Chapter 9: Numerical Solution of 1st order ODEs

9.2. Euler's Method of $O(h^1)$:

We are dealing with first order ordinary differential equations:

$$\left.\begin{aligned} y' &= f(t,y) \\ y(t_0) &= y_0 \end{aligned}\right\} \Rightarrow \text{Initial Value problem (IVP).}$$

Let $[a,b]$ be the Interval we want to find the Solution of $y'(t) = f(t,y)$ with $y(a) = y_0$.

We will approximate the Solution Using set of points (t_k, y_k) where $y(t_k) \approx y_k$.

How to Construct this set of points ??

1) We will divide the Interval $[a,b]$ into M subIntervals such that $h = \dfrac{b-a}{M}$.

and select the mesh points to be:

$$t_k = a + kh, \quad \text{for } k = 0, 1, \ldots, M$$

h is called step size.

2) We take taylor expansion of $y(t_1)$ about $t = t_0$

$$y(t_1) = y(t_0 + h) = y(t_0) + h\, y'(t_0) + \underbrace{\frac{h^2}{2!}\, y''(c_1)}_{\text{step error.}}$$

$$\rightsquigarrow \quad y(t_1) \approx y(t_0) + h\, y'(t_0)$$

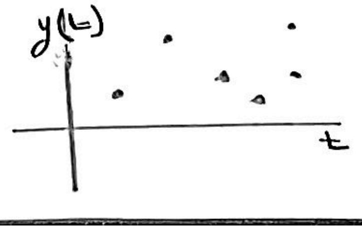
$$y(t_1) \approx y_0 + h\, f(t_0, y_0).$$

We repeat the process and generates a sequence of points that approximates the solution curve.

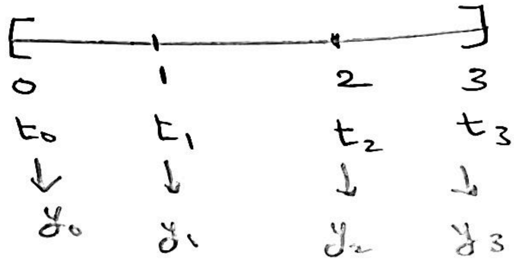
$$\rightsquigarrow \quad \underbrace{y(t_{k+1}) = y_k + h\, f(t_k, y_k)}_{\text{Euler Method.}}, \quad k = 0, 1, \ldots, M-1$$

So we get the points (t_k, y_k), $k = 0, 1, \ldots, M$.

We plot the points and then approximate $y(t)$ by a suitable method.



Example: Estimate the Solution of $y' = \frac{t-y}{2}$ with $y(0)=1$ on $[0,3]$ with $h=1$.



$$y_1 = y_0 + h\, f(t_0, y_0)$$

$$y_1 = 1 + 1\left(\frac{0-1}{2}\right) = \boxed{\frac{1}{2}}$$

$$y_2 = y_1 + h\, f(t_1, y_1) = 0.5 + 1\left(\frac{1-0.5}{2}\right) = 0.5 + 0.25 = \boxed{0.75}$$

$$y_3 = y_2 + h\, f(t_2, y_2) = 0.75 + 1\left(\frac{2-0.75}{2}\right) = \boxed{1.375}$$

So the points are: $(0,1)$, $(1, 0.5)$, $(2, 0.75)$, $(3, 1.375)$.

Total Error:

$$E(y,h) = \frac{h^2}{2} y''(c_1) + \frac{h^2}{2} y''(c_2) + \cdots + \frac{h^2}{2} y''(c_M).$$

$$= \frac{h^2}{2}\left(y''(c_1) + \cdots + y''(c_M)\right)$$ Take the average.

$$\approx \frac{h^2}{2}\left(M\, y''(c)\right)$$

$$= \left(\frac{b-a}{2}\right) y''(c) \cdot h \approx \overset{\text{constant}}{C} \cdot h = O(h^1)$$

Note: The smaller the Value of h, the error get smaller.

9.4. Taylor's Method $O(h^2)$:

Expand $y(t_1)$ about $t = t_0$.

$$y_1 = y_0 + h\,y'(t_0) + \frac{h^2}{2!}\,y''(t_0) + \frac{h^3}{3!}\,y'''(c).$$

$$= y_0 + h\,f(t_0, y_0) + \frac{h^2}{2!}\,f'(t_0, y_0) + \frac{h^3}{3!}\,f''(c, y_c).$$

$$\Rightarrow \quad y_1 \approx y_0 + h\,f(t_0, y_0) + \frac{h^2}{2!}\,f'(t_0, y_0)$$

with step error $= \dfrac{h^3}{3!}\,f''(c, y_c)$.

repeat the process:

$$y_{k+1} \approx y_k + h\,f(t_k, y_k) + \frac{h^2}{2}\,f'(t_k, y_k)$$

for $k = 0, 1, \ldots, M-1$.

