Engineering statistics "ENEE2307"

Chapter 3

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Notes, questions and forms

Chapter 3
Two or more Random Variables

① Discrete

Example - 1

let X and Y be two RV's with the following Joint-PMF

$$P(X=x, Y=y) = \begin{cases} 1/16 & x=-1, y=0 \\ K & x=0, y=0 \\ 1/16 & x=1, y=0 \\ 2/16 & x=-1, y=1 \\ 1/8 & x=-1, y=2 \\ 1/4 & x=1, y=2 \\ 0 & o.w \end{cases}$$

X \ y	0	1	2
-1	1/16	2/16	2/16
0	6/16	0	0
1	1/16	0	4/16

* we always do the Joint PMF on this way to make it easier

* when we don't have a match, we put "0"

① Find K ?

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} P(X=x, Y=y) = 1 \implies \frac{1}{16} + K + \frac{2}{16} \text{ -----} = 1$$

$$\frac{10}{16} + K = 1 \implies K = \frac{6}{16}$$

② $P(X \le 0, y \le 1)$

$y \le 1$ $x \le 0$

X \ y	0	1	2
-1	1/16	2/16	2/16
0	6/16	0	0
1	1/16	0	4/16

$$P(X \le 0) \cap P(y \le 1) = \frac{1}{16} + \frac{2}{16} + \frac{6}{16} + 0$$

$= 9/16$

③ $P(X \leq 0 / y \leq 1)$

$$P(X \leq 0 / y \leq 1) = \frac{P(X \leq 0 , y \leq 1)}{P(y \leq 1)} = \frac{9/16}{10/16} = \boxed{\frac{9}{10}}$$

④ $F_{X,y}(0,1)$ "0 for X , 1 for y" حسب الترتيب

$$F_{X,y}(0,1) = P(X \leq 0 , y \leq 1) = \boxed{9/16}$$

⑤ $F_{y,X}(0,1)$

$$F_{y,X}(0,1) = P(y \leq 0 , X \leq 1) = \boxed{8/16}$$

⑥ $F_{X,y}(-\infty, -\infty)$

$$F_{X,y}(-\infty,-\infty) = P(X \leq -\infty , y \leq -\infty) = \boxed{0}$$

⑦ $F_{X,y}(\infty , \infty)$

$$F_{X,y}(\infty,\infty) = P(X \leq \infty , y \leq \infty) = \boxed{1}$$

⑧ $F_{X,y}(\infty, -\infty)$

$$F_{X,y}(\infty,-\infty) = P(X \leq \infty , y \leq -\infty) = \boxed{0}$$

(⑥–⑧: Rules *)

⑨ $P(X > 0 , y \leq 0)$

$$P(X > 0 , y \leq 0) \neq P(X \leq 0 , y \leq 0)$$

* مهم

$$P(X > 0 , y \leq 0) = \boxed{\frac{1}{16}}$$

X \ y	0	1	2
-1	1/16	2/16	2/16
0	6/16	0	0
1	1/16	0	4/16

⑩ $P(x=1)$ this means find P when $x=1$ on all y "$P(x=1, y=y)$"

$$P(x=1) = \frac{1}{16} + 0 + \frac{4}{16} = \frac{5}{16}$$

X \ y	0	1	2
-1	1/16	2/16	2/16
0	6/16	0	0
1	1/16	0	4/16

⑪ $P(y \leq 1)$

$$P(y \leq 1) = \frac{1}{16} + \frac{2}{16} + \frac{1}{16} + \frac{6}{16} = \frac{10}{16}$$

X \ y	0	1	2
-1	1/16	2/16	2/16
0	6/16	0	0
1	1/16	0	4/16

⑫ Find the marginal PMF of X   لما أكون بدي RV واحد فقط

$$P(X=x) = \begin{cases} 5/16 & x=-1 \\ 6/16 & x=0 \\ 5/16 & x=1 \\ 0 & \text{o.w} \end{cases}$$

X \ y	0	1	2
-1	1/16	2/16	2/16
0	6/16	0	0
1	1/16	0	4/16

⑬ Find the marginal PMF of y

$$P(Y=y) = \begin{cases} 8/16 & y=0 \\ 2/16 & y=1 \\ 6/16 & y=2 \\ 0 & \text{o.w} \end{cases}$$

