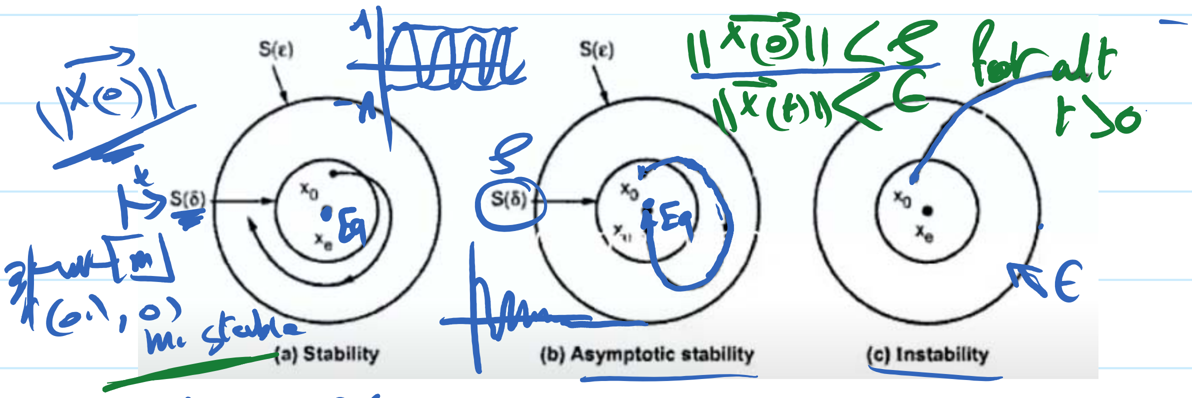
Lyapunov Stability theorem for Linear and non-linears Systems

* Assume $\vec{x}(0)$ represent the vector of IC for given Sys. with a stable equilibrium point if any arbitrary number ϵ that exist some the number δ such that whenever:

* $\|\vec{x}(0)\| < \delta$ then $\|\vec{x}(t)\| < \epsilon$ for all $t > 0$ "stability Condition"

- if $\lim_{t \to \infty} \|\vec{x}(t)\| = 0$, then the Sys. is said Asy. stable.



(a) Stability (b) Asymptotic stability (c) Instability

$\dot{x}(t) = f(x, u, t) \Rightarrow \dot{x}(t) = Ax + \cancel{Bu}$ (internal Stability)

non linear

$u(t) = 0$ $\quad x(0) =$

* According to Lyapunov theory, one can check the stability of the Sys by finding some Scalar fun of the Vector $\vec{x}$ denoted by $V(x)$, which has continuous first partial derivative and satisfy

the following Conditions :-

① $V(\vec{x}) > 0$ for all Values of $\vec{x}(t) \neq \vec{0}$

② $\dot{V}(\vec{x}) = \frac{\partial V}{\partial x}\dot{x} \le 0$ for all Values of $\vec{x}(t) \neq \vec{0}$

$\frac{\partial V}{\partial x}\frac{dx}{dt} \le 0$

③ $\vec{V}(0) = 0$

if you can find such fun satisfied the previous Condition in this Case the Sys. is Asy. stable but if the fun. $V(\vec{x})$ doesn't satisfy the previous Condition in this Case you Can't know if the Sys is Asy. stable / m.stable / unstable.

* Study the Stability of Nonlinear Sys by L.V.

Ex:- Study the stability of the following Sys-

$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 - e^{-t}x_2$

Non linear + t (*) Varient Sys

$\dot{x} = f(x, u, t)$, let $\boxed{V(x) = x_1^2 + x_2^2}$

① $V(\vec{x}) > 0$ for all Value of $\vec{x}(t) \neq \vec{0}$

$\vec{V}(x) > 0$ (ok)

$\frac{\partial V}{\partial x_1} = 2x_1$

$\frac{\partial V}{\partial x_2} = 2x_2$

② $\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} \le 0 =$

$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$ (**) Sub Eq (*) in (**)

$\dot{V}(x) = \cancel{2x_1}x_2 + \cancel{2x_2}(-x_1 - e^{-t}x_2) \le 0$??

$\dot{V}(x) = -2x_2^2 e^{-t} < 0$ satisfied

Tuesday, May 4, 2021 11:29 AM

$x_1 = 0$
$x_2 = 0$

(3) $V(0) = 0 \Rightarrow$ $v(x) = x_1^2 + x_2^2$
$v(0) = (0)^2 + (0)^2$ Satisfied

$v(x)$ is Lyapunov fun, Thus the sys is Asy. stable

Ex: Check the stability $\dot{x}_1 = x_2$
$\dot{x}_2 = -(2+\cos(x_1)) + \sin(x_1)$

let $v(x) = a\, x_1^2 + b\, x_2^2$ where $a, b > 0$

(3) $v(0) = 0$ for $x_1 = 0$, $x_2 = 0$ Satisfied

(1) $v(\vec{x}) > 0$ for all $\vec{x(t)} \neq \vec{0}$
Satisfied

(2) $\frac{\partial v}{\partial x} \dot{x} = 2a x_1 \dot{x}_1 + 2b x_2 \dot{x}_2$ — $\frac{\partial v}{\partial x} \dot{x} \leq 0$ (X)

$= 2a x_1 x_2 - 2b x_2^2 (2 + \cos x_1) + 2b x_2 \sin x_1$

$v(x)$ is not Lyapunov fun. So we can determine any thing

let try $v(x) = a + \cos x_1 + b x_2^2$ for $\begin{cases} a > 1 \\ b > 0 \end{cases}$

