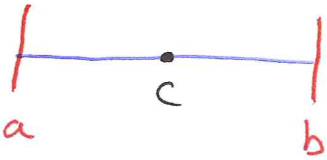
2.5 Continuity

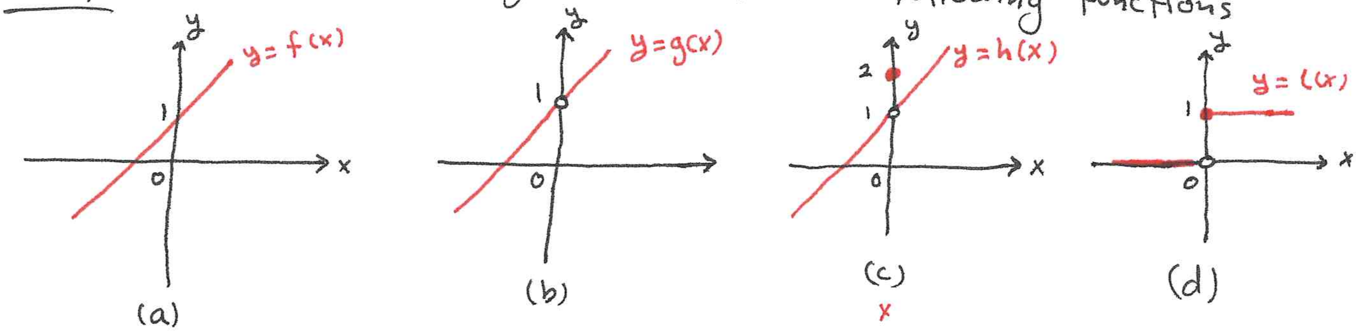
* Points are 3 kinds: ① interior points (c)
② left endpoints (a)
③ right endpoints (b)

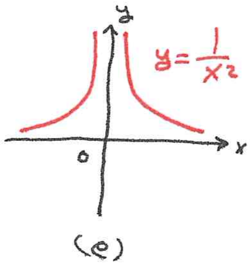


Definition: (Continuity at point)

A function f is continuous at an interior point $x = c$ of its domain if $\lim_{x \to c} f(x) = f(c)$

Example: Discuss the continuity at $x = 0$ to the following functions





a) f is continuous at $x = 0$ because $\lim_{x \to 0} f(x) = f(0) = 1$

b) g is not continuous at $x = 0$ because $\lim_{x \to 0} g(x) = 1 \neq g(0) \rightarrow$ DNE

c) h is not continuous at $x = 0$ because $\lim_{x \to 0} h(x) = 1 \neq h(0) = 2$

d) l is not continuous at $x = 0$ because $1 = f(0) \neq \lim_{x \to 0} l(x) \rightarrow$ DNE

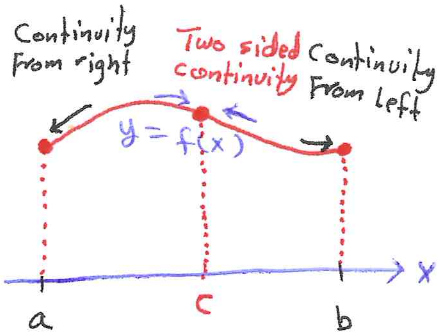
e) $y = \frac{1}{x^2}$ is not continuous at $x = 0$ because $\lim_{x \to 0} \frac{1}{x^2} = \infty$ and $f(0)$ is not defined.

(b) and (c) are called removable discontinuity because in
(b) g would be continuous if $g(0) = 1$
(c) h would be continuous if $h(0) = 1$ instead of 2

(d) is called jump discontinuity.

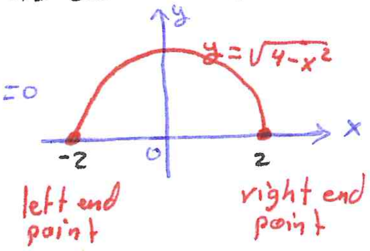
(e) is called infinite discontinuity.

