Electromagnetic Theory I

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Chapter 3: Potentials

- Laplace Equation
- The Method of Images
- Separation of Variables
- Multipole Expansion

3.3 The Separation of Variables

We would like to solve Laplace equation directly using solutions that are products of functions each of which depends on one of the coordinates

Cartesian Coordinates: $V(x,y,z) = X(x)Y(y)Z(z)$

Cylindrical Coordinates: $V(s,\phi,z) = S(s)\Phi(\phi)Z(z)$

Spherical Coordinates: $V(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$

This approach is not always possible as it requires appropriate symmetry, boundary conditions and charge source distributions

3.3.2 Spherical Coordinates

Laplace equation is given by

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

The solution is assumed to be of the form

$$V(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

Boundary Conditions are usually given at:

- Sphere surface
- Origin
- Infinity

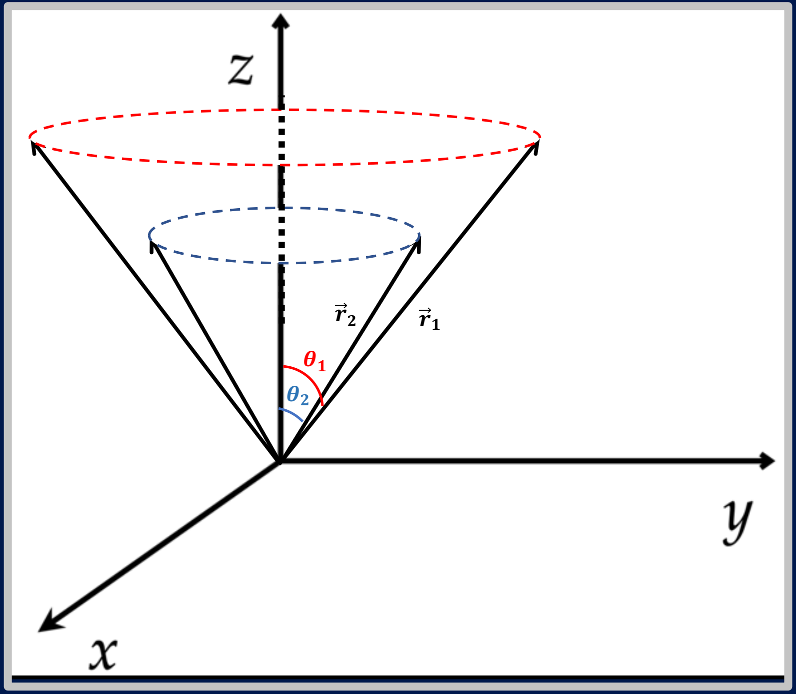
3.3.2 Spherical Coordinates

Azimuthal symmetry, we assume that the potential is independent of ϕ

$$V(r,\theta,\phi) = V(r,\theta) = R(r)\Theta(\theta)$$

The Laplacian is then of the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) = 0$$



3.3.2 Spherical Coordinates

Hence

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\Theta+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)R=0$$

Multiply by $\frac{r^2}{R\Theta}$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0$$

The first term depends on r only, while the second term depends on θ only, therefore both terms must be constant

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=c_1,\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=-c_1$$

3.3.2 Spherical Coordinates

Now the first equation, for an integer $l > 0$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = c_1 = l(l+1)$$

has two solutions

$$R(r) = A_l r^l + \frac{B_l}{r^{l+1}}$$

While the second equation

$$\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = -l(l+1)$$

Using ($x = cos\,\theta$) can be rewritten as

3.3.2 Spherical Coordinates

$$(1-x^2)\frac{\partial^2\Theta}{\partial x^2}-2x\frac{\partial\Theta}{\partial x}+l(l+1)\Theta=0$$

which is called Legendre differential equation, or

$$\frac{\partial}{\partial x}\left((1-x^2)\Theta\right)+l(l+1)\Theta=0$$

Which have two solutions that can be obtained using Power series

Legendre functions of first kind (polynomials when l is integer) $\Theta_1(\theta)=P_l(\cos\theta)$

