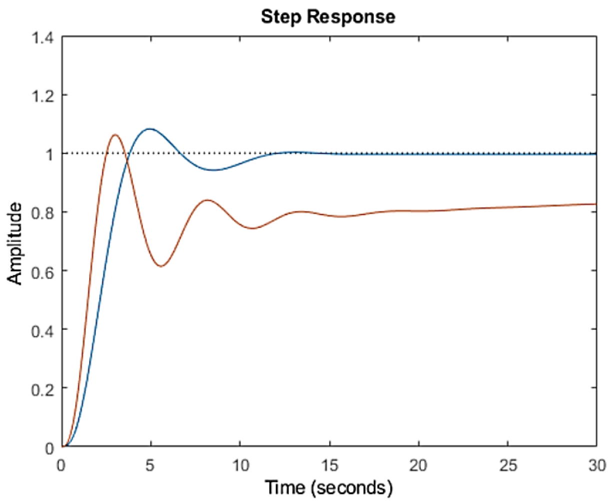
Design of a Servo System (Tracking System):

The tracking system is a control action aims to force the output response y (t) to follow the desired input r(t) with a required performance.



There are two cases for design the tracking system:

- Find the eigenvalues for the open loop system $|sI - A| = 0$
- Check if there is any eigenvalues at the origin or not i.e. find the Type number?
- Remember this:

TABLE 7.2 Relationships between input, system type, static error constants, and steady-state errors

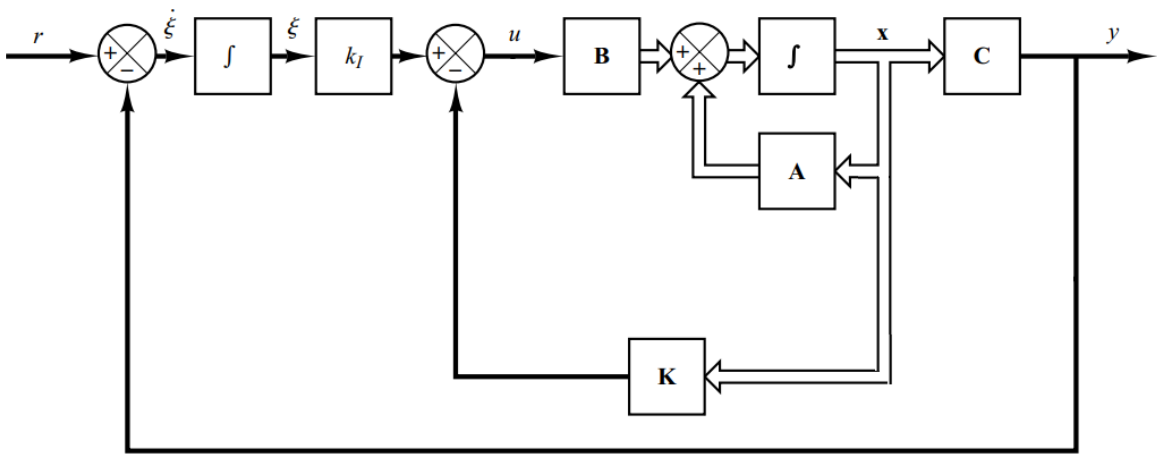
Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

- If the steady state error is equal zero based on the Type number use the first case. Otherwise use the second case.

Case 2 : Design of Type 1 Servo System when the Plant Has No an Integrator:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad 1$$

$$y = \mathbf{C}\mathbf{x} \qquad 2$$



Here we assumed that $y = x_1$

$$u = -\mathbf{K}\mathbf{x} + k_I \xi \qquad 3$$

$$\dot{\xi} = r - y = r - \mathbf{C}\mathbf{x} \qquad 4$$

Assume that the reference input (step function) is applied at t=0. Then, for t >0, the system dynamics can be described by an equation that is a combination of Equations (1) and (4):