Tuesday, May 4, 2021 11:29 AM

* Lyapnouv Stability for LTI

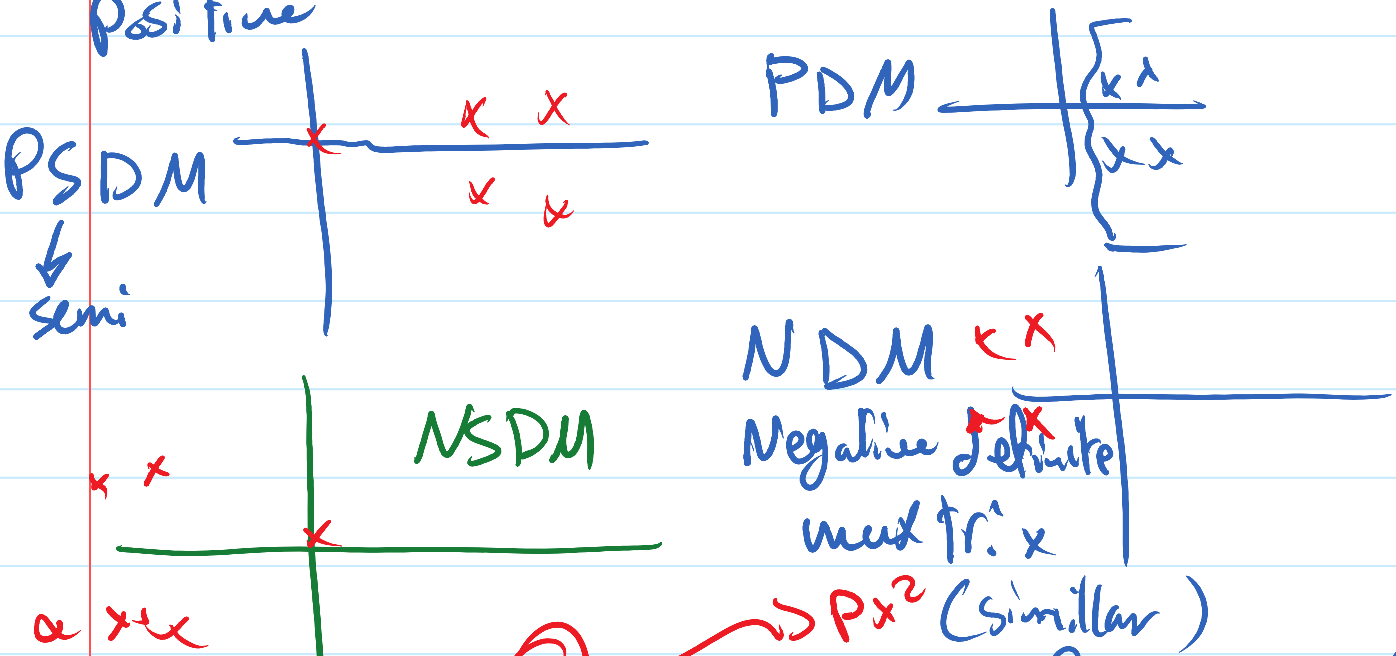
- A Sys is stable in the sense of Lyapnouv if it is able to find lyapunov fun.

* for all LTI Sys $\dot{x}(t) = A x(t)$, $\vec{x}(0) = \vec{x}_0$

the lyapunov fun $\vec{V}(x) = x^T P x > 0$

P: symetric definite matrix $P = P^T > 0$

Positive



$\rightarrow P x^2$ (simillar)

$\vec{V}(x) = x^T P x$, Lyapnouv fun or not??

(1) $V(0) = 0$ satisfied

(2) $V(x) > 0$ for all $\vec{x}(t) \neq \vec{0}$

$\dot{V}(x) = \dot{\vec{x}}^T P \vec{x} + \vec{x}^T P \dot{\vec{x}}$ — (1) $\qquad \dot{\vec{x}} = A\vec{x}$

$\dot{V}(x) = (A\vec{x})^T P \vec{x} + \vec{x}^T P (A\vec{x})$

$= \vec{x}^T A^T P \vec{x} + x^T P A x = \vec{x}^T \underbrace{(A^T P + P A)} x$

Tuesday, May 4, 2021 11:29 AM

$Q \longrightarrow \{x, \; PDM$

$\dot{V}(x) = x^T \left(A^T P + P A \right) x \leq 0 \; ??$

if

$$\boxed{A^T P + P A = -Q}$$

(ii) $Q = Q^T > 0$

is any symatrix

Positive definite matrix

let suppose and Q matrix which is PDM and if you find P matrix is PDM in this case the second condition is satisfied

Ex: $\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} x$ Check the Stability

$|\lambda I - A| = \longleftarrow$ Lyapunov.

$V(x) = x^T P x$ ① $V(0) = 0$ ② $V(x) > 0$

$\dot{V}(x) = x^T \left(A^T P + P A \right) x \leq ?$

$= -Q$

let $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Leftarrow (\lambda I - Q) = \lambda_{1,2} = 1$

PDM

$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$P = \begin{bmatrix} & \end{bmatrix} \longrightarrow eig(P)$

if P PDM then the Asy. Stable

6

Tuesday, May 4, 2021 11:29 AM