Legendre function of second kind $\Theta_2(\theta)$ diverge $\cos(\theta)=\pm 1$

e.g. for l=0, $\Theta_2(\theta)=\ln\left(\tan\frac{\theta}{2}\right)$

3.3.2 Spherical Coordinates

Generating function

$$g(t,x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Rodrigo's Formula

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

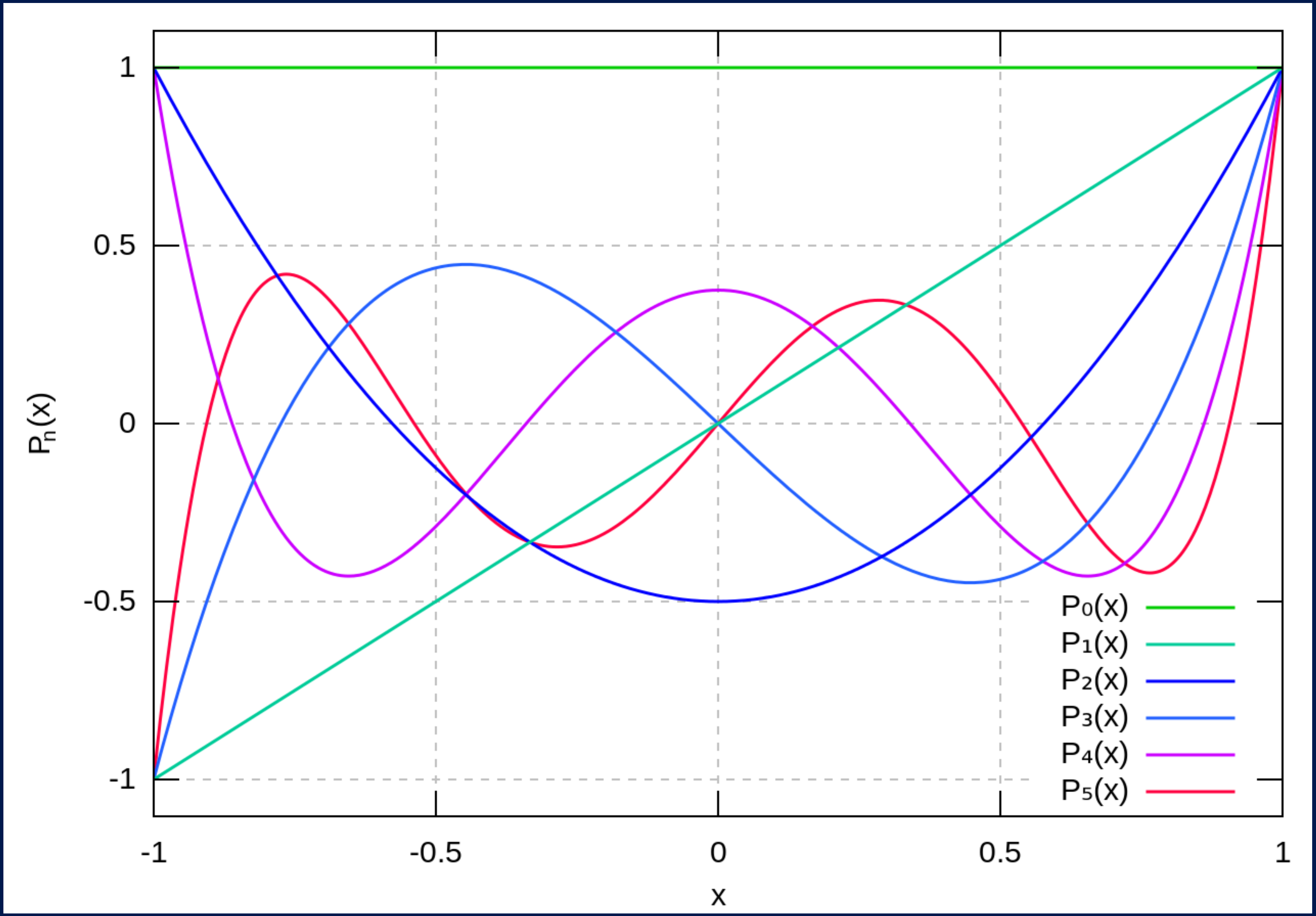
$$P_0(x) = 1 \qquad P_1(x), = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1) \qquad P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_l(-x) = (-1)^l P_l(x) \qquad P_{2l}(\pm 1) = 1, P_{2l+1}(\pm 1) = \pm 1$$

