Ch4 Part 1

Wednesday, October 13, 2021 12:23 PM

Applications of Derivatives

increasing ↑

decreasing ↓

Maximum Max

Minimum Min

Def f defined on interval I

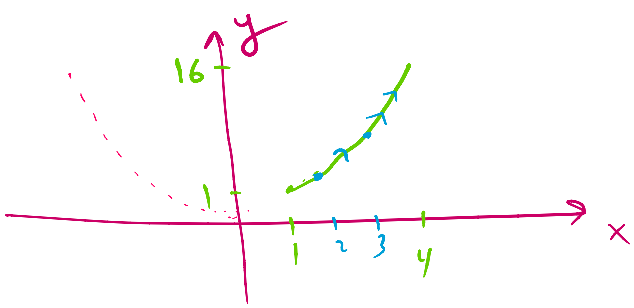
① f ↑ on I if whenever

$x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$ $\forall x_1, x_2 \in I$

② f ↓ on I if whenever

$x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$ $\forall x_1, x_2 \in I$

Exp ① $f(x) = x^2$ on $[1, 4]$

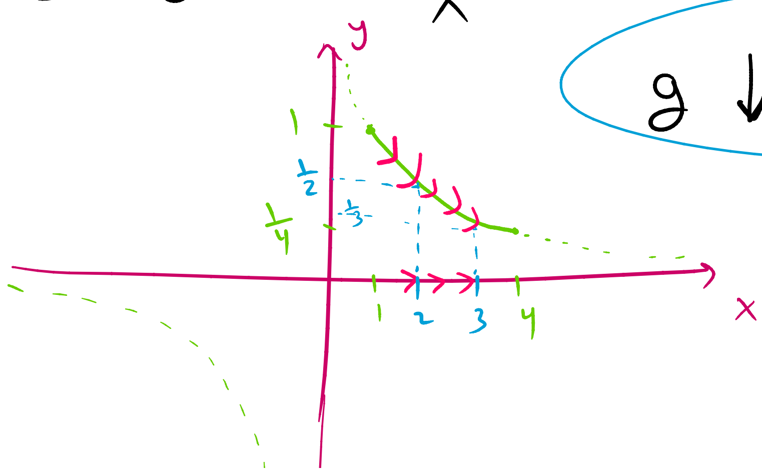


$x_1 = 2$, $x_2 = 3$

$3 > 2 \Rightarrow f(3) > f(2)$

f ↑ on $[1, 4]$

② $g(x) = \frac{1}{x}$ on $[1, 4]$



$\boxed{g \downarrow \text{ on } [1, 4]}$

$x_1 = 2$, $x_2 = 3$

$3 > 2 \Rightarrow f(3) < f(2)$

$\frac{1}{3} < \frac{1}{2}$

Q. Can we use derivatives to know f is $\uparrow$ or $\downarrow$

A. Yes

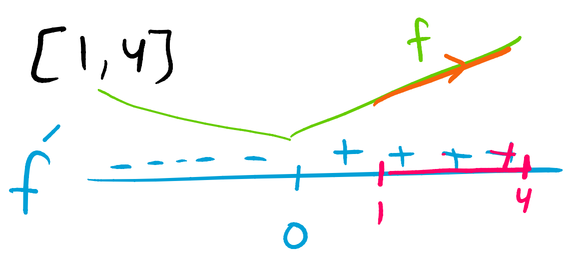
Th f cont on $[a,b]$
f diff on (a,b) Then

1. If $f'(x) > 0 \ \forall x \in (a,b) \Rightarrow f \uparrow$ on $[a,b]$
2. If $f'(x) < 0 \ \forall x \in (a,b) \Rightarrow f \downarrow$ on $[a,b]$

