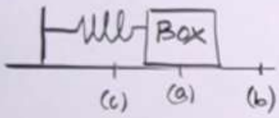
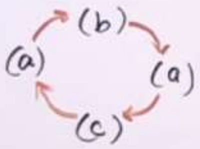
Oscillations of Springs

① A solid object attached to a spring that moves the object back and forth along a frictionless horizontal surface is said to oscillate.

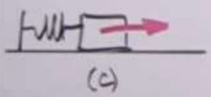


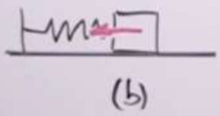
One cycle is defined as:



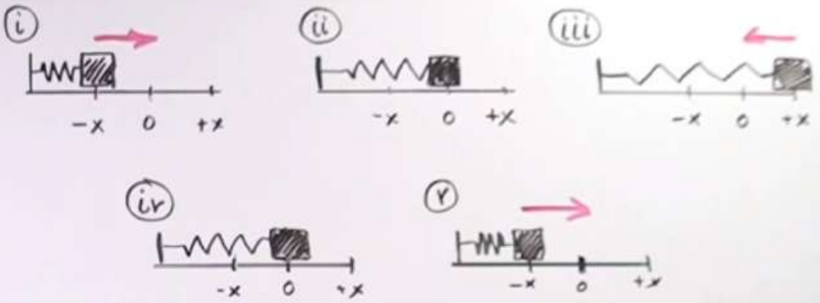
When the object oscillates over and over with the same period, the oscillation is called <u>periodic</u>.

Point (a) is called the <u>equilibrium position</u>.





$$F = -k\Delta x$$



(i) We compress the spring a distance $-x$. The spring exerts a force on mass, accelerating it in (+) direction

(ii) Object has inertia, so it passes the equilibrium point. Note at $x = 0$, spring does <u>not</u> exert a force on mass. Also, the mass reaches a maximum velocity at $x = 0$.

(iii) As object travels past $x = 0$, the force in spring slow the mass down and it stops for an instant at $+x$.

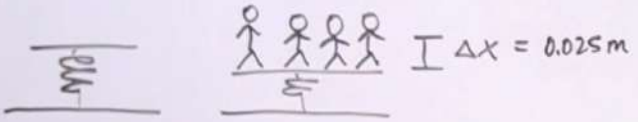
<u>Displacement</u>: distance from equilibrium position

<u>Frequency</u>: # of cycles per second

<u>Amplitude</u>: greatest displacement from equilibrium point

Four people get into a car that has a mass of 2000 kg and the springs in the car compress a distance of 2.5 cm. Assuming that the car has one spring, find

ⓐ the mass of the four people if the spring constant is 7.0×10^4 N/m.



$$mg = k\Delta x$$

$$m = \frac{k\Delta x}{g}$$

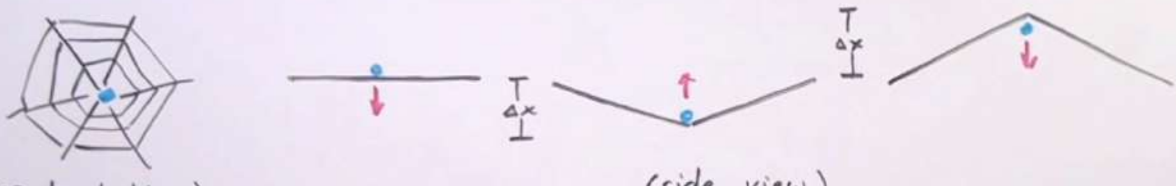
$$m = \frac{(7.0\times 10^4 \text{ N/m})(0.025\text{m})}{9.8 \text{ m/s}^2} = \boxed{179 \text{ kg}}$$

ⓑ the spring constant if the four people weigh a total of 250 kg.

$$mg = k\Delta x \Rightarrow k = \frac{mg}{\Delta x} = \frac{(250\text{ kg})(9.8\text{ m/s}^2)}{0.025\text{m}} = \boxed{9.8\times 10^4 \text{ N/m}}$$

A small insect is caught on the web of a spider and the web oscillates with a frequency of 5.0 Hz.

