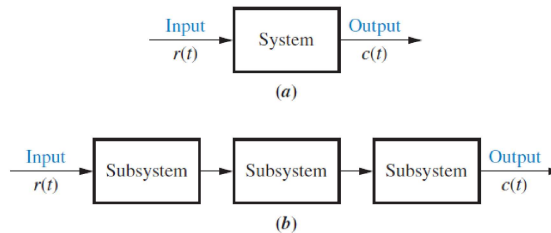


Modeling in the Frequency Domain

2

FIGURE 2.1 a. Block diagram representation of a system; b. block diagram representation of an interconnection of subsystems



Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

2.2 Laplace Transform Review

TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at} u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt$	Definition
2.	$\mathcal{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathcal{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^n s^{n-k}f^{(k-1)}(0-)$	Differentiation theorem
10.	$\mathcal{L}\left[\int_{0-}^t f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \rightarrow 0} sF(s)$	Final value theorem ¹
12.	$f(0+) = \lim_{s \rightarrow \infty} sF(s)$	Initial value theorem ²

¹For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

²For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

Partial-Fraction Expansion

$$F_1(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

$$F_1(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

Case 1. Roots of the Denominator of $F(s)$ Are Real and Distinct

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{2}{s+1} + \frac{-2}{s+2}$$

$$f(t) = (2e^{-t} - 2e^{-2t})u(t)$$

Case 2. Roots of the Denominator of $F(s)$ Are Real and Repeated

$$(s+1)(s+2)^2 \left[F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \right], \boxed{A=2} \checkmark$$

$$2 = 2(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

$$2 = 2s^2 + 8s + 8 + B(s^2 + 3s + 2) + Cs + C$$

$$2 = \underbrace{(2+B)}_{=0} s^2 + \underbrace{(8+3B+C)}_{=0} s + \underbrace{(8+C+2B)}_{=2}$$

$$\boxed{B=-2}$$

$$\boxed{C=-2}$$

$$F(s) = \frac{2}{s+1} - \frac{2}{s+2} - \frac{2}{(s+2)^2}$$

$$\therefore f(t) = 2e^{-t} - 2e^{-2t} - 2te^{-2t} \checkmark$$

$$t^2 \leftrightarrow \frac{2!}{s^3}$$

$$\frac{1}{(s+2)^3} \leftrightarrow \frac{1}{2} t^2 e^{-2t}$$

Case 3. Roots of the Denominator of $F(s)$ Are Complex or Imaginary

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5}$$

$$\left(\frac{3}{s(s^2 + 2s + 5)} = \frac{3/5}{s} + \frac{Bs + C}{s^2 + 2s + 5} \right) \times s(s^2 + 2s + 5)$$

$$3 = \frac{3}{5}(s^2 + 2s + 5) + Bs^2 + Cs$$

$$B = -\frac{3}{5}, \quad C = -\frac{6}{5}$$

$$F(s) = \frac{3/5}{s} - \frac{3}{5} \frac{s+2}{s^2 + 2s + 5}$$

$$= \frac{3/5}{s} - \frac{3}{5} \frac{s+2}{(s+1)^2 + 4}$$

$$= \frac{3/5}{s} - \frac{3}{5} \frac{(s+1) + 1}{(s+1)^2 + (2)^2}$$

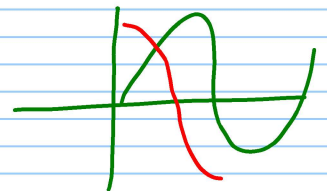
$\sin \omega t \quad \frac{\omega}{s^2 + \omega^2}$
 $\cos \omega t \quad \frac{s}{s^2 + \omega^2}$

$$= \frac{3/5}{s} - \frac{3}{5} \left[\frac{(s+1)}{(s+1)^2 + (2)^2} + \frac{1}{2} \frac{2}{(s+1)^2 + (2)^2} \right]$$

$$o.f(t) = \frac{3}{5} - \frac{3}{5} e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right)$$

$$1 \angle 0 + \frac{1}{2} \angle -90$$

$$\left(1 - \frac{j}{2} \right) \rightarrow 1.118 \angle -26.56^\circ$$



$$\begin{aligned}
 f(t) &= \frac{3}{5} - \frac{3}{5} e^{-t} (1.118 \cos(2t - 26.56)) \\
 &= \frac{3}{5} - 0.6708 e^{-t} \cos(2t - 26.56)
 \end{aligned}$$

2.3 The Transfer Function

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \dots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0)}$$

$C(s) \rightarrow$ output
 $R(s) \rightarrow$ input

Notice that Eq. (2.53) separates the output, $C(s)$, the input, $R(s)$, and the system, which is the ratio of polynomials in s on the right. We call this ratio, $G(s)$, the transfer function and evaluate it with zero initial conditions.

$$C(s) = R(s) G(s)$$

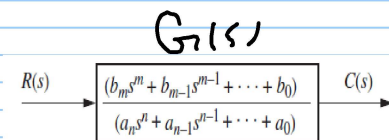
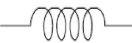


FIGURE 2.2 Block diagram of a transfer function

2.4 Electrical Network Transfer Functions

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor 	$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$	$i(t) = C \frac{dv(t)}{dt}$	$v(t) = \frac{1}{C} q(t)$	$\frac{1}{Cs}$	Cs
Resistor 	$v(t) = Ri(t)$	$i(t) = \frac{1}{R} v(t)$	$v(t) = R \frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor 	$v(t) = L \frac{di(t)}{dt}$	$i(t) = \frac{1}{L} \int_0^t v(\tau) d\tau$	$v(t) = L \frac{d^2 q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

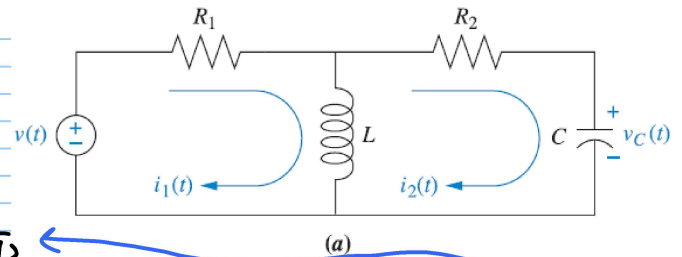
Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

$$\underline{s} \equiv \underline{j\omega}$$

Example 2.10

Transfer Function—Multiple Loops

PROBLEM: Given the network of Figure 2.6(a), find the transfer function, $I_2(s)/V(s)$.

