

Natural logarithms

Exp sketch

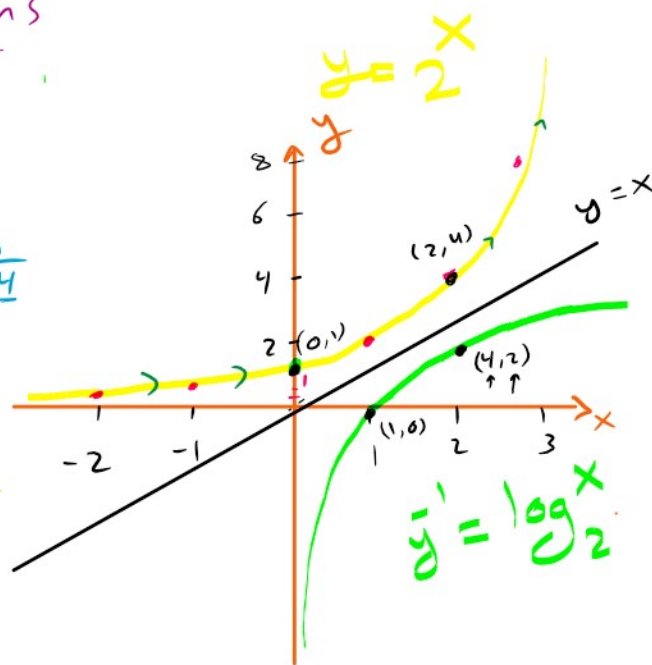
$$y = 2^x$$

$\Rightarrow$

x	2^x
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

increasing on $\mathbb{R}$
concave up on $\mathbb{R}$

Exp sketch the
inverse of 2^x



$$\log_2^4 = 2$$

$$\frac{2}{2} = 4$$

$$y = \log_a^{u(x)} = \frac{\ln u(x)}{\ln a}$$

$$u(x) > 0$$

$$a > 0$$

↑ base $\checkmark$ y_1

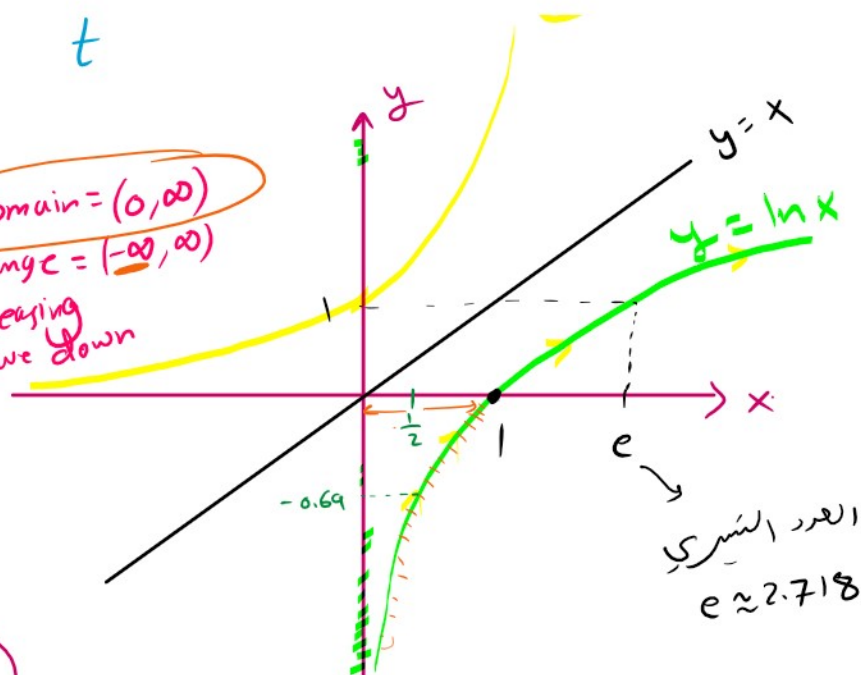
$$y = \log_e^{u(x)} = \frac{\ln u(x)}{\ln e} = \ln u(x) \quad \ln e = 1$$