ⓐ Calculate the value of spring stiffness constant k, if the insect is 0.40 grams.

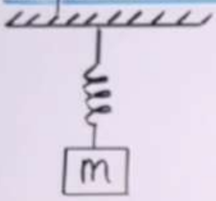


$$f = \frac{1}{2\pi}\sqrt{k/m} \Rightarrow (2\pi f)^2 m = k \Rightarrow k = (2\pi \cdot 5s^{-1})^2(0.0004\,kg) = \boxed{0.39\ N/m}$$

ⓑ Find the frequency if another insect landed ~~with~~ on the web w/ a mass of 0.8 grams.

$$f = \frac{1}{2\pi}\sqrt{k/m} = \frac{1}{2\pi}\sqrt{\frac{0.39\ N/m}{0.0008\ kg}} = \boxed{3.51\ Hz}$$

Simple Harmonic Motion



We would like to determine a function for position of mass with respect to time

$x(t) = ?$

① $\sum \vec{F} = m\vec{a} \Rightarrow -kx = m\dfrac{d^2x}{dt^2}$

② $\boxed{m\dfrac{d^2x}{dt^2} + kx = 0}$ [equation of motion for SHO]

Guess: $x(t) = A\cos(\omega t + \phi)$

Let $x = x(t)$:

$$\frac{dx}{dt} = x'(t) = -A\sin(\omega t + \phi)\,\omega = -\omega A\sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = x''(t) = -A\omega\cos(\omega t + \phi)\,\omega = -A\omega^2\cos(\omega t + \phi)$$

$$m(-A\omega^2\cos(\omega t + \phi)) + kA\cos(\omega t + \phi)$$

$$\Rightarrow -A\omega^2\cos(\omega t + \phi) + \frac{k}{m}A\cos(\omega t + \phi)$$

$$\Rightarrow \left(\frac{k}{m} - \omega^2\right)A\cos(\omega t + \phi) = 0 \ ?$$

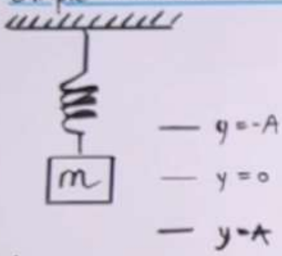
Solution: If $\boxed{\dfrac{k}{m} = \omega^2}$, then equation is equal to zero.

② Since $\omega = 2\pi f \,/\, T = 1/f$:

ⓐ $f = \dfrac{1}{2\pi}\sqrt{k/m}$ ⓑ $T = 2\pi\sqrt{m/k}$

* frequency does not depend on amplitude!

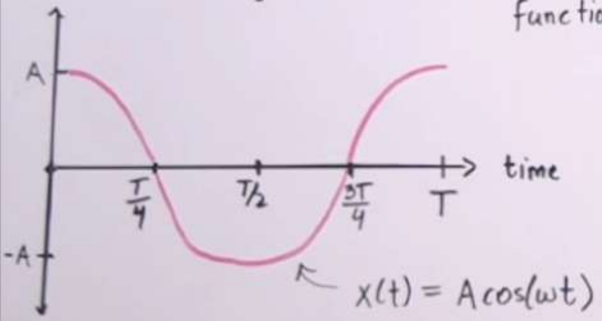
Simple Harmonic Oscillation



$$x(t) = A\cos(\omega t + \phi)$$

A := highest displacement

ϕ := how far to the right or left the cosine function begins



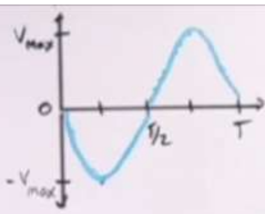
velocity:

$$v = x'(t) = -A\omega \sin(\omega t)$$

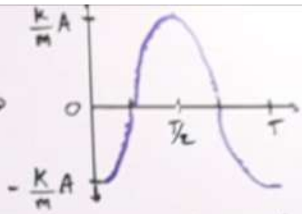
acceleration:

$$a = x''(t) = -A\omega^2 \cos(\omega t)$$

velocity and acceleration also vary sinusoidally



velocity vs time



acceleration vs time

Maximum Velocity:

$$v = -A\omega \sin(\omega t) \Rightarrow V_{max} = \pm A\omega$$

$$\text{Since } \omega^2 = K/m \Rightarrow \boxed{V_{max} = A \cdot \sqrt{K/m}}$$