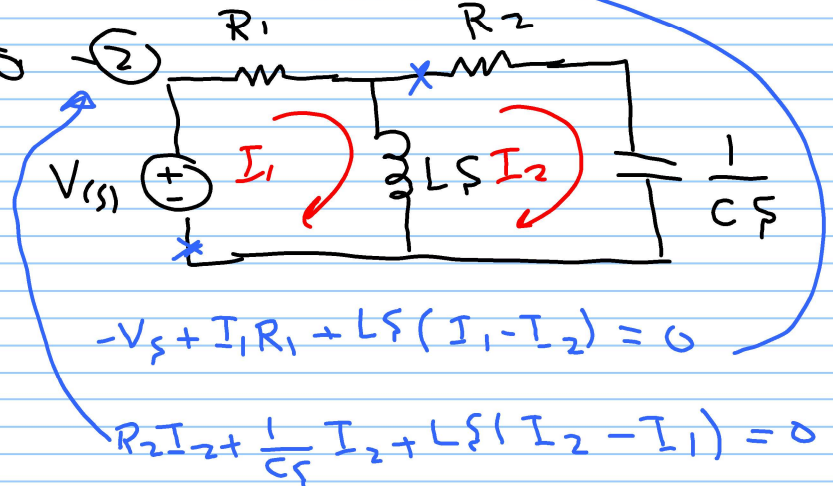


$$(R_1 + Ls)I_1(s) - (Ls)I_2(s) = V(s) \quad \text{--- (1)}$$

$$-(Ls)I_1(s) + (R_2 + Ls + \frac{1}{Cs})I_2(s) = 0 \quad \text{--- (2)}$$

$$I_2(s) = \frac{\begin{vmatrix} R_1 + Ls & V(s) \\ -Ls & 0 \end{vmatrix}}{\begin{vmatrix} R_1 + Ls & -Ls \\ -Ls & R_2 + Ls + \frac{1}{Cs} \end{vmatrix}}$$

$$I_2(s) = \frac{\begin{vmatrix} R_1 + Ls & -Ls \\ -Ls & R_2 + Ls + \frac{1}{Cs} \end{vmatrix}}{\begin{vmatrix} R_1 + Ls & -Ls \\ -Ls & R_2 + Ls + \frac{1}{Cs} \end{vmatrix}}$$



$$I_2(s) = \frac{Ls V(s)}{(R_1 + Ls)(R_2 + Ls + \frac{1}{Cs}) - (Ls)^2}$$

$$\lim_{s \rightarrow \infty} \frac{I_2(s)}{V(s)} \Rightarrow$$

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

Nodal

$$G(s) = \frac{V_o(s)}{V(s)} \quad , \quad V_o(s) = V_1(s)$$

$$\frac{V_1(s) - V(s)}{9/s} + \frac{V_1(s) - V_2(s)}{4 + 6s} + \frac{V_1(s)}{8} = 0$$

$$\left(\frac{1}{9/s} + \frac{1}{4 + 6s} + \frac{1}{8} \right) V_1(s) - \frac{1}{4 + 6s} V_2(s) = \frac{V(s)}{9/s}$$

$$\frac{V_2(s) - V(s)}{2} + \frac{V_2(s)}{2 + 4s} + \frac{V_2(s) - V_1(s)}{4 + 6s} = 0$$

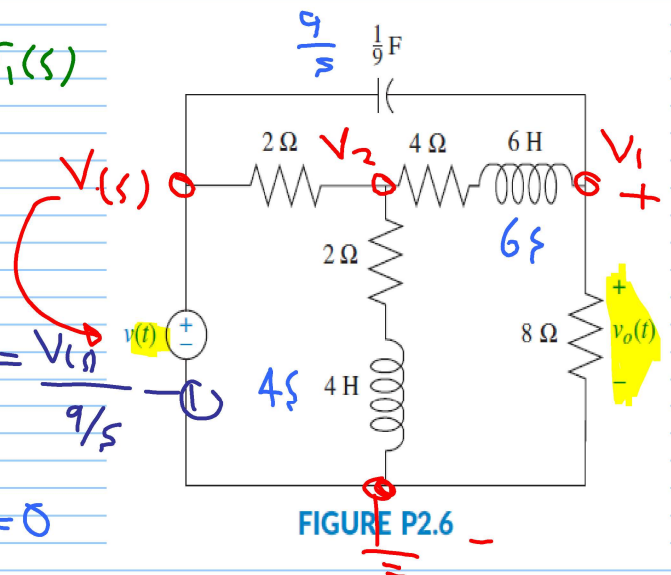


FIGURE P2.6

$$-\frac{1}{4+6s} V_1(s) + \left(\frac{1}{2} + \frac{1}{2+4s} + \frac{1}{4+6s} \right) V_2(s) = \frac{V(s)}{2} \quad (2)$$

2.5 Translational Mechanical System Transfer Functions

→ Mechanical systems → 3 passive linear components

$\left. \begin{matrix} -m \\ -1/t \end{matrix} \right\} \rightarrow \begin{matrix} \text{spring} \\ \text{Mass} \end{matrix} \right\} \text{energy storage elements}$

$-m \rightarrow \text{viscous damper} \rightarrow \text{dissipates energy.}$

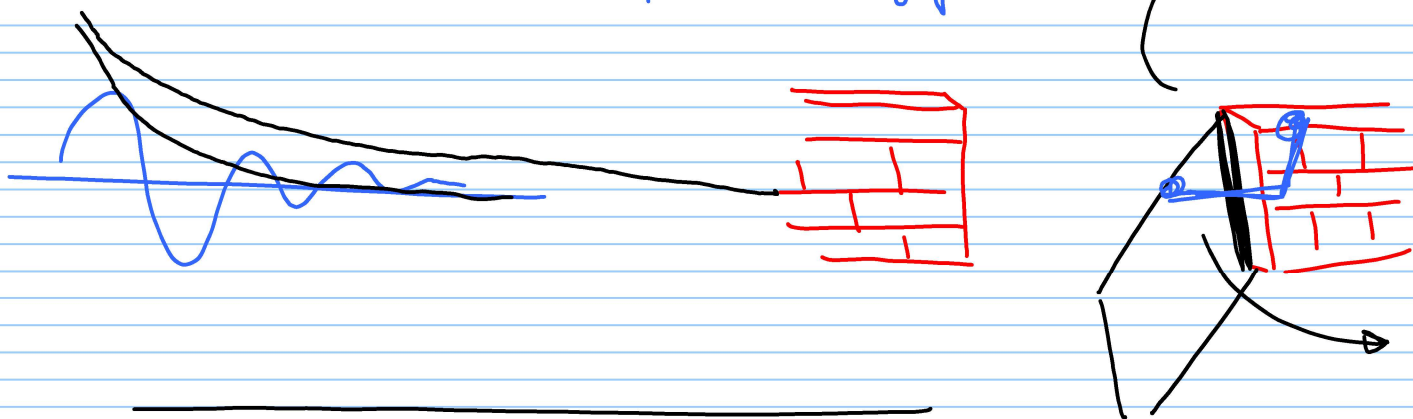


TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K \int_0^t v(\tau) d\tau$ $V = \frac{1}{K} \int I dt$	$f(t) = Kx(t)$ $F(s) = K X(s)$	K
Viscous damper 	$f(t) = f_v v(t)$ $V = R I$	$f(t) = f_v \frac{dx(t)}{dt}$ $F(s) = f_v s X(s)$	$f_v s$
Mass 	$f(t) = M \frac{dv(t)}{dt}$ $V = L \frac{di}{dt}$	$f(t) = M \frac{d^2x(t)}{dt^2}$ $F(s) = M s^2 X(s)$	$M s^2$

Note: The following set of symbols and units is used throughout this book: $f(t) = \text{N (newtons)}$, $x(t) = \text{m (meters)}$, $v(t) = \text{m/s (meters/second)}$, $K = \text{N/m (newtons/meter)}$, $f_v = \text{N-s/m (newton-seconds/meter)}$, $M = \text{kg (kilograms = newton-seconds}^2/\text{meter)}$.

