4.8 Exercises.

Q95: if $\bar{x}$ is the mean of a Random sample of size n from a Normal dist. with mean μ and variance 100, find n so that $Pr(\mu-5 < \bar{x} < \mu+5) = 0.954$.

$\bar{x} \sim N(\mu, \frac{100}{n})$

$Pr(\mu-5 < \bar{x} < \mu+5) = \phi\left(\frac{\cancel{\mu}+5-\cancel{\mu}}{\sqrt{\frac{100}{n}}}\right) - \phi\left(\frac{\mu-5-\mu}{\sqrt{\frac{100}{n}}}\right)$

$0.954 = \phi\left(\frac{5}{\frac{10}{\sqrt{n}}}\right) - \phi\left(\frac{-5}{\frac{10}{\sqrt{n}}}\right)$

$0.954 = \phi\left(\frac{5}{\frac{10}{\sqrt{n}}}\right) - 1 + \phi\left(\frac{5}{\frac{10}{\sqrt{n}}}\right)$

$0.954 = 2\phi\left(\frac{5}{\frac{10}{\sqrt{n}}}\right) \cancel{-1}$
$+1 \qquad \cancel{+1}$

$\frac{1.954}{2} = \frac{\cancel{2}}{\cancel{2}}\phi\left(\frac{5}{\frac{10}{\sqrt{n}}}\right)$

$\phi^{-1}\ 0.977 = \phi^{-1}\ \phi\left(\frac{\sqrt{n}}{2}\right)$

$\frac{2}{1} = \times \frac{\sqrt{n}}{2}$

$(\sqrt{n} = 4)^2$

$\boxed{n = 16}$

∴ size of Random sample = 16.

Q97: Find the mean and variance of $S^2 = \sum_{1}^{n} \frac{(x_i - \bar{x})^2}{n}$ where $x_1, \dots, x_n$ is a Random sample from $N(\mu, \sigma^2)$.

use $\frac{nS^2}{\sigma^2}$

$$\frac{nS^2}{\sigma^2} \sim \chi^2(n-1)$$

$\rightarrow E\left(\frac{nS^2}{\sigma^2}\right) = n-1$ But we need $E(S^2)$

$\Rightarrow \left(\frac{n}{\sigma^2} E(S^2) = n-1\right) \frac{\sigma^2}{n}$

$\Rightarrow \boxed{E(S^2) = \frac{(n-1)\sigma^2}{n}}$

$\rightarrow Var\left(\frac{nS^2}{\sigma^2}\right) = 2(n-1)$

$\Rightarrow \left(\frac{n}{\sigma^2}\right)^2 Var(S^2) = 2(n-1)$

$\Rightarrow \left(\frac{n^2}{\sigma^4} Var(S^2) = 2(n-1)\right) \cdot \frac{\sigma^4}{n^2}$

$\Rightarrow \boxed{Var(S^2) = \frac{2(n-1)\sigma^4}{n^2}}$

Note:

$\rightarrow E(c\,u(x)) = c\,E(u(x))$

$\rightarrow Var(c\,u(x)) = c^2\,Var(u(x))$.

Q98: let S^2 be the variance of a Random sample of size 6 from the Normal distribution $N(\mu, 12)$. Find $Pr(2.30 < S^2 < 22.2)$.

$$\frac{nS^2}{\sigma^2} \sim \chi^2(n-1)$$

$$\frac{6S^2}{12} \sim \chi^2(5) \implies \frac{S^2}{2} \sim \chi^2(5)$$

$$\implies Pr(2.3 < S^2 < 22.2) = Pr\left(\frac{2.3}{2} < \frac{S^2}{2} < \frac{22.2}{2}\right)$$

$$= Pr\left(1.15 < \frac{S^2}{2} < 11.1\right)$$

$$= Pr\left(\frac{S^2}{2} < 11.1\right) - Pr\left(\frac{S^2}{2} < 1.15\right) \quad \text{with } df=5$$

By chi-square table with $df = 5$

$$= 0.95 - 0.05$$

$$= \underline{0.9}$$

ما فهمت
بس أول

Q101: let $X_1, X_2, \ldots, X_5$ be a Random sample of size $n=5$ From $N(0, \sigma^2)$

a. Find the constant c so that $\dfrac{c(X_1 - X_2)}{\sqrt{X_3^2 + X_4^2 + X_5^2}}$ has a t-distribution.

$$\rightarrow \quad X_1 - X_2 \sim N(0, 2\sigma^2)$$

$$\frac{X_1 - X_2}{\sqrt{2}\,\sigma} \sim N(0,1) \qquad\qquad \frac{X_i}{\sigma} \sim N(0,1) \Rightarrow \frac{X_i^2}{\sigma^2} \sim \chi^2(1)$$

$$\rightarrow \quad \frac{X_3^2 + X_4^2 + X_5^2}{\sigma^2} \sim \chi^2(3)$$

$$\rightarrow \quad \frac{X_1 - X_2 / \sqrt{2}\,\sigma}{\sqrt{X_3^2 + X_4^2 + X_5^2 / 3\sigma^2}} \implies C = \frac{\sqrt{3}\,\sigma}{\sqrt{2}\,\sigma} \implies C = \underline{\underline{\sqrt{\frac{3}{2}}}}.$$

b. How many degrees of freedom are associated with this T?

$\underline{r = 3}$

Q63: let $\bar{X}$ and S^2 be the mean and the variance of a Random sample of size 25 from a distribution that is $N(3, 100)$. then evaluate $Pr(0 < \bar{X} < 6, \; 55.2 < S^2 < 145.6)$.

$Pr(0<\bar{x}<6, \; 55.2 < S^2 < 145.6) = \underbrace{Pr(0<\bar{x}<6)}_{\text{Normal}} \; \underbrace{Pr(55.2 < S^2 < 145.6)}_{\text{rewrite as } \chi^2}$

$\rightarrow Pr(0<\bar{X}<6) = \phi\left(\frac{6-3}{\frac{10}{5}}\right) - \phi\left(\frac{0-3}{\frac{10}{5}}\right)$

$= \phi(1.5) - (1 - \phi(1.5))$

$= 2\phi(1.5) - 1$

$= 2(0.933) - 1$

$= 1.866 - 1$

$= 0.866$

$\rightarrow Pr(55.2 < S^2 < 145.6)$.

$\frac{nS^2}{\sigma^2} \sim \chi^2(n-1)$

$\frac{25 S^2}{100} \sim \chi^2(24) \Rightarrow \frac{S^2}{4} \sim \chi^2(24)$.

$\Rightarrow Pr(55.2 < S^2 < 145.6) = Pr\left(\frac{55.2}{4} < \frac{S^2}{4} < \frac{145.6}{4}\right)$

$= Pr\left(\frac{S^2}{4} < 36.4\right) - Pr\left(\frac{S^2}{4} < 13.8\right)$

with $df = 24$ $= 0.95 - 0.05$

$= 0.9$

$\Rightarrow Pr(0<\bar{X}<6, \; 55.2<S^2<145.6) = (0.866)(0.9)$

$= 0.7794$.

$= 0.78$