Maximum Acceleration:

$$a = -A\omega^2 \cos(\omega t) \Rightarrow a_{max} = \pm A\omega^2$$

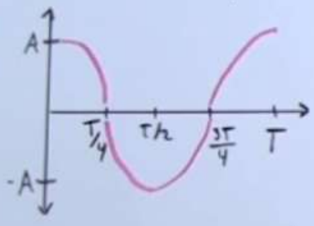
$$\text{Since } \omega^2 = K/m \Rightarrow \boxed{a_{max} = \frac{AK}{m}}$$

Example: If the floor vibrates with a frequency of 20 Hz and the amplitude of the floor is 4 mm, calculate the maximum acceleration.

$$a_{max} = A\omega^2 = A(2\pi f)^2 = (0.004m)(2\pi \cdot 20s^{-1})^2 = 63 \frac{m}{s^2}$$

A certain loud speaker experiences simple harmonic oscillation at a frequency of 300 Hz. The amplitude at center of loudspeaker is 2.0×10^{-4} m and at time of 0 seconds, begins at $y = A$.

ⓐ What equation describes motion of loudspeaker?



$$x(t) = A\cos(\omega t + \phi)$$

$$\boxed{x(t) = 0.0002\cos(600\pi t)}$$

$\underline{\phi}$: Since at $t = 0$, the displacement is equal to the amplitude, the $\boxed{\phi = 0}$

$\underline{\omega}$: $\omega = 2\pi f = 2\pi(300\ s^{-1}) = \boxed{600\pi\ \frac{rad}{s}}$

$\underline{A}$: $\boxed{0.0002\ m}$

ⓑ Find the maximum velocity and maximum acceleration.

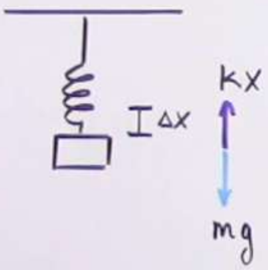
$$V_{max} = A \cdot \omega = (2 \times 10^{-4}\ m)(600\pi\ \tfrac{rad}{s}) = 0.12\pi\ m/s \qquad a_{max} = A\omega^2 = (2 \times 10^{-4}\ m)(600\pi)^2 = 72\pi^2\ \frac{m}{s^2}$$

ⓒ Find the position at $t = 3.0$ seconds.

$$x(t) = 0.0002\cos(600\pi t) \Rightarrow x(3) = 0.0002\cos(1800\pi) = 2 \times 10^{-4}\ m$$

A certain spring stretches 0.2m when a 0.4 kg mass is attached to it (vertically). We then set up the spring horizontally with the same mass resting on a frictionless table. The mass is pushed so that the spring compresses a distance of 0.12 m and released. Assume SHM.

(a) Calculate the spring stiffness constant and angular frequency?



$$mg = kx$$

$$k = \frac{mg}{\Delta x}$$

$$k = \frac{(0.4\,kg)(9.8\,m/s^2)}{(0.2\,m)} = \boxed{19.6\ N/m}$$

$$\omega^2 = k/m$$

$$\omega = \sqrt{k/m}$$

$$\omega = \sqrt{\frac{19.6\ N/m}{0.4\ kg}} = \boxed{7\ \frac{rad}{s}}$$

(B) Find the maximum velocity and maximum acceleration:

$$V_{max} = A\omega = (0.12\,m)(7\,rad/s) = \boxed{0.84\ m/s}$$

$$a_{max} = A\omega^2 = (0.12\,m)(7\,rad/s)^2 = \boxed{5.88\ m/s^2}$$

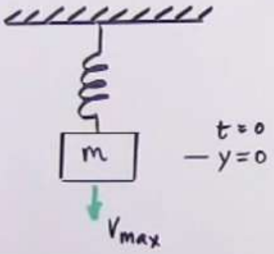
(c) Find the frequency and period of oscillation:

$$f = \frac{\omega}{2\pi} = \frac{7\,rad/s}{2\pi} = \boxed{1.11\ Hz}$$

$$T = 1/f = \frac{1}{1.11\ Hz} = \boxed{0.9\ sec}$$

A vertical spring with a stiffness constant 400 N/m oscillates with an amplitude of 30 cm when a mass of 0.4 kg hangs from it. If the mass passes through the equilibrium point with a positive velocity at $t = 0$ seconds (Assume SHM),

ⓐ Find the equation that describes the motion.

