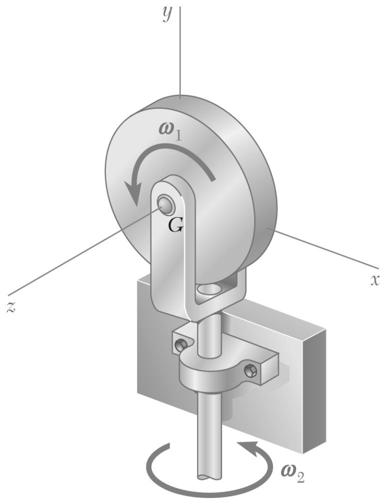
# CHAPTER 18




## PROBLEM 18.1

A thin, homogeneous disk of mass $m$ and radius $r$ spins at the constant rate $\omega_1$ about an axle held by a fork-ended vertical rod, which rotates at the constant rate $\omega_2$. Determine the angular momentum $\mathbf{H}_G$ of the disk about its mass center $G$.

## SOLUTION

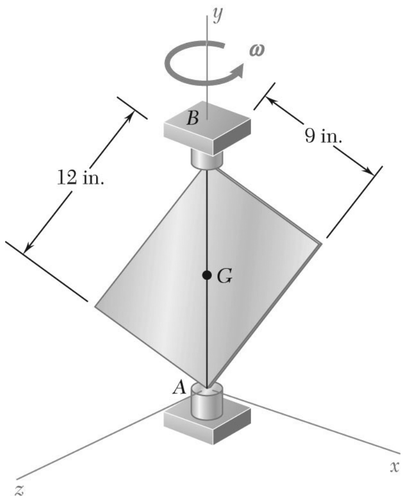
Angular velocity: $\boldsymbol{\omega} = \omega_2 \mathbf{j} + \omega_1 \mathbf{k}$

Moments of inertia: $\bar{I}_x = \frac{1}{4}mr^2, \quad \bar{I}_y = \frac{1}{4}mr^2, \quad \bar{I}_z = \frac{1}{2}mr^2$

Products of inertia: by symmetry, $\bar{I}_{xy} = \bar{I}_{yz} = \bar{I}_{zx} = 0$

Angular momentum: $\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k}$

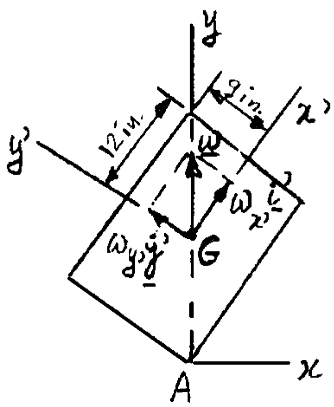
$$\mathbf{H}_G = 0 + \frac{1}{4}mr^2\omega_2 \mathbf{j} + \frac{1}{2}mr^2\omega_1 \mathbf{k} \qquad \mathbf{H}_G = \frac{1}{4}mr^2\omega_2 \mathbf{j} + \frac{1}{2}mr^2\omega_1 \mathbf{k} \blacktriangleleft$$




## PROBLEM 18.2

A thin rectangular plate of weight 15 lb rotates about its vertical diagonal $AB$ with an angular velocity $\omega$. Knowing that the $z$ axis is perpendicular to the plate and that $\omega$ is constant and equal to 5 rad/s, determine the angular momentum of the plate about its mass center $G$.

## SOLUTION



$$h = \sqrt{(9\text{ in.})^2 + (12\text{ in.})^2} = 15\text{ in.}$$

Resolving ω along the principal axes $x'$, $y'$, $z$:

$$\omega_{x'} = \frac{12}{15}\omega = 0.8(5\text{ rad/s}) = 4\text{ rad/s}$$

$$\omega_{y'} = \frac{9}{15}\omega = 0.6(5\text{ rad/s}) = 3\text{ rad/s}$$

$$\omega_z = 0$$

Moments of inertia:

$$I_{x'} = \frac{1}{12}\left(\frac{15\text{ lb}}{32.2}\right)\left(\frac{9}{12}\text{ ft}\right)^2 = 0.021836\text{ slug}\cdot\text{ft}^2$$

$$I_{y'} = \frac{1}{12}\left(\frac{15\text{ lb}}{32.2}\right)\left(\frac{12}{12}\text{ ft}\right)^2 = 0.038820\text{ slug}\cdot\text{ft}^2$$

From Eqs. (18.10):

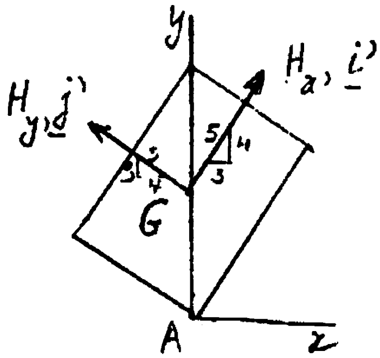
$$H_{x'} = I_{x'}\omega_{x'} = (0.021836)(4) = 0.087345\text{ slug}\cdot\text{ft}^2/\text{s}$$

$$H_{y'} = I_{y'}\omega_{y'} = (0.038820)(3) = 0.11646\text{ slug}\cdot\text{ft}^2/\text{s}$$

$$H_{z'} = I_z\omega_z = 0$$

$$\mathbf{H}_G = (0.087345\text{ slug}\cdot\text{ft}^2/\text{s})\mathbf{i}' + (0.11646\text{ slug}\cdot\text{ft}^2/\text{s})\mathbf{j}'$$

## PROBLEM 18.2 (Continued)




Components along $x$ and $y$ axes:

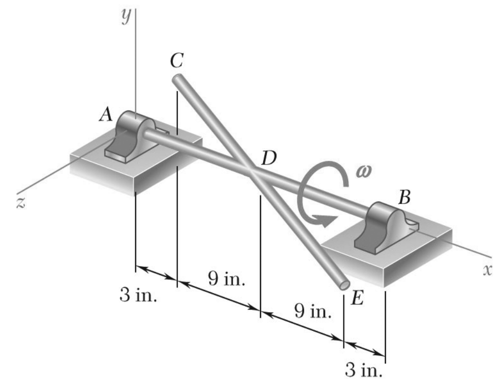
$$H_x = \frac{3}{5}H_{x'} - \frac{4}{5}H_{y'} = \left(\frac{3}{5}\right)(0.087345) - \left(\frac{4}{5}\right)(0.11646)$$

$$= -0.040761$$

$$H_y = \frac{4}{5}H_{x'} + \frac{3}{5}H_{y'}$$

$$= \left(\frac{4}{5}\right)(0.087345) + \left(\frac{3}{5}\right)(0.11646) = 0.13975$$

