4.4: The Beta, t and F distribution.

→ Beta Distribution.

Assume $X_1 \sim Gamma(\alpha, 1)$

$X_2 \sim Gamma(\beta, 1)$

X_1, X_2 independent.

Define $Y_1 = X_1 + X_2$, $Y_2 = \frac{X_1}{X_1 + X_2}$

then using change of variable method we get

$Y_1 \sim Gamma(\alpha+\beta, 1)$

$Y_2 \sim Beta(\alpha, \beta)$.

$$g_1(y_1) = \begin{cases} \frac{1}{\Gamma(\alpha+\beta)} y_1^{\alpha+\beta-1} e^{-y_1} & , y_1 > 0 \\ 0 & , \text{elsewhere}. \end{cases}$$

$$g_2(y_2) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y_2^{\alpha-1} (1-y_2)^{\beta-1} & , 0 < y_2 < 1 \\ 0 & , \text{elsewhere}. \end{cases}$$

check Q4.35.

$\rightsquigarrow$ t ~ distribution

assume $W \sim N(0,1)$

$V \sim \chi^2(r)$

W and V independent.

Define $T = \dfrac{W}{\sqrt{\frac{V}{r}}}$, $U = V$

using change of variable method we get

$T \sim$ t-distribution with r degrees of freedom.

$T \sim t(r)$.

$$\rightsquigarrow g(t) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\ \Gamma\left(\frac{r}{2}\right)} \frac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}} , \quad t \in \mathbb{R}.$$ مش حفظ

Remark:

$r = 1 \Rightarrow T \sim$ cauchy distribution

$r \to \infty \Rightarrow T \sim N(0,1)$.

Remark:

$$\int_{-\infty}^{\infty} \frac{1}{\left(1+\frac{x^2}{r}\right)^{\frac{r+1}{2}}} dx = \frac{\sqrt{\pi r}\ \Gamma\left(\frac{r}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)}.$$

→ F-distribution.

assume $U \sim \chi^2(r_1)$

$V \sim \chi^2(r_2)$

U and V independent.

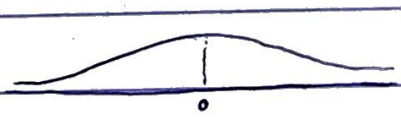
Define $W = \dfrac{\frac{U}{r_1}}{\frac{V}{r_2}}$, $Z = V$

then $W \sim$ F distribution with degrees of fredom $df_1 = r_1$ and $df_2 = r_2$

→ $W \sim F(r_1, r_2)$.

→ $g(w) = \dfrac{\Gamma\left(\frac{r_1+r_2}{2}\right)\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)} \cdot \dfrac{w^{\frac{r_1}{2}-1}}{\left(1+\frac{r_1}{r_2}w\right)^{\frac{r_1+r_2}{2}}}$, $w > 0$.

→ $T \sim t(r)$



→ $W \sim F(r_1, r_2)$