3.6 Row Space and Column Space

Def Let A be $m\times n$ matrix.

- The subspace of $\mathbb{R}^{1\times n} = \mathbb{R}^n$ spanned by the row vectors of A is called the row space of A
- The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the column space of A

Exp Find row and column spaces of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

- Row space of $A = \{\vec{x} \in \mathbb{R}^3 : \alpha(1,0,0) + \beta(0,1,0) = \vec{x}\}$

$$= \{\vec{x} \in \mathbb{R}^3 : \vec{x} = (\alpha, \beta, 0), \ \alpha, \beta \in \mathbb{R}\}$$

Hence, Row space of A is two-dimensional subspace of $\mathbb{R}^{1\times 3}$

- Coulmn space of $A = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} + \gamma\begin{pmatrix}0\\0\end{pmatrix}\right\}$

$$= \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \ \alpha, \beta \in \mathbb{R}\right\} = \mathbb{R}^2$$

Hence, the dimension of the column space is 2.
That is, the column space of A is $\mathbb{R}^2$.

Th 6.1 Two row-equivalent matrices have the same row space.

Proof • If B is row equivalent to A, then B can be formed from A by a finite sequence of row operations.

- Thus, the vectors row of B are linear combinations of the vectors row of A

✓ Hence, the row space of B is a subspace of the row space of A.

- But if B is row equivalent to A, then A is row equivalent to B.

✓ Hence, the row space of A is a subspace of the row space of B.

Def The rank of a matrix is the dimension of the row space of A, denoted by $\operatorname{rank}(A)$.

* To find $\operatorname{rank}(A)$, we reduce A to row echelon form. The nonzero rows of the row echelon matrix will form a basis for the row space.

Exp Find the $\operatorname{rank}(A)$ if $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix}$

- The row echelon form of A is

$$U = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

- The set $\{(1,-2,3),(0,1,5)\}$ form a basis for the row space of U
- Since U and A are row equivalent $\Rightarrow$ they have the same row space (Th 6.1)
- Hence, $\operatorname{rank}(A) = 2$

Remark Recall that a consistent linear system $Ax=b$ can be written as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \cdots (*)$$

Th 6.2 (Consistency Th for Linear Systems)

A linear System $Ax=b$ is consistent iff b is in the column space of A

Proof $\Rightarrow$ If $Ax=b$ is consistent then $(*)$ holds $\Rightarrow$ b is in the column space of A

$\Leftarrow$ If b is in the column space of A, then $(*)$ holds for some $(x_1, x_2, \dots, x_n)^T = X$. Hence, X is a solution to $Ax=b$.

Exp For each of the following choices of A and b, determine whether b is in the column space of A and state whether the system $Ax=b$ is consistent.

1) $A=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$, $b=\begin{bmatrix}4\\ 8\end{bmatrix}$

$$\left[\begin{array}{cc|c}1 & 2 & 4\\ 2 & 4 & 8\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 2 & 4\\ 0 & 0 & 0\end{array}\right]$$

The system is consistent $\Rightarrow$ b is in the column space of A.

" infinitly many solution since there is one free variable "

2) $A=\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}$, $b=\begin{bmatrix}4\\ 6\end{bmatrix}$

$$\left[\begin{array}{cc|c}2 & 1 & 4\\ 3 & 4 & 6\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}2 & 1 & 4\\ 0 & 5/3 & 0\end{array}\right]$$

The system is consistent $\Rightarrow$ b is in the column space of A

" Unique solution since there is no free variables "

3) $A=\begin{bmatrix}3 & 6\\ 1 & 2\end{bmatrix}$, $\begin{bmatrix}1\\ 1\end{bmatrix}$

$$\left[\begin{array}{cc|c}3 & 6 & 1\\ 1 & 2 & 1\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}3 & 6 & 1\\ 0 & 0 & 2\end{array}\right]$$

The system is inconsistent $\Rightarrow$ b is not in the column space of A

" no solution "

That is, $\nexists$ α_1, α_2 s.t

$$\alpha_1\begin{pmatrix}3\\ 1\end{pmatrix} + \alpha_2\begin{pmatrix}6\\ 2\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}$$

Remark': If $b=0$, then $Ax=0$ can be written as

$$x_1a_1 + x_2a_2 + \cdots + a_nx_n = 0 \quad \cdots\cdots * ^2$$

• Thus, $Ax=0$ has only the trivial solution, $\vec{x}=\vec{0}$, iff the column vectors of A are linearly independent.

