10.3 Integral Test.

The integral test:

let $\{a_n\}$ be a sequence of positive terms.
suppose that $a_n = f(n)$, where f is a continuous ~~f~~ positive, decreasing function for all $x \geq N$ then the series $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

Questions: 6, 13, 20, 22, 28, 32, 38, 40, 42

pages: 557 - 558.

$\boxed{6}$ $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ use the integral test

~~integral~~

let $f(x) = \frac{1}{x(\ln x)^2}$, $f(x)$ is continuous and positive for $x \geq 2$

$$f'(x) = \frac{-(\cancel{x} \cdot 2\ln x \cdot \frac{1}{\cancel{x}} + (\ln x)^2)}{x^2(\ln x)^4} = \frac{-(2\ln x + (\ln x)^2)}{x^2(\ln x)^4}$$

$f'(x) < 0$ for $x \geq 2$

so $f(x)$ is decreasing

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$$

let $u = \ln x$

$du = \frac{dx}{x}$

$$\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\ln x}$$

$$\therefore \int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{\ln x} \Big|_2^b \right]$$

$$= \lim_{b\to\infty} -\left[\frac{1}{\ln b} - \frac{1}{\ln 2} \right] = -\left[0 - \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$$

the integral converges

So by the integral test $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

$\boxed{13}$ $\sum_{n=1}^{\infty} \frac{n}{n+1}$ use any method.

n^{th} term test :

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0$$

so the series diverges.

$\boxed{20}$ $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$ use any method.

The integral test:

$f(x) = \dfrac{\ln x}{\sqrt{x}}$, positive , continuous for $x \geq 2$

$$f'(x) = \frac{\sqrt{x} \cdot \frac{1}{x} - \ln x \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} = \frac{\frac{1}{\sqrt{x}} - \frac{\ln x}{2\sqrt{x}}}{x}$$

$$= \frac{2 - \ln x}{2\,(x)^{3/2}} \rightarrow f'(x) = 0 \text{ if } 2 = \ln x \text{ if } x = e^2 \simeq 7.3$$

$$\overset{0}{\underset{e^2}{|}} - - - - - \rightarrow$$

$f(x)$ decreasing for $x \geq 8$

$$\rightarrow \int_2^{\infty} \frac{\ln x}{\sqrt{x}}\, dx$$

let $u = \ln x$

$du = \dfrac{dx}{x}$

$\downarrow$

$x = e^u$

$\downarrow$

$\sqrt{x} = e^{u/2}$

u	$e^{u/2}$
1	$2e^{u/2}$
0	$4e^{u/2}$

$$\int \frac{\ln x}{\sqrt{x}}\, dx = \int \frac{u}{e^{u/2}} \cdot e^u\, du$$

$$= \int u e^{u/2}\, du$$

$$= 2ue^{u/2} - 4e^{u/2}$$

$$= 2 \cdot \ln x \cdot \sqrt{x} - 4 \cdot \sqrt{x}$$

$$= 2\sqrt{x}\,(\ln x - 2)$$

$$\int_2^{\infty} \frac{\ln x}{\sqrt{x}}\, dx = \lim_{b\to\infty} \left[2\sqrt{x}(\ln x - 2) \right]\Big|_2^b = \lim_{b\to\infty} \left[2 \cdot \sqrt{b}(\ln b - 2) - 2\sqrt{2}(\ln 2 - 2) \right]$$

$= \infty$ so the integral diverges and

then the series diverges

$\boxed{22}$ $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$ use any method.

n^{th} term test :-

$$\lim_{n\to\infty} \frac{5^n}{4^n+3} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{5^n \ln 5}{4^n \ln 4} = \frac{\ln 5}{\ln 4} \lim_{b\to\infty} \left(\frac{5}{4}\right)^n$$

$= \infty$

so the series diverges.

