Chapter

3

Chapter 3: Fourier Series

Trigonometric Fourier Series | Exponential Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n(\cos n w_0 t) + \sum_{n=1}^{\infty} b_n \sin(n w_0 t)$$

where a_n, a_0 and b_n are trigonometric coefficient fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jnw_0 t}$$

where X_n: complex exponential fourier series.

1 Trigonometric Fourier Series

To evaluate a_0, a_n, b_n

To evaluate "a_0"

$$\int_{T_0} x(t)\,dt = \int_{T_0} a_0\,dt + \sum_{n=1}^{\infty} a_n \underbrace{\cancel{\int_{T_0} \cos(n w_0 t)\,dt}}_{0} + \sum_{n=1}^{\infty} b_n \underbrace{\cancel{\int_{T_0} \sin(n w_0 t)\,dt}}_{0 \text{ periodic}}$$

$$\boxed{a_0 = \frac{1}{T_0}\int x(t)\,dt} \Rightarrow \text{average value}$$

To evaluate a_n:

$*\cos(m w_0 t)$

$$\int_{T_0} x(t)\,dt\,\cos(m w_0 t) \Rightarrow$$

$$= \underbrace{\cancel{\int_{T_0} a_0 \cos(m w_0 t)\,dt}}_{=0,\ \text{periodic}} + \int_{T_0} \sum_{n=1}^{\infty} a_n \cos(n w_0 t)\cos(m w_0 t)\,dt + \underbrace{\cancel{\int_{T_0} \sum_{n=1}^{\infty} \sin(n w_0 t)\cos(m w_0 t)\,dt}}_{=0,\ \text{orthognality}}$$

$$\int_{T_0} x(t)\,dt\,\cos(m w_0 t) = \sum_{n=1}^{\infty} \frac{a_n}{2}\left[\int_{T_0} \cos((n+m) w_0 t)\,dt + \int_{T_0} \cos((n-m) w_0 t)\,dt\right.$$

if $m \neq n \Rightarrow$ a_n undefined

if $m = n \Rightarrow \int_{T_0} x(t)\cos(n\omega_0 t)\,dt = \frac{a_n}{2} T_0$

$$\boxed{a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt}$$

To evaluate b_n :

$* \sin(m\omega_0 t)$

$$\int_{T_0} x(t)\sin(n\omega_0 t)\,dt =$$

$$= \underbrace{\cancel{\int_{T_0} a_n \sin(m\omega_0 t)\,dt}}_{=0,\text{ periodic}} + \underbrace{\cancel{\int_{T_0} \sum a_n \cos(n\omega_0 t)\sin(m\omega_0 t)\,dt}}_{=0,\text{ orthognality}} + \int_{T_0} \sum b_n \sin(n\omega_0 t)\sin(m\omega_0 t)\,dt$$

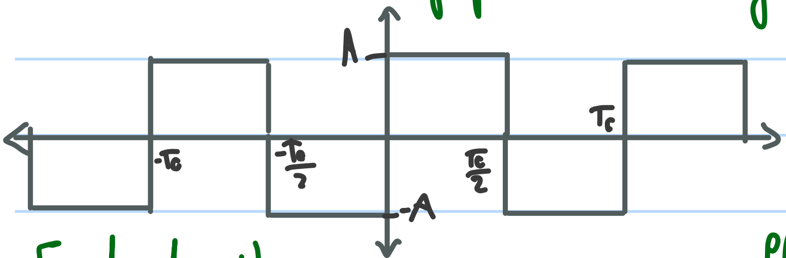
$$\int_{T_0} x(t)\sin(n\omega_0 t)\,dt = \sum_{n=1}^{\infty} \frac{b_n}{2}\left[\int_{T_0} \cos((n+m)\omega_0 t)\,dt - \int_{T_0} \cos((n-m)\omega_0 t)\,dt\right]$$

if $m \neq n \Rightarrow$ b_n undefined

if $m = n \Rightarrow \int_{T_0} x(t)\sin(n\omega_0 t)\,dt = b_n \cdot \frac{T_0}{2}$

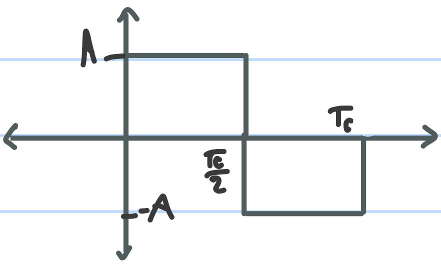
$$\boxed{b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt}$$

Ex. for the following periodic signal shown below



Evaluate the trigonometric coefficient fourier series

Ans.



$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$= \frac{1}{T_0}\left[\int_0^{\frac{T_0}{2}} A\,dt + \int_{\frac{T_0}{2}}^{T} -A\,dt\right] = 0$$

$\boxed{a_0 = 0}$ → اذا كانت الاشارة تتكرر بس قسمين متماثلين الاول موجب والثاني سالب

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt = \frac{2}{T_0}\left[\int_0^{\frac{T_0}{2}} A\cos(n\omega_0 t)\,dt + \int_{\frac{T_0}{2}}^{T_0} -A\cos(n\omega_0 t)\,dt\right]$$

$$= \frac{2}{T_0}\left[\frac{A}{n\omega_0}\sin(n\omega_0 t)\Big|_0^{\frac{T_0}{2}} - \frac{A}{n\omega_0}\sin(n\omega_0 t)\Big|_{\frac{T_0}{2}}^{T_0}\right]$$