Th 6.3 Let A be $m\times n$ matrix. Then,

$\boxed{1}$ $Ax=b$ is consistent for every $b\in\mathbb{R}^m$ iff the column vectors of A span $\mathbb{R}^m$.

$\boxed{2}$ $Ax=b$ has at most one solution for every $b\in\mathbb{R}^m$ iff the column vectors of A are linearly independent.

Proof $\boxed{1}$ By Th6.2, $Ax=b$ is consistent iff b is in the column space of A.

$Ax=b$ is consistent for every $b\in\mathbb{R}^m$ $\Leftrightarrow$ the column vecters of A span $\mathbb{R}^m$

$\boxed{2}$ $\Rightarrow$ Take $b=0$. If $Ax=b$ has at most one solution, then $Ax=0$ can have only the trivial solution. Hence, the column of A must be linearly independent.

$\Leftarrow$ • If the column vectors of A are linearly independent, then $Ax=0$ has only the trivial solution.

• If x_1 and x_2 were two solutions for $Ax=b$, then x_1-x_2 is a solution for $Ax=0$ since $A(x_1-x_2) = Ax_1 - Ax_2 = b-b=0$

• Thus, $x_1-x_2=0 \Rightarrow x_1=x_2$

It follows from Th 6.3 that: The column vectors of $A_{n\times n}$ form a basis for $\mathbb{R}^n$ iff $Ax=b$ has a unique solution for each $b\in\mathbb{R}^n$.

Remark² Let A be $m\times n$ matrix.

$\boxed{1}$ If the column vectors of A span $\mathbb{R}^m$, then $n \geq m$.
since no set of fewer than m vectors could span $\mathbb{R}^m$.

$\boxed{2}$ If the column vectors of A are linearly independent, then $n \leq m$. since every set of more than m vectors in $\mathbb{R}^m$ are linearly dependent.

$\boxed{3}$ Thus, If the column vectors of A form a basis for $\mathbb{R}^m$, then $n = m$

Corollary 6.4: The matrix $A_{n\times n}$ is nonsingular iff the column vectors of A form a basis for $\mathbb{R}^n$.

Proof: Th 6.3 provides that:

The column vectors of A form a basis for $\mathbb{R}^n$ iff $Ax=b$ has a unique solution for each $b \in \mathbb{R}^n$.

$\Rightarrow$ If A is nonsingular, then $Ax=b$ has a unique solution $x = A^{-1}b$. Thus, the column vectors of A form a basis for $\mathbb{R}^n$ (Th 6.3)

$\Leftarrow$ If the column vectors of A form a basis for $\mathbb{R}^n$, then $Ax=b$ has a unique solution for each $b \in \mathbb{R}^n$ (Th 6.3). Hence,

$Ax=0$ has the only trivial solution $x=0$.

Thus, by Th 1.4.2, A is nonsingular.

Th 6.5 (The Rank - Nullity Th)

If A is $m \times n$ matrix, then $\boxed{Rank(A) + \dim(N(A)) = n}$.

Nullity $= \dim(N(A))$: the dimension of the null space of A = # of free variables.

n : the number of columns.

$R(A) = \dim(\text{row space of } A)$.

Proof
- Let U be the reduced row echelon form of A.
- $Ax = 0$ is equivelent to $Ux = 0$
- If $Rank(A) = r$, then U has r nonzero rows

$\Rightarrow$ $Ux = 0$ has r leading variables and $n - r$ free variables.