Exp ① $f(x) = x^2$ on $[1,4]$

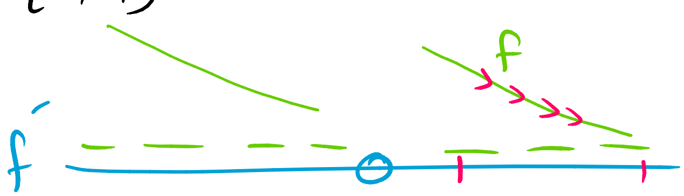
$f'(x) = 2x$

$f' = 0 \Rightarrow 2x = 0$
$x = 0$



f is $\uparrow$ on $[1,4]$

② $g(x) = \dfrac{1}{x}$ on $[1,4]$



$g'(x) = -\left(\dfrac{1}{x^2}\right)$

$g'(x) = -\frac{1}{x^2}$

$+$

f' : $- - - - - \; \circ \; - - - -$ at 0, 1, 4

$g \downarrow$ on $[1, 4]$

③ $f(x) = x^3 - 12x - 5$

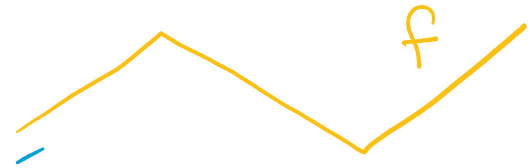
$f'(x) = 3x^2 - 12 = 0$

$3(x^2 - 4) = 0$

$x^2 - 4 = 0 \quad \rightarrow \quad x^2 = 4, \quad x = \pm 2$

$(x-2)(x+2) = 0$

$x = 2, \quad x = -2$



f' : $+ + + \; | \; - - - - - \; | \; + + +$ at -2, 0, 2

$f'(0) = 3(0)^2 - 12$
$= 0 - 12$
$= -12 < 0$

$f \uparrow$ on $(-\infty, -2] \cup [2, \infty)$

$f \downarrow$ on $[-2, 2]$

Def (Critical Point) - CP

CP's are interior points نقاط داخلية of f such that

$f'(cp) = 0$ ✓ or $f'(cp)$ undefined ✓

CP's $\in D(f)$ → interior

$\searrow$ CPs $\in D(f)$

Exp ① $f(x) = x^2$ on $[1,4]$ Find CP's

$f'(x) = 2x = 0$

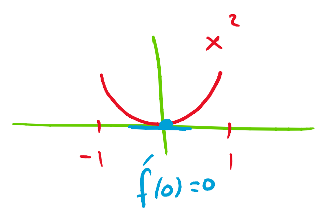
$x = 0 \Rightarrow 0 \notin [1,4]$

f has no CP's on $[1,4]$

② $f(x) = x^2$ on $[-1,1]$ Find CP's

$f'(x) = 2x = 0 \Rightarrow x = 0 \Rightarrow 0 \in [-1,1]$

$x = 0$ is CP $\Rightarrow (0, f(0)) = (0,0)$



③ $g(x) = \dfrac{1}{x}$ Find CP

$D(g) = \mathbb{R} \setminus \{0\}$

$g'(x) = -\dfrac{1}{x^2} \Rightarrow g'$ is undefined at $x=0$

but $x = 0 \notin D(g)$

$g'(0)$ undefined

g has no CP's

④ $f(x) = x^3 - 12x - 5$

$D(f) = \mathbb{R} = (-\infty, \infty)$

(4)

