# Compound Interest - Geometric Sequences



- Recall that the simple interest is given by $I = Prt$
  where $P$ is principal (present value) in dollars
  $r$ is interest rate per year
  $t$ is time in years

- Another method of paying interest is the compound interest method where the interest $I$ for each period is added to the principal $P$ before we calculate the interest $I$ for the next period.

---

**Exp** Assume \$ 20,000 in invested for 3 years at 10% compounded annually: ① how much interest is earned?

First year $\Rightarrow P_1 = \$20,000 \Rightarrow$ simple interest $I_1 = P_1 r t = (20,000)(\frac{10}{100})(1) = \$2000$

Second year $\Rightarrow P_2 = \$22000 \Rightarrow$ Simple interest $I_2 = P_2 r t = (22,000)(\frac{10}{100})(1) = \$2200$

Third year $\Rightarrow P_3 = \$24,200 \Rightarrow$ Simple interest $I_3 = P_3 r t = (24,200)(\frac{10}{100})(1) = \$2420$

Total simple interest is $I = I_1 + I_2 + I_3 = 2000 + 2200 + 2420 = \$6,620$

② Find the future value $S$

$S = P_3 + I_3 = 24200 + 2420 = \$26,620$

③ Is there any other method to find the future value $S$?

First year $\Rightarrow S_1 = P\left(1 + \frac{10}{100}\right)^1 = (20,000)(1 + 0.1) = (20,000)(1.1) = \$22,000$

Second year $\Rightarrow S_2 = P\left(1 + \frac{10}{100}\right)^2 = (20,000)(1 + 0.1)^2 = (20,000)(1.1)^2 = \$24,200$

Third year $\Rightarrow S_3 = P\left(1 + \frac{10}{100}\right)^3 = (20,000)(1 + 0.1)^3 = (20,000)(1.1)^3 = \$26,620$

④ Write general form to find the future value $S$ if $P$ is invested at interest rate $r$ per year compounded annually.

$$S = P(1 + r)^n$$

**Exp** If \$3000 is invested for 4 years at 9% compounded annually

① How much the future value will be?

$P = \$3000 \ , \ n = 4 \ , \ r = \frac{9}{100} = 0.09$

$S = P(1+r)^n = 3000(1+0.09)^4 = 3000(1.09)^4 = 3000(1.412)$

$\approx \$4236$

② How much interest is earned?

$S = P + I \quad \Rightarrow 4236 = 3000 + I$

$\Rightarrow I = 4236 - 3000$

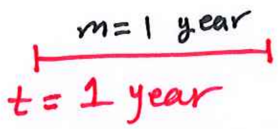
$= \$1236$

---

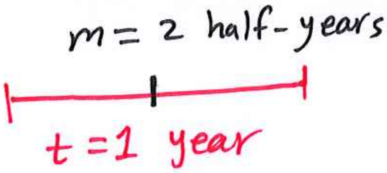
**Remark** Recall $t$ is time in years

- Sometimes the time $t$ is compounded into $m$ times per year

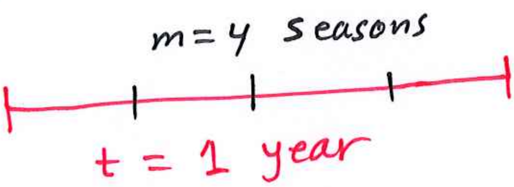
→ If $m=1$ ⇒ the time $t$ is compounded annually




→ If $m=2$ ⇒ the time $t$ is compounded semiannually




→ If $m=4$ ⇒ the time $t$ is compounded quarterly




→ If $m=12$ ⇒ the time $t$ is compounded monthly


⋮

- The interset per year $r$ is also called nominal annual rate

- The interset rate per period is denoted by $i$ and defined by $i = \frac{r}{m}$