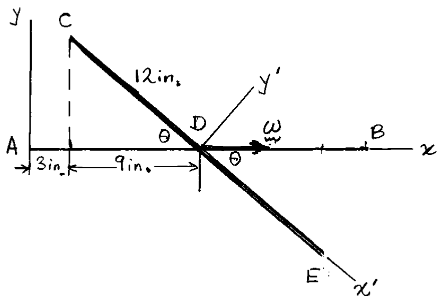
$$\mathbf{H}_G = -(0.0408 \text{ slug}\cdot\text{ft}^2/\text{s})\mathbf{i} + (0.1398 \text{ slug}\cdot\text{ft}^2/\text{s})\mathbf{j} \blacktriangleleft$$




## PROBLEM 18.3

Two uniform rods *AB* and *CE*, each of weight 3 lb and length 2 ft, are welded to each other at their midpoints. Knowing that this assembly has an angular velocity of constant magnitude $\omega = 12$ rad/s, determine the magnitude and direction of the angular momentum $\mathbf{H}_D$ of the assembly about *D*.

## SOLUTION




$$m = \frac{W}{g} = \frac{3}{32.2} = 0.093168 \text{ lb}\cdot\text{s}^2/\text{ft}, \qquad l = 2 \text{ ft},$$

$$\boldsymbol{\omega} = (12 \text{ rad/s})\mathbf{i}$$

For rod *ADB*, $\quad \mathbf{H}_D = \bar{I}_x\omega\mathbf{i} \approx 0, \quad$ since $\bar{I}_x \approx 0.$

For rod *CDE*, use principal axes $x'$, $y'$ as shown.

$$\cos\theta = \frac{9}{12}, \qquad \theta = 41.410°$$

$$\omega_{x'} = \omega\cos\theta = 9 \text{ rad/s}^2$$

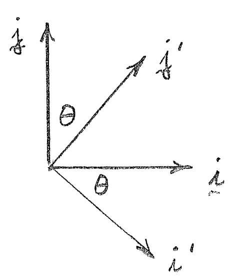
$$\omega_{y'} = \omega\sin\theta = 7.93725 \text{ rad/s}^2$$

$$\omega_{z'} = 0$$

$$\bar{I}_{x'} \approx 0$$

$$\bar{I}_{y'} = \frac{1}{12}ml^2 = \frac{1}{12}(0.093168)(2)^2$$

$$= 0.0310559 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$




$$\mathbf{H}_D = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}' + \bar{I}_{z'}\omega_{z'}\mathbf{k}'$$

$$= 0 + (0.0310559)(7.93725)\mathbf{j}' + 0$$

$$= 0.246498\mathbf{j}'$$

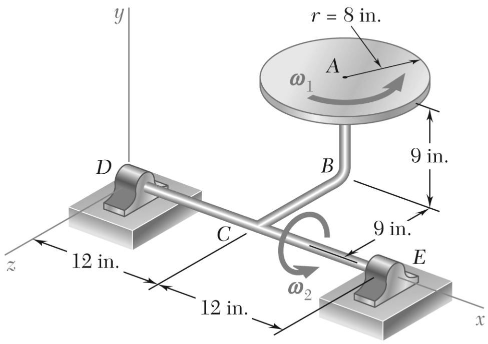
$$H_D = 0.246 \text{ lb}\cdot\text{s}\cdot\text{ft} \blacktriangleleft$$

$$\mathbf{H}_D = 0.246498(\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = 0.163045\mathbf{i} + 0.184874\mathbf{j}$$

$$\cos\theta_x = \frac{0.163045}{0.246498} \qquad \theta_x = 48.6° \blacktriangleleft$$

$$\cos\theta_y = \frac{0.184874}{0.246498} \qquad \theta_y = 41.4° \blacktriangleleft$$

$$\cos\theta_z = 0 \qquad \theta_z = 90° \blacktriangleleft$$




## PROBLEM 18.4

A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm $ABC$, which is welded to a shaft $DCE$ rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the angular momentum $\mathbf{H}_A$ of the disk about its center $A$.

## SOLUTION

$$\boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{i} = (8 \text{ rad/s})\mathbf{i} + (16 \text{ rad/s})\mathbf{j}$$

For axes $x'$, $y'$, $z'$ parallel to $x$, $y$, $z$ with origin at $A$,

$$m = \frac{W}{g} = \frac{6}{32.2} = 0.186335 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\bar{I}_{x'} = \frac{1}{4}mr^2 = \frac{1}{4}(0.186335)\left(\frac{8}{12}\right)^2$$

$$= 0.020704 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_{z'} = \bar{I}_{x'} = 0.020704 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

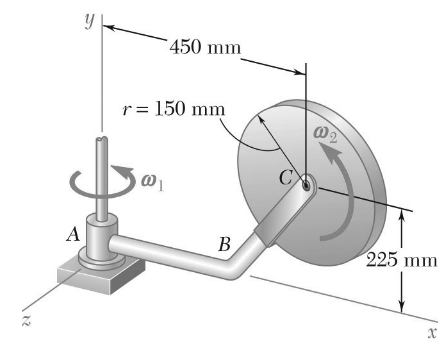
$$\bar{I}_{y'} = \bar{I}_{x'} + \bar{I}_{z'} = 0.041408 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\mathbf{H}_A = \bar{I}_{x'}\omega_{x'}\mathbf{i} + \bar{I}_{y'}\omega_{y'}\mathbf{j} + \bar{I}_{z'}\omega_{z'}\mathbf{k}$$

$$= (0.020704)(8)\mathbf{i} + (0.041408)(16)\mathbf{j}$$

$$= 0.1656\mathbf{i} + 0.6625\mathbf{j}$$

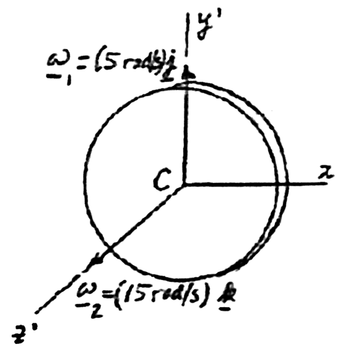
$$\mathbf{H}_A = (0.1656 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{i} + (0.663 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{j} \blacktriangleleft$$




## PROBLEM 18.5

A thin disk of mass $m = 4$ kg rotates at the constant rate $\omega_2 = 15$ rad/s with respect to arm $ABC$, which itself rotates at the constant rate $\omega_1 = 5$ rad/s about the $y$ axis. Determine the angular momentum of the disk about its center $C$.