$D(f) = \mathbb{R} = (-\infty, \infty)$

$f' = 3x^2 - 12 = 0$

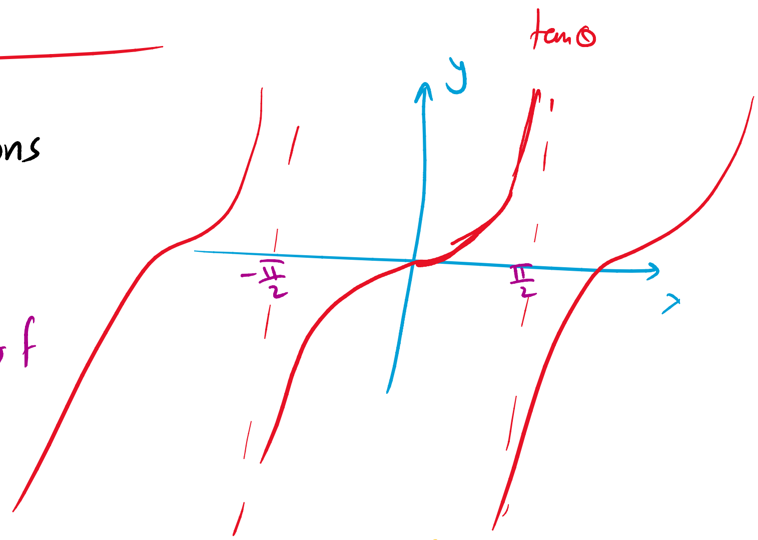
$x^2 - 4 = 0 \Rightarrow x = \pm 2$

$f'(2) = 0 \Rightarrow 2 \in D(f) \Rightarrow x = 2$ is CP

$f'(-2) = 0 \Rightarrow -2 \in D(f) \Rightarrow x = -2$ is CP

<u>Exp</u> Examples of $\uparrow$ functions

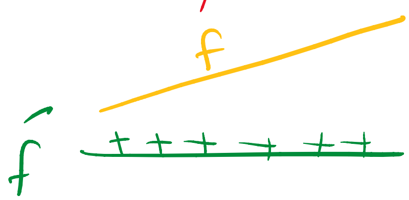
① $f(x) = \tan\theta$



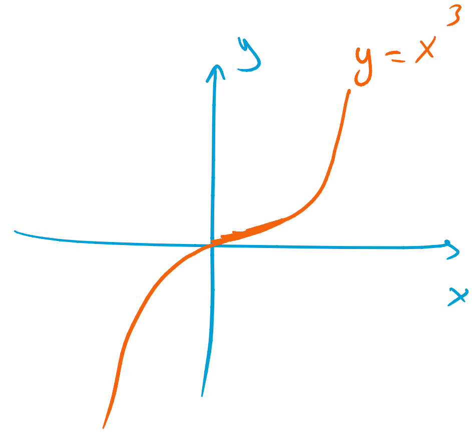
f is $\uparrow$ on domain of $\tan\theta$