$$y = \log_e^x = \frac{\ln x}{\ln e} = \frac{\ln x}{1} = \ln x$$

Def $y = \ln x = \int_1^x \frac{dt}{t} \Rightarrow y' = \frac{1}{x}$

x	ln x
0.5	-0.69
1	0
2	0.69
e	1
4	1.39
10	2.3

Domain = $(0, \infty)$
 Range = $(-\infty, \infty)$
 increasing
 concave down



ln x
 $x > 0$
 domain

when $0 < x < 1 \Rightarrow \ln x < 0$

when $1 < x \Rightarrow \ln x > 0$

$x = 1 \Rightarrow \ln x = 0$

If $y = \ln x \Rightarrow \dot{y} = \frac{1}{x}$

$x > 0$

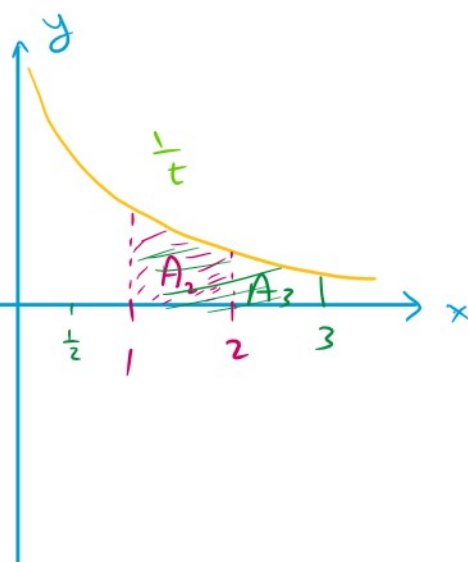
$$\ln x = \int_1^x \frac{1}{t} dt$$

$\ln 2 = \int_1^2 \frac{1}{t} dt \approx 0.69 > 0$

$\ln 3 = \int_1^3 \frac{1}{t} dt \approx$

$\ln \frac{1}{2} = \int_1^{\frac{1}{2}} \frac{1}{t} dt = -$

$\int_1^{\frac{1}{2}} \frac{1}{t} dt$



$$\ln \frac{1}{2} = \ln 1 - \ln 2$$

$$= 0 - \ln 2 = -\ln 2 = -0.69$$

Properties of $\ln x = \log_e x = \frac{\ln x}{\ln e} \rightarrow 1$
 $a > 0, b > 0$

① $\ln ab = \ln a + \ln b$ Product Rule

② $\ln \frac{a}{b} = \ln a - \ln b$ Quotient Rule

③ $\ln \frac{1}{b} = \ln 1 - \ln b$
 $= 0 - \ln b$
 $= -\ln b$ Reciprocal Rule

④ $\ln b^r = r \ln b$ Power Rule

① $\log_a xy = \log_a x + \log_a y$

② $\log_a \frac{x}{y} = \log_a x - \log_a y$

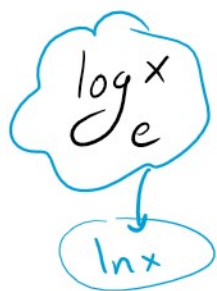
③ $\log_a y^r = r \log_a y$

inverse of 2^x is $\log_2^x$

inverse of $\log_3^x$ is 3^x

$$\ln e = 1$$

inverse of $\log_e^x$ is e^x



inverse of e^x is $\ln x$

$$\log_e^x \checkmark$$

inverse $\sqrt{5}^x$ is $\log_{\sqrt{5}}^x$

Ex write $\ln \sqrt{13.5}$ in terms $\ln 2$ and $\ln 3$

$$\ln \sqrt{13.5} = \ln \sqrt{\frac{27}{2}} = \ln \frac{\sqrt{27}}{\sqrt{2}} = \ln \sqrt{27} - \ln \sqrt{2}$$

$$= \ln 27^{\frac{1}{2}} - \ln 2^{\frac{1}{2}}$$

$$= \frac{1}{2} \ln 27 - \frac{1}{2} \ln 2$$

$$= \frac{1}{2} \ln 3^3 - \frac{1}{2} \ln 2$$

$$= \frac{1}{2} \ln 3^3 - \frac{1}{2} \ln 2$$

$$= \frac{3}{2} \ln 3 - \frac{1}{2} \ln 2$$

Exp Find $\int_1^2 \frac{dx}{x} = \ln x \Big|_1^2$

$$= \ln 2 - \ln 1$$

$$= \ln 2 - 0$$

$$= \ln 2$$

$$y = \ln(u(x)), \quad \begin{matrix} u(x) > 0 \\ u(x) \text{ diff} \end{matrix}$$

