Calculus (2) (10.1)

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- A sequence. Is a list of numbers $a_1, a_2, \ldots\ldots, a_n$ → الحد العام
- a_i numbers with index i "order"
- It can be finite or infinite
- It is a function that sends 1 to a_1 , 2 to a_2, n to a_n
- Sequence
  - Converge
  - Diverge
  - Infinite
  - Finite

Exp(1): $a_n=\sqrt{n}$, n =1,2,3,...

n=1⇒ $a_1 = \sqrt{1}$

n=2⇒ $a_2 = \sqrt{2}$

n=100⇒ $a_{100} = \sqrt{100}$

الارقام بتكبر و رايحة الى المالانهاية

$$\lim_{n\to\infty} \sqrt{n} = \infty \Rightarrow a_n = \sqrt{n}(diverge)$$

Exp(2) : $b_n = \frac{1}{n}, \text{n} = 1,2,3,\ldots.$

$b_1 = 1 , b_2 = \frac{1}{2}, b_\infty = \frac{1}{\infty}$=0

الارقام بتصغر

$\lim_{n\to\infty} \frac{1}{n} = 0$ (converge)

- Find the n th term of the following sequences:
- 1,-4,9,-16,25,....   N=1,2,3,..
- $a_n = n^2(-1)^{n+1}$

حطيت +1 لانه اول حد عندي موجب

حطينا هاد الحد عشان الاشارة

العلاقة بين الاعداد دون الانتباه الى الاشارة

- 1,-4,9,-16,25,....   N=0,1,2,3,.....
- $a_n = (-1)^n(n+1)^2$
- 0,3,8,15,24.   N =1,2,3,.....
- $a_n = n^2 - 1$
- -3,-2,-1,0,1,....   N=1,2,3,...
- $a_\eta = n - 4$

- Exp(3): $C_n = (-1)^n \frac{1}{n}$, alternating sequence. (الحد مرة موجب و مرة سالب)
- N=1,2,3,…….
- N=1 $\Rightarrow (-1)^1 \frac{1}{1}$ =-1
- N=2 $\Rightarrow (-1)^2 \frac{1}{2} = \frac{1}{2}$
- $C_n \to 0 \; as \; n \to \infty, \; \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

- Th: $\lim_{n \to \infty} a_n$ = A→ number , $a_n \to A, n \to \infty$ (converge)
- $\lim_{n\to\infty} b_n = B \to number$, $b_n \to B, n \to \infty$ (converge)
- Than :- 1) $\lim_{n\to\infty} (a_n + b_n) = A + B$
- 2) $\lim_{n\to\infty} (a_n - b_n) = A - B$
- 3) $\lim_{n\to\infty} k(a_n) = \text{kA}$
- 4) $\lim_{n\to\infty} (a_n b_n) = AB$
- 5) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, $B \neq 0$

- Exp(1): $\lim_{n\to\infty} \frac{-\sqrt{5}}{n} = -\sqrt{5} \lim_{n\to\infty} \frac{1}{n}$ =-$\sqrt{5}(0) = 0$
- Exp(2): $\lim_{n\to\infty} \frac{7\text{n}-3}{5+14\text{n}}$= $\frac{7}{14} = \frac{1}{2}$
- Exp (3): $\lim_{n\to\infty} \frac{7\text{n}-3}{5+14n^2}$=zero
- Exp(4) : $\lim_{n\to\infty} \frac{7n^3-3}{5+14n}$=∞

