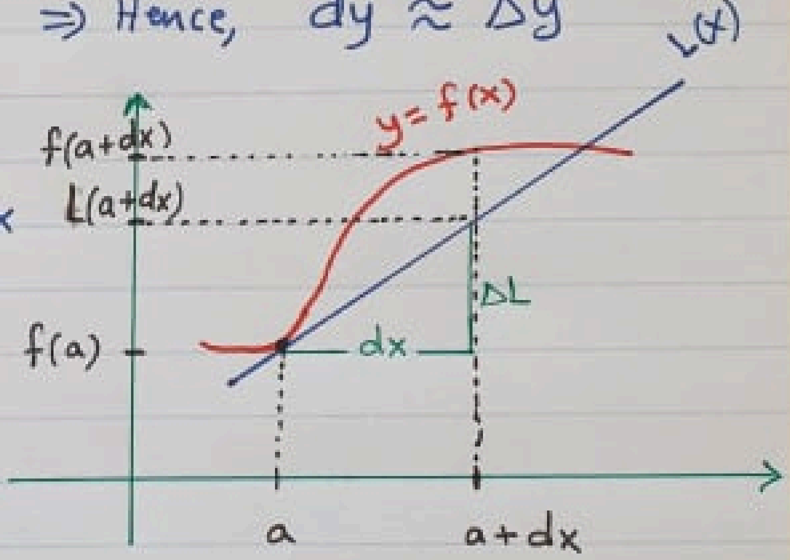
If $y=f(x)$ is diff at $x=a$, then $\frac{dy}{dx} = f'(x)\Big|_{x=a} = f'(a)$

$$dy = f'(a)\,dx$$

dy: differential or "estimated value"

$dx = \Delta x$
$= x_2 - x_1$

- $\Delta y = \Delta f = f(x_2) - f(x_1) = y_2 - y_1$ is True Value
- we will see $dy = \Delta L \Rightarrow$ Hence, $dy \approx \Delta y$
- Suppose we move from $x_1 = a$ to nearby point $x_2 = a + dx$



- True change in $y=f(x)$ is

$$\Delta y = \Delta f = f(x_2) - f(x_1)$$
$$= f(a+dx) - f(a)$$

$\Delta x = x_2 - x_1$
$= a + dx - a$
$= dx$

- Estimated value "differential" is

$$dy = f'(a)\,dx$$
$$= \frac{\Delta L}{\cancel{\Delta x}}\,\cancel{dx}$$
$$dy = \Delta L$$

Hence, we use differential dy to estimate the true change Δy

- Error $= |\text{True value} - \text{Estimated value}|$

Exp Find the differential of

① $y = x^3 + 2x$ at $x = 1$ if $dx = \frac{1}{5}$

$dy = f'(a)\,dx$ $\qquad y' = 3x^2 + 2$

$= f'(1)\left(\frac{1}{5}\right)$ $\qquad y'(1) = 3(1)^2 + 2$

$= (5)\left(\frac{1}{5}\right)$ $\qquad = 3 + 2$

$= 1$ $\qquad = 5$

② $y = \tan(2x)$ at $x = 0$ if $dx = \frac{1}{4}$

$dy = f'(0)\,dx$ $\qquad y' = 2\sec^2(2x)$

$= (2)\left(\frac{1}{4}\right)$ $\qquad y'(0) = 2\sec^2(0)$

$= \frac{1}{2}$ $\qquad = (2)(1)$

$= 2$

Exp Find the error in Exp ①

- Estimated value is $dy = 1$
- True value is $\Delta y = f(x_2) - f(x_1)$

$= f(1.2) - f(1)$

$= 4.128 - 3$

$= 1.128$

- Error $= |\text{True Value} - \text{Estimated Value}|$

$= |\Delta y - dy|$

$= |1.128 - 1|$

$= 0.128$

$x_1 = 1 = a$

$x_2 = a + dx$

$= 1 + \frac{1}{5}$

$= \frac{6}{5}$

$= 1.2$

$f(1) = 1^3 + 2(1) = 1 + 2 = 3$

$f(1.2) = (1.2)^3 + 2(1.2)$

$= 1.728 + 2.4$

$= 4.128$

Exp The radius r of circle increases from 10 cm to 10.1 cm.

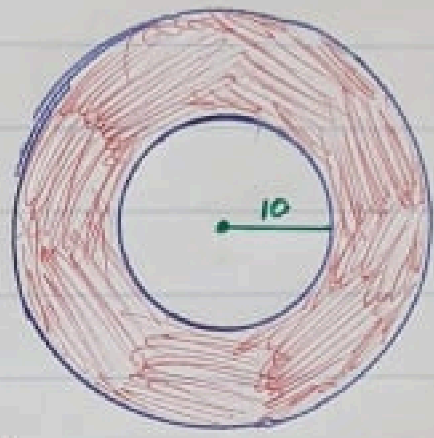
① Estimate the change in the circle's area.

$A = r^2 \pi$

$dA = 2r\pi \; dr$

$= 2(10)\pi \; (0.1)$

$= 2\pi$



② Find the true change in the area.

$r_1 = 10$

$r_2 = 10.1$

$dr = r_2 - r_1$

$= 10.1 - 10$

$= 0.1$

$\Delta A = A_2 - A_1$

$= A(r_2) - A(r_1)$

$= A(10.1) - A(10)$

$= (10.1)^2\pi - (10)^2\pi$

$= 102.01\pi - 100\pi$

$= 2.01\pi$

③ Find the error

$\text{Error} = |\Delta A - dA| = |2.01\pi - 2\pi| = 0.01\pi$

④ Estimate the area of the large circl.

$A(r_1) + dA = 100\pi + 2\pi = 102\pi$