$$\sum f(t) = \text{Zero}$$

$$Z_M(s) = \frac{F(s)}{X(s)}$$

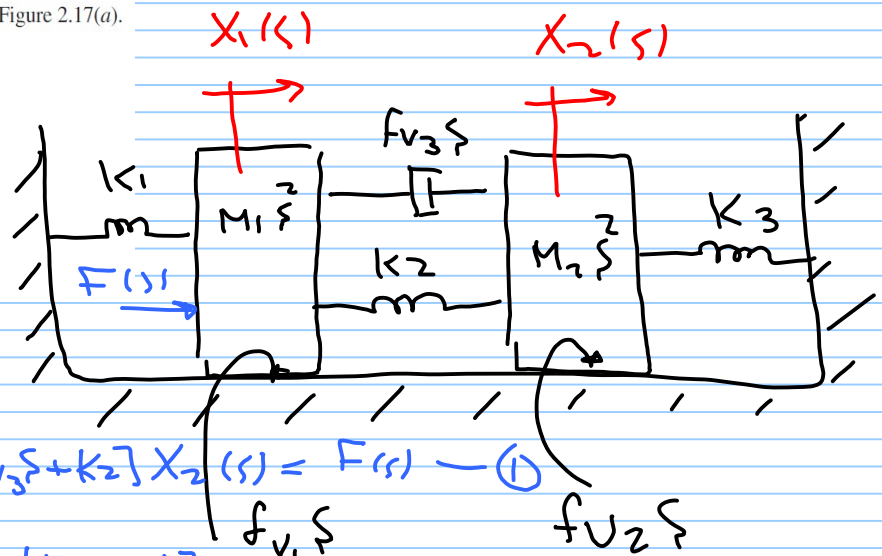
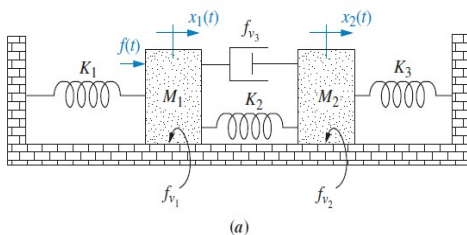
$$F(s) = Z_M(s) X(s)$$

[sum of impedances] $X(s)$ = sum of applied forces

Example 2.17

Transfer Function—Two Degrees of Freedom

PROBLEM: Find the transfer function, $X_2(s)/F(s)$, for the system of Figure 2.17(a).



$$[M_1 \ddot{x}_1 + (f_{v1} + f_{v2})\dot{x}_1 + (K_1 + K_2)x_1 - [f_{v3}\dot{x}_2 + K_2x_2] = F(t) \quad \text{--- (1)}$$

$$- [f_{v3}\dot{x}_1 + K_2x_1] + [M_2 \ddot{x}_2 + (f_{v2} + f_{v3})\dot{x}_2 + (K_2 + K_3)x_2] = 0 \quad \text{--- (2)}$$

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{(f_{v3}s + K_2)}{\Delta}$$

$$\left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_1 \end{array} \right] X_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \text{ } x_1 \text{ and } x_2 \end{array} \right] X_2(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_1 \end{array} \right] \quad (2.120a)$$

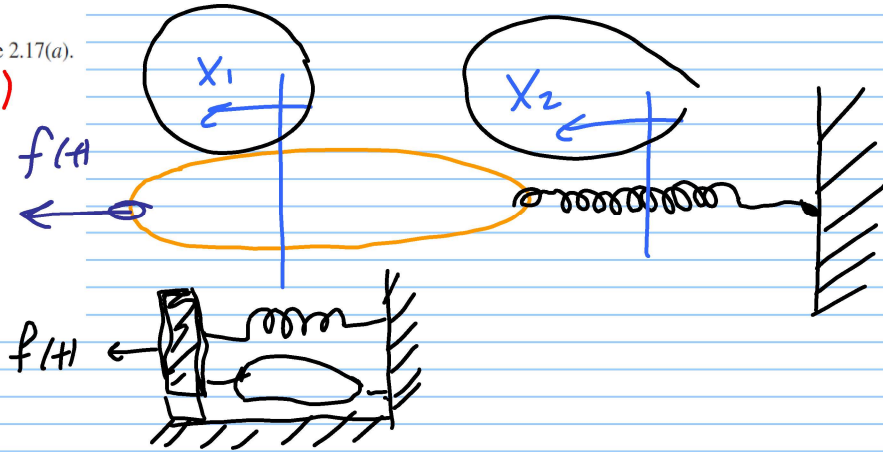
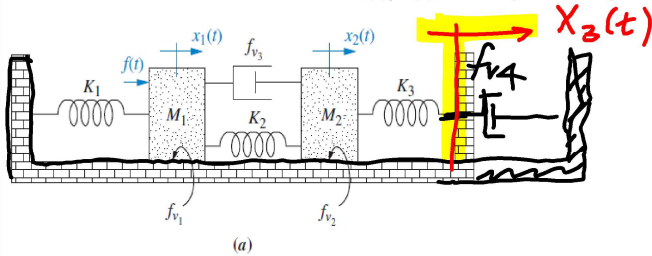
$$- \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \text{ } x_1 \text{ and } x_2 \end{array} \right] X_1(s) + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_2 \end{array} \right] X_2(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_2 \end{array} \right] \quad (2.120b)$$

Note: # of eq. of motion required is equal to the number of linearly independent motions.

Example 2.17

Transfer Function—Two Degrees of Freedom

PROBLEM: Find the transfer function, $X_2(s)/F(s)$, for the system of Figure 2.17(a).



3 equations

$$[M_1 s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) - [f_{v3}s + K_2]X_2(s) + 0 \cdot X_3(s) = F(s) \quad \text{--- (1)}$$

$$- [f_{v3}s + K_2]X_1(s) + [M_2 s^2 + (f_{v2} + f_{v4})s + (K_2 + K_3)]X_2(s) - K_3 X_3(s) = 0 \quad \text{--- (2)}$$

$$- 0 \cdot X_1(s) - K_3 X_2(s) + (f_{v4}s + K_3)X_3(s) = 0 \quad \text{--- (3)}$$

Example 2.18

Equations of Motion by Inspection

PROBLEM: Write, but do not solve, the equations of motion for the mechanical network of Figure 2.20.

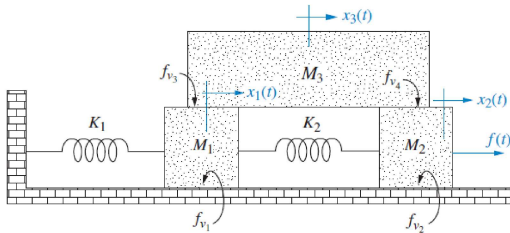


FIGURE 2.20 Three-degrees-of-freedom translational mechanical system

$$[M_1 s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) - K_2 X_2(s) - f_{v3}s X_3(s) = 0 \quad \text{--- (1)}$$

$$- K_2 X_1(s) + [M_2 s^2 + (f_{v2} + f_{v4})s + K_2]X_2(s) - f_{v4}s X_3(s) = F(s) \quad \text{--- (2)}$$

$$- f_{v3}s X_1(s) - f_{v4}s X_2(s) + [M_3 s^2 + (f_{v3} + f_{v4})s]X_3(s) = 0 \quad \text{--- (3)}$$

Skill-Assessment Exercise 2.8

PROBLEM: Find the transfer function, $G(s) = X_2(s)/F(s)$, for the translational mechanical system shown in Figure 2.21.