- If $P$ is invested for $t$ years at nominal annual rate $r$ with $m$ compounded times per 1 year, then
  - the total number of compounding periods in $t$ years is $n = mt$
  - and the future value is

$$S = P(1+i)^{n} = P\left(1+\frac{r}{m}\right)^{mt}$$

---

**Exp** If \$ 5000 is invested at 6 % nominal annual rate, compounded quarterly, for 5 years. Find the

① number of compounding periods per year

quarterly $\Rightarrow m = 4$

② number of compounding periods for the investment

$n = mt = 4(5) = 20$

③ interest rate for each compounding period




$i = \frac{r}{m} = \frac{0.06}{4} = 0.015$

④ future value of the investment

$S = P(1+i)^{n} = 5000(1+0.015)^{20} = 5000(1.015)^{20} \approx \$\,6,734.28$ to the nearest cent

⑤ interest that is totally earned

$S = P + I \Rightarrow 6,734.28 = 5000 + I$

$\Rightarrow I = 6,734.28 - 5000 = \$\,1,734.28$

**Exp** Find the interest per period $i$ and the number of compounding periods $n$ for the following investments

① 12 % compounded monthly for 7 years

$r = \frac{12}{100} = 0.12$ , $m = 12$ , $t = 7$

$i = \frac{r}{m} = \frac{0.12}{12} = 0.01$ and $n = mt = 12(7) = 84$

② 7.2 % compounded quarterly for 11 quarters

1 year → 4 quarters
$t$ → 11 quarters

$r = \frac{7.2}{100} = 0.072$ , $m = 4$ , $t = \frac{11}{4}$

$i = \frac{r}{m} = \frac{0.072}{4} = 0.018$ and $n = mt = 4\left(\frac{11}{4}\right) = 11$

---

**Exp** • What amount must be invested now in order to have \$12,000 after 3 years if 6% interest rat is compounded semiannually.

Find the principal $P$ if $S = \$12,000$

$t = 3$ years

$r = \frac{6}{100} = 0.06$

$m = 2$

$n = mt = 2(3) = 6$

$i = \frac{r}{m} = \frac{0.06}{2} = 0.03$

$S = P(1+i)^n$

$12,000 = P(1+0.03)^6$

$12,000 = P(1.03)^6$

$P = \frac{12,000}{(1.03)^6} = \frac{12,000}{1.194052} \approx \$10,049.81$ to the nearest cent

• How much interest is earned through this investment

$S = P + I \Rightarrow 12,000 = 10,049.81 + I$

$\Rightarrow I = 12,000 - 10,049.81$

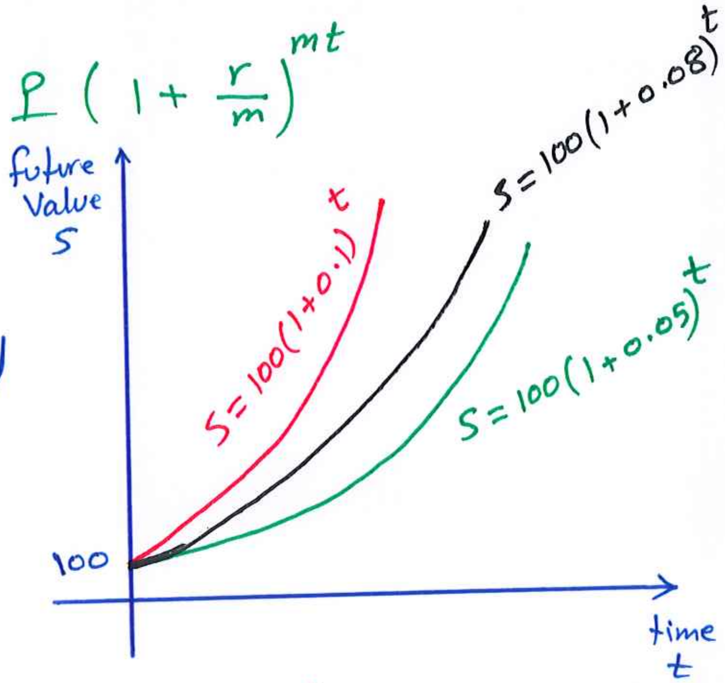
$= \$1950.19$

**Remark**

- If we invest $P = \$100$, then the higher interest rate $r \Rightarrow$ the higher future value $S$ since

$$S = P(1+i)^n = P\left(1+\frac{r}{m}\right)^{mt}$$

- This Figure shows how $S \uparrow$ as $r \uparrow$ when interest compounded annually $(m=1)$