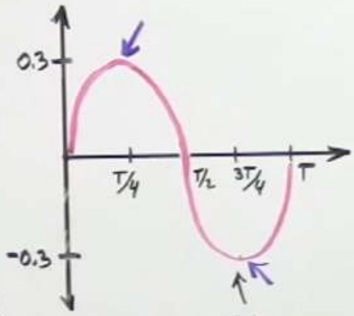


$$x(t) = A\sin(\omega t + \phi)$$

$$x(t) = 0.3\sin(\omega t)$$

since $\omega^2 = k/m$, then

$$\omega = (k/m)^{1/2} = \left(\frac{400\ \text{N/m}}{0.4\ \text{kg}}\right)^{1/2} = 31.6\ \tfrac{\text{rad}}{\text{s}}$$



$$x(t) = 0.3\sin(31.6t)$$

ⓑ At what time will spring be longest and shortest?

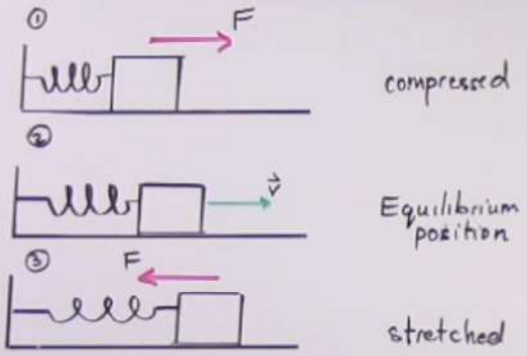
$$\omega = 2\pi f \Rightarrow \omega = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{31.6\ \frac{\text{rad}}{\text{s}}} = 0.2\ \text{sec}$$

$$T/4 = \frac{0.2}{4} = 0.05\ \text{sec}$$

$$3T/4 = \frac{3(0.2)}{4} = 0.15\ \text{sec}$$

$$\begin{bmatrix} x(0.05) = 0.3\sin(31.6 \cdot 0.05) = 0.3 \\ x(0.15) = 0.3\sin(31.6 \cdot 0.15) = -0.3 \end{bmatrix} \underline{\text{check}}$$

Energy in Simple Harmonic Motion



① Mass is fully compressed, so all the energy is stored in the spring as elastic potential energy.

$$U = \frac{1}{2}kx^2$$

② The displacement is zero, so all elastic potential energy has been transformed into kinetic energy

$$\frac{1}{2}mv_{MAX}^2$$

③ Spring is fully stretched and all kinetic energy has been transformed back into elastic potential energy.

(i) $$\boxed{\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E}$$

As long as there is no friction, energy is conserved.

elastic energy $\rightleftarrows$ kinetic energy

Recall: $x(t) = A\cos(\omega t + \phi)$ and $v(t) = -A\omega\sin(\omega t + \phi)$

$$\frac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi) + \frac{1}{2}kA^2\cos^2(\omega t + \phi) = E \qquad \boxed{\omega^2 = k/m}$$

$$\Rightarrow \frac{1}{2}\frac{mk}{m}A^2\sin^2(\omega t + \phi) + \frac{1}{2}kA^2\cos^2(\omega t + \phi) = E$$

$$\Rightarrow \frac{1}{2}kA^2\left[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right] = E$$

$$\Rightarrow \boxed{\frac{1}{2}kA^2 = E} \quad \text{(ii)}$$

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \Rightarrow \boxed{v = \pm\sqrt{\frac{k}{m}\left(A^2 - x^2\right)}}$$

A 2.0 kg mass oscillates according to equation $x(t) = 0.7\cos(8t)$.