## SOLUTION




$$r = 150 \text{ mm}$$

Angular velocity of disk:

$$\boldsymbol{\omega} = \omega_1 \mathbf{j} + \omega_2 \mathbf{k}$$
$$= (5 \text{ rad/s})\mathbf{j} + (15 \text{ rad/s})\mathbf{k}$$

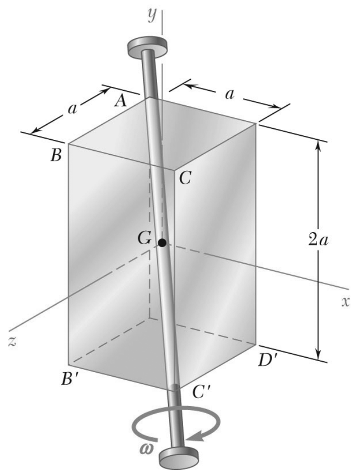
Centroidal moments of inertia:

$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4}mr^2$$
$$= \frac{1}{4}(4)(0.150 \text{ m})^2 = 0.0225 \text{ kg} \cdot \text{m}^2$$
$$\bar{I}_{z'} = \frac{1}{2}mr^2 = 0.045 \text{ kg} \cdot \text{m}^2$$

Angular momentum about Point $C$.

$$\mathbf{H}_C = \bar{I}_{x'}\omega_{x'}\mathbf{i} + \bar{I}_{y'}\omega_{y'}\mathbf{j} + I_{z'}\omega_{z'}\mathbf{k}$$
$$= 0 + (0.0225)(5)\mathbf{j} + (0.045)(15)\mathbf{k}$$
$$= (0.1125 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{j} + (0.6750 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{k}$$

$$\mathbf{H}_C = (0.1125 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{j} + (0.675 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{k} \blacktriangleleft$$




## PROBLEM 18.6

A solid rectangular parallelepiped of mass $m$ has a square base of side $a$ and a length $2a$. Knowing that it rotates at the constant rate $\omega$ about its diagonal $AC'$ and that its rotation is observed from $A$ as counterclockwise, determine ($a$) the magnitude of the angular momentum $\mathbf{H}_G$ of the parallelepiped about its mass center $G$, ($b$) the angle that $\mathbf{H}_G$ forms with the diagonal $AC'$.

## SOLUTION

Body diagonal: $$d = \sqrt{a^2 + (2a)^2 + a^2} = \sqrt{6}a$$

$$\boldsymbol{\omega} = \frac{\omega}{d}(-a\mathbf{i} + 2a\mathbf{j} - a\mathbf{k}) = -\frac{\omega}{\sqrt{6}}\mathbf{i} + \frac{2\omega}{\sqrt{6}}\mathbf{j} - \frac{\omega}{\sqrt{6}}\mathbf{k}$$

$$I_x = \frac{1}{12}m[(2a)^2 + a^2] = \frac{5}{12}ma^2$$

$$I_y = \frac{1}{12}m[a^2 + a^2] = \frac{1}{6}ma^2$$

$$I_z = \frac{1}{12}m[a^2 + (2a)^2] = \frac{5}{12}ma^2$$

($a$) $$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= \left(\frac{5}{12}ma^2\right)\left(-\frac{\omega}{\sqrt{6}}\right)\mathbf{i} + \left(\frac{1}{6}ma^2\right)\left(\frac{2\omega}{\sqrt{6}}\right)\mathbf{j} + \left(\frac{5}{12}ma^2\right)\left(-\frac{\omega}{\sqrt{6}}\right)\mathbf{k}$$

$$= \frac{ma^2\omega}{12\sqrt{6}}(-5\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$$

$$H_G = \frac{ma^2\omega}{12\sqrt{6}}\sqrt{5^2 + 4^2 + 5^2} = \frac{\sqrt{11}ma^2\omega}{12}$$

$\mathbf{H}_G = 0.276\, ma^2\omega$ ◀

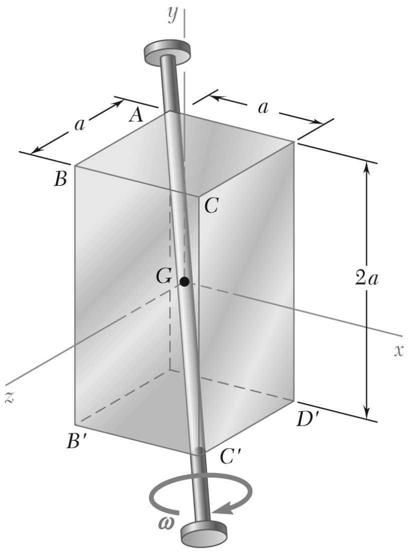
($b$) $$\mathbf{H}_G \cdot \boldsymbol{\omega} = \frac{ma^2\omega^2}{(12)(6)}(-5\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

$$= \frac{18ma^2\omega^2}{(12)(6)} = \frac{1}{4}ma^2\omega$$

$$H_G\omega = \frac{\sqrt{11}}{12}ma^2\omega^2$$

$$\cos\theta = \frac{\mathbf{H}_G \cdot \boldsymbol{\omega}}{H_G\omega} = \frac{12}{4\sqrt{11}} = 0.90453$$

$\theta = 25.2°$ ◀




## PROBLEM 18.7

Solve Problem 18.6, assuming that the solid rectangular parallelepiped has been replaced by a hollow one consisting of six thin metal plates welded together.

**PROBLEM 18.6** A solid rectangular parallelepiped of mass $m$ has a square base of side $a$ and a length $2a$. Knowing that it rotates at the constant rate $\omega$ about its diagonal $AC'$ and that its rotation is observed from $A$ as counterclockwise, determine ($a$) the magnitude of the angular momentum $\mathbf{H}_G$ of the parallelepiped about its mass center $G$, ($b$) the angle that $\mathbf{H}_G$ forms with the diagonal $AC'$.

## SOLUTION

Body diagonal:
$$d = \sqrt{a^2 + (2a)^2 + a^2} = \sqrt{6}a$$

$$\boldsymbol{\omega} = \frac{\omega}{d}(-a\mathbf{i} + 2a\mathbf{j} - a\mathbf{k}) = -\frac{\omega}{\sqrt{6}}\mathbf{i} + \frac{2\omega}{\sqrt{6}}\mathbf{j} - \frac{\omega}{\sqrt{6}}\mathbf{k}$$

$$\text{Total area} = 2(a^2 + 2a^2 + 2a^2) = 10a^2$$

For each square plate:
$$m' = \frac{1}{10}m$$

$$I_x = \frac{1}{12}m'a^2 + m'a^2 = \frac{13}{12}m'a^2 = \frac{13}{120}ma^2$$

$$I_y = \frac{1}{6}m'a^2 = \frac{1}{60}ma^2$$

$$I_z = I_x = \frac{13}{120}ma^2$$

For each plate parallel to the $yz$ plane:
$$m' = \frac{1}{5}m$$

$$I_x = \frac{1}{12}m'[a^2 + (2a)^2] = \frac{5}{12}m'a^2 = \frac{1}{12}ma^2$$

$$I_y = \frac{1}{12}m'a^2 + m'\left(\frac{a}{2}\right)^2 = \frac{1}{3}m'a^2 = \frac{1}{15}ma^2$$

$$I_z = \frac{1}{12}m(2a)^2 + m'\left(\frac{a}{2}\right)^2 = \frac{7}{12}m'a^2 = \frac{7}{60}ma^2$$