- Sandwich th: I need to know $\lim_{n\to\infty} b_n$ if $a_n \le b_n \le c_n$ for all n and
- $\lim_{n \to \infty} a_n$ = $\lim_{n \to \infty} c_n$ =L , then $\lim_{n \to \infty} b_n$ =L
- Exp(1): Check convergence / divergence
- $\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$, $\lim_{n\to\infty} \frac{1}{n} = 0$, $\lim_{n\to\infty} \frac{-1}{n} = 0$ then by ST $\lim_{n\to\infty} \frac{\sin n}{n}$=0
- Exp(2): $\lim_{n\to\infty} \frac{\sin^2 n}{2^n}$, $\frac{0}{2^n} \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$
- $\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \frac{1}{2^n} = 0$ then by ST $\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0$
- Exp(3): $\lim_{n\to\infty} (-1)^n \frac{1}{n}$, $\frac{-1}{n} \le (-1)^n \frac{1}{n} \le \frac{1}{n}$
- $\lim_{n\to\infty} \frac{-1}{n} = o$ $\lim_{n\to\infty} \frac{1}{n} = 0$ then by ST $\lim_{n\to\infty} (-1)^n \frac{1}{n} = 0$

- Th(5) (6 cases)
- 1) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
- Exp : $\lim_{n\to\infty} \frac{\ln n^3}{3n} \Rightarrow \lim_{n\to\infty} \frac{3\ln n}{3n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0$
- 2) $\lim_{n\to\infty} n\sqrt{n} = 1$
- Exp(1): $\lim_{n\to\infty} \frac{\ln n}{\sqrt[n]{n}} = \frac{\lim_{n\to\infty} \ln n}{\lim_{n\to\infty} \sqrt[n]{n}} = \frac{\infty}{1} = \infty$
- Exp(2): $\lim_{n\to\infty} \sqrt[n]{n^3} = \lim_{n\to\infty} (n^3)^{\frac{1}{n}} = \lim_{n\to\infty} \left(n^{\frac{1}{n}}\right)^3 = (1)^3 = 1$
- Exp(3) : $\lim_{n\to\infty} \sqrt[n]{\pi n} = \lim_{n\to\infty} (\pi)^{\frac{1}{n}} (n)^{\frac{1}{n}} = \lim_{n\to\infty} (\pi)^{\frac{1}{n}} = 1$
- 3) $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$ when x>0

- 4)If -1<x<1 ⇒ then $\lim_{n\to\infty}(x)^n = 0$
- Exp(1): $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$
- Exp(2): $\lim_{n\to\infty}\left(\frac{-2}{3}\right)^n = 0$
- Exp(3): $\lim_{n\to\infty}\frac{\pi^{-n}}{e^{-n}} \Rightarrow \lim_{n\to\infty}\left(\frac{\pi}{e}\right)^{-n} \Rightarrow \lim\left(\frac{e}{\pi}\right)^n = 0$
- 5) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$
- $\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-1}=\frac{1}{e}$
- Exp : $\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^3$
- 6) $\lim_{n\to\infty}\frac{x^n}{n!} = 0$ for any x
- Exp: $\lim_{n\to\infty}\frac{2^n}{n!}=0$, $\lim_{n\to\infty}\frac{\frac{1}{2}^n}{n!}=0$, $\lim_{n\to\infty}\frac{\pi^n}{n!}=0$
- Exp(2): $\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n \Rightarrow \lim_{n\to\infty}\left(\frac{n-1+2}{n-1}\right)^n \Rightarrow \lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^n$, let u=n-1 ⇒n =u+1
- $\lim_{n\to\infty}\left(1+\frac{2}{u}\right)^{u+1} \Rightarrow \lim_{u\to\infty}\left(1+\frac{2}{u}\right)\left(1+\frac{2}{u}\right)^u$ =(1+0)(e^2) $= e^2$

Recursive sequence

- Exp :$a_1 = 1, a_{n+1} = \frac{1}{2} a_n$
- Assume this sequence converges find its limit
- $a_2 = a_{1+1} = \frac{1}{2} a_1 \Rightarrow a_2 = \frac{1}{2}$
- $a_3 = a_{2+1} = \frac{1}{2} a_2 \Rightarrow \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
- $a_n = \left(\frac{1}{2}\right)^{n-1} \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{-1} \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n}$=2.0=0

- Def:a sequence is bounded from above if ∃ number M such that $a_n \leq$M for all n

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- A sequence $\{a_n\}$ is bounded from below if ∃ a number m such that $a_n \geq$m for all n

