Problem

Let X and Y be sets, let A and B be any subsets of X, and let C and D be any subsets of Y. Determine which of the properties are true for all functions F from X to Y and which are false for at least one function F from X to Y. Justify your answers.

Exercise

For all subsets C and D of Y,

F-1($C \cap D$) = F-1(C) $\cap$ F-1(D).

Step-by-step solution

Step 1 of 2

Let F be a function from set X to set Y, and suppose, $C \subseteq y$, and $D \subseteq y$.

The objective is to determine whether the following statement,

$F^{-1}(C \cap D) = F^{-1}(C) \cap F^{-1}(D)$ is true or false.

The proof can be divided by two parts.

Part 1:-

The statement of the part 1 is,

$F^{-1}(C \cap D) \subseteq F^{-1}(C) \cap F^{-1}(D)$.

Let $x \in F^{-1}(C \cap D)$.

To show that, $x \in F^{-1}(C)$ and $x \in F^{-1}(D)$.

$x \in F^{-1}(C \cap D)$

$\Leftrightarrow F(x) \in (C \cap D)$

$\Leftrightarrow F(x) \in C$ and $F(x) \in D$

$\Leftrightarrow x \in F^{-1}(C)$ and $x \in F^{-1}(D)$

$\Leftrightarrow x \in F^{-1}(C) \cap F^{-1}(D)$

Hence,

$F^{-1}(C \cap D) \subseteq F^{-1}(C) \cap F^{-1}(D)$. ...... (1)

Step 2 of 2

The statement of the part 2 is,

$$F^{-1}(C \cap D) \subseteq F^{-1}(C) \cap F^{-1}(D).$$

Let $x \in F^{-1}(C) \cap F^{-1}(D).$

To show that, $x \in F^{-1}(C \cap D)$

$x \in F^{-1}(C) \cap F^{-1}(D)$

$\Leftrightarrow x \in F^{-1}(C)$ and $x \in F^{-1}(D)$

$\Leftrightarrow F(x) \in C$ and $F(x) \in D$

$\Leftrightarrow F(x) \in C \cap D$ By the definition of intersection

$\Leftrightarrow x \in F^{-1}(C \cap D)$

Hence,

$$F^{-1}(C) \cap F^{-1}(D) \subseteq F^{-1}(C \cap D) \quad \text{...... (2)}$$

By combining equation (1) and (2), obtained as,

$$\boxed{F^{-1}(C \cap D) = F^{-1}(C) \cap F^{-1}(D)}.$$