$(-\frac{\pi}{2}, \frac{\pi}{2})$

$f'(x) = \sec^2\theta \rightarrow +$

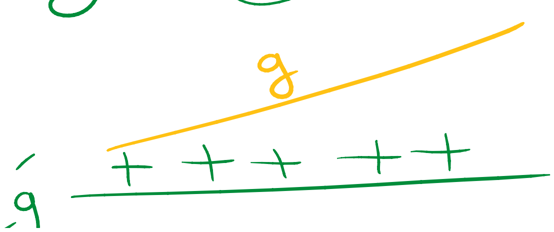


② $g(x) = x^3$ on $\mathbb{R}$



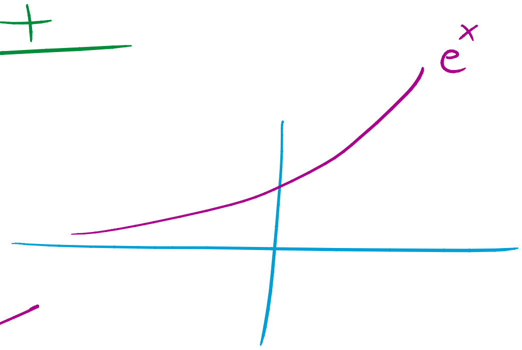
g $\uparrow$ on $\mathbb{R}$

$g'(x) = 3x^2 \rightarrow +$



e^x

g' $+ + + + +$



③ $y = e^x$

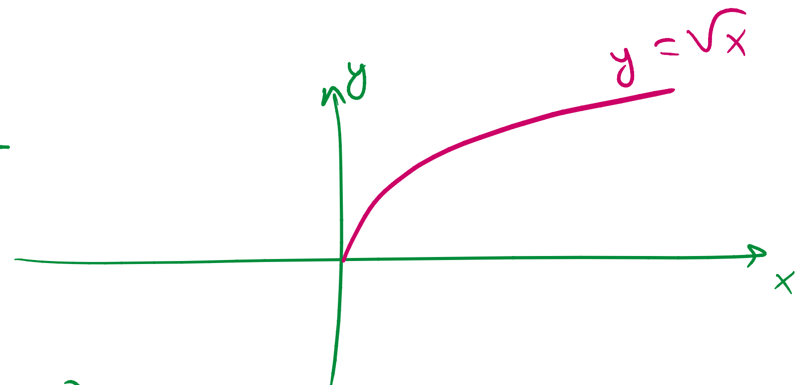
$y' = e^x$

y' $+ + + +$ y

$y \uparrow$ on $\mathbb{R}$

④ $y = \sqrt{x}$ on $[0, \infty)$

$y \uparrow$ on $[0, \infty)$



$y' = \dfrac{1}{2\sqrt{x}}$

$y'_{+}(0) = \infty$ undefined

y' is positive on $(0, \infty)$

y' $\cancel{////}\ 0\ + + + + +$ y

$y \uparrow$ on $[0, \infty)$

Extreme Values (EV's)

القيم القصوى

Max — Min

Max — Min

Max: Abs. Max (من كل المجال) → Local Max (محلي، فترة مفتوحة)

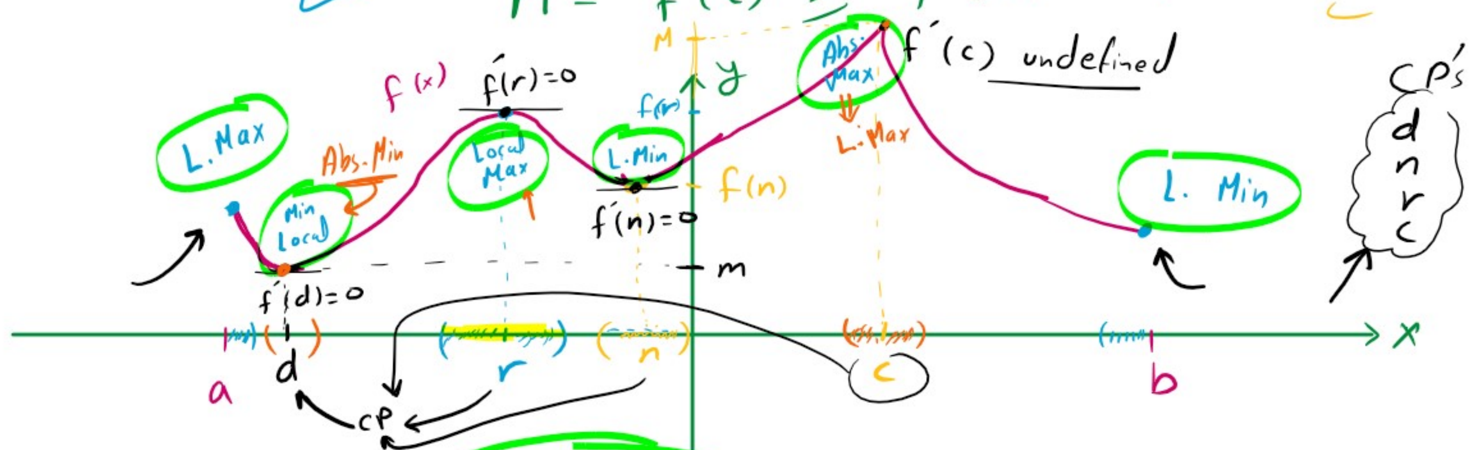
Min: Abs Min (من كل الفترة) → Local Min (محلي، فترة مفتوحة)