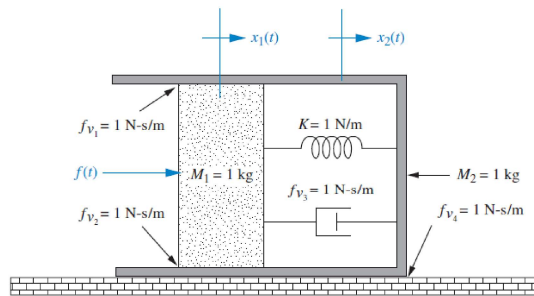


FIGURE 2.21 Translational mechanical system for Skill-Assessment Exercise 2.8

ANSWER: $G(s) = \frac{3s + 1}{s(s^3 + 7s^2 + 5s + 1)}$

The complete solution is at www.wiley.com/college/nise.

$$(s^2 + 3s + 1)X_1(s) - (3s + 1)X_2(s) = F(s) \quad \text{--- (1)}$$

$$-(3s + 1)X_1(s) + [s^2 + 4s + 1]X_2(s) = 0 \quad \text{--- (2)}$$

$$G(s) = \frac{X_2(s)}{F(s)}, \quad X_2(s) = \frac{\begin{vmatrix} s^2 + 3s + 1 & F(s) \\ -(3s + 1) & 0 \end{vmatrix}}{\begin{vmatrix} s^2 + 3s + 1 & -(3s + 1) \\ -(3s + 1) & (s^2 + 4s + 1) \end{vmatrix}}$$

$$X_2(s) = \frac{(3s + 1)F(s)}{s^4 + 7s^3 + 14s^2 + 7s + 1 - (3s + 1)^2}$$

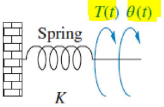
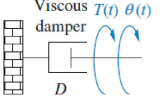
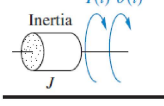
$s^4 + 4s^3 + s^2$
 $3s^3 + 12s^2 + 3s$
 $s^2 + 4s + 1$

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{3s + 1}{s^4 + 7s^3 + 5s^2 + s}$$

$9s^2 + 6s + 1$

2.6 Rotational Mechanical System Transfer Functions

TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K \int_0^t \omega(\tau) d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D \frac{d\theta(t)}{dt}$	Ds
	$T(t) = J \frac{d\omega(t)}{dt}$	$T(t) = J \frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ —N-m (newton-meters), $\theta(t)$ —rad (radians), $\omega(t)$ —rad/s (radians/second), K —N-m/rad (newton-meters/radian), D —N-m-s/rad (newton-meters-seconds/radian), J —kg-m² (kilograms-meters²—newton-meters-seconds²/radian).

Example 2.19

Transfer Function—Two Equations of Motion

PROBLEM: Find the transfer function, $\theta_2(s)/T(s)$, for the rotational system shown in Figure 2.22(a). The rod is supported by bearings at either end and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.

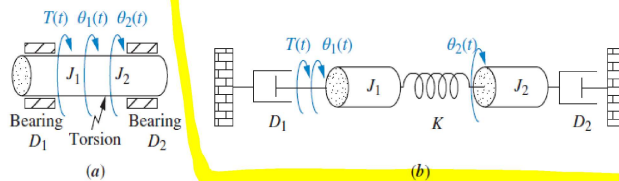
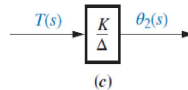


FIGURE 2.22 a. Physical system; b. schematic; c. block diagram



$$(J_1 s^2 + D_1 s + K) \theta_1(s) - K \theta_2(s) = T(s) \quad (1)$$

$$-K \theta_1(s) + (J_2 s^2 + D_2 s + K) \theta_2(s) = 0 \quad (2)$$

$$\text{then } G(s) = \frac{\theta_2(s)}{T(s)} \quad (\text{Do it})$$

Example 2.20

Equations of Motion by Inspection

PROBLEM: Write, but do not solve, the Laplace transform of the equations of motion for the system shown in Figure 2.25.

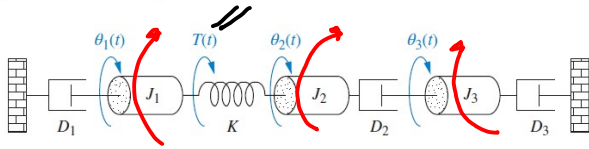


FIGURE 2.25 Three-degrees-of-freedom rotational system

$$(J_1 s^2 + D_1 s + K) \theta_1(s) - K \theta_2(s) - 0 \theta_3(s) = T(s) \quad (1)$$

$$-(K) \theta_1(s) + (J_2 s^2 + D_2 s + K) \theta_2(s) - D_2 s \theta_3(s) = 0 \quad (2)$$

$$0 \theta_1(s) - D_2 s \theta_2(s) + (J_3 s^2 + (D_3 + D_2) s + 0) \theta_3(s) = 0 \quad (3)$$

~

Skill-Assessment Exercise 2.9

PROBLEM: Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system shown in Figure 2.26.

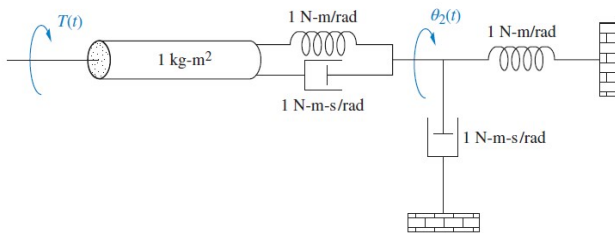
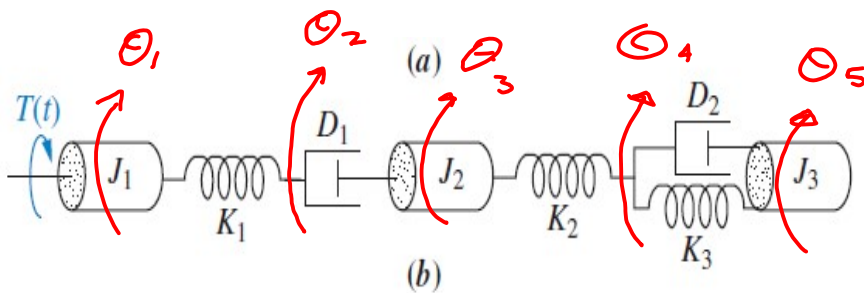


FIGURE 2.26 Rotational mechanical system for Skill-Assessment Exercise 2.9

ANSWER: $G(s) = \frac{1}{2s^2 + s + 1}$

The complete solution is at www.wiley.com/college/nise.

H.W.