X \ y	0	1	2
-1	1/16	2/16	2/16
0	6/16	0	0
1	1/16	0	4/16

⑭ Are X and Y Statically independent ?

$$P(X=0, y=0) \stackrel{?}{=} P(X=0)P(y=0)$$

$$6/16 \neq (6/16)(8/16)$$

they are not S.I

Note

X and Y are said to be Statically independent if

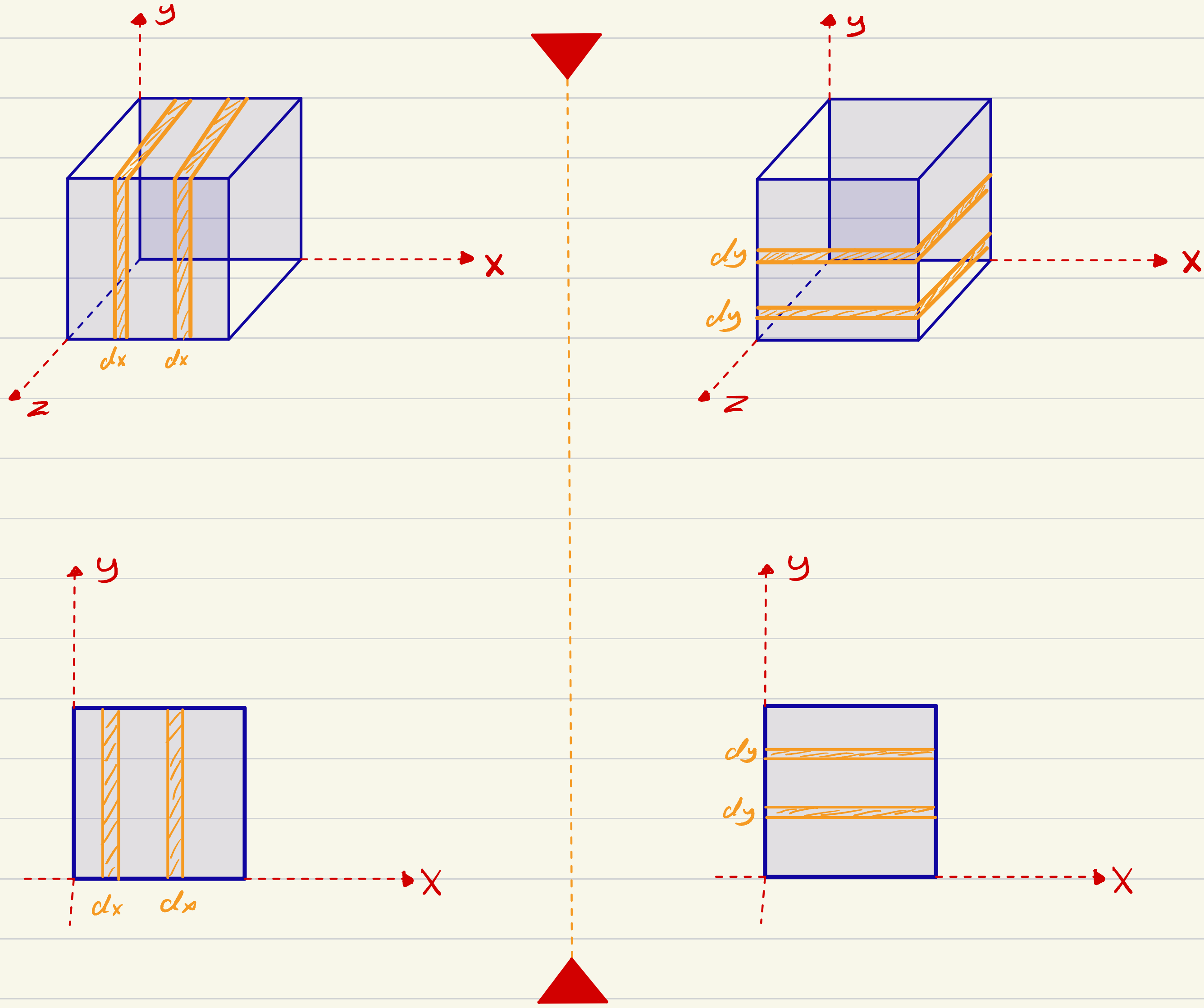
$$P(X=x, Y=y) = P(X=x)P(Y=y)$$

for all x and y

② Continues

Example - 2

X and Y are RV's with $f_{x,y} = \begin{cases} K & 0 \leq x \leq 2 \quad , \quad 0 \leq y \leq 4 \\ 0 & \text{o.w} \end{cases}$