Absolutely
مطلق

<u>Def</u> f defined domain $D = D(f)$

1. f has Abs. Max M at point c $\in D$ if

$$M = f(c) \geq f(x) \quad \forall x \in D$$



2. f has Abs. Min m at point d $\in D$ if

$$m = f(d) \leq f(x) \quad \forall x \in D$$

3. f has Local Max of f(r) at point r $\in D$

③ f has Local max of f(r) at point r∈D

since $f(r) \geq f(x)$ $\forall$ x in small interval about r

④ f has local min of f(n) at point n∈D

since $f(n) \leq f(x)$ $\forall$ x in small interval about n

EV's occurs at ① critical points or

may ② End point

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check if $\Leftarrow$ EV's للبحث عن

القيم القصوى

they occur at

① CP's or

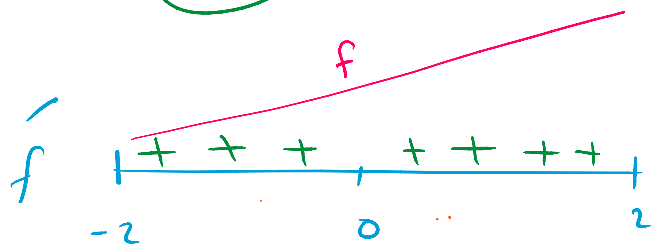
② End points

Exp Find EV's of $f(x) = x^3$ on $[-2, 2]$

CP's ⟶ $f'(x) = 3x^2 = 0 \Rightarrow x = 0 \in D$

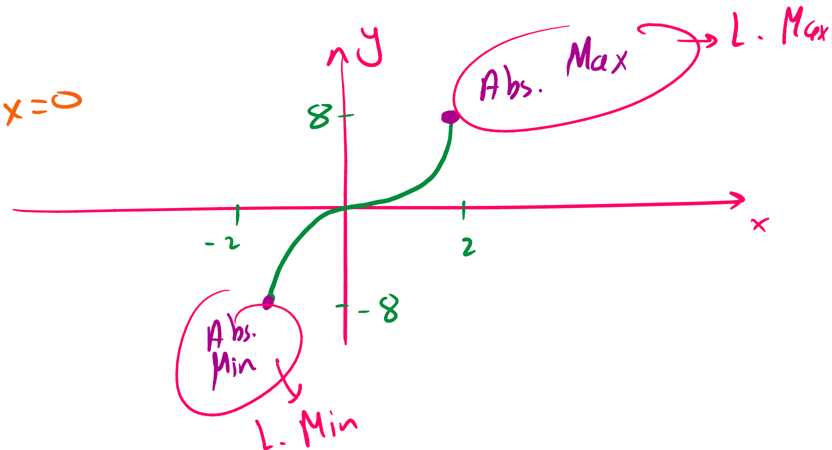
$\times$ Check at CP's $\longrightarrow$ $f'(x) = 3x^2 = 0 \Rightarrow x = 0 \in D$

Check at Endpoints



$x = 0 \in D$
$f'(0) = 0$ $\Big\} \Rightarrow x = 0$ is CP

but f has no EV at $x = 0$



Endpoint $f(2) = 2^3 = 8$

$f(-2) = (-2)^3 = -8$

L. Max

f has Abs. Max of 8 at $x = 2$

f has Abs. Min of -8 at $x = -2$

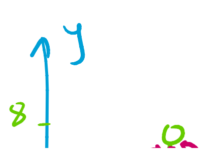
L. Min

Th (Extreme Value Theorem - EVT)