32. For each of the rotational mechanical systems shown in Figure P2.17, write, but do not solve, the equations of motion. [Section: 2.6]

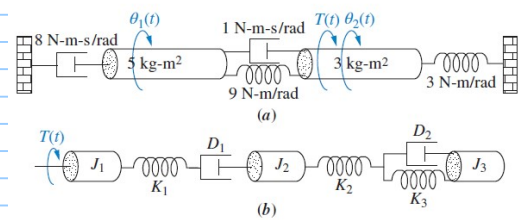


FIGURE P2.17

$$(J_1 s^2 + K_1) \theta_1(s) - K_1 \theta_2(s) = T(s) \quad (1)$$

$$-K_1 \theta_1(s) + (D_1 s + K_1) \theta_2(s) - D_1 s \theta_3(s) = 0 \quad (2)$$

$$D_1 s \theta_1(s) - D_1 s \theta_2(s) + (J_2 s^2 + D_1 s + K_2) \theta_3(s) - K_2 \theta_4(s) = 0 \quad (3)$$

$$D_1 s \theta_1(s) - D_1 s \theta_2(s) - K_2 \theta_3(s) + (D_2 s + [K_2 + K_3]) \theta_4(s) - (D_2 s + K_3) \theta_5(s) = 0 \quad (4)$$

$$-(D_2 s + K_3) \theta_4(s) + (J_3 s^2 + D_2 s + K_3) \theta_5(s) = 0 \quad (5)$$

Defining

$\theta_1(s)$ = rotation of J_1

$\theta_2(s)$ = rotation between K_1 and D_1

$\theta_3(s)$ = rotation of J_3

$\theta_4(s)$ = rotation of right-hand side of K_2

the equations of motion are

$$(J_1 s^2 + K_1) \theta_1(s) - K_1 \theta_2(s) = T(s)$$

$$-K_1 \theta_1(s) + (D_1 s + K_1) \theta_2(s) - D_1 s \theta_3(s) = 0$$

$$-D_1 s \theta_2(s) + (J_2 s^2 + D_1 s + K_2) \theta_3(s) - K_2 \theta_4(s) = 0$$

$$-K_2 \theta_3(s) + (D_2 s + (K_2 + K_3)) \theta_4(s) = 0$$

Sol. manual.

2.7 Transfer Functions for Systems with Gears

→ The distance traveled along each gear circumference is the same

$$r_1 \Theta_1 = r_2 \Theta_2$$

$$\frac{\Theta_2}{\Theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

→ The energy into G_1 = energy out of G_2

$$T_1 \Theta_1 = T_2 \Theta_2$$

$$\frac{T_2}{T_1} = \frac{\Theta_1}{\Theta_2} = \frac{N_2}{N_1}$$

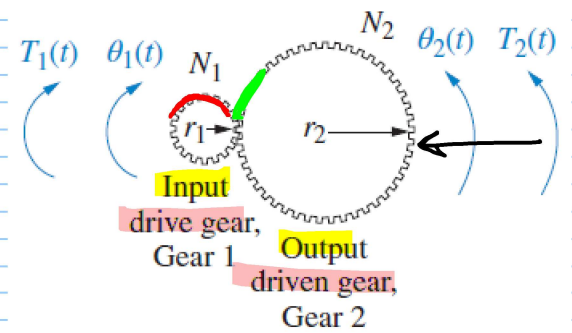
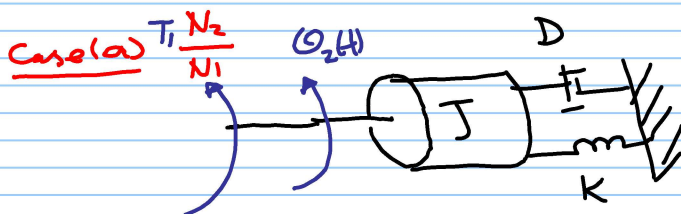
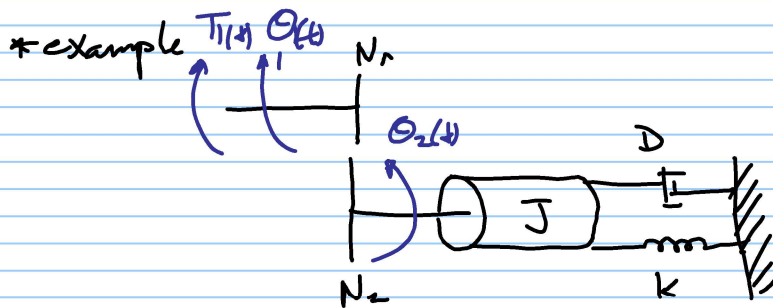


FIGURE 2.27 A gear system

→ what about mechanical impedances driven by gears?



$$\frac{\Theta_2}{\Theta_1} = \frac{N_1}{N_2}$$

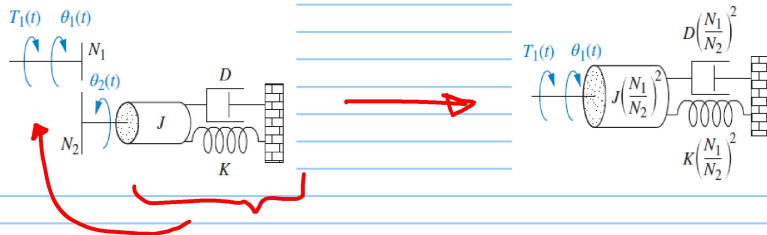
Case (b) The eq. of motion:-

$$(J s^2 + D s + K) \Theta_2(s) = T_1(s) \frac{N_2}{N_1}$$

$$\left((J s^2 + D s + K) \Theta_1(s) \frac{N_1}{N_2} = T_1(s) \frac{N_2}{N_1} \right) \times \frac{N_1}{N_2}$$

$$(J \dot{\theta}_1^2 + D \dot{\theta}_1 + K) \theta_1(s) \left(\frac{N_1}{N_2} \right)^2 = T_1(s)$$

$$\left[J \left(\frac{N_1}{N_2} \right)^2 \dot{\theta}_1^2 + D \left(\frac{N_1}{N_2} \right)^2 \dot{\theta}_1 + K \left(\frac{N_1}{N_2} \right)^2 \right] \theta_1(s) = T_1(s)$$



Generalizing the results, we can make the following statement: Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio

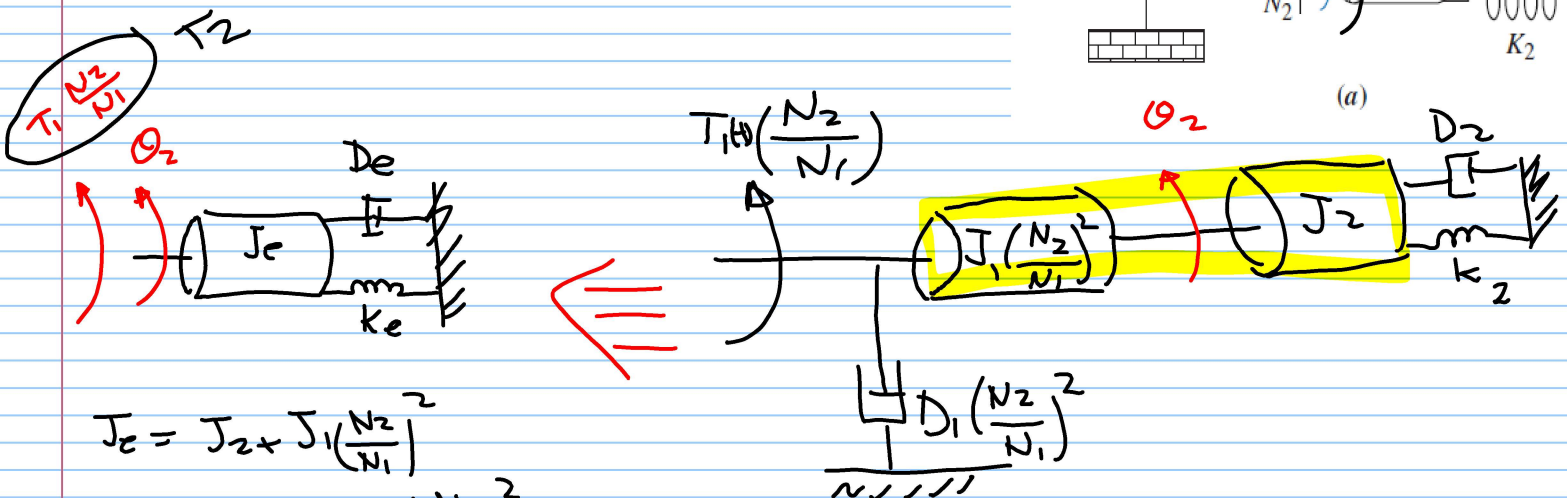
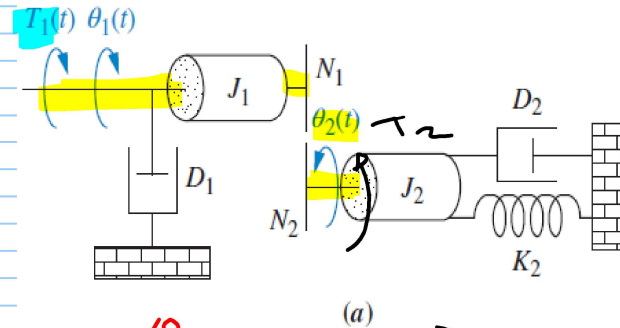
$$\left(\frac{\text{Number of teeth of gear on destination shaft}}{\text{Number of teeth of gear on source shaft}} \right)^2$$

$$\left(\frac{N_{\text{destination}}}{N_{\text{source}}} \right)^2$$

Example 2.21

Transfer Function—System with Lossless Gears

PROBLEM: Find the transfer function, $\theta_2(s)/T_1(s)$, for the system of Figure 2.30(a).



$$J_e = J_2 + J_1 \left(\frac{N_2}{N_1} \right)^2$$

$$D_e = D_2 + D_1 \left(\frac{N_2}{N_1} \right)^2$$

$$K_e = K_2$$

$$(J_e \dot{\theta}_2^2 + D_e \dot{\theta}_2 + K_e) \theta_2(s) = T_1(s) \frac{N_2}{N_1}$$

$$\rightarrow G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_e}$$

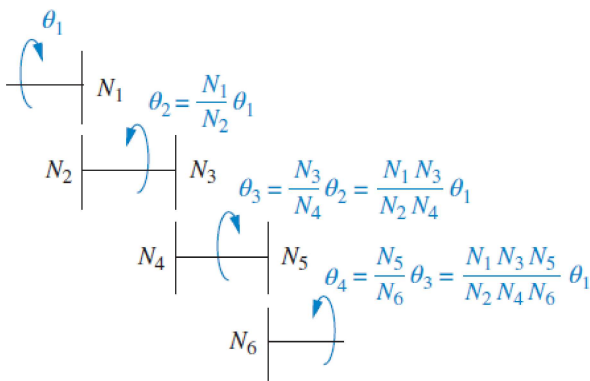


FIGURE 2.31 Gear train

$$\theta_4 = \frac{N_1 N_3 N_5}{N_2 N_4 N_6} \theta_1$$

Example 2.22

Transfer Function—Gears with Loss

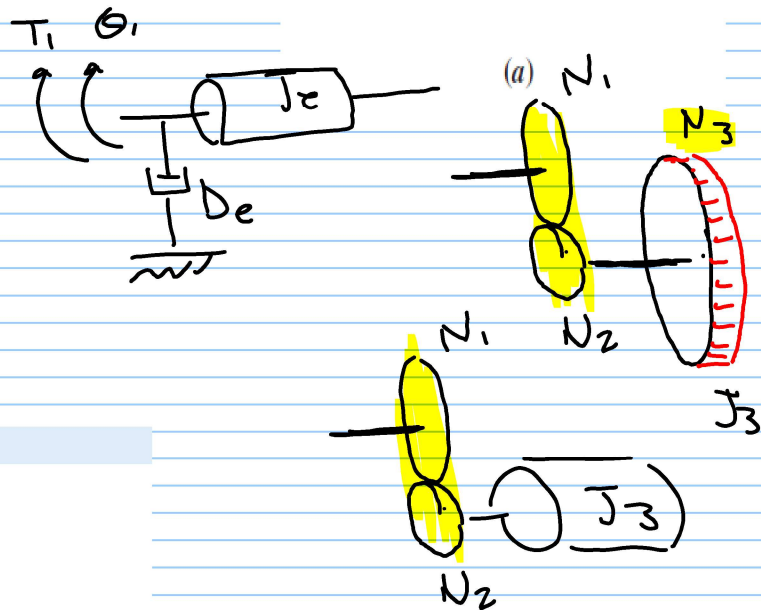
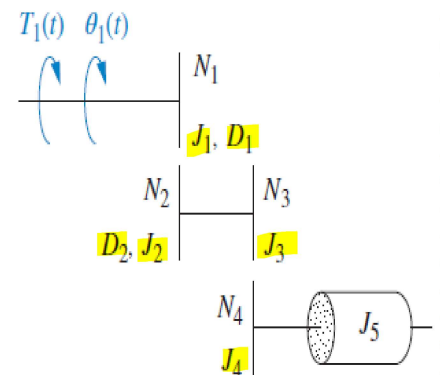
PROBLEM: Find the transfer function, $\theta_1(s)/T_1(s)$, for the system of Figure 2.32(a).

$$J_e = J_1 + (J_2 + J_3) \left(\frac{N_1}{N_2} \right)^2 + (J_4 + J_5) \left(\frac{N_3}{N_4} \frac{N_1}{N_2} \right)^2$$

$$D_e = D_1 + D_2 \left(\frac{N_1}{N_2} \right)^2$$

$$(J_e s^2 + D_e s) \Theta(s) = T_1(s)$$

$$G(s) = \frac{\Theta(s)}{T_1(s)} = \frac{1}{s(J_e s + D_e)}$$



Skill-Assessment Exercise 2.10

PROBLEM: Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system with gears shown in Figure 2.33.