## PROBLEM 18.7 (Continued)

For each plate parallel to the $xy$ plane: $m' = \frac{1}{5}m$

$$I_x = \frac{1}{12}m'(2a)^2 + m'\left(\frac{a}{2}\right)^2 = \frac{7}{12}m'a^2 = \frac{7}{60}ma^2$$

$$I_y = \frac{1}{12}m'a^2 + m'\left(\frac{a}{2}\right)^2 = \frac{1}{3}m'a^2 = \frac{1}{15}ma^2$$

$$I_z = \frac{1}{12}m'[a^2 + (2a)^2] = \frac{5}{12}m'a^2 = \frac{1}{12}ma^2$$

Total moments of inertia:

$$I_x = 2\left(\frac{13}{120} + \frac{1}{12} + \frac{7}{60}\right)ma^2 = \frac{37}{60}ma^2$$

$$I_y = 2\left(\frac{1}{60} + \frac{1}{15} + \frac{1}{15}\right)ma^2 = \frac{3}{10}ma^2$$

$$I_z = 2\left(\frac{13}{120} + \frac{7}{60} + \frac{1}{12}\right)ma^2 = \frac{37}{60}ma^2$$

(*a*)

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k} = \frac{ma^2\omega}{60\sqrt{6}}(-37\mathbf{i} + 36\mathbf{j} - 37\mathbf{k})$$

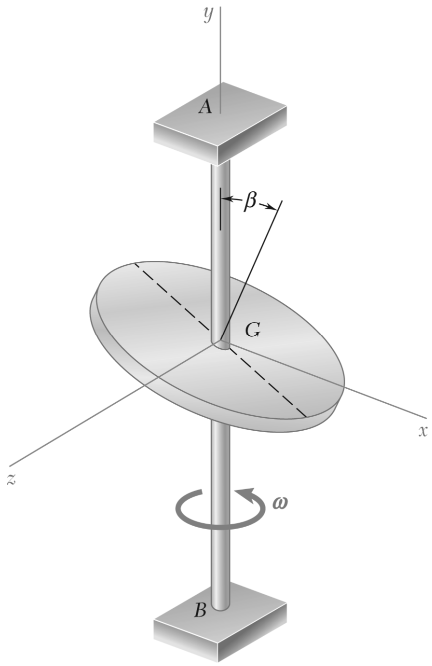
$$H_G = \frac{ma^2\omega}{60\sqrt{6}}\sqrt{(37)^2 + (36)^2 + (37)^2} = 0.43216\, ma^2\omega$$

$$H_G = 0.432\, ma^2\omega \blacktriangleleft$$

(*b*)

$$\mathbf{H}_G \cdot \boldsymbol{\omega} = \frac{ma^2\omega}{(60)(6)}(-37\mathbf{i} + 36\mathbf{j} - 37\mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 0.40556\, ma^2\omega^2$$

$$\cos\theta = \frac{\mathbf{H}_G \cdot \boldsymbol{\omega}}{H_G\omega} = \frac{0.40556}{0.43216} = 0.93844 \qquad \theta = 20.2° \blacktriangleleft$$

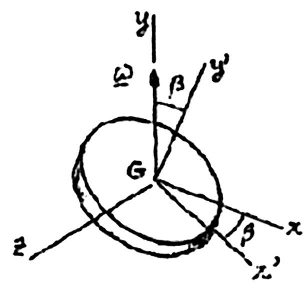



## PROBLEM 18.8

A homogeneous disk of mass $m$ and radius $r$ is mounted on the vertical shaft $AB$. The normal to the disk at $G$ forms an angle $\beta = 25°$ with the shaft. Knowing that the shaft has a constant angular velocity $\omega$, determine the angle $\theta$ formed by the shaft $AB$ and the angular momentum $\mathbf{H}_G$ of the disk about its mass center $G$.

## SOLUTION

Use the principal centroidal axes $Gx'$, $y'$ $z$.




Moments of inertia:

$$\bar{I}_{x'} = \bar{I}_z = \frac{1}{4}mr^2$$

$$\bar{I}_{y'} = \frac{1}{2}mr^2$$

Angular velocities:

$$\omega_{x'} = -\omega \sin\beta$$

$$\omega_{y'} = \omega\cos\beta$$

$$\omega_z = 0$$

Using Eq. (18.10):

$$H_{x'} = \bar{I}_{x'}\omega_{x'} = -\frac{1}{4}mr^2\omega\sin\beta$$

$$H_{y'} = \bar{I}_{y'}\omega_{y'} = \frac{1}{2}mr^2\omega\cos\beta$$

$$H_z = \bar{I}_z\omega_z = 0$$

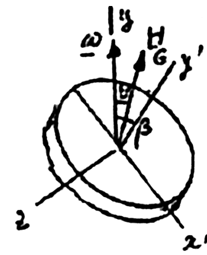
We have

$$\mathbf{H}_G = H_{x'}\mathbf{i}' + H_{y'}\mathbf{j}' + H_z\mathbf{k}$$

where $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}$ are the unit vectors along the $x'$, $y'$, $z$ axes.

$$\mathbf{H}_G = -\frac{1}{4}mr^2\omega\sin\beta\mathbf{i}' + \frac{1}{2}mr^2\omega\cos\beta\mathbf{j}' \tag{1}$$

$$\mathbf{H}_G = \frac{1}{4}mr^2\omega(-\sin\beta\mathbf{i}' + 2\cos\beta\mathbf{j}')$$




Forming the scalar product,

$$\mathbf{H}_G \cdot \boldsymbol{\omega} = |\mathbf{H}_G|\,\omega\cos\theta$$

$$\cos\theta = \frac{\mathbf{H}_G \cdot \boldsymbol{\omega}}{|\mathbf{H}_G|\,\omega} \tag{2}$$