$\boxed{a_n = 0}$ → Since x(t) odd function

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt = \frac{2}{T_0}\left[\int_0^{\frac{T_0}{2}} A\sin(n\omega_0 t) - \int A\sin(n\omega_0 t)\,dt\right]$$

$$= \frac{2}{T_0}\left[\frac{-A}{n\omega_0}\cos(n\omega_0 t)\Big|_0^{T_0/2} + \frac{A}{n\omega_0}\cos(n\omega_0 t)\Big|_{T_0/2}^{T_0}\right]$$

$$= \frac{2}{T_0}\cdot\left[\frac{A\,T_0}{n2\pi}\left\{\cos\left(n\cdot\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right) - \cos(0)\right\}\right] + \frac{2}{T_0}\cdot\left[\frac{A\,T_0}{n2\pi}\left[\cos\left(n\frac{2\pi}{T_0}\cdot T_0\right) - \cos\left(n\cdot\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right)\right.\right.$$

$$\Rightarrow b_n = \frac{-A}{n\pi}\left(\cos(n\pi) - 1\right) + \frac{A}{n\pi}\left(\cos(2\pi n) - \cos(n\pi)\right)$$

Note

if n even $\Rightarrow b_n = 0$

if n odd $\Rightarrow$

$$b_n = \frac{4A}{n\pi}$$

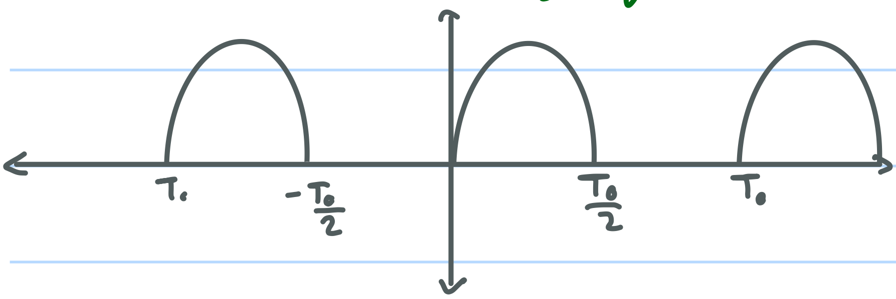
$$x(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4A}{n\pi} \sin(n w_0 t)$$

Note:-

if $x(t)$ even $\Rightarrow$ $a_0 = ??$, $a_n = \checkmark$, $b_n = 0$

if $x(t)$ odd $\Rightarrow$ $a_0 = ??$, $a_n = 0$, $b_n = \checkmark$

Ex. Consider the following signal shown below



Evaluate the coefficients of trigonometric fourier Series

Ans.

$$x(t) = \begin{cases} A\sin(\omega_0 t) & 0 < t < \frac{T_0}{2} \\ 0 & \frac{T_0}{2} < t < T_0 \end{cases}$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{T_0}\int_0^{T_0/2} A\sin(\omega_0 t)\,dt$$

$$= \frac{-A}{T_0}\cdot\frac{1}{\omega_0}\cos(\omega_0 t)\Big|_0^{T_0/2}$$ Since $\omega_0 = \frac{2\pi}{T_0}$

$$= \frac{-A}{2\pi}\left[\cos\Big(\underbrace{\frac{2\pi}{\cancel{T_0}}\cdot\frac{\cancel{T_0}}{\cancel{2}}}_{=-1}\Big) - \underbrace{\cos(0)}_{=1}\right]$$

$$\boxed{a_0 = \frac{A}{2\pi}}$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt = \frac{2}{T_0}\int_0^{T_0/2} A\,\underline{\sin(\omega_0 t)\cos(n\omega_0 t)}\,dt$$

$* \ \frac{1}{2}\sin((1+n)\omega_0 t) + \frac{1}{2}\sin((1-n)\omega_0 t)$

$$= \frac{\cancel{2}A}{T_0}\left[\frac{1}{\cancel{2}}\int_0^{T_0/2}\sin((1+n)\omega_0 t)\,dt + \int_0^{T_0/2}\sin((1-n)\omega_0 t)\,dt\right.$$

$$= \frac{A}{T_0}\left[\frac{-1}{\underbrace{(1+n)\omega_0}_{(n=-1)}}\cos((1+n)\omega_0 t)\Big|_0^{T_0/2} - \frac{1}{\underbrace{(1-n)\omega_0}_{(n=1)}}\cos((1-n)\omega_0 t)\Big|_0^{T_0/2}\right.$$

Since $\omega_0 = \frac{2\pi}{T_0}$

$$= \frac{A}{\cancel{T_0}} \cdot \frac{\cancel{T_0}}{((1+n)2\pi)} \left(\cos\left((1+n) \cdot \frac{\cancel{2}\pi}{\cancel{T_0}} \cdot \frac{\cancel{T_0}}{\cancel{2}} \right) - \cos(0) \right) - \frac{A}{\cancel{T_0}} \cdot \frac{\cancel{T_0}}{((1-n)2\pi)} \left(\cos\left((1-n) \cdot \frac{\cancel{2}\pi}{\cancel{T_0}} \cdot \frac{\cancel{T_0}}{\cancel{2}} \right) - \cos(0) \right)$$

$$= \frac{A}{2\pi} \left[\frac{1}{(1+n)} \left(\underbrace{\cos((1+n)\pi)}_{=-1} - 1 \right) - \frac{1}{(1-n)} \left(\underbrace{\cos((1-n)\pi)}_{=-1} - 1 \right) \right.$$

$$= \frac{A}{2\pi} \left[\frac{2}{(1+n)} + \frac{2}{(1-n)} \right]$$

$$a_n = \frac{A}{\pi} \cdot \frac{1}{(1+n)} + \frac{A}{\pi} \cdot \frac{1}{(1-n)} = \frac{A}{\pi} \left[\frac{1+n-n+1}{1-n^2} \right]$$