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(t) \qquad 5$$

We shall design an asymptotically stable system such that **x**(t), ξ(t), and u(t) approach constant values, respectively. Then, at steady state, $\dot{\xi}$(t)=0, and we get y(t)=r

$$\begin{bmatrix} \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(\infty) \\ \xi(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(\infty) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(\infty) \qquad 6$$

Noting that r(t) is a step input, we have $r(\infty) = r(t) = r$ (constant) for t>0. By subtracting Equation (5) from Equation (6), we obtain:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(t) - \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) - \mathbf{x}(\infty) \\ \xi(t) - \xi(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} [u(t) - u(\infty)] \quad 7$$

Define

$$\mathbf{x}(t) - \mathbf{x}(\infty) = \mathbf{x}_e(t)$$
$$\xi(t) - \xi(\infty) = \xi_e(t)$$
$$u(t) - u(\infty) = u_e(t)$$

Then Equation (7) can be written as

$$\begin{bmatrix} \dot{\mathbf{x}}_e(t) \\ \dot{\xi}_e(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u_e(t) \quad 8$$

where

$$u_e(t) = -\mathbf{K}\mathbf{x}_e(t) + k_I \xi_e(t)$$

Define a new $(n + 1)$th-order error vector $\mathbf{e}(t)$ by

$$\mathbf{e}(t) = \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} = (n + 1)\text{-vector}$$

Then Equation (8) becomes

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}} u_e \quad 9$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix}, \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}$$

and Equation (9) becomes

$$u_e = -\hat{\mathbf{K}}\mathbf{e} \quad 10$$

where

$$\hat{\mathbf{K}} = [\mathbf{K} \;\vdots\; -k_I]$$

The state error equation can be obtained by substituting Equation (10) into Equation (9):

$$\dot{\mathbf{e}} = (\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}})\mathbf{e} \quad 11$$

- The design of the type 1 servo system here is converted to the design of an asymptotically stable regulator system such that $\mathbf{e}(t)$ approaches zero, given any initial condition $\mathbf{e}(0)$.
- If the system defined by Equation (1) is completely state controllable, then, by specifying the desired eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ for the matrix $(\hat{A} - \hat{B}\hat{K})$, matrix $\hat{K}$ can be determined by the pole-placement technique or Linear Quadratic Regulator (LQR).

Example : Consider the system below:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$
$$y = \boldsymbol{C}\boldsymbol{x} + Du$$

Where:

$A = \begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\quad C = [1 \quad 0]$

a) Study the stability for the system.
b) Design a tracking system for the plant by using the pole-placement **(<u>Bass-Gura Approach -</u>second method)**. The desired input for the control scheme is a step input and the controller must achieve the following requirements:
-settling time (T_s) is one second.
-critical damped (ζ).
-the steady state error (ess) must be equal to zero.

Note: in the case you need to approximate the system to second order system use (s_3= -30) if it is necessary.

c) Draw the control scheme for the system.

a) Check the stability

$$|sI - A| = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} s & -1 \\ -1 & s-5 \end{bmatrix} = s^2 - 5\,s - 1 = 0$$

s_1 =-0.19, $s_2 =$ 5.19. The system is unstable, also we must use case 2 for the tracking system because the type number is equal zero.

b) Design the tracking the system for the system.

The dynamics for the tracking system (case 2) is shown below:

$$\dot{e}(t) = \left(\hat{A} - \hat{B}\hat{K}\right)e(t)$$

Where:

$$\hat{A} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ -\boldsymbol{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 5 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \boldsymbol{B} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{Also Define: } \hat{K} = [k_1 \quad k_2 \quad k_3]$$

Note : $\hat{A}, \hat{B}$ are not in the first companion form. So it is case 2 in the Bass-Gura Approach

1. Check the controllability:

$$M = \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 5 & 26 \\ 0 & 0 & -1 \end{bmatrix}$$

$|M| = 1$ Thus, the system is fully state controllable.