$$y' = \frac{u'(x)}{u(x)}$$

$$\Rightarrow \int \frac{u'(x)}{u(x)} dx = \ln|u(x)| + c$$

Exp ① $y = \ln(x) \Rightarrow y' = \frac{1}{x}$

② $y = \ln(\sqrt{x}) \Rightarrow \text{Find } y'(4)$

$$= \frac{1}{2} - \frac{1}{2} \ln x$$

$$y = \ln x^{\frac{1}{2}} = \frac{1}{2} \ln x$$

$$y' = \frac{1}{2} \cdot \frac{1}{x}$$

$$\Rightarrow y'(4) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

or $y' = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}} = \frac{1}{2\sqrt{x}} \cdot \frac{1}{\sqrt{x}} = \frac{1}{2x}$

$$y'(4) = \frac{1}{2(4)} = \frac{1}{8}$$

$$\int \frac{dx}{x} = \ln|x| + C$$

Exp ① $\int_{-3}^{-2} \frac{dx}{x} = \ln|x| \Big|_{-3}^{-2} = \ln|-2| - \ln|-3|$
 $= \ln 2 - \ln 3$
 $= \ln \frac{2}{3}$

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 ② $\int_2^{16} \frac{dx}{2x \sqrt{\ln x}}$

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

$$\int_{\ln 2}^{\ln 16} \frac{du}{2\sqrt{u}}$$

$$du = \frac{1}{x} \cdot x$$

$$x = 2 \Rightarrow u = \ln 2$$

$$x = 16 \Rightarrow u = \ln 16$$

$$\left. \sqrt{u} \right|_{\ln 2}^{\ln 16} = \sqrt{\ln 16} - \sqrt{\ln 2}$$

$$= \sqrt{\ln 2^4} - \sqrt{\ln 2}$$

$$= \sqrt{4 \ln 2} - \sqrt{\ln 2}$$

$$= \sqrt{4} \sqrt{\ln 2} - \sqrt{\ln 2}$$

$$= 2 \sqrt{\ln 2} - \sqrt{\ln 2}$$

$$= \sqrt{\ln 2}$$

25 Exp Find $\dot{y}$ if $y = \ln(\ln x)$

Find $\dot{y}(e)$

$$\dot{y} = \frac{1}{x}$$

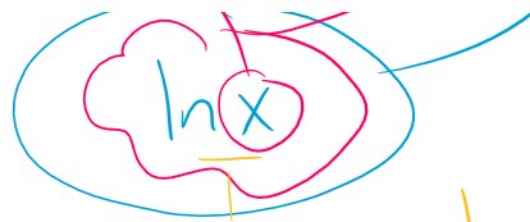
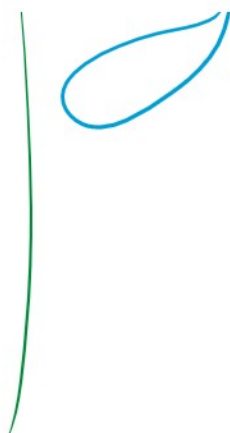
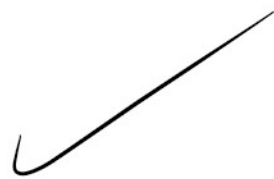
Find $y(e)$

$$= \frac{1}{\ln e}$$

$$= \frac{1}{e \ln e}$$

$$= \frac{1}{e(1)}$$

$$= \frac{1}{e}$$



$$= \frac{1}{x} \cdot \frac{1}{\ln x} = \frac{1}{x \ln x}$$

$$= \frac{1}{\ln^x x}$$

Exp Use logarithmic diff to find $\frac{dy}{dx}$ if

$$(1) y = \sqrt{x(x+1)}$$

$$\frac{\sqrt{x}}{\frac{1}{2\sqrt{x}}} \cdot \frac{\sqrt{x+1}}{\frac{1}{2\sqrt{x+1}}} + \dots$$

$$\ln y = \ln \sqrt{x(x+1)}$$

$$= \ln \sqrt{x} \sqrt{x+1}$$

$$= \ln \sqrt{x} + \ln \sqrt{x+1}$$

$$= \ln x^{\frac{1}{2}} + \ln (x+1)^{\frac{1}{2}}$$

$$= \ln x^2 + \ln(x+1)$$

$$\ln y = \frac{1}{2} \ln x + \frac{1}{2} \ln(x+1)$$

$$\frac{y'}{y} = \frac{1}{2} \frac{1}{x} + \frac{1}{2} \frac{1}{x+1}$$

$$\frac{y'}{y} = \frac{1}{2} \left(\frac{1}{x} + \frac{1}{x+1} \right)$$

$$y' = \frac{y}{2} \left(\frac{1}{x} + \frac{1}{x+1} \right)$$

$$= \frac{\sqrt{x(x+1)}}{2} \left(\frac{1}{x} + \frac{1}{x+1} \right)$$

$$(2) \quad y = t(t+1)(t+2)(t+3)$$

$$\ln y = \ln t(t+1)(t+2)(t+3)$$

$$= \ln t + \ln(t+1) + \ln(t+2) + \ln(t+3)$$

$$\frac{y'}{y} = \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2} + \frac{1}{t+3}$$