$$\boxed{a_n = \frac{A}{\pi} \cdot \frac{2}{1-n^2}}$$ where n even

$\{-1, 1\}$

for $n = 1$

$$a_1 = \frac{2}{T_0} \int_{T_0} x(t) \cos(\omega_0 t)\, dt = \frac{2A}{T_0} \int_0^{T_0/2} \sin(\omega_0 t) \cos(\omega_0 t)\, dt$$

$$= \frac{A}{T_0} \int_0^{T_0/2} \sin(2\omega_0 t)\, dt$$ Since: $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= \frac{A}{T_0} \cdot \frac{-1}{2\omega_0} \left[\cos(\omega_0 t) \Big|_0^{T_0/2} \right]$$

$$= \frac{A}{\cancel{T_0}} \cdot \frac{-\cancel{T_0}}{4\pi} \left[\underbrace{\cos\left(\frac{2\pi}{\cancel{T_0}} \cdot \frac{\cancel{T_0}}{\cancel{2}} \right)}_{=-1} - \underbrace{\cos(0)}_{=1} \right]$$

$$\boxed{a_1 = \frac{A}{\pi}} = \boxed{a_{-1} = \frac{A}{\pi}}$$

Note

$a_n = a_{-n}$

$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n w_0 t)\,dt \quad , \quad a_{-n} = \frac{2}{T_0}\int_{T_0} x(t)\cos(-n w_0 t)\,dt$

$\therefore$ Since "cos" even function $\Rightarrow \cos(n w_0 t) = \cos(-n w_0 t)$

$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n w_0 t)\,dt = \frac{2}{T_0}\int_0^{\frac{T_0}{2}} A\sin(w_0 t)\sin(n w_0 t)\,dt$

$= \frac{\cancel{2}}{T_0}\cdot\frac{A}{\cancel{2}}\left[\int_0^{\frac{T_0}{2}}\cos((1+n)w_0 t)\,dt - \int_0^{\frac{T_0}{2}}\cos((1-n)w_0 t)\,dt\right]$

$= \frac{A}{T_0}\left[\frac{1}{(1+n)w_0}\cdot\sin((1+n)w_0 t)\Big|_0^{\frac{T_0}{2}} - \frac{1}{(1-n)w_0}\sin((1-n)w_0 t)\Big|_0^{\frac{T_0}{2}}\right]$

Since $w_0 = \frac{2\pi}{T_0}$

$\frac{A}{\cancel{T_0}}\left[\frac{\cancel{T_0}}{\underset{n=-1}{(1+n)2\pi}}\left\{\sin\left((1+n)\cdot\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right) - \underbrace{\sin(0)}_{=0}\right\} - \left(\frac{T_0}{\underset{n=1}{(1+n)2\pi}}\cdot\left\{\sin\left((1+n)\cdot\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right) - \underbrace{\sin(0)}_{=0}\right\}\right)\right]$

$b_n = 0$ for all values of n (even, odd) $- \{-1, 1\}$

$b_1 = \frac{2}{T_0}\int_0^{\frac{T_0}{2}} A\sin(w_0 t)\sin(w_0 t)\,dt = \frac{2}{T_0}\int_0^{\frac{T_0}{2}}\left(\frac{1}{2} - \frac{1}{2}\cos(2w_0 t)\right)dt$

$= \frac{A}{T_0}\left[\int_0^{\frac{T_0}{2}} 1\,dt - \int_0^{\frac{T_0}{2}}\cos(2w_0 t)\,dt\right]$

$= \frac{A}{T_0}\left[\frac{T_0}{2} - \frac{T_0}{4\pi}\left(\underbrace{\cancel{\sin\left(2\cdot\frac{2\pi}{T_0}\cdot\frac{T_0}{2}\right)}}_{0} - \underbrace{\cancel{\sin(0)}}_{0}\right)\right]$

$b_1 = \frac{A}{2}$, $b_{-1} = -b_1 \Rightarrow b_{-1} = \frac{-A}{2}$

② Complex Exponential Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)\,dt$$

Since: $\cos(n\omega_0 t) = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$, $\sin(n\omega_0 t) = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{j2}$

Remember:-

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$e^{-j\theta} = \cos(\theta) - j\sin(\theta) \quad +$$

$$* \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$e^{-j\theta} = \cos(\theta) - j\sin(\theta) \quad -$$

$$* \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{j2}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) + \sum_{n=1}^{\infty} \frac{b_n}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2j}\right)e^{jn\omega_0 t} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} - \frac{b_n}{j2}\right)e^{-jn\omega_0 t}$$

$\Rightarrow$ let "$n = -n$"

$$= a_0 + \sum_{n=1}^{\infty}\underbrace{\left(\frac{a_n - jb_n}{2}\right)}_{X_n} e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1}\underbrace{\left(\frac{a_n + jb_{-n}}{2}\right)}_{X_{-n}} e^{jn\omega_0 t}$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$$; X_n: Complex Coefficient Fourier Series

$$X_n = \begin{cases} \frac{1}{2}(a_n - jb_n) & n>0 \\ \frac{1}{2}(a_n + jb_{-n}) & n<0 \\ a_0 & n=0 \end{cases}$$