If f cont. on $[a, b]$

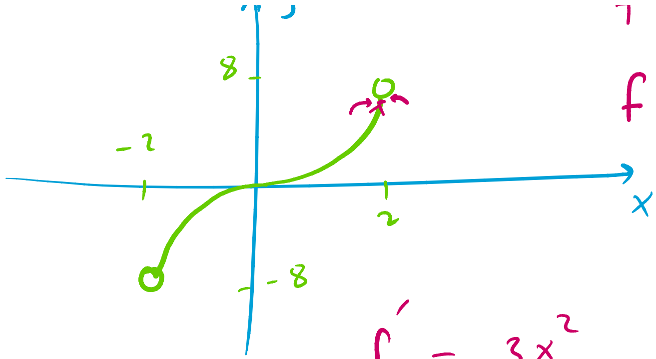
Then f has Abs. Max and Abs. Min

Exp $f(x) = x^3$ on $(-2, 2)$



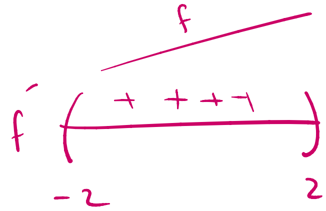
f cont. on $(-2, 2)$

f has ... EV's

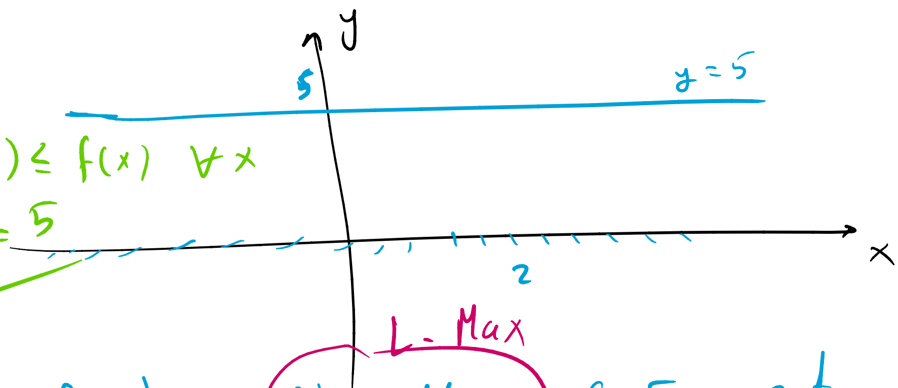


f has EV's
↓
no

$f' = 3x^2$
$= 0$
$x = 0$



Exp $f(x) = 5$ on $\mathbb{R}$



$x = 2 \Rightarrow f(2) = 5$

$f(2) \geq f(x) \quad \forall x \in \mathbb{R}$

$5 \geq 5$ ✓

$f(5) \leq f(x) \quad \forall x$

$5 \leq 5$ ✓

f has Abs. Max of 5 at $x = 5, 4, 3, 2, \ldots$

" " " " " " = all points in $\mathbb{R}$

L. Max

f has Abs. Min of 5 at $x = 5, 4, 3, 2 \ldots$

" " " " " " = all points in $\mathbb{R}$

L. Min

Exp

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

Find EV's