Definition: A function f is



- continous at left endpoint $x=a$ of its domain if $\lim_{x\to a^+} f(x) = f(a)$
- continous at right endpoint $x=b$ of its domain if $\lim_{x\to b^-} f(x) = f(b)$

Example : $f(x)=\sqrt{4-x^2}$ is continuous at every point of its domain $[-2,2]$

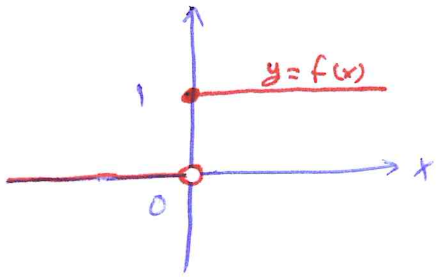


→ f is continous at -2 because $\lim_{x\to -2^+}\sqrt{4-x^2}=f(-2)=0$

→ f is continous at 2 because $\lim_{x\to 2^-}\sqrt{4-x^2}=f(2)=0$

→ f is continous on an interval $[-2,2]$.

Example : $f(x)=\begin{cases} 1, & x\geq 0 \\ 0, & x<0 \end{cases}$



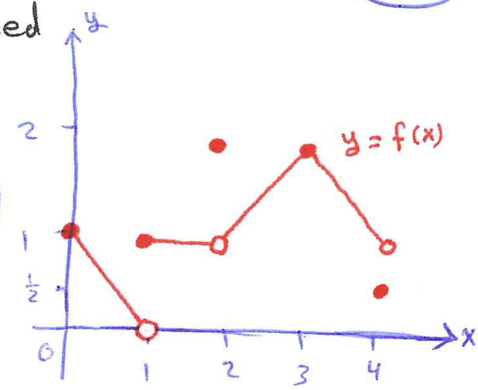
- f is right continuous at $x=0$ because $\lim_{x\to 0^+} f(x) = f(0) = 1$
- f is not left continuous at $x=0$ because $\lim_{x\to 0^-} f(x) = 0 \neq f(0)=1$
- f is not continuous at $x=0$ because $\underset{\text{DNE}}{\lim_{x\to 0} f(x)} \neq f(0)=1$

Test of Continuity:

$f(x)$ is continuous at $x=c$ (an interior point) iff the following three conditions hold:

1) $f(c)$ exists where $c\in D(f)$ domain of f
2) $\lim_{x\to c} f(x)$ exists
3) $\lim_{x\to c} f(x) = f(c)$

Example : Discuss the continuity of f at $x = 0, 1, 2, 3, 4$, where f is as given in the graph defined over the domain $[0, 4]$.



(a) f is continuous at $x = 0$ because

$\lim_{x \to 0^+} f(x) = f(0) = 1$ (right-hand limit exists at the left endpoint)

(b) f is discontinuous at $x = 1$ because

$\lim_{x \to 1} f(x)$ DNE

(c) f is discontinuous at $x = 2$ because $\lim_{x \to 2} f(x) = 1 \neq f(2) = 2$

(d) f is continuous at $x = 3$ because $\lim_{x \to 3} f(x) = f(3) = 2$

e) f is discontinuous at $x = 4$ because $\lim_{x \to 4^-} f(x) = 1 \neq f(4) = \frac{1}{2}$ (left-hand limit exists at the right end point).

Theorem : If the functions f and g are continuous at $x = c$, then the following functions are continuous at $x = c$:

1) $f \pm g$ 2) fg 3) Kf, where $K \in \mathbb{R}$

4) $\frac{f}{g}$ where $g(c) \neq 0$

5) $[f(x)]^{\frac{m}{n}}$ where $(f(x))^{\frac{m}{n}}$ is defined on an interval containing c, and $m, n \in \mathbb{Z}$

Example : • Every polynomial is continuous at every point of the real line.

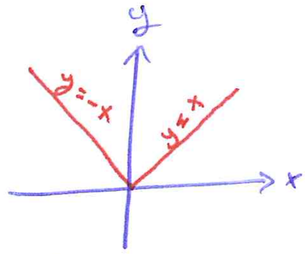
• Every rational function is continuous at every point where it's defined (denominator is different from zero).