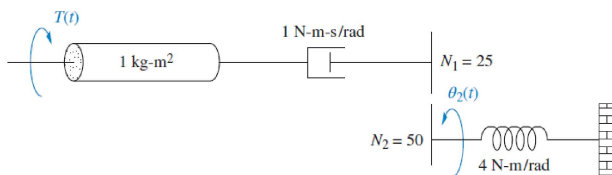


FIGURE 2.33 Rotational mechanical system with gears for Skill-Assessment Exercise 2.10

ANSWER: $G(s) = \frac{1/2}{s^2 + s + 1}$

The complete solution is at www.wiley.com/college/nise.

$$J_e = 150 + 3 \times \left(\frac{50}{5}\right)^2 + 100 \times \left(\frac{5}{25} \cdot \frac{50}{5}\right)^2$$

$$D_e = 500 \left(\frac{50}{25}\right)^2$$

$$K_e = 300 + 3 \times \left(\frac{50}{5}\right)^2$$

$$(J_e s^2 + D_e s + K_e) \Theta_2(s) = T(s) \times \frac{50}{5}$$

37. Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system shown in Figure P2.22. [Section: 2.7]

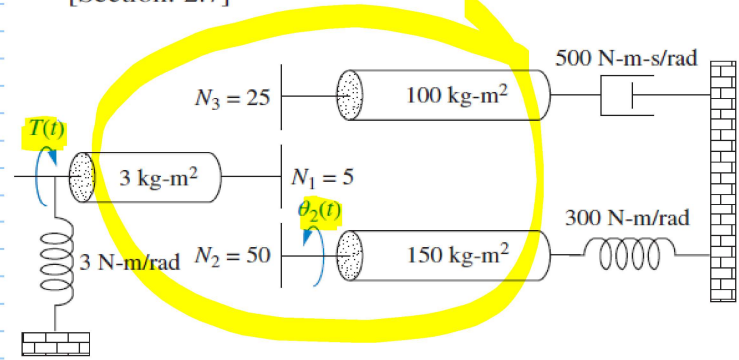
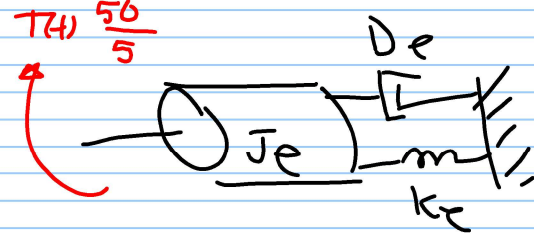
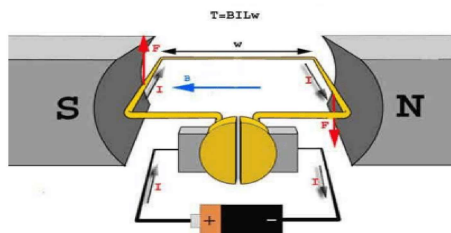
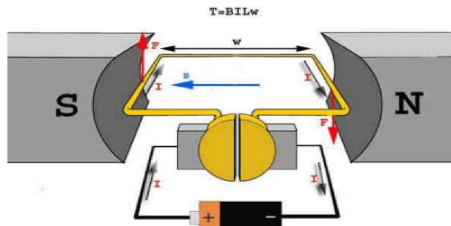
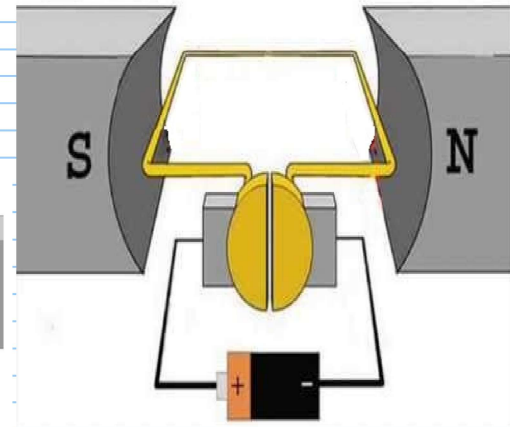
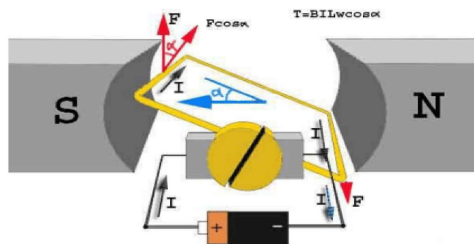
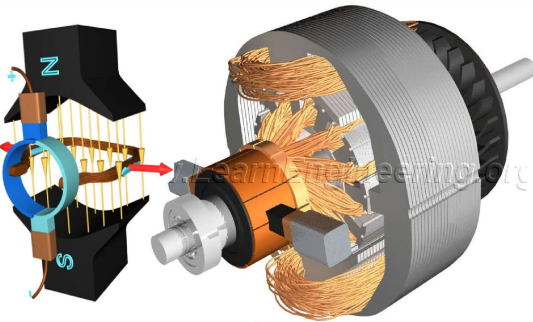
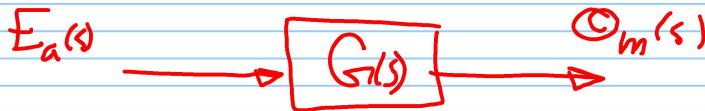


FIGURE P2.22



2.8 Electromechanical System Transfer Functions



$$F = B l i_a(t) \text{ (N)}$$

mag. field strength

$$e = B l v \text{ (V)}$$

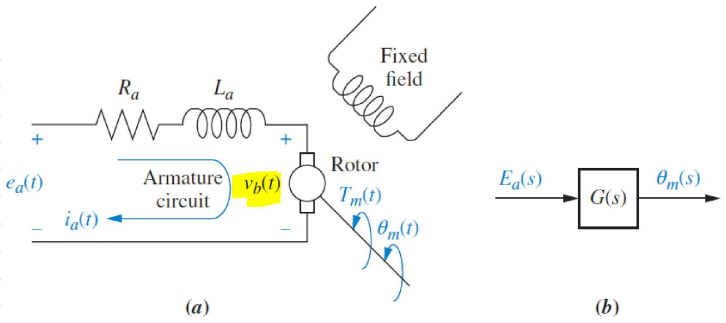
speed

* back emf $\propto$ speed

$$V_b(t) = K_b \frac{d}{dt} \theta_m(t)$$

$\downarrow$

$$V_b(s) = K_b s \theta_m(s) \quad \text{--- (A)}$$



DC motor: a. schematic; b. block diagram

$V_b(t)$: back electromotive force (back emf)

* developed torque $\propto$ armature current

$$T_m(s) = K_t I_a(s)$$

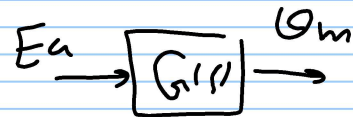
$$I_a(s) = \frac{1}{K_t} T_m(s) \quad \text{--- (B)}$$

$$\frac{d}{dt} \theta_m(t) = \omega_m(t)$$

* KVL

$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s) \quad \text{--- (1)}$$

$$(A) \& (B) \rightarrow (1)$$



$$(R_a + L_a s) \frac{1}{K_t} T_m(s) + K_b s \theta_m(s) = E_a(s) \quad \text{--- (2)}$$