- بتكامل عشان أوجد المساحة , ومن ثم بكامل عشان اوجد الحجم .
- عندي حرية في اختيار مين بدي أكامل أول dx او dy .

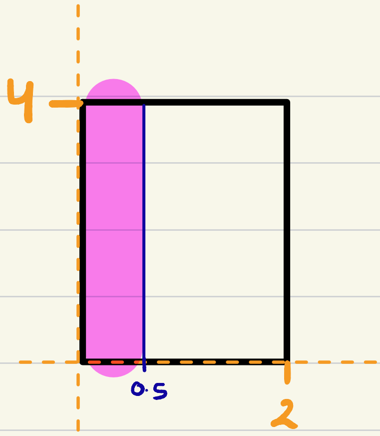
① Find K

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x,y)\, dy\, dx \Rightarrow \int_0^2 \int_0^4 K = \int_0^2 4K = 8K = 1 \Rightarrow K = \frac{1}{8}$$

② $P(X \leq 0.5)$

from the graph $(x: 0 - 0.5 \mid y: 0 - 4)$

$$\int_0^{0.5}\int_0^4 1/8 \, dy \, dx \Rightarrow \int_0^{0.5} 1/2 \, dx = \frac{1}{4}$$

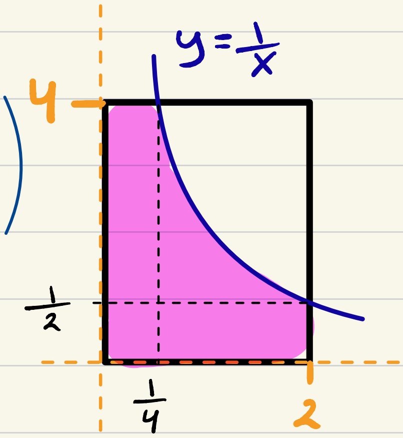


③ $P(xy \leq 1)$

$$xy \leq 1 \Rightarrow y \leq \frac{1}{x}$$

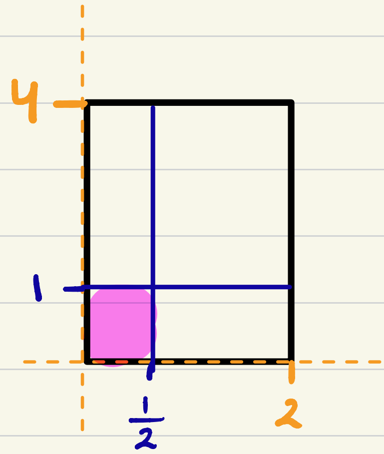
(when $x = 2 \Rightarrow y = 0.5$
$y = 4 \Rightarrow x = 0.25$)

$$\int_0^{\frac{1}{4}}\int_0^4 \frac{1}{8} \, dy \, dx + \int_{\frac{1}{4}}^{2}\int_0^{\frac{1}{x}} \frac{1}{8} \, dy \, dx = \frac{1}{8}(1 + \ln x)\Big|_{\frac{1}{4}}^{2}$$



④ $P(0 \leq x \leq 0.5 \ , \ 0 \leq y \leq 1)$

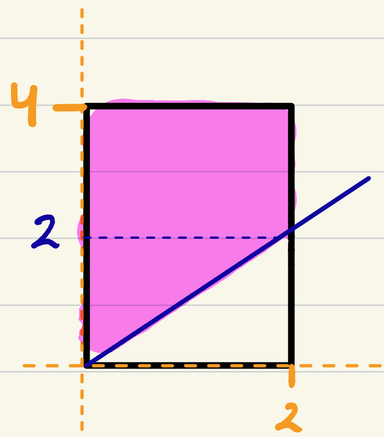
$$\int_0^{0.5}\int_0^1 \frac{1}{8} \, dy \, dx = \int_0^{0.5} \frac{1}{8} \, dx = \frac{1}{16}$$