ⓐ Determine the amplitude of oscillation

$x(t) = A\cos(\omega t + \phi)$ $\boxed{A = 0.7\,m}$

↑ maximum displacemet

ⓑ Determine the period.

$\omega = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{8\,rad/s} = \boxed{\frac{\pi}{4}\text{ seconds}}$

ⓒ Determine the total energy:

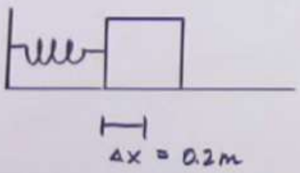
$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2 \Rightarrow \begin{cases} \omega^2 = k/m \\ k = m\omega^2 \end{cases} \Rightarrow E = \frac{1}{2}(m\omega^2)A^2 = \boxed{31.36\,J}$

ⓓ Find velocity at $x = 0.2m$

$v = \sqrt{k/m(A^2 - x^2)} = \sqrt{\omega^2(A^2 - x^2)} = \boxed{5.4\ m/s}$

An object with an unknown mass is resting on a frictionless surface and is attached to a coil spring. 4.0 Joules of work is required to compress the spring a distance of 0.2 m. If the mass is compressed to that distance and released, it reaches a maximum acceleration of 17 m/s².

ⓐ Calculate spring stiffness constant



$$U = \frac{1}{2} K x^2 \Rightarrow K = \frac{2U}{x^2} = \frac{2(4.0\,J)}{(0.2m)^2} = \boxed{200\ N/m}$$

ⓑ Calculate the mass.

$$\Sigma F_{max} = m a_{max} \Rightarrow KX = ma \Rightarrow m = \frac{KX}{a}$$

$$m = \frac{(200\,N/m)(0.2m)}{17\,m/s^2} = \boxed{2.35\ kg}$$

A certain spring is attached to a mass of 0.500 kg. If it has a spring constant of 25.0 N/m and the oscillation has an amplitude of 0.15 m,

ⓐ calculate the total energy

$$E_{total} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \underbrace{\frac{1}{2}kA^2} \Rightarrow \frac{1}{2}kA^2 = \frac{1}{2}(25\,N/m)(0.15\,m)^2 = \boxed{0.28\ J}$$

ⓑ find the potential and kinetic energy equations w/ respect to time.

$$x(t) = A\cos(\omega t + \phi) \Rightarrow x(t) = 0.15\cos(7.1t)$$

$$v(t) = -\omega A\sin(\omega t + \phi) \Rightarrow v(t) = -1.06\sin(7.1t)$$

$$\left[\omega^2 = k/m \Rightarrow \omega = \sqrt{k/m}\right] \Rightarrow \omega = 7.1\,\frac{rad}{s}$$

$$U = \frac{1}{2}kx^2 = \boxed{0.28\cos^2(7.1t)}$$

$$K = \frac{1}{2}mv^2 = \boxed{0.28\sin^2(7.1t)}$$

ⓒ find the velocity when mass is 0.06 m from equilibrium point.

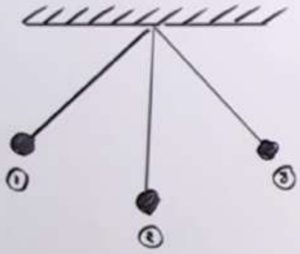
$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \Rightarrow v = \sqrt{\frac{k}{m}(A^2 - x^2)} = \boxed{0.97\ m/s}$$

ⓓ Find the kinetic energy at A/2.

$$U = \frac{1}{2}kx^2 \Rightarrow U = \frac{1}{2}(25)(0.075)^2 = 0.07\ J \qquad K = E_{total} - U = \boxed{0.21\ J}$$

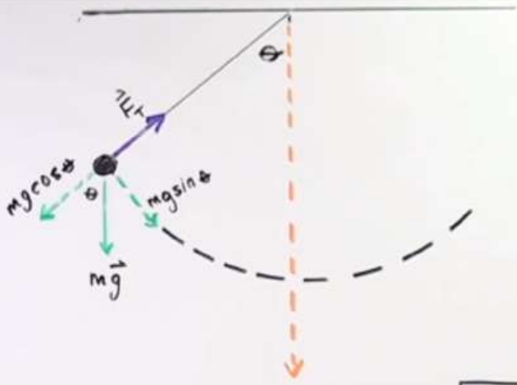
Simple Pendulum

① A SIMPLE PENDULUM CONSISTS OF A SMALL MASS SUSPENDED BY A ROPE.
IF WE NEGLECT THE MASS OF ROPE AND ANY FRICTION, THEN THE MOTION THE MASS MAKES IS SIMPLE HARMONIC.