$\boxed{28}$ $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ use any method

n^{th} term test :-

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e^1 = e \neq 0$$

So the series diverges

$\boxed{32}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$ use any method.

integral test $f(x) = \dfrac{1}{x(1+\ln^2 x)}$, positive , continuous for $x \geq 1$

$$f'(x) = \frac{-\left(\cancel{x} \cdot 2\ln x \cdot \frac{1}{\cancel{x}} + \ln^2 x\right)}{\left[x\left(1+\ln^2 x\right)\right]^2}$$

$$= \frac{-(2\ln x + \ln^2 x)}{\left[x(1+\ln^2 x)\right]^2} < 0 \quad \text{for } x > 1$$

since $\ln x > 0$ for $x > 1$

So $f(x)$ decreasing for $x > 1$

$$\int_1^\infty \frac{dx}{x(1+\ln^2 x)}$$

let $u = \ln x$
$du = \dfrac{dx}{x}$

$$\int \frac{dx}{x(1+\ln^2 x)} = \int \frac{du}{1+u^2} = \tan^{-1}(u) = \tan^{-1}(\ln x)$$

$$\int_1^\infty \frac{dx}{x(1+\ln^2 x)} = \lim_{b\to\infty}\left[\tan^{-1}(\ln x)\Big|_1^b\right]$$

$$= \lim_{b\to\infty}\left[\tan^{-1}(\ln b) - \tan^{-1}(\ln 1)\right] = \frac{\pi}{2} - 0$$

So the integral converges and then the series converges.

$\boxed{38}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ use any method

integral test $f(x) = \dfrac{x}{x^2+1}$ continuous, positive and decreasing for $x \geq 1$

(Note: $f'(x) = \dfrac{x^2+1-2x^2}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$ $\quad -1 \quad 1$)

$$\int_1^\infty \frac{x}{x^2+1}\,dx$$

let $u =$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1|$$

$$\therefore \int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{b\to\infty}\left[\frac{1}{2}\ln(x^2+1)\Big|_1^b\right]$$

$$= \cancel{\lim_{b\to\infty}\frac{1}{2}}\ \frac{1}{2}\lim_{b\to\infty}\left[\ln(b^2+1) - \ln(2)\right] = \frac{1}{2}\left[\infty - \ln 2\right] = \infty$$

so the series diverges.

$\boxed{40}$ $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$ use any method.

$$f(x) = \operatorname{sech}^2 x = \frac{1}{\cosh^2 x} = \frac{1}{\left(\frac{e^x + e^{-x}}{2}\right)^2} = \frac{4}{(e^x + e^{-x})^2}$$

positive, continuous and decreasing

$$\int_1^\infty \operatorname{sech}^2 x \, dx = \lim_{b \to \infty} \left[\tanh x \Big|_1^b \right]$$

$$= \lim_{b \to \infty} \left[\tanh b - \tanh 1 \right] = 1 - \tanh(1)$$

Converges since $\tanh(1) = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}}$

So the series converges.

$\boxed{42}$ For what values of a do the series converge?

$$\sum_{n=3}^{\infty} \left(\frac{1}{n-1} - \frac{2a}{n+1} \right)$$

$$\int_3^\infty \left(\frac{1}{x-1} - \frac{2a}{x+1} \right) dx = \cancel{\lim_{b \to \infty} \left[\frac{1}{x-1} - \frac{2a}{x+1} \right] \Big|_3^b}$$

$$= \lim_{b \to \infty} \left[\ln|x-1| - 2a \ln|x+1| \right] \Big|_3^b$$

$$= \lim_{b \to \infty} \left[\ln \left| \frac{x-1}{(x+1)^{2a}} \right| \right] \Big|_3^b$$

$$= \lim_{b \to \infty} \left[\ln \left| \frac{b-1}{(b+1)^{2a}} \right| - \ln \left| \frac{2}{4^{2a}} \right| \right]$$

$$\lim_{b\to\infty} \ln \frac{b-1}{(b+1)^{2a}} \doteq \lim_{b\to\infty} \ln \frac{1}{2a(b+1)^{2a-1}}$$

$$= \begin{cases} 0 & a=\frac{1}{2} \\ \infty & a<\frac{1}{2} \end{cases}$$

the series converges if $a=\frac{1}{2}$

the series diverges if $a<\frac{1}{2}$

note that for $a>\frac{1}{2}$, the terms becomes negative and we can't apply the integral test

$\left(\text{take } a=1 \text{ , } a_n = \frac{1}{x-1} - \frac{2}{x+1} = \frac{x+1-2x+2}{x^2-1} = \frac{3-x}{x^2-1} \right)$