⑤ $P(x \leq y)$

$$\int_0^2\int_x^4 \frac{1}{8} \, dy \, dx = \int_0^2 \frac{1}{2} - \frac{x}{8} = \int_0^2 \frac{1}{2} - \int_0^2 \frac{x}{8}$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$



⑥ find the marginal PDF of X

$$f(x) = \int_{y_0}^{y_1} f(x,y)\,dy \implies \int_0^4 \frac{1}{8}\,dy = \frac{1}{2}$$

$$f(X=x) = \begin{cases} \frac{1}{2} & 0 \leq x \leq 2 \\ 0 & \text{o.w} \end{cases} \quad \#$$

⑦ find the marginal PDF of y

$$f(y) = \int_{x_0}^{x_1} f(x,y)\,dx \implies \int_0^2 \frac{1}{8}\,dx = \frac{1}{4}$$

$$f(Y=y) = \begin{cases} \frac{1}{4} & 0 \leq y \leq 4 \\ 0 & \text{o.w} \end{cases} \quad \#$$

⑧ Are they Statically independent ?

$$\begin{cases} \frac{1}{8} & 0 \leq x \leq 2,\ 0 \leq y \leq 4 \\ 0 & \text{o.w} \end{cases} \stackrel{?}{=} \begin{cases} \frac{1}{2} & 0 \leq x \leq 2 \\ 0 & \text{o.w} \end{cases} * \begin{cases} \frac{1}{4} & 0 \leq x \leq 4 \\ 0 & \text{o.w} \end{cases}$$

$$\frac{1}{8} \stackrel{?}{=} \frac{1}{2} * \frac{1}{4}$$, Yes $$\frac{1}{8} = \frac{1}{2} * \frac{1}{4}$$

∴ They are S. I

Note

X and Y are said to be Statically independent if

$$f(x,y) = f(x)\,f(y)$$

for all X and y

⑨ $P(0 \leq x \leq 0.5 \ , \ 0 \leq y \leq 1 \ / \ y \leq 2)$

$$= \frac{P(0 \leq x \leq 0.5 \ , \ 0 \leq y \leq 1)}{P(y \leq 2)} = \frac{\int_0^{0.5} \int_0^1 \frac{1}{8} \, dy \, dx}{\int_0^2 \frac{1}{4} \, dy} = \frac{1}{8}$$

⑩ $P(y \leq 1 \ , \ x = 0.5)$

- Conditional PDF of X : $f_{y/x=x} = \left. \frac{f(x,y)}{f(x)} \right|_{x=x}$
- Conditional PDF of Y : $f_{x/y=y} = \left. \frac{f(x,y)}{f(y)} \right|_{y=y}$

$$P(y \leq 1 \ , \ x = 0.5) = f_{y/x=0.5} = \frac{1/8}{1/2} = \frac{1}{4}$$

$\Rightarrow f_{y/x=0.5} = \frac{1}{4}$ then we integrate on y interval

$$\int_0^1 f_{y/x=0.5} \Rightarrow \int_0^1 \frac{1}{4} \, dy = \frac{1}{4}$$

سؤال :- شو الفرق بين ٩ و١٥ ، ليش كل واحدة بطريقة ؟

الأولى : معطى y اقل من شيء " ≤ " وفي هذه الحالة نحل عادي بلا أية شروط و قوانين .