But $$\mathbf{H}_G \cdot \boldsymbol{\omega} = \frac{1}{4}mr^2\omega(-\sin\beta\mathbf{i}' + 2\cos\beta\mathbf{j}')\cdot\omega\mathbf{j}$$

or observing that $$\mathbf{i}'\cdot\mathbf{i} = -\sin\beta \quad \text{and} \quad \mathbf{j}'\cdot\mathbf{j} = \cos\beta$$

$$\mathbf{H}_G \cdot \boldsymbol{\omega} = \frac{1}{4}mr^2\omega^2(\sin^2\beta + 2\cos^2\beta)$$

$$= \frac{1}{4}mr^2\omega^2(1+\cos^2\beta) \tag{3}$$

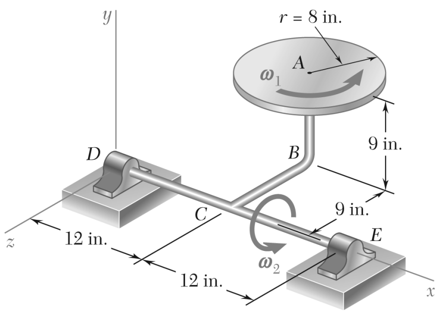
Also, $$|\mathbf{H}_G|\,\omega = \frac{1}{4}mr^2\omega^2\sqrt{\sin^2\beta + 4\cos^2\beta}$$

$$= \frac{1}{4}mr^2\omega^2\sqrt{1+3\cos^2\beta} \tag{4}$$

Substituting from Eqs. (3) and (4) into Eq. (2),

$$\cos\theta = \frac{1+\cos^2\beta}{\sqrt{1+3\cos^2\beta}}$$

For $\beta = 25°$, $\cos\theta = 0.9786$ $\qquad \theta = 11.88°$ ◀




## PROBLEM 18.9

Determine the angular momentum $\mathbf{H}_D$ of the disk of Problem 18.4 about Point $D$.

**PROBLEM 18.4** A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm $ABC$, which is welded to a shaft $DCE$ rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the angular momentum $\mathbf{H}_A$ of the disk about its center $A$.

## SOLUTION

$$\boldsymbol{\omega} = \omega_2\mathbf{i} + \omega_1\mathbf{j} = (8 \text{ rad/s})\mathbf{i} + (16 \text{ rad/s})\mathbf{j}$$

For axes $x'$, $y'$, $z'$ parallel to $x$, $y$, $z$ with origin at $A$,

$$m = \frac{W}{g} = \frac{6}{32.2} = 0.186335 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\bar{I}_{x'} = \frac{1}{4}mr^2 = \frac{1}{4}(0.186335)\left(\frac{8}{12}\right)^2$$

$$= 0.020704 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_{z'} = \bar{I}_{x'} = 0.020704 \text{ lb}\cdot\text{s}^2\cdot\text{ft},$$

$$\bar{I}_{y'} = \bar{I}_{x'} + \bar{I}_{z'} = 0.041408 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\mathbf{H}_A = \bar{I}_{x'}\omega_{x'}\mathbf{i} + \bar{I}_{y'}\omega_{y'}\mathbf{j} + \bar{I}_{z'}\omega_{z'}\mathbf{k}$$

$$= (0.020704)(8)\mathbf{i} + (0.041408)(16)\mathbf{j}$$

$$= (0.1656 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{i} + (0.6625 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{j}$$

Point $A$ is the mass center of the disk.

$$\mathbf{r}_{A/D} = (1.0 \text{ ft})\mathbf{i} + (0.75 \text{ ft})\mathbf{j} - (0.75 \text{ ft})\mathbf{k}$$

$$\bar{\mathbf{v}} = \mathbf{v}_A = \omega_2\mathbf{i} \times \mathbf{r}_{A/D}$$

$$= 8\mathbf{i} \times (1.0\mathbf{i} + 0.75\mathbf{j} - 0.75\mathbf{k})$$

$$= (6 \text{ ft/s})\mathbf{j} + (6 \text{ ft/s})\mathbf{k}$$

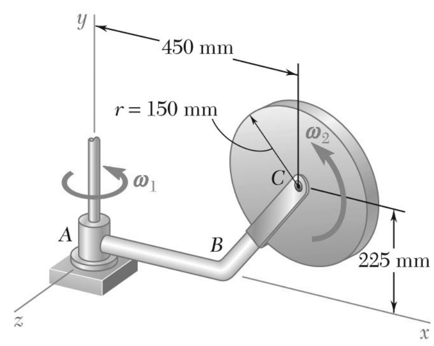
$$m\bar{\mathbf{v}} = (1.118 \text{ lb}\cdot\text{s})\mathbf{j} + (1.118 \text{ lb}\cdot\text{s})\mathbf{k}$$

$$\mathbf{r}_{A/D} \times m\bar{\mathbf{v}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.0 & 0.75 & -0.75 \\ 0 & 1.118 & 1.118 \end{vmatrix}$$

$$= (1.677 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{i} - (1.118 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{j} + (1.118 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{k}$$

$$\mathbf{H}_D = \mathbf{H}_A + \mathbf{r}_{A/D} \times m\bar{\mathbf{v}}$$

$$\mathbf{H}_D = (1.843 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{i} - (0.455 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{j} + (1.118 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{k} \blacktriangleleft$$

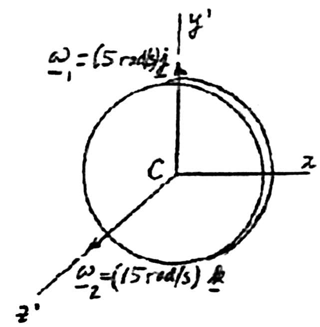



## PROBLEM 18.10

Determine the angular momentum of the disk of Problem 18.5 about Point *A*.

**PROBLEM 18.5** A thin disk of mass $m = 4$ kg rotates at the constant rate $\omega_2 = 15$ rad/s with respect to arm *ABC*, which itself rotates at the constant rate $\omega_1 = 5$ rad/s about the *y* axis. Determine the angular momentum of the disk about its center *C*.