- The curves of $S$ are growth exponentials

- Assume the interest results from continuous compounding (compounding every instant). Assume also $P = \$1$ is invested for one year $(t=1)$ at 100% interest rate $(r=1)$.
  Then, $S = P\left(1+\frac{r}{m}\right)^{mt} = \left(1+\frac{1}{m}\right)^{m}$ and when interest is compounded:

[1] Annually $\Rightarrow$ Number of periods per year $m=1$ $\Rightarrow$ future value $S = \left(1+\frac{1}{1}\right)^1 = 2$

[2] Monthly $\Rightarrow$ $m = 12$ $\Rightarrow S = \left(1+\frac{1}{12}\right)^{12} = 2.6130\ldots$

[3] Daily $\Rightarrow m = 360$ (Business Year) [$(12)(30)$] $\Rightarrow S = \left(1+\frac{1}{360}\right)^{360} = 2.7145\ldots$

[4] Hourly $\Rightarrow m = (360)(24) = 8{,}640$ $\Rightarrow S = \left(1+\frac{1}{8640}\right)^{8640} = 2.71812\ldots$

[5] Each minute $\Rightarrow m = (8640)(60) = 518{,}400$ $\Rightarrow S = \left(1+\frac{1}{518{,}400}\right)^{518{,}400} = 2.71827\ldots$

$e \simeq 2.718$

As $m \uparrow \Rightarrow S$ approaches $e \simeq 2.718$

$\Downarrow$

**Def** (Future Value for continuous Compounding)

If $ P is invested now for t years at nominal rate r compounded continuously, then the future value is given by the exponential function:

$$S = P e^{rt} \quad , \quad e \approx 2.718$$

---

**Exp** Find the future value if $ 1000 is invested for 20 years at 8% compounded continuously

$P = \$1000 \quad , \quad t = 20 \text{ years} \quad , \quad r = 0.08$

$$S = P e^{rt} = 1000 e^{(0.08)(20)} = 1000 e^{1.6} = 1000(4.95303)$$
$$= \$4953.03$$

---

**Exp** • What amount must be invested now at 6.5% compounded continuously, so that it will be worth $ 25,000 after 8 years

$P = ?? \quad , \quad t = 8 \text{ years} \quad , \quad r = \frac{6.5}{100} = 0.065 \quad , \quad S = \$25,000$

$$S = P e^{rt}$$
$$25,000 = P e^{(0.065)(8)}$$
$$25,000 = P e^{0.52}$$
$$25,000 = P (1.68202765)$$
$$P = \frac{25,000}{1.68202765}$$
$$= \$14,863.01$$

• How much interest the investor has earned?

$$S = P + I$$
$$25,000 = 14,863.01 + I$$
$$I = 25,000 - 14,863.01$$
$$= \$10,136.99$$

**Exp** [1] How much will you earn if you invest \$1000 for 5 years at 8% compounded continuously?

$P = \$1000$, $t = 5$ years, $r = 0.08$

$$S = P e^{rt} = 1000 e^{(0.08)(5)} = 1000 e^{0.4} \approx 1000(1.49182)$$
$$= \$1491.82$$

[2] How much will you earn if you invest \$1000 for 5 years at 8% compounded quarterly?

$P = \$1000$, $t = 5$ years, $r = 0.08$, $m = 4$

$$S = P(1+i)^n = P\left(1 + \frac{r}{m}\right)^{mt} = 1000\left(1 + \frac{0.08}{4}\right)^{(4)(5)}$$
$$= 1000(1+0.02)^{20} = 1000(1.02)^{20} \approx 1000(1.485947)$$
$$= \$1485.95$$

[3] Compare investments in [1] and [2]

Investment by compounding continuously has extra interest by \$1491.82 − \$1485.95 = \$5.87

[4] How much interest is earned in (1) and (2)

Compounded Continuously $\Rightarrow S = P + I_c$

$1491.82 = 1000 + I_c \Rightarrow I_c = \$491.82$

Compounded quarterly $\Rightarrow S = P + I_q$

$1485.95 = 1000 + I_q \Rightarrow I_q = \$485.95$

Note that $I_c - I_q = \$491.82 - \$485.95 = \$5.87$