الثانية : معطى y تساوي شيء " = " وفي هذه الحالة يجب استخدام قوانين • Conditional

Example - 3

X and y are two RV's with this PMF

x \ y	-1	0	1
-1	1/8	1/2	0
1	0	1/4	1/8

① $E\{XY\}$ — $g(x,y)$

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} g(x,y)\, P(X=x, Y=y) \Rightarrow \sum_{-1}^{1}\sum_{-1}^{1} xy\, P(x,y)$$

$$= (-1*-1)\left(\tfrac{1}{8}\right) + (0) + (0) + (0) + (0) + (1*1)\left(\tfrac{1}{8}\right) = \frac{2}{8}$$

② $E\{X^2 y\}$

$$-1/8 + 0 + 0 + 0 + 0 + 1/8 = 0$$

③ $E\{(X+1)y\}$

$$0 + 0 + 0 + 0 + 0 + 2*1/8 = 1/4$$

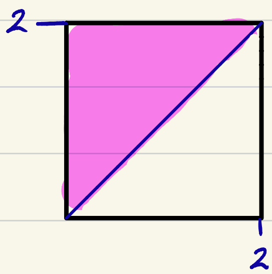
Example - 4

X and y are two RV's with this PDF. $f_{x,y} = \begin{cases} kx^2y & 0 \le x \le y \le 2 \\ 0 & \text{o.w} \end{cases}$

$x \le y$
$0 \le y \le 2$
$0 \le x \le 2$

① Find K

$$\int_0^2 \int_x^2 kx^2y\, dy\, dx = 1 \Rightarrow \int_0^2 2kx^2 - \frac{kx^4}{2}\, dx = 1$$



$$\frac{16k}{3} - \frac{16k}{10} = 1 \Rightarrow k = \frac{15}{32}$$

② $E\{X(y+1)\}$

$$\int_0^2 \int_x^2 g(x,y)\, f_{x,y} \Rightarrow \int_0^2 \int_x^2 x(y+1) * \frac{15}{32} x^2 y \Rightarrow \int_0^2 \int_x^2 \frac{15}{32} x^3 y(y+1)$$

then we continue as before

Important Rules:

① $E\{ax + by\} = aE\{x\} + bE\{y\}$

② $E\{axy\} = aE\{x\}E\{y\}$ iff they are staticlly indep.

Correlation Coefficient

Rules

- Covariance : $\mu_{xy} = E\{(X-\mu_x)(Y-\mu_y)\}$
- Variance : $\sigma_x^2 = \mu_{xx} = E\{(X-\mu_x)^2\}$
- Correlation Coefficient : $\rho_{xy} = \dfrac{\mu_{xy}}{\sigma_x \sigma_y}$, $-1 \leq \rho_{xy} \leq 1$

if $\rho_{xy} = 0 \Rightarrow$ X and Y are uncorrelated لا يوجد بينهم علاقة

if $\rho_{xy} = 1 \Rightarrow$ X and Y are fully correlated يوجد بينهم علاقة قوية

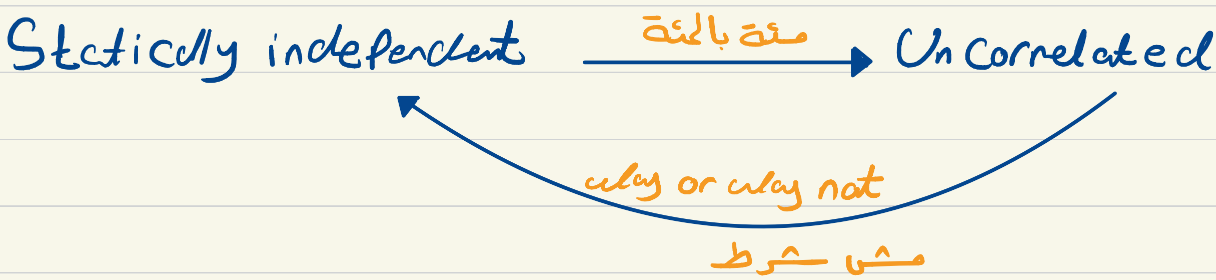
- let $R = a_1X + a_2Y$ "لتسهيل الحفظ: هو شبيه بالحالة التربيعية"

$\sigma_R^2 = a_1^2\sigma_x^2 + a_2^2\sigma_y^2 + 2a_1a_2\sigma_x\sigma_y\rho_{xy}$