$$P_{2l+1}(0) = 0 \qquad P_{2l}(0) = \frac{(-1)^l}{2^{2l}}\frac{(2l)!}{l!^2}$$

3.3.2 Spherical Coordinates



3.3.2 Spherical Coordinates

Recurrence Relations

$$(l+1)P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x)$$

$$(x^2-1)\frac{d}{dx}P_l(x) = lxP_l(x) - P_{l-1}(x)$$

$$(x^2-1)P_l(x) = \frac{d}{dx}\left((P_{l+1}(x) - P_{l-1}(x)\right)$$

3.3.2 Spherical Coordinates

Orthonormality

$$\int_{-1}^{1} P_l(x)P_m(x)dx = \int_{0}^{\pi} P_l(\cos\theta)P_m(\cos\theta)\sin\theta\, d\theta = \frac{2}{2l+1}\delta_{lm}$$

Completeness on interval $[-1,1]$

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

3.3.2 Spherical Coordinates

The general solution is a superposition of all solutions

$$V(\mathrm{r},\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

3.3.2 Spherical Coordinates

Example: Find the potential inside and outside a hollow sphere of radius R that has a potential $V_0(\theta)$ is specified on its surface.

$$V(\mathrm{r},\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

Boundary conditions

$$V(r,\theta) \to 0 \;\; as\, r \to \infty$$

$$V(r,\theta)\; is\; finite\;\; as\, r \to 0$$

$$V(R,\theta) = V_0(\theta)$$

$$V(r,\theta)\; is\; continuous\; at\, r = R$$

3.3.2 Spherical Coordinates

Inside, for $r < R$: $V(r,\theta) \rightarrow finite \;\; as\, r \rightarrow 0$

$$V_{inside}(\mathrm{r},\theta) = \sum_{l=0}^{\infty} A_l r^l \, P_l(\cos\theta)$$

outside, for $r > R$: $V(r,\theta) \rightarrow 0 \;\; as\, r \rightarrow \infty$

$$V_{outside}(\mathrm{r},\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

3.3.2 Spherical Coordinates

$V(r,\theta)$ *is continuous at* $r = R$

$$V_{inside}(\mathrm{R},\theta) = V_{outside}(\mathrm{R},\theta)$$

$$\sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = \sum_{l=0}^{\infty} A_l R^l \, P_l(\cos\theta)$$

$$\frac{B_l}{R^{l+1}} = A_l R^l \qquad\qquad B_l = A_l R^{2l+1}$$

3.3.2 Spherical Coordinates

$$V_0(\theta) = V_{inside}(\mathrm{R}, \theta) = \sum_{l=0}^{\infty} A_l R^l\, P_l(\cos\theta)$$

$$\int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta d\theta$$

$$= \int_0^{\pi} \left(\sum_{m=0}^{\infty} A_m R^m\, P_m(\cos\theta) \right) P_l(\cos\theta) \sin\theta d\theta$$

3.3.2 Spherical Coordinates

$$\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta d\theta$$

$$= \sum_{m=0}^{\infty} A_m R^m \int_0^\pi P_m(\cos\theta)P_l(\cos\theta)\sin\theta d\theta$$

$$= \sum_{m=0}^{\infty} A_m R^m \frac{2}{2l+1}\delta_{lm} = \frac{2}{2l+1}A_l R^l$$

$$A_l = \frac{2l+1}{2R^l}\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta d\theta$$

$$B_l = A_l R^{2l+1} = \frac{2l+1}{2}R^{l+1}\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta d\theta$$

3.3.2 Spherical Coordinates

Example: Find the potential inside and outside a hollow sphere of radius R that has a potential $V_0(\theta) = \cos^3\theta$ is specified on its surface.