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- A sequence. $\{a_n\}$ is bounded If it is bounded from above and from below
- A sequence. $\{a_n\}$ is not bounded If it is not bounded from above or from below
- Exp : ....,-2,-1,0,1,2,3,..... (not bounded sequence)
- A sequence $\{a_n\}$ is non decreasing if $a_n \leq a_{n+1}$ for all n $a_1 \leq a_2 \leq a_3$ ...
- A sequence $\{a_n\}$ is non increasing if $a_n \leq a_{n+1}$ for all n $a_1 \geq a_2 \geq a_3$ ...
- The. sequence $\{a_n\}$ monotomic if it is either non decreasing or non increasing

- Exp(1) : 1,2,3,4,5,….N. $a_n = n$, n=1,2,3,…..
- الحدود مع الوقت تزيد$\Rightarrow a_n (non\ decreasing\ \&monotomic)$
- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$ (Diverges)
- m=1,0,-1,-2,-3,…. a_n is only bounded from below so its not bounded

Greatest lower bound

- EXP(2): $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots..$ $b_n = \left(\frac{1}{2}\right)^n$ n =0,1,2,3,…(converges)
- الحدود مع الوقت تقل⇒(NON INCREASING &monotomic)
- b_n is bounded from above and below (bounded)
- M=1,2,3,….   m=0,-1,-2,…

Least upper. Bound

Greatest lower bound

- Exp(3): 3,3,3,3,.... $c_n = 3.$ n=1,2,3,... (converges to 3)
- c_n is non decreasing & non increasing & monotomic
- And its bounded from below and above (Bounded)
- M=3,4,5,..   m=3,2,1,,....

Greatest. Lower bound

Least upper bound

Exp(4) : 1,-1,1,-1,... (not monotomic)

- Th: if a sequence. $\{a_n\}$ $is\ both\ bounded$ & $monotomic\ then\ a_n\ is\ converges$
- Exp : $a_n = \frac{1}{n}$ $\quad n = 1,2,3, \ldots$
- = $1, \frac{1}{2}, \frac{1}{3}, \ldots$
- 1)Find m , M
- M=1,2,3,,....
- m=0,-1,-2
- 2) does $a_n monotomic$?
- $a_n\ is\ non\ increasing\ , so\ yes\ its\ monotomic$
- 3) does $a_n\ bounded$?
- Yes , since we found M , m
- 4) does $a_n\ converges$?
- $Yes\ , since\ a_n\ monotomic\ and\ bounded$
- $\lim_{n \to \infty} a_n \Rightarrow \lim_{n \to \infty} \frac{1}{n} = \frac{1}{\infty} = 0$

- OUTLINE SOLUTION

FIND THE VALUES OF A1 ,A2,A3 AND A4.

4. $a_n = 2 + (-1)^n$

$a_1 = 2 + (-1)^1 = 1$, $a_2 = 2 + (-1)^2 = 3$, $a_3 = 2 + (-1)^3 = 1$, $a_4 = 2 + (-1)^4 = 3$

WRITE OUT THE FIRST TEN TERMS OF THE SEQUENCE.

10. $a_1 = -2, \quad a_{n+1} = na_n/(n+1)$

$a_1 = -2$, $a_2 = \frac{1\cdot(-2)}{2} = -1$, $a_3 = \frac{2\cdot(-1)}{3} = -\frac{2}{3}$, $a_4 = \frac{3\cdot\left(-\frac{2}{3}\right)}{4} = -\frac{1}{2}$, $a_5 = \frac{4\cdot\left(-\frac{1}{2}\right)}{5} = -\frac{2}{5}$, $a_6 = -\frac{1}{3}$,
$a_7 = -\frac{2}{7}$, $a_8 = -\frac{1}{4}$, $a_9 = -\frac{2}{9}$, $a_{10} = -\frac{1}{5}$