Example : $f(x) = x^3 - 2x^2 + 1$ is continuous at every point x.

$g(x) = \frac{f(x)}{x^2 - 4} = \frac{x^3 - 2x^2 + 1}{(x-2)(x+2)}$ is continuous at every value of x except $x = 2$ and $x = -2$ where the denominator is zero.

Example $f(x) = |x|$ is continuos

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



if $x > 0$ then $f(x) = x$ polynomial which is continuous

if $x < 0$ then $f(x) = -x$ polynomial which is continuous

if $x = 0$ then $\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = f(0) = 0$

Continuity of trigonometric functions:

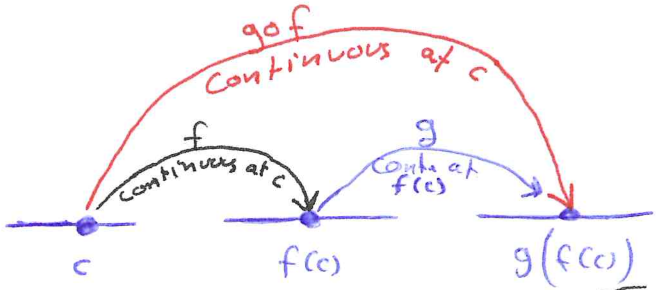
* The functions $\sin x$ and $\cos x$ are continuous at every value x.

* The functions $\tan x = \frac{\sin x}{\cos x}$, $\cot = \frac{\cos x}{\sin x} = \frac{1}{\tan}$, $\sec x = \frac{1}{\cos x}$ and $\csc x = \frac{1}{\sin x}$

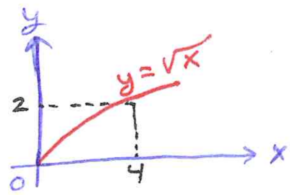
are continuous at every point except where they are not defined.

Theorem (Continuity of Composition)

If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

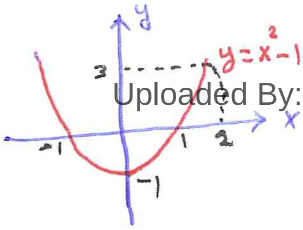


Example: Let $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$ show that $g \circ f$ is continuous at $x = 4$



1. $f(x)$ is continuous at $x = 4$ because

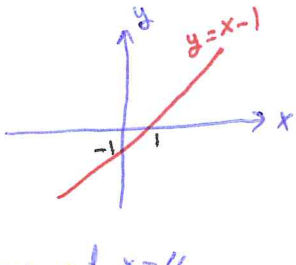
$$\lim_{x \to 4} \sqrt{x} = f(4) = 2 \quad \text{and}$$



$g(x)$ is continuous at $x = f(4) = 2$ because

$$\lim_{x \to 2} (x^2 - 1) = g(2) = 3.$$

Thus by Theorem above $g \circ f$ is continuous at $x = 4$.



2. Note that $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 - 1 = x - 1$.

This is polynomial and continuous every where. Thus continuous at $x = 4$.

Continuous Extension to point

* A rational function f may have a limit L at point $x=c$ even if $f(c)$ is not defined (the denominator is zero).

Example: $f(x) = \dfrac{x^2-4}{x-2}$

- If $x=2 \Rightarrow f(2)$ is not defined but
- If $x \neq 2 \Rightarrow f(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x-2)(x+2)}{(x-2)} = x+2$

The function $F(x) = x+2$ is the same as $f(x) = \dfrac{x^2-4}{x-2}$ for all $x \neq 2$

The only difference is that $F(x)$ is continuous at $x=2$ because

$$\lim_{x\to 2} F(x) = \lim_{x\to 2}(x+2) = 4 = F(2)$$

but $f(x)$ is not continuous at $x=2$ because

$$\lim_{x\to 2} f(x) = 4 \neq f(2)$$

Thus, $F(x)$ is called the continuous extension of $f(x)$ at $x=c$, and we write

$$F(x) = \begin{cases} f(x) & \text{if } x \neq c \text{ and } x \in D(f) \\ L & \text{if } x = c \end{cases}$$

where $\lim_{x\to c} f(x) = L$.