⇒ if Trigon. coefficient are known

$a_n = 2Re\{X_n\}$

$b_n = -2\,Im\{X_n\}$

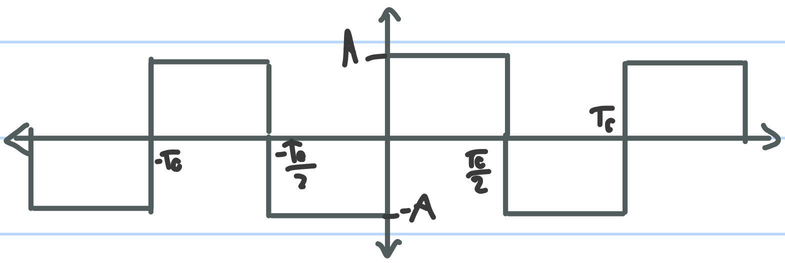
$a_0 = X_0$

Note

if $x(n)$ even ⇒ a_n ✓, $b_n = 0$ X_n ⇒ real part

if $x(n)$ odd → $a_n = 0$, b_n ✓ X_n ⇒ Im part

Ex. Consider the following signal



a) Evaluate the trigon coefficient Fourier Series

b) Evaluate the complex Exponential Fourier Series

a) $a_0 = 0$, $a_n = 0$, & $b_n = \begin{cases} \frac{4A}{n\pi} & , n \text{ odd} \\ 0 & , \text{o.w} \end{cases}$

b) $X_n \begin{cases} -j\frac{4A}{n\pi}\cdot\frac{1}{2} & n \text{ odd } \& \ n>0 \\ +j\frac{4A}{n\pi}\cdot\frac{1}{2} & n \text{ odd } \& \ n<0 \end{cases}$

* بعرف القيمة في الحالتين
على انها موجبة

$b_{-n} = -b_n$

Ex. $x(t) = 3 + (2)\cos(20\pi t) - j2\sin(20\pi t)$

a) Evaluate the trigonometric coefficient Fourier Series

b) Sketch the line Spectra amplitude and phase

$\omega_0 = 2\pi f_0 = 20\pi \Rightarrow f_0 = 10\text{ Hz}$

a)

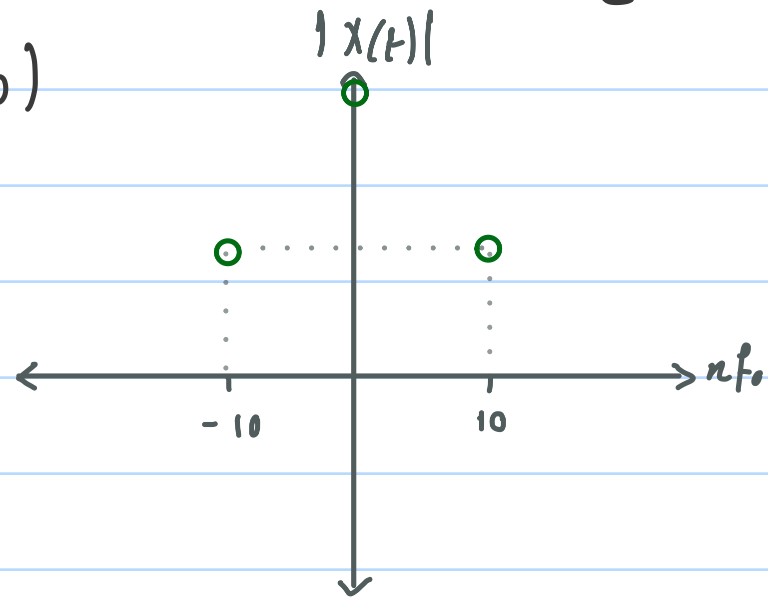
$a_0 = 3$ $\qquad a_1 = 2$ $\qquad b_1 = -j2$

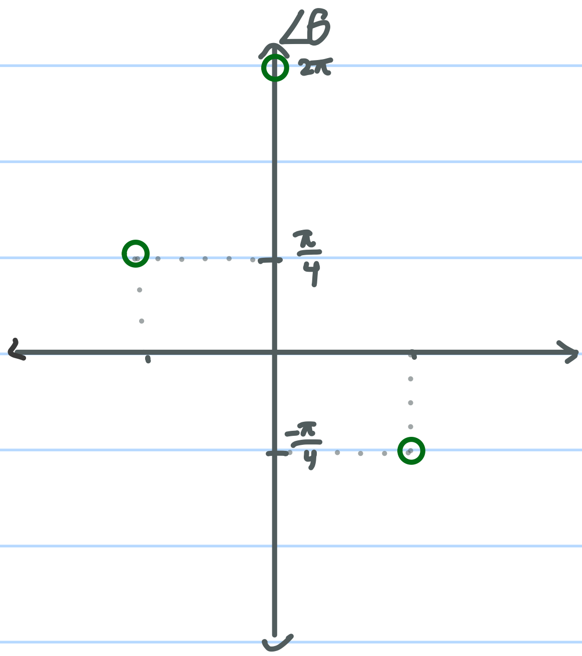
$a_{-n} = a_n \Rightarrow$ even function $\qquad b_{-n} = -b_n \Rightarrow$ odd function

$a_{-n} = 2$ $\qquad b_n = 2j$

$$X_n \begin{cases} \frac{2-j2}{2} & n=1 \\ \frac{2+j2}{2} & n=-1 \\ 3 & n=0 \end{cases} \Rightarrow \begin{cases} \sqrt{1^2+(-1)^2} & n=1 \\ \sqrt{1^2+1^2} & n=-1 \\ 3 & n=0 \end{cases} = \begin{cases} \sqrt{2} & n=1 \\ \sqrt{2} & n=-1 \\ 3 & n=0 \end{cases}$$

b)





Ex. Consider the following signal $x(t) = -2 + \sum_{\substack{n=1 \\ n,\text{odd}}}^{\infty} \frac{4A}{n\pi} \sin(40n\pi t)$