Find EV's

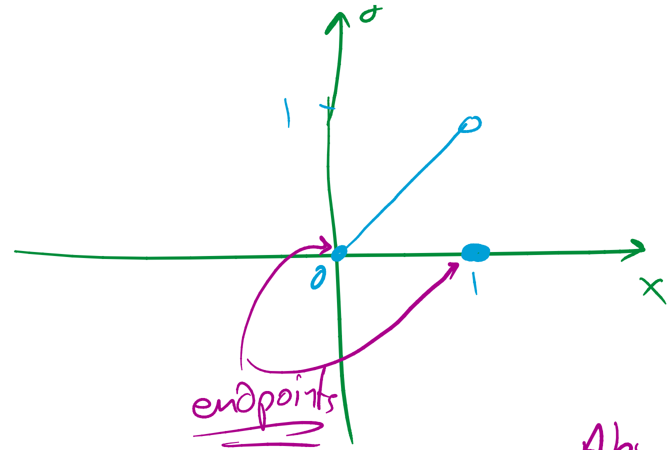
CP's End point

⇓

$$f'(x) = \begin{cases} 1 & ; \ 0 < x < 1 \\ 0 & ; \ x = 1 \end{cases}$$

$x = 1$ → endpoint

$f' \neq 0$



$x = 0 \Rightarrow f(0) = 0$

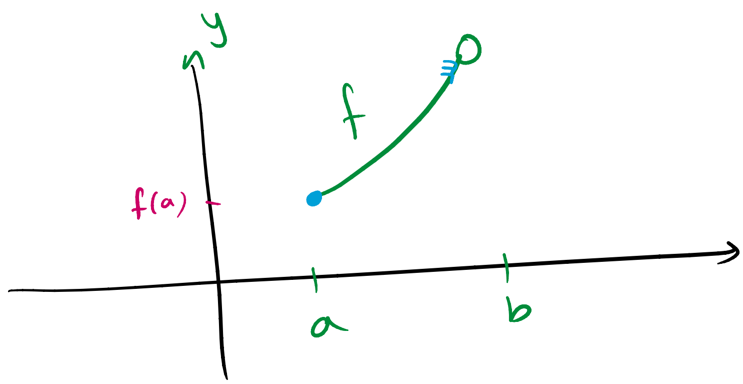
$x = 1 \Rightarrow f(1) = 0$

Abs Min

⇓

L. Min

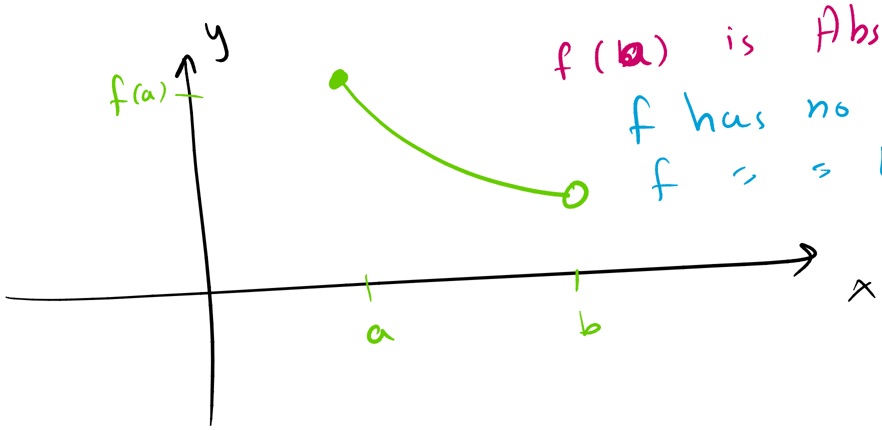
Exp



f(a) is Abs. Min ⇒ f(a) is L. Min

f has no Abs. Max

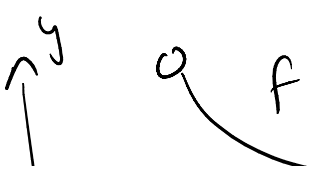
⇒ f " " L. Max



f(a) is Abs. Max ⇒ L. Max

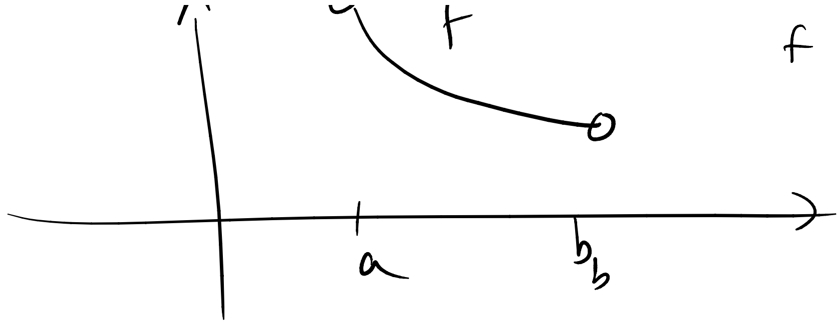
f has no Abs. Min

f " " L. Min



f has no L. Max

f has no L. Min



f has no L. Min $\Downarrow$ no Abs.

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f on [a, b]

$f'(a) \Rightarrow f'_+(a) = f'_-(a)$??