② EQUILIBRIUM POSITION

① & ③ MAXIMUM DISPLACEMENT



$\theta = \frac{x}{\ell} \Rightarrow \boxed{x = \theta \ell}$

angle in radians rope makes w/ y-axis

length of rope

Force:

$F = -mg\sin\theta$ { the force is directly proportional to sine of the angular displacent

If we let θ be very small, then we can assume $\theta \approx \sin\theta$.

$$F = -mg\sin\theta \approx -mg\theta = \boxed{-\frac{mgx}{\ell}}$$

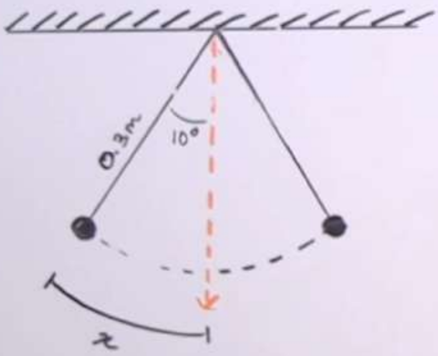
* For small θ displacement, we have SHM.

$$\Rightarrow F \approx \underbrace{-\frac{mg}{\ell}}\cdot x = \underbrace{-K}\, x \qquad \Rightarrow K = \frac{mg}{\ell}$$

Since:

$$\omega = \sqrt{\frac{K}{m}} \Rightarrow \boxed{\omega = \sqrt{\frac{g}{\ell}}} \Rightarrow \boxed{f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}}$$

A simple pendulum is 0.30 m long. At $t = 0$ s, it is released from rest at an angle of 10°. Assuming SHM, calculate the angular position at (a) $t = 0.35$ s (b) $t = 3.0$ sec



$$x(t) = \underbrace{A}_{\uparrow} \cos(\underbrace{\omega}_{\uparrow} t + \underbrace{\phi}_{\uparrow})$$

$\phi = \boxed{0}$ (because at $t=0$, the oscillation begins at maximum disp.)

$A = x = \boxed{\theta \ell}$

$\omega = \sqrt{K/m} = \boxed{\sqrt{g/\ell}}$

$$x(t) = \theta \ell \cos\left(\sqrt{g/\ell}\, t\right) = \frac{0.3\pi}{18} \cos(5.72t)$$

$$\begin{bmatrix} 180° & - & \pi \text{ radians} \\ 10° & - & \frac{10°}{180°}\pi \text{ radians} \end{bmatrix}$$

(a) $x(0.35\text{s}) = -0.022$ m

$\theta = \frac{x}{\ell} = \frac{-0.022}{0.3} = -0.73$ rad

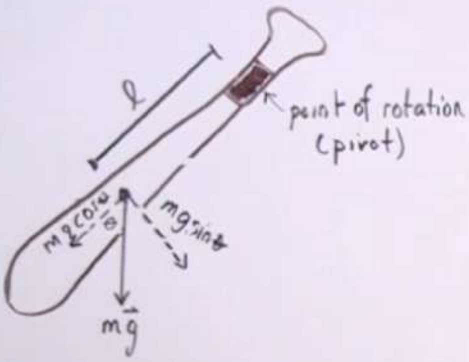
$\Rightarrow \boxed{-4.02°}$

(b) $x(3) = -0.0062$ m

$\theta = \frac{x}{\ell} = \frac{-0.0062}{0.3} = -0.021$ rad

$\Rightarrow \boxed{-1.18°}$

Physical Pendulum



① $\tau = -\ell F_{\perp} = -\ell mg \sin\theta$

② Since $\Sigma\tau = I\alpha = I\dfrac{d^2\theta}{dt^2}$

③ $\Rightarrow I\dfrac{d^2\theta}{dt^2} = -\ell mg\sin\theta$

④ $I\dfrac{d^2\theta}{dt^2} + \ell mg\sin\theta = 0$

If we assume θ is small, $\theta \approx \sin\theta$

⑤ $\boxed{I\dfrac{d^2\theta}{dt^2} + \ell mg\theta = 0}$ [for small angular displacements]