① Evaluate the trigonometric coefficient FS

② Evaluate the complex exponential coefficient FS

③ Sketch the line spectra amplitude and phase

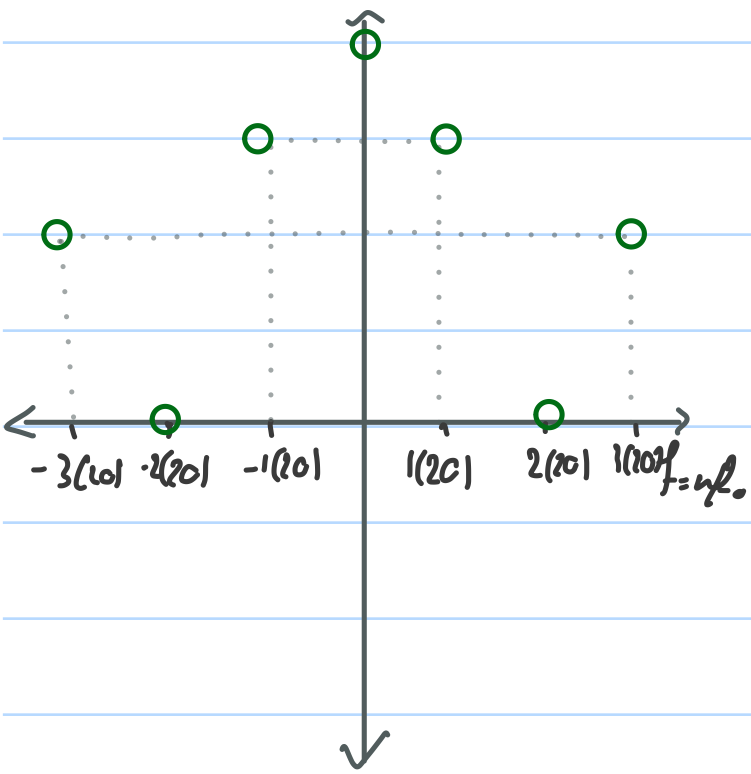
① $a_0 = -2$, $a_n = 0$, $b_n = \begin{cases} \frac{4A}{n\pi} & n>0, n\text{ odd} \\ 0 & \text{o.w} \end{cases}$

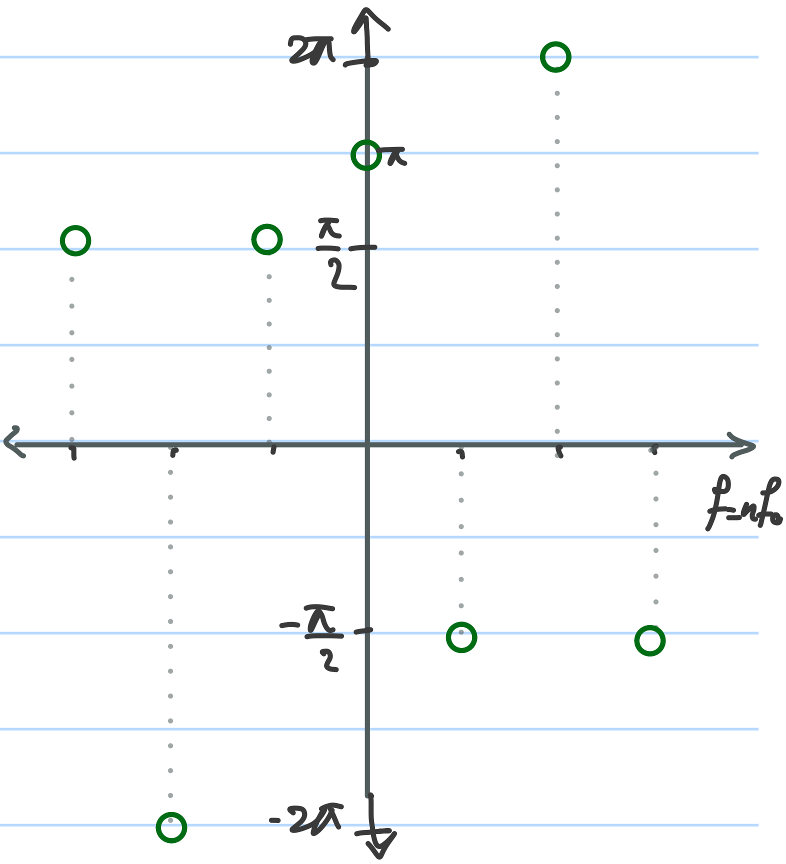
②

$X_n = \begin{cases} -j\frac{4A}{n\pi} \cdot \frac{1}{2} & n>0, n\text{ odd} \\ j\frac{4A}{n\pi} \cdot \frac{1}{2} & n<0\ n\text{ odd} \\ -2 & n=0 \\ 0 & \text{o.w} \end{cases}$

$X_n = \begin{cases} \frac{2A}{n\pi} \angle -\frac{\pi}{2} & n>0, n\text{ odd} \\ \frac{2A}{n\pi} \angle \frac{\pi}{2} & n<0\ n\text{ odd} \\ 2 \angle \pm\pi \text{ or} & n=0 \\ 0 & \text{o.w} \end{cases}$

③





To evaluate X_n

$\cdot\, e^{-jw_0t}$

$$\int_{T_0} x(t) e^{-jmw_0t} = \int_{T_0} \sum_{n=-\infty}^{\infty} X_n e^{jnw_0t} \cdot e^{-jmw_0t}$$

$$\int_{T_0} x(t) e^{-jmw_0t} dt = \sum_{n=-\infty}^{\infty} X_n \int_{T_0} \underline{e^{j(n-m)w_0t}} dt$$