$a_n < 0$ for $x > 3$

so the series behaves as a negative multiple of the harmonic series, so it diverges

$\therefore$ the series converges if $a=\frac{1}{2}$

10.4 Comparison Test

- The comparison test: (D.C.T)

let $\sum a_n$, $\sum c_n$ and $\sum b_n$ be series with nonnegative terms. Suppose that for some integer N.

$$d_n \leq a_n \leq c_n \text{ for all } n > N$$

a) If $\sum c_n$ converges, then $\sum a_n$ also converges

b) if $\sum d_n$ diverges, then $\sum a_n$ also diverges

- The limit comparison test (L.C.T)

Suppose that $a_n > 0$, $b_n > 0$ for $n \geq N$

1) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converges or both diverges

2) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges

3) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Remember that harmonic series $\sum \frac{1}{n}$ diverges and p-series $\sum \frac{1}{n^p}$ converges if $P > 1$ and diverges for $P \leq 1$

Questions: 8, 15, 18, 27, 28, 32, 40, 43, 52

pages: 603 - 604

$\boxed{8}$ use D.C.T to determine if the following sequence converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}} \quad \text{Compare with} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

(Note that $\frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ and $\frac{1}{\sqrt{n}}$ nonnegative for $n \geq 1$)

(also note that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series, $P = \frac{1}{2}$ which is divergent series)

for $n \geq 1$, $\sqrt{n} \geq 1$, $2\sqrt{n} \geq 2$

$2\sqrt{n} + 1 \geq 3$ multiply both sides by n

$2n\sqrt{n} + n \geq 3n \geq 3$ add n^2 for both sides

$n^2 + 2n\sqrt{n} + n \geq n^2 + 3$

$n(n + 2\sqrt{n} + 1) \geq n^2 + 3$

$$\frac{n(\sqrt{n}+1)^2}{n^2+3} \geq 1$$

$$\frac{(\sqrt{n}+1)^2}{n^2+3} \geq \frac{1}{n}$$ take the square root of both sides

$$\frac{\sqrt{n}+1}{\sqrt{n^2+3}} \geq \frac{1}{\sqrt{n}}$$ so $\frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ diverges by L.C.T

15 use L.C.T. to determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} \quad , \text{ compare with } \sum_{n=2}^{\infty} \frac{1}{n}$$

Note that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (harmonic series)

$$\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty$$

So by L.C.T $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges

18 use any method to determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}} \quad , \text{ use L.C.T with } \sum_{n=1}^{\infty} \frac{1}{n}$$

(Note that $\frac{3}{n+\sqrt{n}}$, $\frac{1}{n}$ are positive for $n \geq 1$)

$$\lim_{n\to\infty} \frac{\frac{3}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n}{n+\sqrt{n}} = \lim_{n\to\infty} \frac{3}{1+\frac{1}{2\sqrt{n}}} = 3$$

So both converge or both diverge.

but $\sum \frac{1}{n}$ diverges (harmonic series)

So by L.C.T $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ diverges

$\boxed{27}$ $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ (use any method)

Compare with $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$

(Note that $\frac{1}{n}$, $\frac{1}{\ln(\ln n)}$ are positive for $n \geq 3$)

$n > \ln n$ (take $\ln$ for both sides)

$\ln n > \ln(\ln n)$

so $n > \ln n > \ln(\ln n)$

$\therefore \frac{1}{n} < \frac{1}{\ln(\ln n)}$

so $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n} < \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

but $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (harmonic series)

so by D.C.T $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ diverges.

$\boxed{28}$ $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

use L.C.T, compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Note : $\frac{(\ln n)^2}{n^3}$, $\frac{1}{n^2}$ are positive for $n \geq 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (P-series with $p>1$)

$$\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^{3}} \cdot \cancel{n^2} = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{2(\ln n)(\frac{1}{n})}{1} = \lim_{n\to\infty} \frac{2\ln n}{n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{2(\frac{1}{n})}{1} = 0$$

so by L.C.T $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ converges.