Find a formula for the n th term of the sequence

16. The sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

$a_n = \frac{(-1)^{n+1}}{n^2}$, $n = 1, 2, \ldots$

22. The sequence 2, 6, 10, 14, 18, …

$a_n = 4n - 2$, $n = 1, 2, \ldots$

26. The sequence 0, 1, 1, 2, 2, 3, 3, 4, …

$$a_n = \frac{n - \frac{1}{2} + (-1)^n \left(\frac{1}{2}\right)}{2} = \left\lfloor \frac{n}{2} \right\rfloor, n = 1, 2, \ldots$$

30. $a_n = \frac{2n + 1}{1 - 3\sqrt{n}}$

$$\lim_{n \to \infty} \frac{2n+1}{1-3\sqrt{n}} = \lim_{n \to \infty} \frac{2\sqrt{n} + \left(\frac{1}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}} - 3\right)} = -\infty \Rightarrow \text{diverges}$$

WHICH OF THE SEQUENCES CONVERGE, AND WHICH DIVERGE? FIND THE LIMIT OF EACH CONVERGENT SEQUENCE.

30. $a_n = \dfrac{2n+1}{1-3\sqrt{n}}$

$$\lim_{n\to\infty} \frac{2n+1}{1-3\sqrt{n}} = \lim_{n\to\infty} \frac{2\sqrt{n} + \left(\frac{1}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}} - 3\right)} = -\infty \Rightarrow \text{diverges}$$

36. $a_n = (-1)^n \left(1 - \dfrac{1}{n}\right)$

$\lim_{n\to\infty} (-1)^n \left(1 - \frac{1}{n}\right)$ does not exist $\Rightarrow$ diverges

42. $a_n = \dfrac{1}{(0.9)^n}$

$$\lim_{n\to\infty} \frac{1}{(0.9)^n} = \lim_{n\to\infty} \left(\frac{10}{9}\right)^n = \infty \Rightarrow \text{diverges}$$

47. $a_n = \dfrac{n}{2^n}$

$$\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 0 \Rightarrow \text{converges (using l'Hôpital's rule)}$$

48. $a_n = \dfrac{3^n}{n^3}$

$$\lim_{n \to \infty} \frac{3^n}{n^3} = \lim_{n \to \infty} \frac{3^n \ln 3}{3n^2} = \lim_{n \to \infty} \frac{3^n (\ln 3)^2}{6n} = \lim_{n \to \infty} \frac{3^n (\ln 3)^3}{6} = \infty \Rightarrow \text{ diverges (using l'Hôpital's rule)}$$

50. $a_n = \dfrac{\ln n}{\ln 2n}$

$$\lim_{n \to \infty} \frac{\ln n}{\ln 2n} = \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{2}{2n}\right)} = 1 \Rightarrow \text{ converges}$$

54. $a_n = \left(1 - \dfrac{1}{n}\right)^n$

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \to \infty} \left[1 + \frac{(-1)}{n}\right]^n = e^{-1} \Rightarrow \text{ converges} \quad \text{(Theorem 5, \#5)}$$

59. $a_n = \dfrac{\ln n}{n^{1/n}}$

$$\lim_{n \to \infty} \frac{\ln n}{n^{1/n}} = \frac{\lim_{n \to \infty} \ln n}{\lim_{n \to \infty} n^{1/n}} = \frac{\infty}{1} = \infty \Rightarrow \text{ diverges} \quad \text{(Theorem 5, \#2)}$$

62. $a_n = \sqrt[n]{3^{2n+1}}$

$$\lim_{n \to \infty} \sqrt[n]{3^{2n+1}} = \lim_{n \to \infty} 3^{2+(1/n)} = \lim_{n \to \infty} 3^2 \cdot 3^{1/n} = 9 \cdot 1 = 9 \Rightarrow \text{converges} \qquad \text{(Theorem 5, \#3)}$$

63. $a_n = \frac{n!}{n^n}$ (*Hint:* Compare with $1/n$.)