Total error $= \dfrac{h^3}{3!}\left(y'''(c_1) + \cdots + y'''(c_M)\right)$

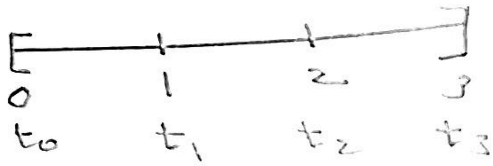
$$= \frac{h^3}{3!}\,M\,y'''(c) = \frac{(b-a)}{3!}\,y'''(c) \cdot h^2$$

$$\approx C \cdot h^2 = O(h^2).$$

Example: Approximate the solution of

$$y' = \frac{t-y}{2} \quad \text{on } [0,3] \text{ with } y_0 = 1 \ \& \ h = 1$$

Using Taylor's Method.



$$y' = \frac{t-y}{2} = f(t,y)$$

$$y'' = \frac{1}{2} - \frac{1}{2}y' = \frac{1}{2} - \frac{1}{2}\left(\frac{t-y}{2}\right) = f'(t,y)$$

$$\Rightarrow y_1 = y_0 + h\,f(t_0,y_0) + \frac{h^2}{2!}\,f'(t_0,y_0)$$

$$= 1 + 1\left(\frac{0-1}{2}\right) + \frac{(1)^2}{2!}\left(\frac{1}{2} - \frac{1}{2}\left(\frac{0-1}{2}\right)\right)$$

$$= 1 + (-0.5) + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4}\right) = \boxed{0.875}$$

$$y_2 = y_1 + h\,f(t_1,y_1) + \frac{h^2}{2}\,f'(t_1,y_1)$$

$$= 0.875 + (1)\left(\frac{1-0.875}{2}\right) + \frac{(1)^2}{2}\left(\frac{1}{2} - \frac{1}{2}\left(\frac{1-0.875}{2}\right)\right)$$

$$= 0.875 + (0.0625) + \frac{1}{2}(0.46875)$$

$$y_2 = \boxed{1.171875}$$

$$y_3 = 1.171875 + \left(\frac{2-1.171875}{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\left(\frac{2-1.171875}{2}\right)\right)$$

$$= 1.171875 + 0.4140625 + 0.146484375$$

$$= \boxed{1.73242}$$

Example: Use Taylor Method of $O(h^4)$ to estimate the solution of $y' = \frac{t-y}{2}$, $y_0 = 1$ on $[0,3]$ with $h=1$.

sol: $$y(t_1) = y_0 + h\,y'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \frac{h^4}{4!} y''''(t_0) + \underbrace{\frac{h^5}{5!} y^{(5)}(c)}_{\text{step error}}$$

repeat the process $\rightsquigarrow$ $y_{k+1} = \cdots$

so the total error $$= \frac{h^5}{5!}\left(y^{(5)}(c_1) + \cdots + y^{(5)}(c_M)\right)$$

$$= \frac{h^5}{5!}\left(M\, y^{(5)}(c)\right) = \frac{(b-a)}{5!} y^{(5)}(c)\, \underbrace{h^4}_{O(h^4)}.$$

Now: $y' = \frac{t-y}{2}$, $y'' = \frac{1}{2} - \frac{1}{2}\left(\frac{t-y}{2}\right)$, $y''' = -\frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\left(\frac{t-y}{2}\right)\right)$

$$y'''' = \frac{1}{8} - \frac{1}{8}\left(\frac{t-y}{2}\right).$$

$$\Rightarrow y_1 = 1 + 1\left(\frac{0-1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\left(\frac{0-1}{2}\right)\right) + \frac{1}{6}\left(-\frac{1}{2}\left[\frac{1}{2} - \frac{1}{2}\left(\frac{0-1}{2}\right)\right]\right)$$

$$+ \frac{1}{24}\left(\frac{1}{8} - \frac{1}{8}\left(\frac{0-1}{2}\right)\right) = \boxed{0.8203125}$$

$$\boxed{y_2 = 1.1045125}.$$

$$\boxed{y_3 = 1.6701860}.$$

9.3. Modified Euler Method (Heun's Method):

Our Aim is to approximate the solution of the IVP : $y' = f(t,y)$ with $y(t_0) = y_0$ over $[a,b]$