$\hookrightarrow$ in terms of $\theta_m(s)$

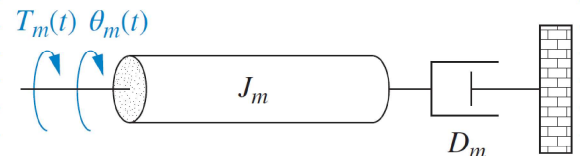
Note

$$J_m = J_a + J_L \text{ reflected to armature}$$

$$D_m = D_a + D_L \quad \text{,, , , ,}$$

$$T_m(s) = (J_m s^2 + D_m s) \theta_m(s) \rightarrow \text{sub in (2)}$$

$$\frac{(R_a + s L_a) (J_m s^2 + D_m s)}{K_t} \theta_m(s) + K_b s \theta_m(s) = E_a(s)$$



Typical equivalent mechanical loading on a motor

→ assume $L_a \ll R_a$ (usual for DC motors)

$$\left[\frac{R_a}{K_t} (J_m s + D_m) + K_b \right] s \Theta_m(s) = E_a(s)$$

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{R_a J_m}}{s \left[s + \frac{1}{J_m} \left(D_m + \frac{K_t K_b}{R_a} \right) \right]} = \frac{K}{s(s + \alpha)}$$

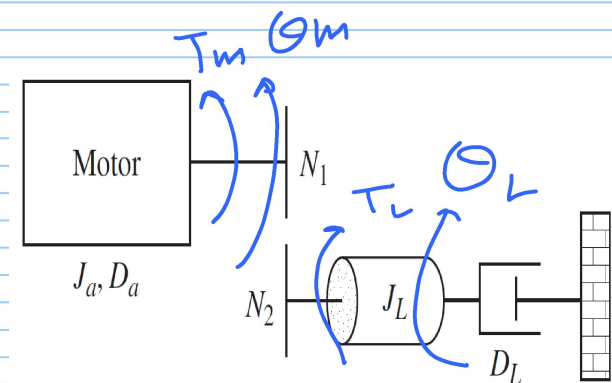
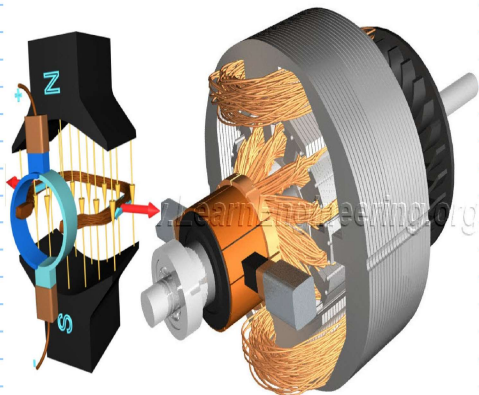
→ How to evaluate mechanical & electrical constants?

$$J_m = J_a + J_L \left(\frac{N_1}{N_2} \right)^2$$

$$D_m = D_a + D_L \left(\frac{N_1}{N_2} \right)^2$$

* (K_t & K_b)?

can be obtained through
dynamometer test



DC motor driving a rotational mechanical load

$$\left\{ \begin{array}{l} \frac{W_m}{E_a} \\ \Theta_m \quad W_m \checkmark \\ W_m(t) = \frac{d}{dt} \Theta_m(t) \\ W_m(s) = s \Theta_m(s) \end{array} \right.$$

→ Dynamometer measures T & W of the motor under the condition of a const. applied voltage (E_a)

→ from 2 $L_a = \text{zero}$

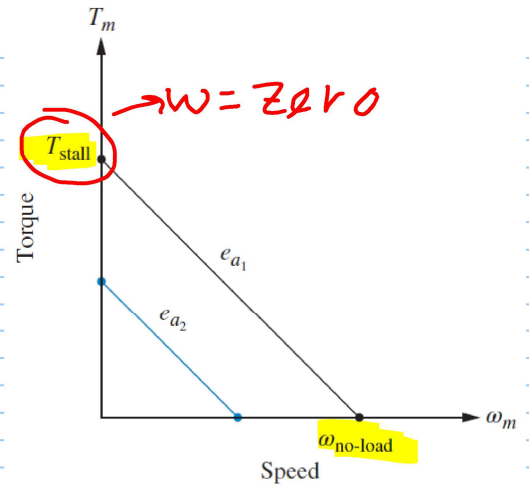
$$(\cancel{R_a + L_a s}) \frac{1}{K_t} T_m(s) + K_b s \Theta_m(s) = E_a(s)$$

$$\frac{R_a}{K_t} T_m(t) + K_b \omega_m(t) = e_a$$

under the S.S. condition

$$\frac{R_a}{K_t} T_m + K_b \omega_m = e_a$$

$$T_m = -\frac{K_b K_t}{R_a} \omega_m + \frac{K_t}{R_a} e_a$$



Torque-speed curves with an armature voltage, e_a , as a parameter

$$T_{stall} = \frac{K_t}{R_a} e_a$$

$$\omega_{nl} = \frac{e_a}{K_b}$$

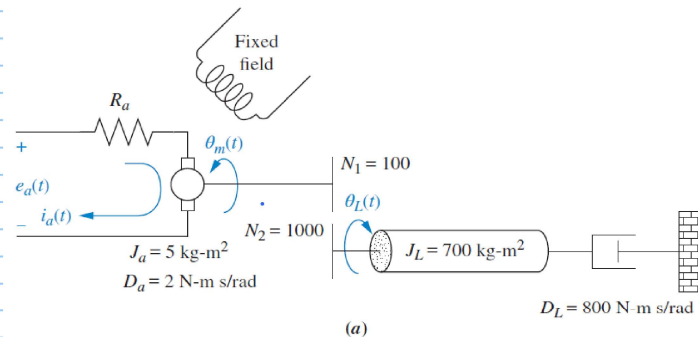
$$\frac{K_t}{R_a} = \frac{T_{stall}}{e_a} *$$

$$K_b = \frac{e_a}{\omega_{nl}} **$$

Transfer Function—DC Motor and Load

PROBLEM: Given the system and torque-speed curve of Figure 2.39(a) and (b), find the transfer function, $\theta_L(s)/E_a(s)$.

$$\begin{cases} J_m = 5 + 700 \left(\frac{100}{1000} \right)^2 = 12 \\ D_m = 2 + 800 \left(\frac{100}{1000} \right)^2 = 10 \end{cases}$$



$$\frac{K_t}{R_a} = \frac{T_{stall}}{e_a} = \frac{500}{100} = 5$$

$$K_b = \frac{e_a}{\omega_{nl}} = \frac{100}{50} = 2$$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{R_a J_m}}{s \left[s + \frac{1}{J_m} \left(D_m + \frac{K_t K_b}{R_a} \right) \right]}$$

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{0.417}{s(s+1.667)}$$

$$\Theta_m(s) \leftrightarrow \Theta_L(s)$$

$$\frac{\Theta_m(s)}{\Theta_L(s)} = \frac{N_2}{N_1} = 10 \Rightarrow \Theta_m(s) = 10 \Theta_L(s)$$

$$\frac{10 \Theta_L(s)}{E_a(s)} = \frac{0.417}{s(s+1.667)}$$

$$\frac{\Theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)}$$

$$\omega \rightarrow \text{rad/sec.}$$

$$n \rightarrow \text{rpm}$$

$$1 \text{ rpm} \leftrightarrow \frac{2\pi}{60} \text{ rad/sec.}$$

$$1 \text{ rad/s} \leftrightarrow \frac{60}{2\pi} \text{ rpm}$$