$$\lim_{n \to \infty} \frac{n!}{n^n} = \lim_{n \to \infty} \frac{1 \cdot 2 \cdot 3 \cdots (n-1)(n)}{n \cdot n \cdot n \cdots n \cdot n} \le \lim_{n \to \infty} \left(\frac{1}{n}\right) = 0 \text{ and } \frac{n!}{n^n} \ge 0 \Rightarrow \lim_{n \to \infty} \frac{n!}{n^n} = 0 \Rightarrow \text{converges}$$

69. $a_n = \left(\frac{3n+1}{3n-1}\right)^n$

72. $a_n = \left(1 - \frac{1}{n^2}\right)^n$

$$\lim_{n \to \infty} \left(\frac{3n+1}{3n-1}\right)^n = \lim_{n \to \infty} \exp\left(n \ln\left(\frac{3n+1}{3n-1}\right)\right) = \lim_{n \to \infty} \exp\left(\frac{\ln(3n+1) - \ln(3n-1)}{\frac{1}{n}}\right)$$

$$= \lim_{n \to \infty} \exp\left(\frac{\frac{3}{3n+1} - \frac{3}{3n-1}}{\left(-\frac{1}{n^2}\right)}\right) = \lim_{n \to \infty} \exp\left(\frac{6n^2}{(3n+1)(3n-1)}\right) = \exp\left(\frac{6}{9}\right) = e^{2/3} \Rightarrow \text{converges}$$

72. $a_n = \left(1 - \frac{1}{n^2}\right)^n$

$$\lim_{n \to \infty} \left(1 - \frac{1}{n^2}\right)^n = \lim_{n \to \infty} \exp\left(n \ln\left(1 - \frac{1}{n^2}\right)\right) = \lim_{n \to \infty} \exp\left(\frac{\ln\left(1 - \frac{1}{n^2}\right)}{\left(\frac{1}{n}\right)}\right) = \lim_{n \to \infty} \exp\left[\frac{\left(\frac{2}{n^3}\right) / \left(1 - \frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}\right]$$

$$= \lim_{n \to \infty} \exp\left(\frac{-2n}{n^2-1}\right) = e^0 = 1 \Rightarrow \text{converges}$$

77. $a_n = \dfrac{n^2}{2n-1}\sin\dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{n^2 \sin\left(\frac{1}{n}\right)}{2n-1} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{2}{n}-\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{-\left(\cos\left(\frac{1}{n}\right)\right)\left(\frac{1}{n^2}\right)}{\left(-\frac{2}{n^2}+\frac{2}{n^3}\right)} = \lim_{n\to\infty} \frac{-\cos\left(\frac{1}{n}\right)}{-2+\left(\frac{2}{n}\right)} = \frac{1}{2} \Rightarrow \text{converges}$$

82. $a_n = \dfrac{1}{\sqrt{n}}\tan^{-1} n$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \tan^{-1} n = 0 \cdot \frac{\pi}{2} = 0 \Rightarrow \text{converges}$$

89. $a_n = \dfrac{1}{n}\displaystyle\int_1^n \frac{1}{x}\,dx$

$$\lim_{n\to\infty} \frac{1}{n}\int_1^n \frac{1}{x}\,dx = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \text{converges} \qquad \text{(Theorem 5, \#1)}$$

Assume that the sequence is converges and find its limit

95. $a_1 = 5, \quad a_{n+1} = \sqrt{5a_n}$

Since a_n converges $\Rightarrow \lim_{n \to \infty} a_n = L \Rightarrow \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{5a_n}$

$L = \sqrt{5L} \Rightarrow L^2 - 5L = 0 \Rightarrow L = 0$ or $L = 5$; since

$a_n > 0$ for $n \geq 1 \Rightarrow L = 5$

- اللهمّ إنّي استودعتك ما قرأت وما حفظت وما تعلّمت، فردّه عند حاجتي اليه، إنك على كل شيء قدير، حسبنا الله ونعم الوكيل. - اللهم ارزقني قوة الحفظ، وسرعة الفهم، وصفاء الذهن، اللهم ألهمني الصواب في الجواب، وبلغني أعلى المراتب في الدين والدنيا والآخرة، وحفظني وأصلحني وأصلح بي الأمة