2. Find the open loop (real) characteristic equation

$$\left|sI - \hat{A}\right| = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 5 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ -1 & s-5 & 0 \\ 1 & 0 & s \end{bmatrix} = s^3 - 5\,s^2 - \,s = 0$$

By the way the general form is:

$$s^3 + a_1\,s^2 + a_2\,s + a_3$$

$$a_1 = -5 \quad a_2 = -1 \quad a_3 = 0$$

3. Find the similarity matrix T

T=MW where $W = \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -5 & 1 \\ -5 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$T = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 5 & 26 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -5 & 1 \\ -5 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

4. Write the desired characteristic equation:

For $\zeta = 1$ $\quad T_s = \frac{4}{\zeta\omega_n} = 1 \rightarrow \quad \omega_n = 4$ rad/s

Based on the desired eigenvalues are:

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$s_{1,2} = -\omega_n = -4$$

$$(s+4)(s+4)(s+30) = s^3 + 38\,s^2 + 256\,s + 480$$

By the way the general form is:

$$s^3 + \alpha_1\,s^2 + \alpha_2\,s + \alpha_3$$

$$\alpha_1 = 38 \quad \alpha_2 = 256 \quad \alpha_3 = 480$$

5. Find the gain matrix:

$$\hat{K} = [k_1 \quad k_2 \quad k_3] = [(\alpha_3 - a_3) \qquad (\alpha_2 - a_2) \quad (\alpha_1 - a_1)]T^{-1}$$

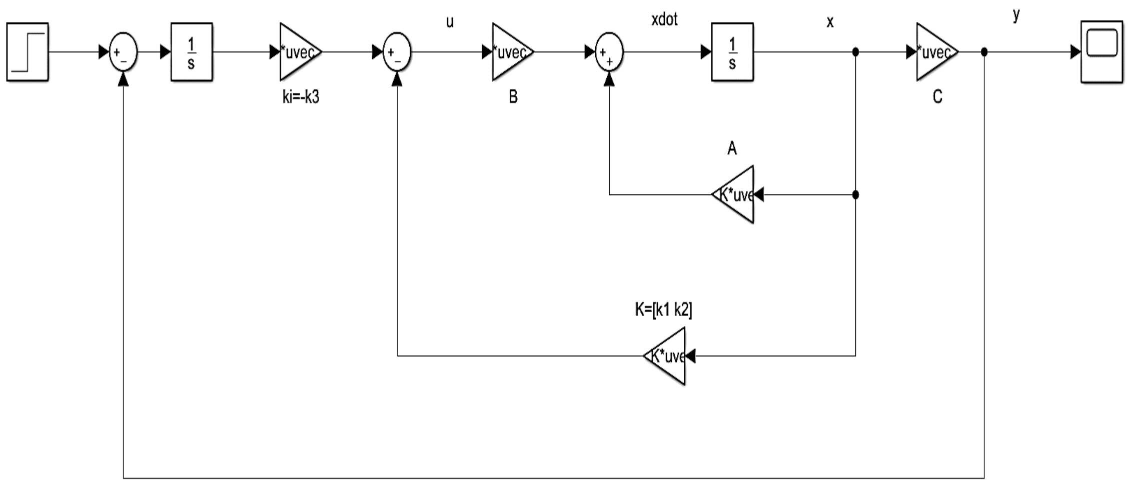
$$\hat{K} = [(480 - 0) \;\; (256 - -1) \qquad (38 - -5)] \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

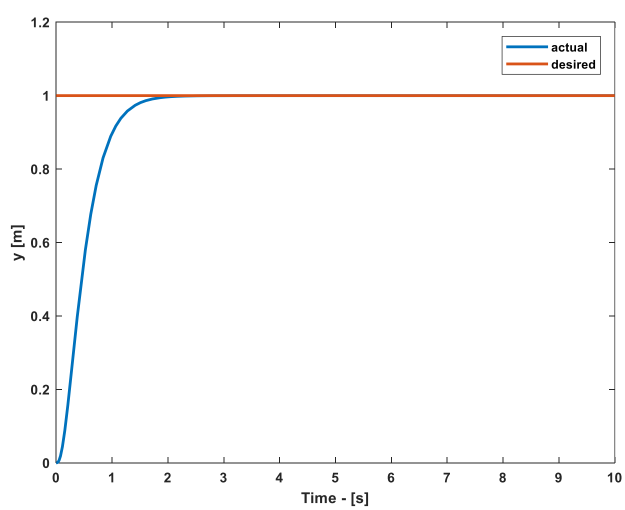
$$K = [\,257.00 \qquad 43.00 \qquad -480.00]$$

