3.5: Bivariate Normal distribution.

Def: X, Y are called Bivariate Normal r.v with parameters "$\mu_1, \mu_2, \sigma_1, \sigma_2, \rho$" if they have the following joint p.d.f.

$$f(x,y) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} e^{-\frac{q}{2}}, \quad (x,y) \in \mathbb{R}^2.$$

Where $q = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]$

and $\mu_1 \in \mathbb{R}$, $\mu_2 \in \mathbb{R}$, $\sigma_1 > 0$, $\sigma_2 > 0$, $-1 < \rho < 1$.

Thm:

1. $f(x,y) \geq 0$, $\forall (x,y) \in \mathbb{R}^2$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

2. $X \sim N(\mu_1, \sigma_1^2)$

3. $Y \sim N(\mu_2, \sigma_2^2)$

4. $\rho = \frac{\text{cov}(X,Y)}{\sigma_1 \sigma_2}$

5. $X|y \sim N\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2),\ \sigma_1^2(1-\rho^2)\right)$.

6. $Y|x \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1),\ \sigma_2^2(1-\rho^2)\right)$.

7. $M(t_1,t_2) = e^{\mu_1 t_1 + \frac{1}{2}\sigma_1^2 t_1^2 + \mu_2 t_2 + \frac{1}{2}\sigma_2^2 t_2^2 + \rho\sigma_1\sigma_2 t_1 t_2}$, $(t_1,t_2) \in \mathbb{R}^2$.

Thm 3: X, Y bivariate Normal Randoms variable with mean μ_1, μ_2 and positive variance σ_1^2, σ_2^2 and correlation coeffeicent ρ then

X, Y independent iff $\rho = 0$

Proof:

$(\Rightarrow)$ X, Y indep $\Rightarrow E(XY) = E(X)\,E(Y)$

$\Rightarrow \operatorname{Cov}(X,Y) = 0$

$\Rightarrow \rho = 0$.

$(\Leftarrow)$ $\rho = 0$ and X, Y bivariate Normal

$\Rightarrow M(t_1, t_2) = e^{\mu_1 t_1 + \frac{1}{2}\sigma_1^2 t_1^2} \cdot e^{\mu_2 t_2 + \frac{1}{2}\sigma_2^2 t_2^2}$

$\Rightarrow M(t_1, t_2) = M_1(t_1) \cdot M_2(t_2)$.

$\Rightarrow$ X, Y independent.

exp 1: X_1 : height of husband

X_2 : height of wife

X_1, X_2 bivariate Normal

$\mu_1 = 5.8$ $\quad \sigma_1 = 0.2$ $\quad \rho = 0.6$.

$\mu_2 = 5.3$ $\quad \sigma_2 = 0.2$

1. Find $Pr(5.28 < X_2 < 5.92) = Pr\left(\frac{5.28 - 5.3}{0.2} < \frac{X_2 - \mu_2}{\sigma_2} < \frac{5.92 - 5.3}{0.2}\right)$

$= Pr(-0.1 < Z < 3.1)$ " $Z = \frac{X_2 - \mu_2}{\sigma_2} \sim N(0,1)$ "

$= \phi(3.1) - \phi(-0.1)$

$= \phi(3.1) - (1 - \phi(0.1))$

$= 0.999 - (1 - 0.540)$

$= 0.539$

2. Find $Pr(5.28 < X_2 < 5.92 \mid X_1 = 6.3)$.

$\rightarrow E(X_2 \mid X_1 = 6.3) = \mu_2 + \rho \dfrac{\sigma_1}{\sigma_2}(x_1 - \mu_1)$

$= 5.3 + 0.6 \dfrac{0.2}{0.2}(6.3 - 5.8)$

$= 5.6$

$\rightarrow Var(X_2 \mid X_1 = 6.3) = \sigma_2^2(1-\rho^2) = (0.2)^2(1-(0.6)^2) = (0.16)^2$

$\rightarrow Pr(5.28 < X_2 < 5.92 \mid X_1 = 6.3) = Pr\left(\dfrac{5.28-5.6}{0.16} < Z < \dfrac{5.92-5.6}{0.16}\right)$ " $Z \sim N(0,1)$ "

$Pr(-2 < Z < 2) = \Phi(2) - \Phi(-2)$

$= \Phi(2) - (1 - \Phi(2))$

$= 2\Phi(2) - 2$

$= 2(0.977) - 1$

$= 0.954$.

Remark:

$(5.28, 5.92)$: 53.9 % Prediction interval for X_2

95.4 % Prediction interval for $X_2 \mid X_1 = 6.3$.

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