$\Rightarrow$ $\dim(N(A)) = n - r$

Exp* Let $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}_{3\times 4}$.

[1] Find Rank (A).

The reduced' row echelon form of A is

$$U = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

': We can consider the row echelon form (Since they are equivalent).

$\{(1,2,0,3), (0,0,1,2)\}$ is a basis for the row space of A.

$Rank(A) = \dim(\text{row space of } A) = 2$

[2] Nullity of A

Nullity of $A = \dim(N(A)) = n - Rank(A) = 4 - 2 = 2$

This is the number of free variables

[3] Find a basis for $N(A)$

The systems $Ax = 0$ and $Ux = 0$ are equivalent $\Rightarrow$

$N(A) = \{x \in \mathbb{R}^4 : Ux = 0\}$ $\Rightarrow$

$$Ux = 0 \iff \left[\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 3 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_4 = \beta$, $x_2 = \alpha$ $\Rightarrow x_3 = -2\beta$

$\Rightarrow x_1 = -3\beta - 2\alpha$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2\alpha - 3\beta \\ \alpha \\ -2\beta \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Hence, $N(A) = \left\{ X \in \mathbb{R}^4 : X = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \alpha, \beta \in \mathbb{R} \right\}$

Thus, $\{(-2,1,0,0)^T, (-3,0,-2,1)^T\}$ is a basis for $N(A)$

Nullity $= \dim(N(A)) = 2$

Remark[3] $\boxed{1}$ If A is $m \times n$ matrix and U is the row echelon form of A, then their column vectors satisfy the same dependency relations.

To see that in Exp*:

→ For the matrix U: u_1 and u_3 are linearly independent

so ⇒ $u_2 = 2u_1$

$u_4 = 3u_1 + 2u_3$

→ The same relations hold for columns of A:

$a_2 = 2a_1$

$a_4 = 3a_1 + 2a_3$

$\boxed{2}$ Two row equivalent matrices may have different column spaces

To see that in Exp*:

Column space of $A = \{X \in \mathbb{R}^3 : X = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \alpha_4 a_4\} = C(A)$

Column space of $U = \{X \in \mathbb{R}^3 : X = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4\} = C(U)$

when $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ $\Rightarrow (3,3,9) \in C(A)$ but $(3,3,9) \notin C(U)$.

Th 6.6 If A is $m \times n$ matrix, then

dimension of the row space of A = dimension of the column space of A.

Proof. Let A be $m \times n$ matrix with dimension of the row space r.

- That is, $\text{rank}(A) = r \Rightarrow$
- The row echelon form U of A will have r leading 1's $\Rightarrow$
- The columns of U corresponding to the lead 1's are linearly independent "They don't form a basis for the column space of A (see Remark[3])"
- Let U_L be the matrix obtained from U by deleting all the columns corresponding to the free variables.

  A_L 〃 〃 〃 〃 〃 A 〃 〃 〃

  〃 〃 〃 〃 〃 〃 〃

- A_L and U_L are row equivalent.
- Hence, if x is a solution for $A_L x = 0$, then

  x 〃 〃 〃 〃 $U_L x = 0$.

- Since the columns of U_L are linearly independent $\Rightarrow$

  x must be 0

Thus, by Remark[1] $\Rightarrow$ the columns of A_L are linearly independent.

$\Rightarrow$ Since A_L has r column

$\Rightarrow$ dimension of the column space of $A \geq \underbrace{r}_{= \text{rank}(A)}$

Hence, $\boxed{\dim(\text{column space of } A) \geq \dim(\text{row space of } A)}$ (1)

- Apply this result for A^T:

$$\boxed{\begin{aligned} \dim(\text{row space of } A) &= \dim(\text{column space of } A^T) \\ &\geq \dim(\text{row space of } A^T) \\ &= \dim(\text{column space of } A) \end{aligned}}$$ (2)

Combining (1) and (2) to conclude the proof.