Notes

- if x and y are Statically indep. then they are uncorrelated

But if x and y are uncorrelated this doesn't mean they are S.I



Example - 5

let X and Y be two RV's with this PMF

X \ Y	-1	1
-1	$\frac{1}{4}$	$\frac{1}{2}$
1	$\frac{1}{4}$	0

① Find the Correlation Coefficient ρ_{xy}

First: we find f_x and f_y

$$P(X=x) = \begin{cases} 3/4 & x=-1 \\ 1/4 & x=1 \\ 0 & \text{o.w} \end{cases} \qquad P(Y=y) = \begin{cases} 1/2 & y=-1 \\ 1/2 & y=1 \\ 0 & \text{o.w} \end{cases}$$

Second: find any thing we need to find ρ_{xy}

$\mu_x = -1/2$, $\mu_y = 0$, $\sigma_x = \sqrt{E\{x^2\} - \mu_x} = \sqrt{3}/2$

$\sigma_y = \sqrt{E\{y^2\} - \mu_y^2} = 1$, $\mu_{xy} = E\{(x+\frac{1}{2})(y)\} = -1/2$

$\rho = \dfrac{-1/2}{\sqrt{3}/2 * 1} \Rightarrow \boxed{\rho = -0.577}$

Example - 6

let X and Y two RV's with $\mu_x = 1$, $\sigma_x^2 = 4$, $\mu_y = -1$, $\sigma_y^2 = 9$

$R = 2x - y$, $\rho_{xy} = 1/2$, Find :

① μ_R

$E\{R\} = E\{2x - y\} = 2E\{x\} - E\{y\} = 2 + 1 = \boxed{3}$

② Var R

$\sigma_R^2 = a_1^2 \sigma_x^2 + a_2^2 \sigma_y^2 + 2a_1 a_2 \sigma_x \sigma_y \rho_{xy}$

$= 4 * 4 + 1 * 9 + 2 * 2 * -1 * 2 * 3 * \frac{1}{2} = \boxed{13}$

Functions of Random Variables

Example - 7

let X and Y two RV's, and $Z = X + Y$ with the following joint-PMF

X \ Y	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

① Find the PMF of Z

$$P(Z=z) = \begin{cases} 0.1 & z=2 \\ 0.3 & z=3 \\ 0.1 & z=4 \\ 0.3 & z=5 \\ 0.2 & z=6 \\ 0 & o.w \end{cases}$$

X	Y	Z	P(Z=z)
1	1	2	0.1
1	2	3	0
1	3	4	0.1
1	4	5	0
2	1	3	0.3
2	2	4	0
2	3	5	0.1
2	4	6	0.2
3	1	4	0
3	2	5	0.2
3	3	6	0
3	4	7	0

we just use the non-zero terms

when we have z value two times we add them together " like z=5"

② $E\{Z\}$

$$E\{Z\} = E\{X+Y\} \Rightarrow \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}(x+y)f_{x,y} \quad \text{OR} \quad \sum_{-\infty}^{\infty} z f_z$$

③ Find the PDF of Z

this is called Convolutional integral

See the next Page ♥

فـي بطن الحوت كان صالك أمل
ما عذر إحباطك ؟ وَتَوَكَّلْ عَلَى الْعَزِيزِ الرَّحِيمِ (٢١٧)

Convolutional Integral

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx \quad \text{if } X \text{ and } Y \text{ are S.I}$$

Example-8

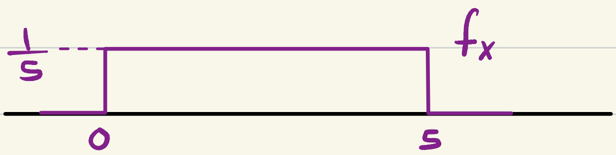
X is a R.V with a Uniform distribution over $[0,5]$, and Y is a R.V with a Uniform distribution over $[2,4]$, Z is a new R.V such that $Z = X + Y$, Find

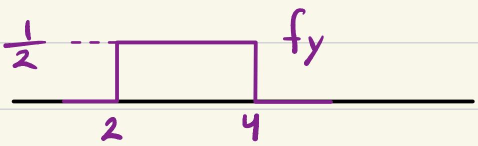
① PDF of Z at $z = 4$

First: we find $f_X(x)$ and $f_Y(y)$ and Plot them

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{o.w} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{b-a} & a \leq y \leq b \\ 0 & \text{o.w} \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{5} & 0 \leq x \leq 5 \\ 0 & \text{o.w} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{2} & 2 \leq y \leq 4 \\ 0 & \text{o.w} \end{cases}$$





then: we use the low above to find the PDF

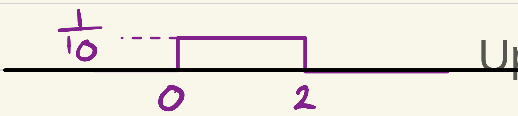
$$f_Z(4) = \int_{-\infty}^{\infty} f(x)\, f(4-x)$$