## SOLUTION




$$r = 150 \text{ in.}$$

Angular velocity of disk:

$$\boldsymbol{\omega} = \omega_1 \mathbf{j} + \omega_2 \mathbf{k} = (5 \text{ rad/s})\mathbf{j} + (15 \text{ rad/s})\mathbf{k}$$

Centroidal moments of inertia:

$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4} m r^2$$

$$= \frac{1}{4}(4)(0.150 \text{ m})^2 = 0.0225 \text{ kg} \cdot \text{m}^2$$

$$I_{z'} = \frac{1}{2} m r^2 = 0.045 \text{ kg} \cdot \text{m}^2$$

<u>Angular momentum about Point *C*</u>.

$$\mathbf{H}_C = \bar{I}_{x'} \omega_{x'} \mathbf{i} + \bar{I}_{y'} \omega_{y'} \mathbf{j} + I_{z'} \omega_{z'} \mathbf{k}$$

$$= 0 + (0.0225)(5)\mathbf{j} + (0.045)(15)\mathbf{k}$$

$$= (0.1125 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{j} + (0.6750 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{k}$$

<u>Location of mass center</u>.

$$\mathbf{r}_{C/A} = (0.450 \text{ m})\mathbf{i} + (0.225 \text{ m})\mathbf{j}$$

<u>Velocity of mass center</u>.

$$\bar{\mathbf{v}} = \boldsymbol{\omega}_1 \times \mathbf{r}_{C/A} = 5\mathbf{j} \times (0.45\mathbf{i} + 0.225\mathbf{j})$$

$$= -(2.25 \text{ m/s})\mathbf{k}$$

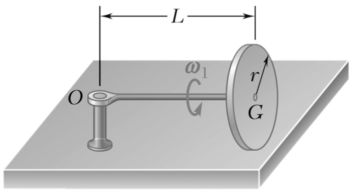
<u>Angular momentum about Point *A*</u>.

$$\mathbf{H}_A = \mathbf{H}_C + \mathbf{r}_{C/A} \times (m\bar{\mathbf{v}})$$

$$\mathbf{H}_A = 0.1125\mathbf{j} + 0.675\mathbf{k} + (0.45\mathbf{i} + 0.225\mathbf{j}) \times [-(4)(2.25)\mathbf{k}]$$

$$= 0.1125\mathbf{j} + 0.675\mathbf{k} + 4.05\mathbf{j} - 2.025\mathbf{i}$$

$$\mathbf{H}_A = -(2.03 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{i} + (4.16 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{j} + (0.675 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{k} \blacktriangleleft$$




## PROBLEM 18.11

Determine the angular momentum $\mathbf{H}_O$ of the disk of Sample Problem 18.2 from the expressions obtained for its linear momentum $m\bar{\mathbf{v}}$ and its angular momentum $\mathbf{H}_G$ using Eq. (18.11). Verify that the result obtained is the same as that obtained by direct computation.

**PROBLEM 18.2** A homogeneous disk of radius $r$ and mass $m$ is mounted on an axle $OG$ of length $L$ and negligible mass. The axle is pivoted at the fixed Point $O$, and the disk is constrained to roll on a horizontal floor. Knowing that the disk rotates counterclockwise at the rate $\omega_1$ about the axle $OG$, determine (*a*) the angular velocity of the disk, (*b*) its angular momentum about $O$, (*c*) its kinetic energy, (*d*) the vector and couple at $G$ equivalent to the momenta of the particles of the disk.

## SOLUTION

Using Equation (18.11),

$$\mathbf{H}_O = \bar{\mathbf{r}} \times m\bar{\mathbf{v}} + \bar{\mathbf{H}}_G$$

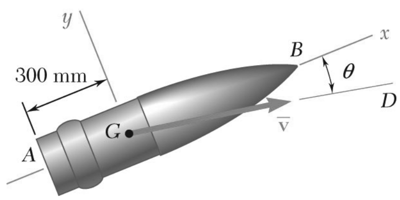
$$= (L\mathbf{i}) \times (mr\omega_1\mathbf{k}) + \frac{1}{2}mr^2\omega_1\left(\mathbf{i} - \frac{r}{2L}\mathbf{j}\right)$$

$$= -mrL\omega_1\mathbf{j} + \frac{1}{2}mr^2\omega_1\mathbf{i} - \frac{1}{4}m\frac{r^3}{L}\omega_1\mathbf{j}$$

$$\mathbf{H}_O = \frac{1}{2}mr^2\omega_1\mathbf{i} - m\left(L^2 + \frac{1}{4}r^2\right)\left(\frac{r\omega_1}{L}\right)\mathbf{j}$$

which is the answer obtained in Part *b* of Sample Problem 18.2.

## PROBLEM 18.12




The 100-kg projectile shown has a radius of gyration of 100 mm about its axis of symmetry $Gx$ and a radius of gyration of 250 mm about the transverse axis $Gy$. Its angular velocity $\boldsymbol{\omega}$ can be resolved into two components; one component, directed along $Gx$, measures the *rate of spin* of the projectile, while the other component, directed along $GD$, measures its *rate of precession*. Knowing that $\theta = 6°$ and that the angular momentum of the projectile about its mass center $G$ is $\mathbf{H}_G = (500\ \text{g}\cdot\text{m}^2/\text{s})\mathbf{i} - (10\ \text{g}\cdot\text{m}^2/\text{s})\mathbf{j}$, determine (*a*) the rate of spin, (*b*) the rate of precession.

## SOLUTION

$$m = 100\ \text{kg},\quad k_x = 100\ \text{mm} = 0.1\ \text{m},\quad k_y = 250\ \text{mm} = 0.25\ \text{m}$$

$$\bar{I}_x = mk_x^2 = (100)(0.1)^2 = 1\ \text{kg}\cdot\text{m}^2$$

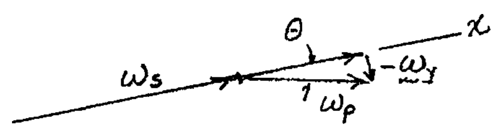
$$\bar{I}_y = \bar{I}_z = mk_y^2 = (100)(0.25)^2 = 6.25\ \text{kg}\cdot\text{m}^2$$

$$\mathbf{H}_G = (H_G)_x\mathbf{i} + (H_G)_y\mathbf{j} + (H_G)_z\mathbf{k} = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

$$\omega_x = \frac{(H_G)_x}{\bar{I}_x} = \frac{0.500\ \text{kg}\cdot\text{m}^2/\text{s}}{1\ \text{kg}\cdot\text{m}^2} = 0.5\ \text{rad/s}$$

$$\omega_y = \frac{(H_G)_y}{\bar{I}_y} = \frac{-0.01\ \text{kg}\cdot\text{m}^2/\text{s}}{6.25\ \text{kg}\cdot\text{m}^2} = -0.0016\ \text{rad/s}$$

$$\omega_z = 0$$




$$\omega_P \sin\theta = -\omega_y$$

$$\omega_P = \frac{-\omega_y}{\sin\theta} = \frac{0.0016}{\sin 6°} = 0.015307\ \text{rad/s}$$

(*a*) *Rate of spin.*

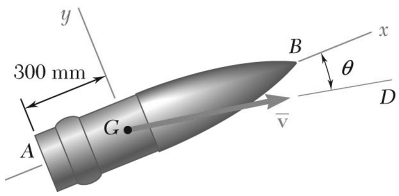
$$\omega_x = \omega_s + \omega_P\cos\theta$$

$$\omega_s = \omega_x - \omega_P\cos\theta$$

$$= 0.5 - 0.15307\cos 6°$$

$$= 0.4847 \qquad \omega_s = 0.485\ \text{rad/s} \blacktriangleleft$$

(*b*) *Rate of precession.* $\qquad \omega_P = 0.01531\ \text{rad/s} \blacktriangleleft$




## PROBLEM 18.13

Determine the angular momentum $\mathbf{H}_A$ of the projectile of Problem 18.12 about the center $A$ of its base, knowing that its mass center $G$ has a velocity $\overline{\mathbf{v}}$ of 750 m/s. Give your answer in terms of components respectively parallel to the $x$ and $y$ axes shown and to a third axis $z$ pointing toward you.