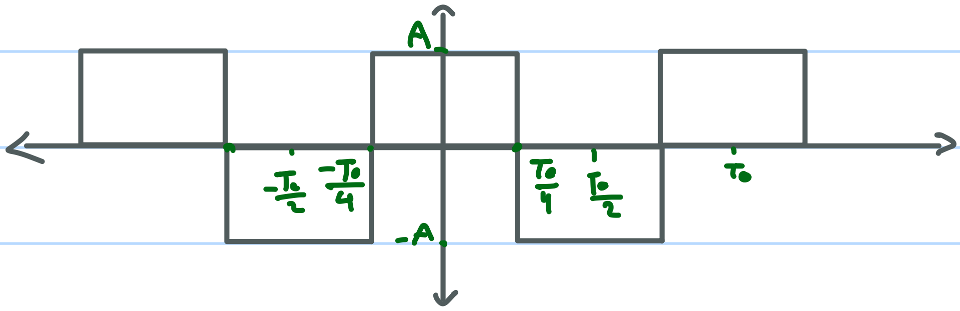
$\cos((n-m)w_0t) + j\sin((n-m)w_0t)$

if $m \neq n \Rightarrow X_n$ undefined

if $m = n$

$$\int_{T_0} x(t) e^{-jnw_0t} dt = X_n \int_{T_0} 1\, dt \Rightarrow \boxed{X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jnw_0t} dt}$$

Ex. For the following signal shown below



ⓐ Evaluate the complex exponential FS

$$= \frac{1}{T_0}\left[-\int_{-\frac{T_0}{2}}^{\frac{T_0}{4}} A e^{-jn\omega_0 t}\, dt + \int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} A e^{-jn\omega_0 t}\, dt + -\int_{\frac{T_0}{4}}^{\frac{T_0}{2}} A e^{-jn\omega_0 t}\, dt \right]$$

$$= \frac{1}{T_0}\left[\frac{\cancel{-}A}{\cancel{-}jn\omega_0} e^{-jn\omega_0 t}\Big|_{-\frac{T_0}{2}}^{-\frac{T_0}{4}} + \frac{A}{-jn\omega_0} e^{-jn\omega_0 t}\Big|_{-\frac{T_0}{4}}^{\frac{T_0}{4}} \cancel{-}\frac{A}{\cancel{-}jn\omega_0} e^{-jn\omega_0 t}\Big|_{\frac{T_0}{4}}^{\frac{T_0}{2}} \right]$$

= Since $\omega_0 = \frac{2\pi}{T_0}$

$$X_n = \frac{1}{\cancel{T_0}}\left[\frac{A \cancel{T_0}}{j2\pi n}\left(e^{-jn\frac{2\pi}{\cancel{T_0}}\cdot\frac{-\cancel{T_0}}{4}} - e^{-jn\frac{2\pi}{\cancel{T_0}}\cdot\frac{-\cancel{T_0}}{2}}\right)\right.$$

$$- \frac{A \cancel{T_0}}{j2\pi n}\left(e^{-jn\frac{2\pi}{\cancel{T_0}}\cdot\frac{\cancel{T_0}}{4}} - e^{-jn\frac{2\pi}{\cancel{T_0}}\cdot\frac{-\cancel{T_0}}{4}}\right)$$

$$\left. + \frac{A \cancel{T_0}}{j2\pi n}\left(e^{-jn\frac{2\pi}{\cancel{T_0}}\cdot\frac{\cancel{T_0}}{2}} - e^{-jn\frac{2\pi}{\cancel{T_0}}\cdot\frac{\cancel{T_0}}{4}}\right)\right]$$

$$= \frac{A}{j2\pi n}\left(e^{jn\frac{\pi}{2}} - e^{jn\pi}\right) - \frac{A}{j2\pi n}\left(e^{-jn\frac{\pi}{2}} - e^{jn\frac{\pi}{2}}\right) + \frac{A}{j2\pi n}\left(e^{-jn\pi} - e^{-jn\frac{\pi}{2}}\right)$$

$$= \frac{2A}{j2\pi n}\left(e^{jn\frac{\pi}{2}}\right) - \frac{A}{j2\pi n}\left(e^{jn\pi}\right) + \frac{A}{j2\pi n}\left(e^{-jn\pi}\right) - \frac{2A}{j2\pi n}\left(e^{-jn\frac{\pi}{2}}\right.$$

$$= \frac{2A}{n\pi}\left(\frac{e^{j\frac{n\pi}{2}} - e^{-j\frac{n\pi}{2}}}{j2}\right) - \frac{A}{n\pi}\left(\frac{e^{jn\pi} - e^{-jn\pi}}{2j}\right)$$

$$\boxed{X_n = \frac{2A}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{A}{n\pi}\sin(n\pi)}$$

at $n=0$

when n even $-\{0\}$ $\Rightarrow$ $X_n = 0$

when n odd $\Rightarrow$ $X_n = \begin{cases} -\frac{2A}{n\pi} & , n = 3, 7, 11, \dots \\ \frac{2A}{n\pi} & , n = 1, 5, 9, \dots \\ -\frac{2A}{n\pi} & , n = -1, -5, -9, \dots \\ \frac{2A}{n\pi} & , n = -3, -7, -11, \dots \end{cases}$

$$X_0 = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\cancel{n}\omega_0 t} = \frac{1}{T_0}\int x(t)\,dt \Rightarrow a_0$$

$$= \frac{1}{T_0}\left[\int_{-\frac{T_0}{2}}^{-\frac{T_0}{4}} -A\,dt + \int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} A\,dt + \int_{\frac{T_0}{4}}^{\frac{T_0}{2}} A\,dt\right] = \frac{1}{\cancel{T_0}}\left[\frac{-A\cancel{T_0}}{4} + \frac{A\cancel{T_0}}{2} - \frac{A T_0}{4}\right]$$