Exp Let $A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix}$.

Find a basis for the column space of A.

- The row echelon form of A is $U = \begin{bmatrix} \boxed{1} & -2 & 1 & 1 & 2 \\ 0 & \boxed{1} & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
- The leading 1's occur in the 1^{st}, 2^{nd}, 5^{th} columns. Thus,

$a_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \\ 2 \end{pmatrix}$, $a_5 = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 5 \end{pmatrix}$ form a basis for the column space of A.

Note $\boxed{1}$. We can use the row echelon form U of A to find a basis for the column space of A.

- We need only to determine the columns of U that correspond to the leading 1's.
- These same columns of A will be linearly independent and form a basis for the column space of A.

$\boxed{2}$
- The row echelon form U tells us only which columns of A to use to form a basis.
- We can not use the column vectors from U, since in general, U and A have different column spaces.

ee Exp

Exp Find the dimension of the subspace of $\mathbb{R}^4$ spanned by

$x_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ 5 \\ -3 \\ 2 \end{pmatrix}$, $x_3 = \begin{pmatrix} 2 \\ 4 \\ -2 \\ 0 \end{pmatrix}$, $x_4 = \begin{pmatrix} 3 \\ 8 \\ -5 \\ 4 \end{pmatrix}$

$X = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 5 & 4 & 8 \\ -1 & -3 & -2 & -5 \\ 0 & 2 & 0 & 4 \end{bmatrix}$ has row echelon form $\begin{bmatrix} \boxed{1} & 2 & 2 & 3 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\{x_1, x_2\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ -3 \\ 2 \end{pmatrix} \right\}$ form a basis for the column spaace of X.

$\Rightarrow$ $\dim(\text{Span}(x_1, x_2, x_3, x_4)) = 2$

Exp Let $A=\begin{bmatrix}1&3&2\\2&1&4\\4&7&8\end{bmatrix}$

[1] Find a basis for the row space.

- The row echelon form of A is $\begin{bmatrix}\boxed{1}&3&2\\0&\boxed{1}&0\\0&0&0\end{bmatrix}$
- $\{(1,3,2)^T, (0,1,0)^T\}$ is basis for the row space.

[2] Find a basis for the column space.

- leading variables are x_1 and x_2
- $\left\{\begin{pmatrix}1\\2\\4\end{pmatrix}, \begin{pmatrix}3\\1\\7\end{pmatrix}\right\}$ is a basis for the column space.

[3] Find a basis for the null space

- Nullity $=\dim(N(A))= 1$ = one free variable

  $x_3=\alpha \Rightarrow x_2=0 \Rightarrow x_1=-2\alpha$
- $N(A)=\left\{x\in \mathbb{R}^3 : x=\begin{pmatrix}-2\alpha\\0\\\alpha\end{pmatrix}=\alpha\begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}$
- $\left\{\begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}$ is a basis for the null space

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Exp Find the dimension of the subspace of $\mathbb{R}^3$ spanned by

$$\underbrace{\begin{pmatrix}1\\-2\\2\end{pmatrix}}_{x_1}, \underbrace{\begin{pmatrix}2\\-2\\4\end{pmatrix}}_{x_2}, \underbrace{\begin{pmatrix}-3\\3\\6\end{pmatrix}}_{x_3} \Rightarrow \begin{vmatrix}1&2&-3\\-2&-2&3\\2&4&6\end{vmatrix}=\begin{vmatrix}1&2&-3\\0&2&-3\\0&0&12\end{vmatrix}=24\neq 0$$

$\Rightarrow \dim(\mathrm{span}(x_1, x_2, x_3))=3$

$\Downarrow$ or we can transform

$\begin{bmatrix}1&2&-3\\-2&-2&3\\2&4&6\end{bmatrix}$ to row echelon form U

and U will not have free variables