To obtain (t_1, y_1), we can use the Fundemental theorem of Calculus and Integrate y' over $[t_0, t_1]$

$$\Rightarrow \int_{t_0}^{t_1} f(t, y(t))\, dt = \int_{t_0}^{t_1} y'(t)\, dt = y(t_1) - y(t_0)$$

$$\Rightarrow y(t_1) = y_1 = y(t_0) + \underbrace{\int_{t_0}^{t_1} f(t, y(t))\, dt}_{} \quad \cdots (*)$$

The Integrating part sometimes is hard to find. so we will approximate the Integrating part Using Trapezoidal Rule. with $h = t_1 - t_0$.

$$(*) \Rightarrow \underline{y_1} \approx y_0 + \frac{h}{2}\left(f(t_0, y_0) + \underbrace{f(t_1, \underline{y_1})}_{\text{Implicit form.}} \right)$$

(7)

Now, to get rid of $y(t_1)$ inside $f(t, y(t))$

we use Euler's Method to estimate $y(t_1)$:

$$\circledast \Rightarrow \quad y_1 \approx y_0 + \frac{h}{2}\left(f(t_0, y_0) + f\left(t_1, \underbrace{y_0 + h f(t_0, y_0)}_{\approx y_1}\right)\right)$$

The General form for <u>Heun's Method</u> is:

$$y_{K+1} \approx y_K + \frac{h}{2}\left(f(t_K, y_K) + f\left(t_{K+1}, y_K + h f(t_K, y_K)\right)\right)$$

<u>Error :</u> The error for trapezoidal is: $-\frac{h^3 y''(c)}{12}$.

The error in Euler's Method is: $\frac{(b-a)}{2} h \, y''(c)$.

so we consider the error for trapezoidal since it's smaller than the error in Euler's Method.

$$\Rightarrow \text{Total error} = -\frac{h^3}{12}\left(y''(c_1) + \cdots + y''(c_M)\right)$$

$$\approx \frac{-h^3}{12} M \, y''(c) = \frac{-h^3}{12}\frac{(b-a)}{h} \cdot y''(c)$$

$$= -\frac{h^2}{12}(b-a)\, y''(c) = O(h^2).$$

Example: Estimate the Solution of $y' = \frac{t-y}{2}$ with $y(0)=1$ on $[0,3]$ with $h=1$.

Using Heun's Method.

$$y_1 = y_0 + \frac{h}{2}\left(f(t_0, y_0) + f(t_1, y_0 + h f(t_0, y_0))\right)$$

$$= 1 + \frac{1}{2}\left(f(0,1) + f(1, \underbrace{1 + \left(\tfrac{0-1}{2}\right)}_{0.5})\right)$$

$$= 1 + \frac{1}{2}\left[\left(\frac{0-1}{2}\right) + \left(\frac{1-0.5}{2}\right)\right] = 0.875.$$

$$y_2 = y_1 + \frac{h}{2}\left(f(t_1, y_1) + f(t_2, y_1 + h f(t_1, y_1))\right)$$

$$= 0.875 + \frac{1}{2}\left(f(1, 0.875) + f(2, \underbrace{0.875 + (1) f(1, 0.875)}_{\overset{\shortparallel}{y_1 : \text{Euler}}})\right)$$

$$= 0.875 + \frac{1}{2}\left(f(1, 0.875) + f(2, 0.9375)\right)$$

$$= 0.875 + \frac{1}{2}\left(0.0625 + 0.53125\right)$$

$$= 1.171875$$

$$y_3 = y_2 + \frac{h}{2}\left(f(t_2, y_2) + f(t_3, \underbrace{y_2 + h f(t_2, y_2)}_{\shortparallel \ \text{Euler}})\right)$$

$$= 1.171875 + \frac{1}{2}\left(f(2, 1.171875) + f(3, 1.58594)\right)$$

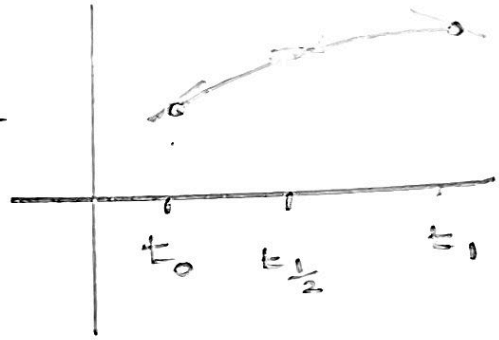
$$= 1.171875 + \frac{1}{2}\left(0.4140625 + 0.70703\right)$$

$$= 1.171875 + 0.560546 = 1.732421$$

9.5. Runge-Kutta Method of order $O(h^4)$ [RK4]

RK4 is also Called Modified Simpson's Method.