$$\mathrm{V}_0 = \cos^3\theta = aP_1(\cos\theta) + bP_3(\cos\theta)$$

$$\cos^3\theta = a\,(\cos\theta) + \frac{b}{2}\,(5\cos^3\theta - 3\cos\theta)$$

$$\cos^3\theta = (a - \frac{3b}{2})\cos\theta + \frac{5b}{2}\cos^3\theta$$

$$b = \frac{2}{5} \qquad\qquad a = \frac{3}{5}$$

$$\mathrm{V}_0 = \cos^3\theta = \frac{3}{5}P_1(\cos\theta) + \frac{2}{5}P_3(\cos\theta)$$

3.3.2 Spherical Coordinates

$$A_1 = \frac{2(1)+1}{2R^1}\int_0^\pi V_0(\theta)P_1(\cos\theta)\sin\theta d\theta$$

$$= \frac{3}{2R}\int_0^\pi \left(\frac{3}{5}P_1(\cos\theta)+\frac{2}{5}P_3(\cos\theta)\right)P_1(\cos\theta)\sin\theta d\theta = \frac{3}{2R}\frac{3}{5}\frac{2}{3} = \frac{3}{5R}$$

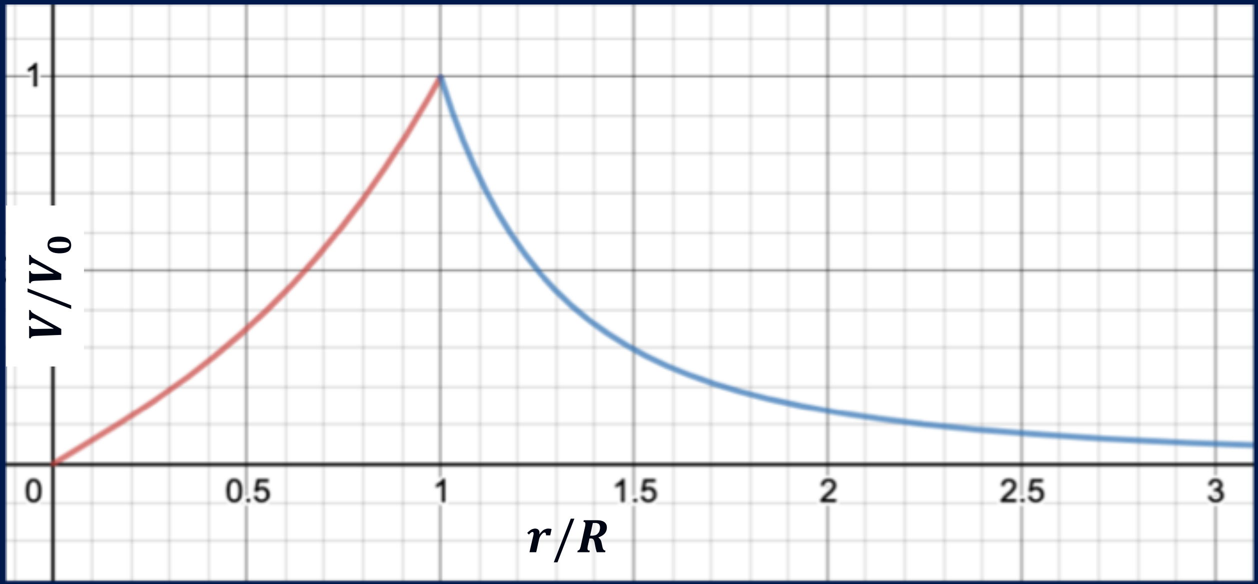
$$B_1 = A_1R^3 = \frac{3}{5}R^2$$

$$A_3 = \frac{2(3)+1}{2R^3}\int_0^\pi V_0(\theta)P_3(\cos\theta)\sin\theta d\theta$$

$$= \frac{7}{2R^3}\int_0^\pi \left(\frac{3}{5}P_1(\cos\theta)+\frac{2}{5}P_3(\cos\theta)\right)P_3(\cos\theta)\sin\theta d\theta = \frac{7}{2R^3}\frac{2}{5}\frac{2}{7} = \frac{2}{5R^3}$$

$$B_3 = A_3R^7 = \frac{2}{5}R^4$$

3.3.2 Spherical Coordinates



$$V(\mathrm{r},\theta) = \begin{cases} \dfrac{3r}{5R}P_1(\cos\theta) + \dfrac{2r^3}{5R^3}P_3(\cos\theta) & r \le R \\ \dfrac{3R^2}{5r^2}P_1(\cos\theta) + \dfrac{2R^4}{5r^4}P_3(\cos\theta) & r \ge R \end{cases}$$