**PROBLEM 18.12** The 100-kg projectile shown has a radius of gyration of 100 mm about its axis of symmetry $Gx$ and a radius of gyration of 250 mm about the transverse axis $Gy$. Its angular velocity $\boldsymbol{\omega}$ can be resolved into two components; one component, directed along $Gx$, measures the *rate of spin* of the projectile, while the other component, directed along $GD$, measures its *rate of precession.* Knowing that $\theta = 6°$ and that the angular momentum of the projectile about its mass center $G$ is $\mathbf{H}_G = (500\ \text{g}\cdot\text{m}^2/\text{s})\mathbf{i} - (10\ \text{g}\cdot\text{m}^2/\text{s})\mathbf{j}$, determine ($a$) the rate of spin, ($b$) the rate of precession.

## SOLUTION

$$m = 100\text{ kg}, \quad \mathbf{r}_{G/A} = (0.300\text{ m})\mathbf{i}$$

$$\overline{\mathbf{v}} = \overline{v}\cos\theta\mathbf{i} - \overline{v}\sin\theta\mathbf{j}$$

$$\mathbf{H}_G = (0.50\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} - (0.10\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j}$$

$$m\overline{\mathbf{v}} = (100)(750)(\cos 6°\mathbf{i} - \sin 6°\mathbf{j})$$

$$= (74589\text{ kg}\cdot\text{m/s})\mathbf{i} - (7839\text{ kg}\cdot\text{m/s})\mathbf{j}$$

$$\mathbf{r}_{G/A} \times m\overline{\mathbf{v}} = 0.3\mathbf{i} \times (74589\mathbf{i} - 7839.6\mathbf{j})$$

$$= -(2351.9\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$$

$$\mathbf{H}_A = \mathbf{H}_G + \mathbf{r}_{G/A} \times m\overline{\mathbf{v}}$$

$$= 0.5\mathbf{i} - 0.1\mathbf{j} + (-2351.9\mathbf{k})$$

$$\mathbf{H}_A = (0.500\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} - (0.100\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} - (2350\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k} \blacktriangleleft$$

## PROBLEM 18.14

($a$) Show that the angular momentum $\mathbf{H}_B$ of a rigid body about Point $B$ can be obtained by adding to the angular momentum $\mathbf{H}_A$ of that body about Point $A$ the vector product of the vector $\mathbf{r}_{A/B}$ drawn from $B$ to $A$ and the linear momentum $m\overline{\mathbf{v}}$ of the body:

$$\mathbf{H}_B = \mathbf{H}_A + \mathbf{r}_{A/B} \times m\overline{\mathbf{v}}$$

($b$) Further show that when a rigid body rotates about a fixed axis, its angular momentum is the same about any two Points $A$ and $B$ located on the fixed axis $(\mathbf{H}_A = \mathbf{H}_B)$ if, and only if, the mass center $G$ of the body is located on the fixed axis.

## SOLUTION

($a$) Angular momenta $\mathbf{H}_A$ and $\mathbf{H}_B$ are related to $\mathbf{H}_G$ and $m\overline{\mathbf{v}}$ by

$$\mathbf{H}_A = \mathbf{r}_{G/A} \times m\overline{\mathbf{v}} + \mathbf{H}_G \quad \text{and} \quad \mathbf{H}_B = \mathbf{r}_{G/B} \times m\overline{\mathbf{v}} + \mathbf{H}_G$$

Subtracting,

$$\mathbf{H}_B - \mathbf{H}_A = \mathbf{r}_{G/B} \times m\overline{\mathbf{v}} - \mathbf{r}_{G/A} \times m\overline{\mathbf{v}}$$

$$\mathbf{H}_B = \mathbf{H}_A + (\mathbf{r}_{G/B} - \mathbf{r}_{G/A}) \times m\overline{\mathbf{v}}$$

$$= \mathbf{H}_A + (\mathbf{r}_{G/B} + \mathbf{r}_{A/G}) \times m\overline{\mathbf{v}}$$

$$\mathbf{H}_B = \mathbf{H}_A + \mathbf{r}_{A/B} \times m\overline{\mathbf{v}}$$

($b$) It follows that $\mathbf{H}_A = \mathbf{H}_B$ if, and only if

$$\mathbf{r}_{A/B} \times m\overline{\mathbf{v}} = 0$$

With Points $A$ and $B$ located on the fixed axis,

$$\boldsymbol{\omega} = \omega\boldsymbol{\lambda}$$

where $\boldsymbol{\lambda}$ is a unit vector along the fixed axis, and

$$\overline{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{r}_{G/A} = \omega\boldsymbol{\lambda} \times \mathbf{r}_{G/A}$$

Then

$$\mathbf{r}_{A/B} \times (m\omega\boldsymbol{\lambda} \times \mathbf{r}_{G/A}) = 0$$

but $\mathbf{r}_{A/B}$ is parallel to $\boldsymbol{\lambda}$, hence,

$$\boldsymbol{\lambda} \times (\boldsymbol{\lambda} \times \mathbf{r}_{G/A}) = 0$$

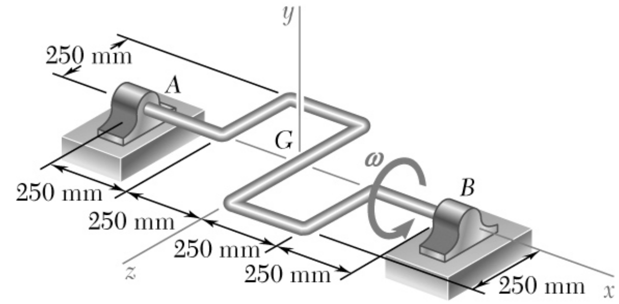
Let $\mathbf{u} = \boldsymbol{\lambda} \times \mathbf{r}_{G/A}$, so that $\boldsymbol{\lambda} \times \mathbf{u} = 0$.

Note that $\mathbf{u}$ must be either perpendicular to $\lambda$ or equal to zero. But if $\mathbf{u}$ is perpendicular to $\boldsymbol{\lambda}$, $\boldsymbol{\lambda} \times \mathbf{u}$ cannot be equal to zero.