$$y' = y \left[\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2} + \frac{1}{t+3} \right]$$

$$= t(t+1)(t+2)(t+3) \left[\frac{1}{t} + \dots + \frac{1}{t+3} \right]$$

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(3)

$$y = \frac{\theta \sin \theta}{\sqrt{\sec \theta}}$$

$$\ln y = \ln \frac{\theta \sin \theta}{\sqrt{\sec \theta}}$$

$$= \ln \theta \sin \theta - \ln \sqrt{\sec \theta}$$

$$= \ln \theta + \ln \sin \theta - \frac{1}{2} \ln \sec \theta$$

$$\frac{y'}{y} = \frac{1}{\theta} + \frac{\cos \theta}{\sin \theta} - \frac{1}{2} \frac{\cancel{\sec \theta} \tan \theta}{\cancel{\sec \theta}}$$

$$= \left[\frac{1}{\theta} + \cot \theta - \frac{1}{2} \tan \theta \right]$$

$$y = y \left[\frac{1}{\theta} + \cot \theta - \frac{1}{2} \tan \theta \right]$$

$$= \frac{\theta \sin \theta}{\sqrt{\sec \theta}} \left[\frac{1}{\theta} + \cot \theta - \frac{1}{2} \tan \theta \right]$$

Recall

$$\int \sec^2 x \, dx = \boxed{\tan x} + C$$

$$\int \csc^2 x \, dx = \boxed{-\cot x} + C$$

$$\int \sec x \tan x \, dx = \boxed{\sec x} + C$$

$$\int \csc x \cot x \, dx = \boxed{-\csc x} + C$$

Q: Show that

$$\textcircled{1} \int \tan x \, dx = -\ln |\cos x| + C \quad \checkmark$$

$$\checkmark \textcircled{2} \int \cot x \, dx = \ln |\sin x| + C \quad \checkmark$$

$$\textcircled{3} \int \sec x \, dx = \ln |\sec x + \tan x| + C \quad \checkmark$$

$$\checkmark \textcircled{4} \int \csc x \, dx = -\ln |\csc x + \cot x| + C \quad \checkmark$$

$$\underline{\text{Ans}} \textcircled{1} \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = - \int \frac{-\sin x}{\cos x} \, dx$$

$$\checkmark = -\ln |\cos x| + C$$

$$= \ln |\cos x|^{-1} + C$$

$$= \ln \frac{1}{|\cos x|} + C$$

$$\checkmark = \ln |\sec x| + C$$

$$\textcircled{3} \int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} \, dx$$

$$= \ln |\tan x + \sec x| + C$$

Exp

$$\int_0^{\frac{\pi}{2}} \tan \frac{x}{2} \, dx$$

$$\int_0^{\frac{\pi}{4}} \tan u \, du$$

$$u = \frac{x}{2}$$

$$du = \frac{1}{2} dx$$

$$2 du = dx$$

$$x=0 \Rightarrow u = \frac{0}{2} = 0$$

$$x = \frac{\pi}{2} \Rightarrow u = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$$

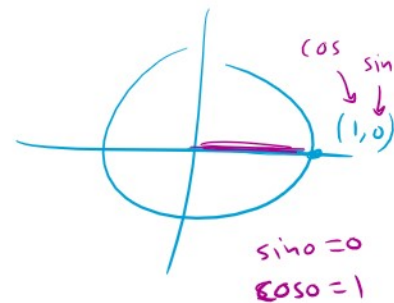
$$2 \int_0^{\frac{\pi}{4}} \tan u \, du$$

$$(2) (-1) \ln |\cos u| \Big|_0^{\frac{\pi}{4}}$$

$$-2 \left[\ln |\cos \frac{\pi}{4}| - \ln |\cos 0| \right]$$

$$-2 \left[\ln \frac{1}{\sqrt{2}} - \ln 1 \right]$$

$$-2 \left[\ln 1 - \ln \sqrt{2} - 0 \right]$$



$$-2 \left[\underset{\downarrow 0}{\ln 1} - \ln \sqrt{2} - 0 \right]$$

$$-2 \left[-\ln \sqrt{2} \right]$$

$$2 \ln \sqrt{2} = \ln (\sqrt{2})^2 = \ln 2$$