Consider the graph of the Solution Curve $y = y(t)$ over the first subinterval $[t_0, t_1]$.



Let f_0 be the slope at the left

f_1, f_2 the estimation of the slope in the middle.

f_3 be the slope at the right

Using FTC: $y(t_1) - y(t_0) = \int_{t_0}^{t_1} f(t, y(t))\, dt$

To estimate the Integral part, Use Simpson's $\frac{1}{3}$ Rule with step size $= \frac{h}{2}$, then:

$$\int_{t_0}^{t_1} f(t, y(t))\, dt \approx \frac{h}{6}\left(f(t_0, y(t_0)) + 4 f\left(t_{\frac{1}{2}}, y(t_{\frac{1}{2}})\right) + f(t_1, y(t_1)) \right)$$

where $t_{\frac{1}{2}}$ is the midpoint of $[t_0, t_1]$.

Let $f(t_0, y(t_0)) = f_0$

& $f(t_{\frac{1}{2}}, y(t_{\frac{1}{2}})) \approx \dfrac{f_1 + f_2}{2}$

where $f_1 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2} f_0)$

$f_2 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2} f_1)$

& $f_3 = f(t_1, y_0 + h f_2)$

Then: $y(t_1) - y(t_0) = \int_{t_0}^{t_1} f(t, y(t))\, dt$

$\approx \frac{h}{6}(f_0 + 2f_1 + 2f_2 + f_3)$

$\Rightarrow$ $y_1 \approx y_0 + \frac{h}{6}(f_0 + 2f_1 + 2f_2 + f_3)$ RK4

Total error $= \left| \dfrac{-h^5}{2880}\left(f^{(4)}(c_1) + \cdots + f^{(4)}(c_M)\right) \right|$

$= \left| \dfrac{-h^5}{2880} \cdot M \cdot f^{(4)}(c) \right| = \left| \dfrac{-h^5}{2880}\left(\dfrac{b-a}{2h}\right) \cdot f^{(4)}(c) \right|$

$\approx \dfrac{h^4}{5760}(b-a) f^{(4)}(c) \approx C \cdot h^4 = O(h^4)$.

Example: Estimate the solution of $y' = \frac{t-y}{2}$ with $y(0) = 1$ and $h = 0.25$ on $[0, 3]$ using RK4.

$$y_1 \approx y_0 + \frac{h}{6}\left(f_0 + 2f_1 + 2f_2 + f_3\right)$$

where $f_0 = f(t_0, y_0) = f(0, 1) = -0.5$.

$$f_1 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2} f_0\right) = f\left(0 + \frac{1}{8}, 1 + \frac{1}{8}(-0.5)\right)$$
$$= f(0.125, 0.9375) = -0.40625$$

$$f_2 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2} f_1\right) = f\left(0.125, 1 + \frac{1}{8}(-0.40625)\right)$$
$$= f(0.125, 0.94922) = -0.4121094$$

$$f_3 = f\left(\overset{t_1}{0.25}, \overset{y_0 + h f_2}{1 + 0.25(-0.4121094)}\right)$$
$$= f(0.25, 0.8969726\overline{5}) = -0.3234863$$

$$\Rightarrow y_1 = 1 + \frac{1/4}{6}\left(-0.5 - 2(0.40625) - 2(0.4121094) - 0.3234863\right)$$
$$y_1 \approx 0.897492$$

Now $y_2 = y_1 + \frac{h}{6}\left(f_0 + 2f_1 + 2f_2 + f_3\right)$, where:

$$f_0 = f(t_1, y_1)$$
$$f_1 = f\left(t_1 + \frac{h}{2}, y_1 + \frac{h}{2} f_0\right)$$
$$f_2 = f\left(t_1 + \frac{h}{2}, y_1 + \frac{h}{2} f_1\right)$$
$$f_3 = f\left(t_2, y_1 + h f_2\right)$$

...

$$y_3 = y_2 + \frac{h}{6}\left(f_0 + 2f_1 + 2f_2 + f_3 \right).$$

where $f_0 = f(t_2, y_2)$

$$f_1 = f\left(t_2 + \frac{h}{2}, y_2 + \frac{h}{2} f_0\right)$$

$$f_2 = f\left(t_2 + \frac{h}{2}, y_2 + \frac{h}{2} f_1\right)$$

$$f_3 = f(t_3, y_2 + h f_2).$$

$\vdots$

Good Luck. :)