Find $f(4-x)$ ⟹

$$f_Y(4-x) = \begin{cases} \frac{1}{2} & 2 \leq 4-x \leq 4 \\ 0 & \end{cases}$$

$2 \leq 4-x \leq 4$ ⟹ $0 \leq x \leq 2$ (what i want)

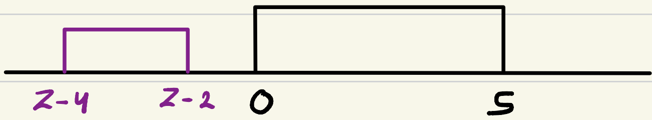
$$f_Z(4) = \int_0^2 \frac{1}{5} * \frac{1}{2} \implies f_Z(4) = 1/5$$



② Find the PDF of Z

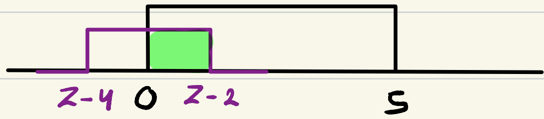
لا توجد منطقة تقاطع عندما $z<0$

$\therefore f_z(z) = 0 \quad , \quad z<0$



$$f_z(z) = \int_0^{z-2} f_x(x)\, f(z-x) \quad = \int_0^{z-2} \frac{1}{5} * \frac{1}{2}\, dx$$

$$f_z(z) = \frac{1}{10}(z-2) \quad , \quad 2 \leq z \leq 4$$

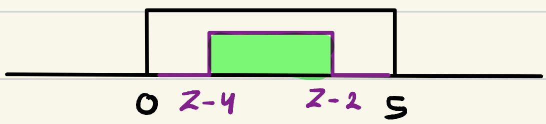


$z-2>0 \Rightarrow z>2$

$z-4<0 \Rightarrow z<4$

$$f_z(z) = \int_{z-4}^{z-2} \frac{1}{10}\, dx \quad = \frac{1}{5}$$

$$f_z(z) = \frac{1}{5} \quad , \quad 4 \leq z \leq 7$$

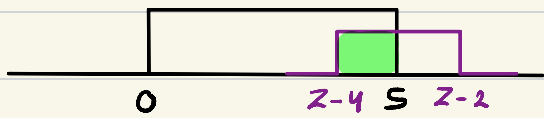


$z-4>0 \Rightarrow z>4$

$z-2<5 \Rightarrow 5<7$

$$f_z(z) = \int_{z-4}^{5} \frac{1}{5} * \frac{1}{2}\, dx \quad = \frac{5}{10} - \frac{1}{10}(z-4)$$

$$f_z(z) = \frac{1}{10}(z-9) \quad , \quad 7 \leq z \leq 9$$

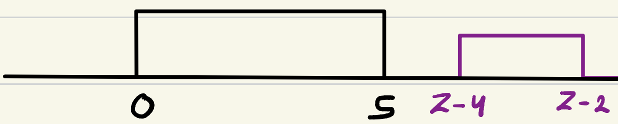


$z-4<5 \Rightarrow z<9$

$z-2>5 \Rightarrow z>7$

لا توجد منطقة تقاطع عندما $z>9$

$\therefore f_z(z) = 0 \quad , \quad z>9$



$$f_z(z) = \begin{cases} 0 & z<0 \text{ or } z>9 \\ 1/10\,(z-2) & 2 \leq z < 4 \\ 1/5 & 4 \leq z < 7 \\ 1/10\,(z-9) & 7 \leq z < 9 \end{cases}$$

— Why i Choose z-2 and z-4 , why 2 and 4 ?

Because we are dealing with $P_Y(y)$ interval which is [2,4]