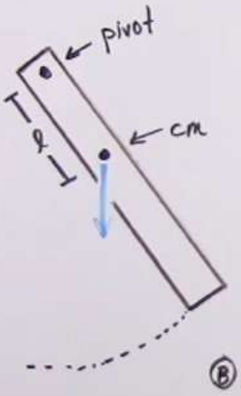
Recall: $m\dfrac{d^2 x}{dt} + kx = 0$ solution: $x(t) = A\cos(\omega t + \phi)$

Hence: $\theta(t) = \theta_{max}\cos(\omega t + \phi)$

In ⑤: $\dfrac{d^2\theta}{dt^2} + \dfrac{\ell mg}{I}\theta = 0 \Rightarrow \dfrac{k}{m} = \dfrac{mg\ell}{I}$

$\Rightarrow \omega = \sqrt{k/m} = \sqrt{\dfrac{mg\ell}{I}} \Rightarrow \boxed{f = \dfrac{1}{2\pi}\sqrt{\dfrac{mg\ell}{I}}}$

Suppose that a nonuniform 2.0 kg is balanced at a point 52 cm from one of its ends. When we pivot this object at that point, it oscillates under SHM with a frequency of 0.5 s^{-1}. (a) Find the moment of inertia of this object.



Recall:

$$f = \frac{1}{2\pi}\sqrt{\frac{mgl}{I}} \Rightarrow I = \frac{mgl}{(2\pi f)^2} \qquad I = \frac{(2\,kg)(9.8\,m/s^2)(0.52\,m)}{(2\pi\ 1/2\,s^{-1})^2}$$

$$I = \boxed{1.03\ kg \cdot m^2}$$

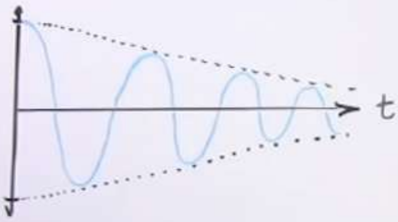
(b) Find the ratio of K/m.

$\omega^2 = K/m$: Since $\omega = 2\pi f$, then $\omega^2 = (2\pi f)^2$

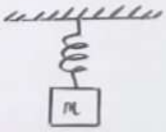
$$(2\pi f)^2 = (2\pi \cdot 0.5\,s^{-1})^2 = \boxed{9.97\ N/kg \cdot m}$$

Damped Harmonic Motion

① A real oscillating system will over time experience a decrease in amplitude as a result of internal friction and air resistance.



{The amplitude decreases since the energy goes into thermal energy



Damping Force

① $F_{damping} = -bv$

② $\sum F = ma \Rightarrow -kx - bv = ma$

$\Rightarrow ma + kx + bv = 0$

$$\Rightarrow m\frac{d^2x}{dt^2} + kx + \frac{b\,dx}{dt} = 0 \quad \text{] equation of motion}$$

Solution: $x(t) = A e^{-\gamma t}\cos(\omega' t)$ [At $t=0$, $x = A$]

Note: $\omega' \neq \omega = \sqrt{k/m}$

① $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

② $\gamma = \frac{b}{2m}$

Therefore: $\underbrace{x(t) = A e^{-\frac{bt}{2m}}\cos(\omega' t)}$

position function for a lightly damped H.O.

Frequency:

$\omega' = 2\pi f' \Rightarrow \boxed{f' = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$

$\Rightarrow \frac{k}{m} - \frac{b^2}{4m^2} \Rightarrow b^2 = 4mk$

ⓐ over damped: $b^2 \gg 4mk$

ⓑ under damped: $b^2 < 4mk$

ⓒ critical damped: $b^2 = 4mk$