Example: show that $f(x) = \dfrac{x^2+x-6}{x^2-4}$ has a continuous extension at $x=2$, and find that extension.

- If $x=2 \Rightarrow f(2)$ is not defined.
- If $x \neq 2 \Rightarrow f(x) = \dfrac{x^2+x-6}{x^2-4} = \dfrac{(x-2)(x+3)}{(x-2)(x+2)} = \dfrac{x+3}{x+2}$

The function $F(x) = \dfrac{x+3}{x+2}$ is the same as $f(x)$ for all $x \neq 2$

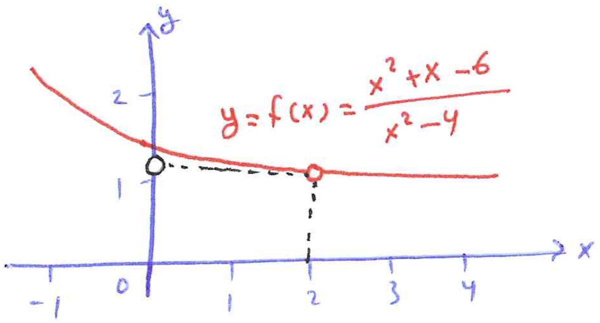
$\Rightarrow$ But $F(x)$ is continuous at $x=2$ because

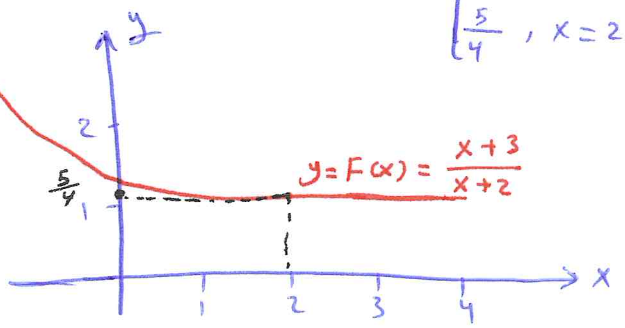
$$\lim_{x\to 2} F(x) = \lim_{x\to 2} \frac{x+3}{x+2} = \frac{5}{4} = F(2)$$

and $f(x)$ is not continuous at $x=2$ because

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2+x-6}{x^2-4} = \lim_{x\to 2} \frac{x+3}{x+2} = \frac{5}{4} \neq f(2)$$

Thus, F is the continuous extension of f to $x=2$. $F(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-4}, & x \neq 2 \\ \dfrac{5}{4}, & x = 2 \end{cases}$





Continuity on Intervals

- Let $D(f)$ be the domain of the function f:

$\rightarrow$ A function f is continuous if it is continuous at every where in $D(f)$

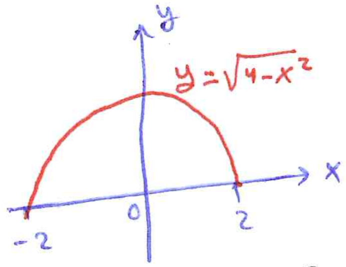
$\rightarrow$ A function f is continuous on an interval $I \subset D(f)$ if f is continuous at every point in I.

x $\rightarrow$ If the function f is continuous on an interval I, then f is continuous on any interval $J \subset I$.

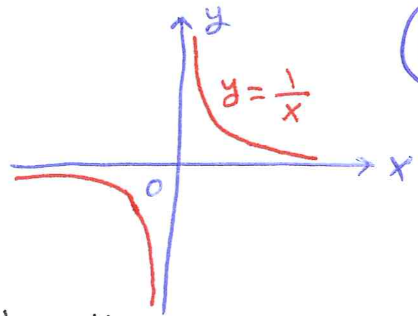
Example: $\rightarrow$ Polynomials functions are continuous on every interval.