3.3.2 Spherical Coordinates

Example: Find the potential inside and outside a hollow sphere of radius R that has a surface charge density $\sigma_0(\theta)$ is specified on its surface.

$$V(\mathrm{r},\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Boundary conditions

$$V(r,\theta)\to 0 \;\; as\, r\to\infty$$

$$V(r,\theta)\; is\; finite \;\; as\, r\to 0$$

$$\sigma_0=-\epsilon_0\left(\left.\frac{\partial V_{out}}{\partial r}\right|_{r=R}-\left.\frac{\partial V_{in}}{\partial r}\right|_{r=R}\right)$$

$$V(r,\theta)\; is\; continuous\; at\; r=R$$

3.3.2 Spherical Coordinates

Inside, for $r < R$: $V(r,\theta) \rightarrow finite \;\; as\, r \rightarrow 0$

$$V_{inside}(\mathrm{r},\theta) = \sum_{l=0}^{\infty} A_l r^l \, P_l(\cos\theta)$$

outside, for $r > R$: $V(r,\theta) \rightarrow 0 \;\; as\, r \rightarrow \infty$

$$V_{outside}(\mathrm{r},\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

3.3.2 Spherical Coordinates

$V(r, \theta)$ *is continuous at* $r = R$

$$V_{inside}(\mathrm{R}, \theta) = V_{outside}(\mathrm{R}, \theta)$$

$$\sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = \sum_{l=0}^{\infty} A_l R^l \, P_l(\cos\theta)$$

$$\frac{B_l}{R^{l+1}} = A_l R^l \qquad\qquad B_l = A_l R^{2l+1}$$

3.3.2 Spherical Coordinates

$$\sigma_0 = -\epsilon_0 \left(\frac{\partial}{\partial r} \sum_{l=0}^{\infty} \frac{A_l R^{2l+1}}{r^{l+1}} P_l(\cos\theta) \bigg|_{r=R} - \frac{\partial}{\partial r} \sum_{l=0}^{\infty} A_l r^l \, P_l(\cos\theta) \bigg|_{r=R} \right)$$

$$\sigma_0 = -\epsilon_0 \left(\sum_{l=0}^{\infty} -(l+1) \frac{A_l R^{2l+1}}{r^{l+2}} P_l(\cos\theta) \bigg|_{r=R} - \sum_{l=0}^{\infty} l A_l r^{l-1} \, P_l(\cos\theta) \bigg|_{r=R} \right)$$

$$\sigma_0 = -\epsilon_0 \left(\sum_{l=0}^{\infty} -(l+1) A_l R^{l-1} \, P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} \, P_l(\cos\theta) \right)$$

$$\sigma_0 = \epsilon_0 \left(\sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} \, P_l(\cos\theta) \right)$$

3.3.2 Spherical Coordinates

$$\int_0^\pi \sigma_0(\theta) P_l(\cos\theta)\sin\theta d\theta$$

$$= \epsilon_0 \sum_{m=0}^{\infty} (2m+1) A_m R^{m-1} \int_0^\pi P_m(\cos\theta) P_l(\cos\theta)\sin\theta d\theta$$

$$= \epsilon_0 \sum_{m=0}^{\infty} (2m+1) A_m R^{m-1} \frac{2}{2l+1} \delta_{lm} = 2\epsilon_0 A_l R^{l-1}$$

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}} \int_0^\pi \sigma_0(\theta) P_l(\cos\theta)\sin\theta d\theta$$

$$B_l = A_l R^{2l+1} = \frac{1}{2\epsilon_0} R^{l+2} \int_0^\pi \sigma_0(\theta) P_l(\cos\theta)\sin\theta d\theta$$

3.3.2 Spherical Coordinates

Example: Find the potential inside and outside a hollow sphere of radius R that has a surface charge density $\sigma_0(\theta) = k\cos\theta$ is specified on its surface.