$\boxed{32}$ $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$

use L.C.T compare with $\sum_{n=2}^{\infty} \frac{1}{n+1}$

$$\lim_{n\to\infty} \frac{\frac{\ln(n+1)}{n+1}}{\frac{1}{n+1}} = \lim_{n\to\infty} \ln(n+1) = \infty$$

note that $\sum_{n=2}^{\infty} \frac{1}{n+1}$ diverges by the integral test

$$\left(\int_2^{\infty} \frac{dx}{x+1} = \lim_{b\to\infty} \left[\ln |x+1| \Big|_2^b \right] = \lim_{b\to\infty} \left[\ln |b+1| - \ln |3| \right] = \infty \right)$$

so by L.C.T $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges.

Note that you can use the integral test to show that $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges.

$\boxed{40}$ $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$

Compare with $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

Note that $\frac{2^n+3^n}{3^n+4^n}$ and $\frac{2^n+3^n}{4^n}$ are positive for $n \geq 1$

Note that $\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{2}{4}\right)^n + \left(\frac{3}{4}\right)^n$

$= \sum_{n=1}^{\infty} \left(\frac{2}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{\frac{2}{4}}{1-\frac{2}{4}} + \frac{\frac{3}{4}}{1-\frac{3}{4}}$

$= \frac{\frac{2}{4}}{\frac{2}{4}} + \frac{\frac{3}{4}}{\frac{1}{4}} = 1 + 3 = 4$

So $\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n}$

by D.C.T since $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n} \leq \sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n}$

~~So~~ $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ Converges.

We can use L.C.T $\left(\lim_{n\to\infty} \frac{\frac{2^n+3^n}{3^n+4^n}}{\frac{2^n+3^n}{4^n}} = 1 \right)$

We can use L.C.T $\left(\lim_{n\to\infty} \frac{\frac{2^n+3^n}{3^n+4^n}}{\frac{3^n}{4^n}} = 1 \right)$

$\boxed{43}$ $\sum_{n=2}^{\infty} \frac{1}{n!}$, Compare with $\sum_{n=2}^{\infty} \frac{1}{2^n}$

$$\sum_{n=2}^{\infty} \frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

$$\leq \underbrace{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots}_{\sum_{n=1}^{\infty} \frac{1}{2^n}}$$

Note that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $|r| = |\frac{1}{2}| < 1$, So it converges.

by D.C.T $\sum_{n=2}^{\infty} \frac{1}{n!}$ converges.

also we can compare with $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$

for $n \geq 2$, $(n-2)! \geq 1$

so $n(n-1)(n-2)! \geq n(n-1)$

$$n! \geq n(n-1)$$

$$\frac{1}{n!} \leq \frac{1}{n(n-1)}$$

$$\sum_{n=2}^{\infty} \frac{1}{n!} \leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$$

Note that $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n}\right]$

which converges by n^{th} partial sum.

$\boxed{52}$ $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$, compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

($\frac{\sqrt[n]{n}}{n^2}$, $\frac{1}{n^2}$ are positive for $n \geq 1$)

$$\lim_{n\to\infty} \frac{\frac{\sqrt[n]{n}}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\cancel{n^2}} \cdot \cancel{n^2}$$

$$= \lim_{n\to\infty} \sqrt[n]{n} = 1$$

- Sec 10.5 : The ratio and root tests. page 585

$\boxed{7}$ Use the ratio test to determine if ~~each~~ the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{n^2 (n+2)!}{n!\, 3^{2n}} \qquad a_n: +ve$$

$$\longrightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2 (n+3)!}{(n+1)!\; 3^{2(n+1)}} \cdot \frac{n!\, 3^{2n}}{n^2 (n+2)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 (n+3)\cancel{(n+2)!}\, \cancel{n!}\, 3^{2n}}{(n+1)\cancel{n!}\, 3^{2n+2} \cdot n^2\, \cancel{(n+2)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 (n+3) \cdot \cancel{3^{2n}}}{(n+1)\, n^2\, \cancel{3^{2n}} \cdot 3^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 (n+3)}{9\,(n+1)\, n^2} = \frac{1}{9} < 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{n^2 (n+2)!}{n!\, 3^{2n}}$$ converges by ratio test.