$\rightarrow$ Rational functions are continuous on every interval on which they are defined.

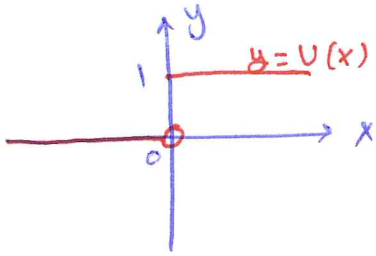
Example*:



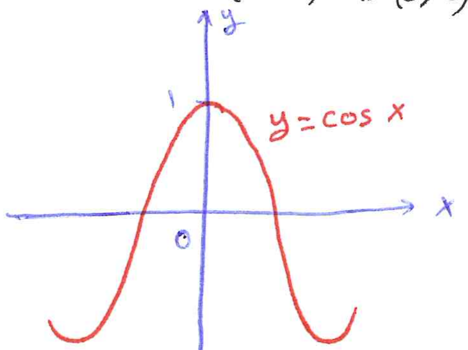
(a) continuous on $[-2,2]$



(b) continuous on $(-\infty,0)$ and $(0,\infty)$



(c) Continuous on $(-\infty,0)$ and $[0,\infty)$

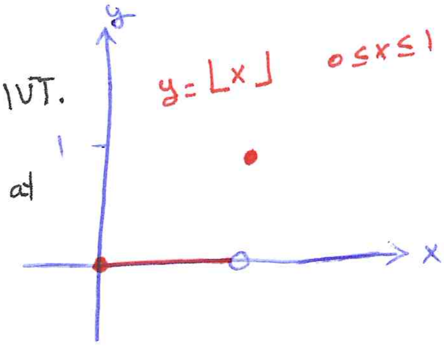


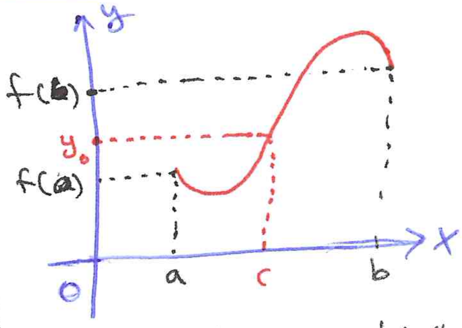
(d) Continuous on $(-\infty,\infty)$

Theorem: The Intermediate Value Theorem (IVT)

Suppose $f(x)$ is continuous on an interval $I=[a,b]$. If y_0 is any number between $f(a)$ and $f(b)$, then there exists a number c between a and b such that $f(c)=y_0$

The continuity of f on I is essential to IVT. If f is discontinuous at even one point of I, then IVT may fail





IVT geometrically

Consequences of the IVT

- IVT is the reason that the graph of a function continuous on an interval I can not have any breaks. It will be connected and single unbroken curve. For instance (a) and (d) in Example*. It will not have jumps like (c) in Example* or separate branches like (b) in Example*.

Theorem: (limits of continuous functions)

If g is continuous at the point b, and

$\lim_{x \to c} f(x) = b$, then $\lim_{x \to c} g(f(x)) = g(b)$

$= g\left(\lim_{x \to c} f(x)\right)$

Example: $\lim_{x \to \frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) = \cos\left(\lim_{x \to \frac{\pi}{2}} 2x + \lim_{x \to \frac{\pi}{2}} \sin\left(\frac{3\pi}{2} + x\right)\right)$

$= \cos\left(\pi + \sin 2\pi\right)$

$= \cos \pi = -1$

Example: Show that $\exists$ a root of the equation $x^3 - x - 1 = \boxed{0}$ between 1 and 2.

let $f(x) = x^3 - x - 1$

$f(1) = 1 - 1 - 1 = -1 < 0$

$f(2) = 8 - 2 - 1 = 5 > 0$

Since $0 = y_0$ is between $f(1)$ and $f(2)$

Since f is continuous (polynomial). Thus, by IVT, $\exists$ a (zero / root) of f between 1 and 2. $\boxed{x = 1.32}$
