Inverse of Square Matrix

Exp Assume $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

1) Find AB

$$AB = \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$$

2) Find BA

$$BA = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & -2+2 \\ 1-1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$$

3) what do you conclude from 1) and 2) ?

$$\left.\begin{array}{l} B \text{ is the inverse of } A \quad (B = A^{-1}) \\ \text{or } A \text{ is the inverse of } B \quad (A = B^{-1}) \end{array}\right\} \text{ since } AB = BA = I$$

Def If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

- the determinante of A is $|A| = a_{11}\, a_{12} - a_{12}\, a_{21}$
- the inverse of A is $A^{-1} = \dfrac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$, $|A| \neq 0$

Exp Assume $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ Find A^{-1} and $(A^{-1})^{-1}$

- $|A| = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = (1)(1) - (2)(1) = 1 - 2 = -1$
- $A^{-1} = \dfrac{1}{\boxed{-1}} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = - \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$
- $|A^{-1}| = \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = (-1)(-1) - (2)(1) = 1 - 2 = -1$

$$(A^{-1})^{-1} = \frac{1}{\boxed{-1}} \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} = - \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = A$$

Remarks

- Determinante of any square matrix $A_{n\times n}$ is real number. That is, $|A| \in \mathbb{R}$
- If $|A| \neq 0$, then A^{-1} exists "A is nonsingular"
- If $|A| = 0$, then A^{-1} does not exist "A is singular"
- $A A^{-1} = A^{-1} A = I$ where $|A| \neq 0$

Exp Given $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

① Find A^{-1} and check

$$|A| = (3)(6) - (5)(4) = 18 - 20 = -2$$

$$A^{-1} = \frac{1}{\boxed{-2}} \begin{bmatrix} 6 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{6}{-2} & \frac{-5}{-2} \\ \frac{-4}{-2} & \frac{3}{-2} \end{bmatrix} = \begin{bmatrix} -3 & \frac{5}{2} \\ 2 & -\frac{3}{2} \end{bmatrix}$$

check

$$A^{-1} A = \begin{bmatrix} -3 & \frac{5}{2} \\ 2 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} -9 + \frac{5}{\cancel{2}}(\cancel{4})^{2} & -15 + \frac{5}{\cancel{2}}(\cancel{6})^{3} \\ 6 + \frac{-3}{\cancel{2}}(\cancel{4})^{2} & 10 + \frac{-3}{\cancel{2}}(\cancel{6})^{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

② Find B^{-1}

$$|B| = \begin{vmatrix} 2 & -1 \\ -4 & 2 \end{vmatrix} = (2)(2) - (-1)(-4) = 4 - 4 = 0$$

$\Rightarrow B^{-1}$ does not exist.

③ Find C^{-1}

C^{-1} does not exist since C is not square matrix

Exp Let $A = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 4 & 3 \\ 1 & -3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ -5 & 8 & 2 \\ 7 & -11 & -3 \end{bmatrix}$

Does $B = A^{-1}$?

If $AB = I_{3\times3}$ then the answer is Yes. Otherwise, no

$$AB = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 4 & 3 \\ 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ -5 & 8 & 2 \\ 7 & -11 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -2-25+28 & 4+40-44 & 2+10-12 \\ -1-20+21 & 2+32-33 & 1+8-9 \\ -1+15-14 & 2-24+22 & 1-6+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ so Yes $B = A^{-1}$ and $A = B^{-1}$

Remark To solve the following linear system for x_1 and x_2

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}\right\} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Rightarrow \quad A X = b$$

we multiply both sides by A^{-1} $\Rightarrow$ $A^{-1}AX = A^{-1}b$

$$IX = A^{-1}b$$

$$\boxed{X = A^{-1}b}$$

Assuming A^{-1} exists

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Exp Given $A^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$. Solve the linear system

$$AX = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$X = A^{-1}b = \underset{2\times2}{\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 5 \\ -1 \end{bmatrix}} = \begin{bmatrix} 5+2 \\ 5-3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\boxed{x_1 = 7}$, $\boxed{x_2 = 2}$

Exp If $B^{-1} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ then solve $AX = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

$$X = B^{-1}b = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3+0-9 \\ -1+2+6 \\ -1+4+3 \end{bmatrix} = \begin{bmatrix} -12 \\ 7 \\ 6 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\boxed{x_1 = -12}$, $\boxed{x_2 = 7}$, $\boxed{x_3 = 6}$