$X_0 = 0$

(b) Evaluate the trigonometric coefficient FS

$a_0 = X_0 = 0$

$a_n = 2\,\mathrm{Re}\{X_n\} = \begin{cases} \frac{4A}{n\pi} & , n = 1, 5, 9, \dots \\ -\frac{4A}{n\pi} & , n = 3, 7, 11, \dots \\ 0 & , \text{o.w} \end{cases}$

y
single-sided

$b_n = 2\,\mathrm{Im}\{X_n\} = 0$

Parseval's theorem

For Periodic Signal

$$P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt$$

Remember

$z = x + jy \quad ; \quad |z| = \sqrt{x^2+y^2} \Rightarrow |z|^2 = x^2 + y^2$

$z^* = x - jy$

$z \cdot z^* = (x+jy)(x-jy) = x^2 - \cancel{jxy} + \cancel{jxy} + y^2 = x^2 + y^2$

$z \cdot z^* = |z|^2$

$$P = \frac{1}{T_0} \int_{T_0} x(t)\, x^*(t) \, dt \quad ; \quad x(t) = \sum_{n=-\infty}^{\infty} X_n e^{-jn\omega_0 t}$$

$$= \frac{1}{T_0} \int_{T_0} \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \, x^*(t) \, dt$$

$$= \sum_{n=-\infty}^{\infty} X_n \cdot \underbrace{\frac{1}{T_0} \int_{T_0} x^*(t) e^{-jn\omega_0 t} \, dt}_{X_n^*}$$

Remember

$$\left(X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} \, dt \right)^* = X_n^* = \frac{1}{T_0} \int_{T_0} x^*(t) e^{jn\omega_0 t} \, dt$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} X_n \cdot \frac{1}{T_0} \int_{T_0} x^*(t) e^{-jn\omega_0 t} \, dt = \sum_{n=-\infty}^{\infty} |X_n|^2$$

$$P = \sum_{n=-\infty}^{\infty} |X_n|^2 \Rightarrow |X_n| = |X_{-n}| \Rightarrow \text{even}$$

$$= X_0^2 + 2\sum_{n=1}^{\infty} |X_n|^2$$

In the example:-

© Evaluate the average power, using parseval's theorem $-4 < n < 4$

$$x_n \begin{cases} -\frac{2A}{n\pi} & , n = 3, 7, 11, \dots \\ \frac{2A}{n\pi} & , n = 1, 5, 9, \dots \\ -\frac{2A}{n\pi} & , n = -1, -5, -9, \dots \\ \frac{2A}{n\pi} & , n = -3, -7, -11, \dots \\ 0 & , o.w \end{cases}$$

$\left.\begin{matrix} n = 3, 7, 11, \dots \\ n = 1, 5, 9, \dots \end{matrix}\right\} \rightarrow$ positive

$\left.\begin{matrix} n = -1, -5, -9, \dots \\ n = -3, -7, -11, \dots \end{matrix}\right\} \rightarrow$ negative

$$P_{avg} = x_0^2 + 2|X_1|^2 + 2|X_2|^2 + 2|X_3|^2 + 2|X_4|^2$$

$$= 0 + 2\left(\frac{2A}{\pi}\right)^2 + 2(0)^2 + 2\left(\frac{2A}{3\pi}\right)^2 + 2(0)^2$$

$$= \frac{8A^2}{\pi^2} + \frac{8A^2}{9\pi^2}$$

Ex. Consider the following signal $x(t)$ is defined as

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{1+j\pi n} e^{\frac{j3\pi n t}{2}} = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}$$

① Determin the fundamental period of the Signal $x(t)$

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+j\pi n} e^{\boxed{\frac{j3\pi n t}{2}}} = \sum_{n=-\infty}^{\infty} x_n e^{\boxed{jn\omega_0 t}}$$

$$\frac{\cancel{j}\,3\pi \cancel{n}\cancel{t}}{2} = \cancel{j}\cancel{n}\omega_0 \cancel{t}$$

$$\omega_0 = \frac{3\pi}{2} \Rightarrow \frac{2\pi}{T_0} = \frac{3\pi}{2} \Rightarrow T_0 = \frac{4}{3}\text{ sec}$$

② Determin the fundamental freq. of the Signal $x(t)$

$$f = \frac{1}{T} \Rightarrow f = \frac{3}{4}\text{ Hz}$$

③ what is the average value of $x(t)$

Since $x_n = \frac{1}{1+j\pi n}$

$\Rightarrow$ average value $= x_0 = \frac{1}{1+j\pi(0)} = 1$

④ Determine the amplitude and phase of third-harmonic component

x_3 or x_{-3}

Since $x_n = \frac{1}{1+j\pi n} \Rightarrow \frac{1\angle 0}{\sqrt{1+(\pi n)^2}\ \angle \tan^{-1}(\frac{\pi n}{1})} = \frac{1e^{j0}}{\sqrt{(1)^2+(\pi n)^2}\,e^{j\tan^{-1}(\pi n)}}$