+

a

CP is interior $\Rightarrow$ $f'(c) = 0$

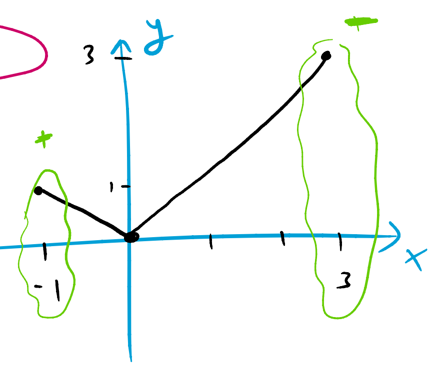
$f'(c)$ defined

Exp $f(x) = |x|$ on $[-1, 3]$ Find EV's

$\Rightarrow$ f has Abs. Min of 0 at $x=0$

$\Rightarrow$ f has L. Max of 1 at $x=-1$

$\Rightarrow$ f has Abs. Max of 3 at $x=3$



$$f(x) = |x| = \begin{cases} x & \text{if } 0 \le x \le 3 \\ -x & \text{if } -1 \le x < 0 \end{cases}$$

$$f(x) = |x| = \begin{cases} x & \ldots \\ -x & \text{if } -1 \le x \le 0 \end{cases}$$

$$f'(x) = \begin{cases} 1 & \text{if } 0 < x < 3 \\ -1 & \text{if } -1 < x < 0 \end{cases}$$

$$= \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 3 \end{cases}$$

$f'(0) \Rightarrow \nexists \qquad f'_+(0) \neq f'_-(0)$

$1 \neq -$

$x = 0$ is CP and $x = 0 \in D(f)$

EV

or

CP   End point

Exp $f(x) = x^{\frac{1}{3}}$ on $[-1, 8]$

① Find CP's

نقطة داخلية

or

$f' = 0$

f' undefined

$f' = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3} \quad \frac{1}{\ldots}$

$f' = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3} \frac{1}{\sqrt[3]{x^2}}$

f' undefined at $\boxed{x=0}$ and $0 \in [-1,8]$

$\Rightarrow$ $x=0$ is CP

② Find EV's

CP's

$x=0$

$\Downarrow$

$f(0) = 0^{\frac{1}{3}} = 0$

Endpoint

$x=-1$, $x=8$

$\Downarrow$

$f(-1) = \sqrt[3]{-1} = -1$

$f(8) = \sqrt[3]{8} = 2$

L.

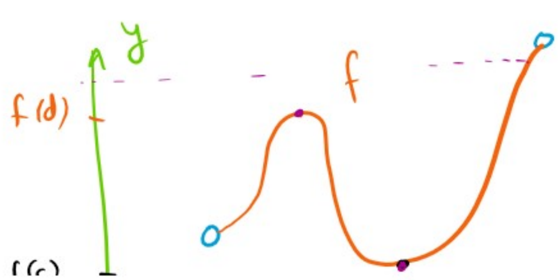
f has Abs. Min of -1 at $x=-1$

f has Abs. Max of 2 at $x=8$

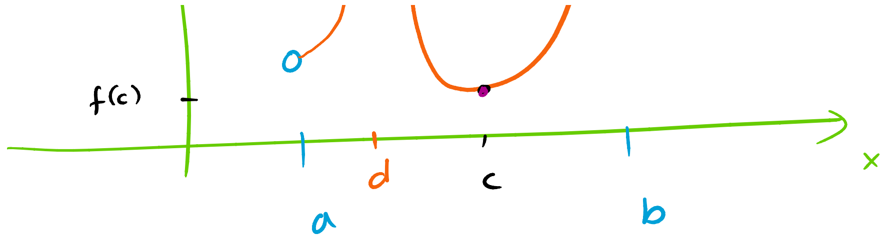
Th f cont. on $[a,b]$ $\Rightarrow$

f has Abs. Max and Abs. Min

Exp



f cont. on (a,b)



f(c) is Abs. Min $\Rightarrow$ L. Min

f(d) is L. Max