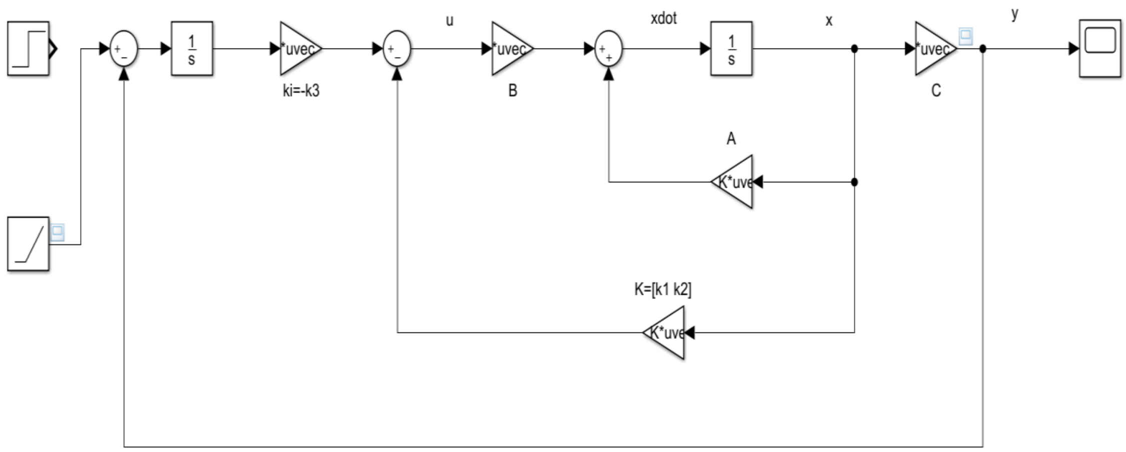
$$k_1 = 257 \quad k_2 = 43 \quad k_3 = -k_i = -480$$

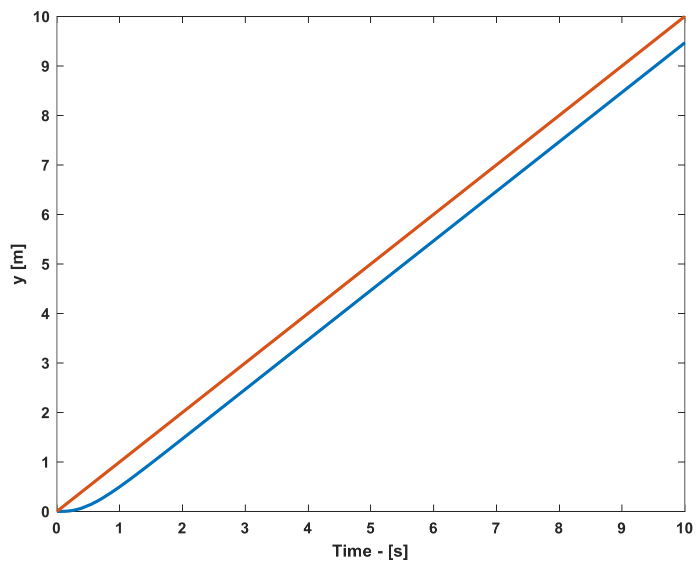
$$k_i = 480$$

c) Draw the control scheme:






u
xdot
x
y
1
s
ki=-k3
B
C
A
K=[k1 k2]
K*uvec


y [m]
Time - [s]
0
1
2
3
4
5
6
7
8
9
10