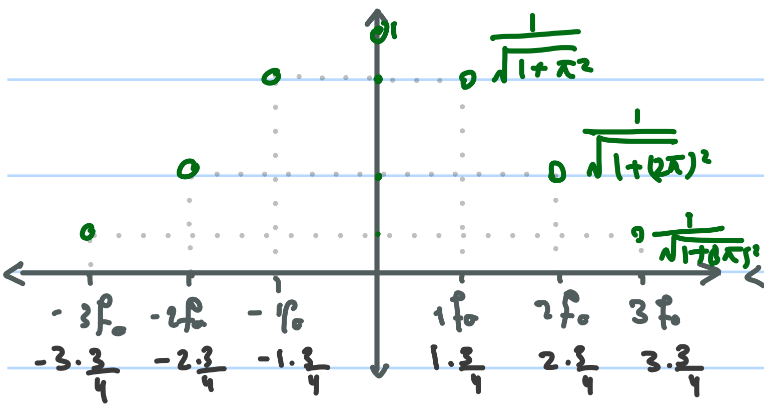
$$|x_n| = \frac{1}{\sqrt{(1)^2+(n\pi)^2}} \quad \text{and} \quad \angle\theta_{x_n} = -\tan^{-1}(\pi n)$$

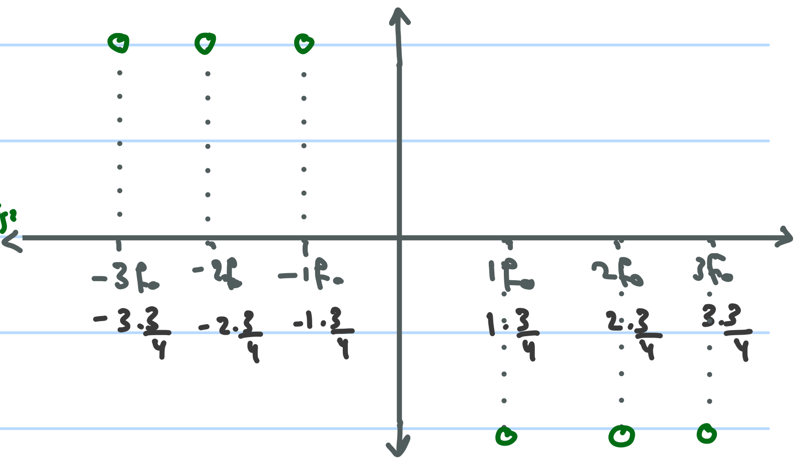
$|X_3| = |X_{-3}| = \frac{1}{\sqrt{1+(3\pi)^2}}$

$\angle\theta_{x_3} = -\tan^{-1}(3\pi)$ and $\angle\theta_{x_{-3}} = -\tan^{-1}(-3\pi)$ → odd ⇒ $-\tan(3\pi)$

$\angle\theta_{x_3} = \tan^{-1}(3\pi)$, $\angle\theta_{x_{-3}} = -\angle\theta_{x_3} = \tan^{-1}(3\pi)$

⑤ plot the line spectra amplitude and phase of signal $x(t)$ in $-3<n<3$





Since $|X_n| = \frac{1}{\sqrt{1^2+(\pi n)^2}}$

$|X_1| = |X_{-1}| = \frac{1}{\sqrt{1+\pi^2}}$

Since $\angle\theta_{x_n} = \tan^{-1}(n\pi)$ $\quad * f = nf_o = n\cdot\frac{3}{4}$

$|X_2| = |X_{-2}| = \frac{1}{\sqrt{1+4\pi^2}}$ $\quad \angle\theta_{x_1} = -\tan^{-1}(\pi) = -\angle\theta_{x_{-1}} = \tan^{-1}(\pi)$

$|X_3| = |X_{-3}| = \frac{1}{\sqrt{1+9\pi^2}}$ $\quad \angle\theta_{x_2} = -\tan^{-1}(2\pi) = -\angle\theta_{x_{-2}} = \tan^{-1}(2\pi)$

$|X_0| = 1$ $\quad \angle\theta_{x_3} = -\tan^{-1}(3\pi) = -\angle\theta_{x_{-3}} = \tan^{-1}(3\pi)$

$\angle\theta_{x_0} = 0, 2\pi, -2\pi$

⑥ Evaluate the average Power using parsevals theorem in range $-3<n<3$

$$P_{avg} = X_0^2 + 2|X_1|^2 + 2|X_2|^2 + 2|X_3|^2$$

$$= (1)^2 + 2\cdot\frac{1}{1+\pi^2} + 2\cdot\frac{1}{1+4\pi^2} + 2\cdot\frac{1}{1+9\pi^2}$$

$$= 1 + \frac{2}{1+\pi^2} + \frac{2}{1+4\pi^2} + \frac{2}{1+9\pi^2}$$

⑥ Evaluate the average Power using parsevals theorem in range $-1<n<3$

$$P_{avg} = X_0^2 + 2|X_1|^2 + |X_2|^2 + |X_3|^2$$

$$= (1)^2 + 2\cdot\frac{1}{1+\pi^2} + \frac{1}{1+4\pi^2} + \frac{1}{1+9\pi^2}$$

$$= 1 + \frac{2}{1+\pi^2} + \frac{1}{1+4\pi^2} + \frac{1}{1+9\pi^2}$$

⑦ Write the expresion in terms of Trigonometric FS

$$X_n = \frac{1}{1+j\pi n}\cdot\frac{1-j\pi n}{1-j\pi n} = \frac{1}{1^2+(n\pi)^2} - \frac{j\pi n}{1+(n\pi)^2}$$

Remember

$$X_n = \begin{cases} \frac{1}{2}(a_n - jb_n) & n>0 \\ \frac{1}{2}(a_n + jb_n) & n<0 \\ a_0 & n=0 \end{cases}$$

$$a_n = 2\,\mathrm{Re}\{X_n\} = 2 \cdot \frac{1}{1+(n\pi)^2}$$

$$b_n = -2\,\mathrm{Im}\{X_n\} = 2 \cdot \frac{\pi n}{1+(n\pi)^2}$$

$$a_0 = X_0 = 1$$