$$V(\mathrm{r},\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Boundary conditions

$$V(r,\theta) \to 0 \;\; as\, r \to \infty$$

$$V(r,\theta)\; is\; finite \;\; as\, r \to 0$$

$$\sigma_0 = -\epsilon_0\left(\left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} - \left.\frac{\partial V_{in}}{\partial r}\right|_{r=R}\right) = k\cos\theta = kP_1(\cos\theta)$$

$$V(r,\theta)\; is\; continuous\; at\; r = R$$

3.3.2 Spherical Coordinates

$$A_1 = \frac{1}{2\epsilon_0 R^{1-1}} \int_0^{\pi} \sigma_0(\theta) P_1(\cos\theta) \sin\theta d\theta$$

$$= \frac{k}{2\epsilon_0} \int_0^{\pi} P_1(\cos\theta) P_1(\cos\theta) \sin\theta d\theta = \frac{k}{2\epsilon_0} \frac{2}{3} = \frac{k}{3\epsilon_0}$$

$$B_1 = A_1 R^3 = \frac{kR^3}{3\epsilon_0}$$

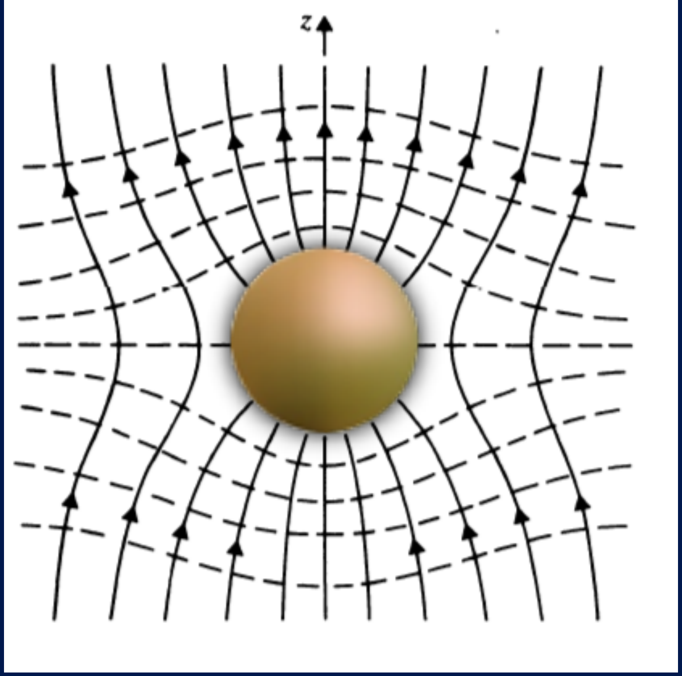
3.3.2 Spherical Coordinates

$$V(\mathrm{r},\theta) = \begin{cases} \dfrac{kr}{\epsilon_0}\cos\theta & r \le R \\ \dfrac{kR^3}{3\epsilon_0 r^2}\cos\theta & r \ge R \end{cases}$$

3.3.2 Spherical Coordinates

Example: An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\vec{E} = E_0\hat{z}$. The induced charge distorts the field in the neighborhood of the sphere. Find the potential outside the sphere.

$$V(\mathrm{r},\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$



Boundary conditions

$$V(r,\theta) \to -E_0 z = -E_0 r\cos\theta \quad as\; r \to \infty$$

The sphere is an equipotential and hence can be set to zero

$$V(R,\theta) = 0$$

3.3.2 Spherical Coordinates

$$V(R,\theta) = 0 = \sum_{l=0}^{\infty} \left(A_l R^l + \frac{B_l}{R^{l+1}} \right) P_l(\cos\theta)$$

⬇

$$\left(A_l R^l + \frac{B_l}{R^{l+1}} \right) = 0$$

⬇

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta)$$

3.3.2 Spherical Coordinates

$$V(r,\theta) \to -E_0 z = -E_0 r\cos\theta \quad as\, r \to \infty$$

⬇

$$A_0 = A_2 = A_3 = \cdots = 0 \qquad A_1 = -E_0$$

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

3.3.2 Spherical Coordinates

Induced Surface Charge Density

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} = \epsilon_0 E_0 \left(1 + 2\frac{R^3}{r^3}\right)\cos\theta\bigg|_{r=R}$$

$$= 3\epsilon_0 E_0 \cos\theta$$