$\boxed{12}$ Use the root test to determine if the series converges:

$$\sum_{n=1}^{\infty} \left(\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1} \qquad a_n: +ve.$$

$$\longrightarrow \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\ln\left(e^2 + \frac{1}{n}\right)^{n+1}}$$

$$= \lim_{n\to\infty} \left(\ln\left(e^2 + \frac{1}{n}\right)\right)^{\frac{n+1}{n}}$$

$$= \lim_{n\to\infty}\left(\ln\left(e^2+\frac{1}{n}\right)\right)^{1+\frac{1}{n}} = \ln(e^2+0)^{1+0} = \ln e^2 = 2\ln e = \boxed{2}$$

[illegible]

$\because \lim \sqrt[n]{a_n} > 1$ so $\sum_{n=1}^{\infty}\left(\ln\left(e^2+\frac{1}{n}\right)\right)^{n+1}$ diverges

by nth root test.

$\boxed{16}$ $\sum_{n=2}^{\infty}\frac{1}{n^{1+n}}$ $\quad a_n$ +ve

$$\rightarrow \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\left(\frac{1}{n^{1+n}}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{1}{n^{\frac{1}{n}+1}}$$

$$= \lim_{n\to\infty}\frac{1}{n^{\frac{1}{n}}\cdot n} = \lim_{n\to\infty}\frac{1}{n^{\frac{1}{n}}}\cdot\lim_{n\to\infty}\frac{1}{n}$$

$$= 1\,(0) = 0 < 1$$

$\therefore \sum_{n=2}^{\infty}\frac{1}{n^{1+n}}$ converges by nth root test.

$\boxed{20}$ Use any method to determine if the series converges or diverges.

$\sum_{n=1}^{\infty}\frac{n!}{10^n}$ $\quad a_n$ +ve

by ratio test:-

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)!}{10^{n+1}}\cdot\frac{10^n}{n!} = \lim_{n\to\infty}\frac{(n+1)\,\cancel{n!}\,\cancel{10^n}}{\cancel{10^n}\cdot 10\cdot\cancel{n!}} = \lim_{n\to\infty}\frac{n+1}{10} = \infty > 1$$

$\rightarrow \sum_{n=1}^{\infty}\frac{n!}{10^n}$ diverges by ratio test.

$\boxed{28}$ $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$ $\quad a_n$ +ve

by nth root test:-

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{(\ln n)^n}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\ln n}{n} = 0 < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$ converges by nth root test.

$\boxed{38}$ $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\quad a_n$ +ve

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)} \cdot \cancel{n!}\, n^n}{(n+1)^n \cancel{(n+1)} \cdot \cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = 1^{\infty}$$ L-Hopital rule.

نستطيع ايجادها

$\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n = e^{-1}$

من القاعدة

$\lim \left(1 + \frac{x}{n}\right)^n = e^x$

$$\ln \left(\frac{n}{n+1}\right)^n = n \ln\left(\frac{n}{n+1}\right)$$

$$\rightarrow \lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{1/n} = \lim_{n\to\infty} \frac{\ln n - \ln n+1}{1/n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{1}{n+1}}{-1/n^2}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^2+n}}{\frac{-1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{-n^2}{n^2+n} = -1$$

$\therefore \lim \left(\frac{n}{n+1}\right)^n = e^{-1} = \frac{1}{e} < 1$ $\quad \therefore \sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by ratio test

43 $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

$$\rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2}$$

$$= \lim_{n\to\infty} \frac{((n+1)\,n!)^2 \cdot (2n)!}{(2n+2)!\,(n!)^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \cancel{(n!)^2}\, \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\,\cancel{(n!)^2}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges by ratio test.

46 $a_1 = 1$, $a_{n+1} = \dfrac{1+\tan^{-1} n}{n}\, a_n$

$\dfrac{a_{n+1}}{a_n}$ نستطيع ايجادها بقسمة الطرفين على a_n

$$\rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1+\tan^{-1} n}{n} = \frac{1+\frac{\pi}{2}}{\lim_{n\to\infty} n} = 0 < 1$$

$\therefore$ the series converges.

60 $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$ $\qquad a_n$ +ve

$$\rightarrow \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n^n}{(2^n)^2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{(n^n)^{\frac{1}{n}}}{(2^{2n})^{\frac{1}{n}}} = \lim_{n\to\infty} \frac{n}{4} = \infty > 1$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$ diverges by nth root test