Hence,

$$\mathbf{u} = \boldsymbol{\lambda} \times \mathbf{r}_{G/A} = 0$$

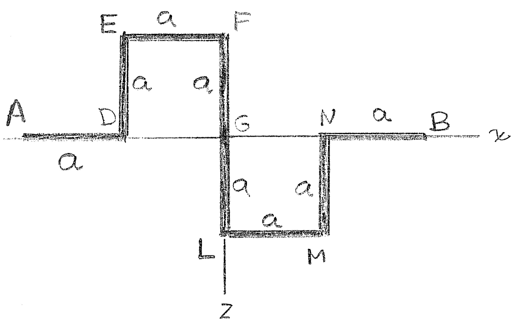
$\mathbf{r}_{G/A}$ is parallel to $\boldsymbol{\lambda}$ and Point $G$ lies on the fixed axis.




## PROBLEM 18.15

A 5-kg rod of uniform cross section is used to form the shaft shown. Knowing that the shaft rotates with a constant angular velocity $\boldsymbol{\omega}$ of magnitude 12 rad/s, determine (*a*) the angular momentum $\mathbf{H}_G$ of the shaft about its mass center $G$, (*b*) the angle formed by $\mathbf{H}_G$ and the axis $AB$.

## SOLUTION




$$\omega = (12 \text{ rad/s})\mathbf{i}, \qquad \omega_y = \omega_z = 0$$

$$(H_G)_x = \bar{I}_x\omega$$

$$(H_G)_y = -\bar{I}_{xy}\omega$$

$$(H_G)_z = -\bar{I}_{xz}\omega$$

The shaft is comprised of 8 sections, each of length $a = 0.25$ m and of mass $m' = \dfrac{m}{8} = 0.625$ kg.

$$\bar{I}_x = (4)\left(\frac{1}{3}m'a^2\right) + (2)(m'a^2) = \frac{10}{3}m'a^2 = \frac{10}{3}(0.625)(0.25)^2 = 0.130208 \text{ kg}\cdot\text{m}^2$$

$$\bar{I}_{xy} = 0$$

$$\bar{I}_{xz} = (4)\left(m'a\frac{a}{2}\right) = 2m'a^2 = (2)(0.625)(0.25)^2 = 0.078125 \text{ kg}\cdot\text{m}^2$$

$$(H_G)_x = (0.130208)(12) = 1.5625 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_G)_y = 0$$

$$(H_G)_z = -(0.078125)(12) = -0.9375 \text{ kg}\cdot\text{m}^2/\text{s}$$

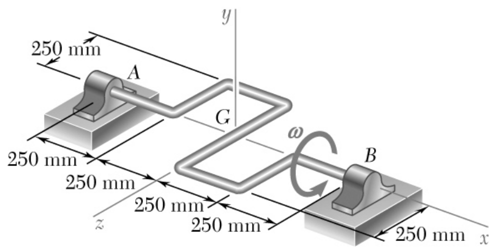
(*a*) $\mathbf{H}_G = (1.563 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} - (0.938 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$ ◀

$$H_G = \sqrt{(1.5625)^2 + (0.9375)^2} = 1.82217 \text{ kg}\cdot\text{m}^2/\text{s}$$

(*b*) $\mathbf{H}_G \cdot \boldsymbol{\omega} = (1.5625\mathbf{i} - 0.9375\mathbf{k}) \cdot 12\mathbf{i} = 18.75 \text{ kg}\cdot\text{m}^2/\text{s}^2$

$$\cos\theta = \frac{\mathbf{H}_G \cdot \boldsymbol{\omega}}{H_G\omega} = \frac{18.75}{(1.82217)(12)} = 0.85749$$

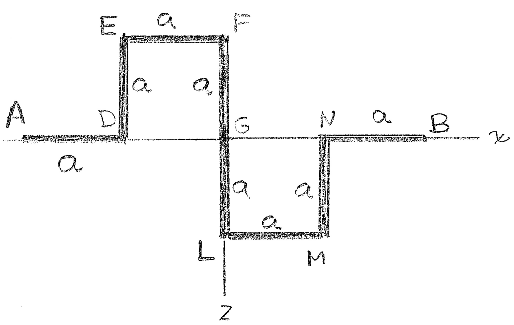
$\theta = 31.0°$ ◀




## PROBLEM 18.16

Determine the angular momentum of the shaft of Problem 18.15 about (*a*) Point *A*, (*b*) Point *B*.

## SOLUTION




$$\omega = (12 \text{ rad/s})\mathbf{i}, \qquad \omega_y = \omega_z = 0, \qquad m = 5 \text{ kg}$$

$$(H_G)_x = \bar{I}_x \omega$$

$$(H_G)_y = -\bar{I}_{xy} \omega$$

$$(H_G)_z = -\bar{I}_{xz} \omega$$

The shaft is comprised of 8 sections, each of length $a = 0.25$ m and of mass $m' = \dfrac{m}{8} = 0.625$ kg.

$$\bar{I}_x = (4)\left(\frac{1}{3}m'a^2\right) + (2)(m'a^2) = \frac{10}{3}m'a^2 = \frac{10}{3}(0.625)(0.25)^2 = 0.130208 \text{ kg}\cdot\text{m}^2$$

$$\bar{I}_{xy} = 0$$

$$\bar{I}_{xz} = (4)\left(m'a\frac{a}{2}\right) = 2m'a^2 = (2)(0.625)(0.25)^2 = 0.078125 \text{ kg}\cdot\text{m}^2$$

$$(H_G)_x = (0.130208)(12) = 1.5625 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_G)_y = 0$$

$$(H_G)_z = -(0.078125)(12) = -0.9375 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$\mathbf{H}_G = (1.5625 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} - (0.9375 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$$

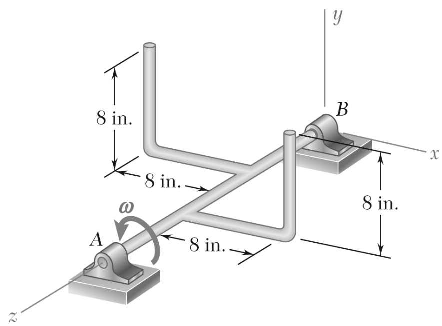
Since Point *G* lies on the axis of rotation, its velocity is zero.

$$\bar{\mathbf{v}} = \bar{\mathbf{v}}_G = 0$$

(*a*) $\mathbf{H}_A = \mathbf{H}_G + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}} = \mathbf{H}_G$ $\qquad \mathbf{H}_A = (1.563 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} - (0.938 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$ ◀

(*b*) $\mathbf{H}_B = \mathbf{H}_G + \mathbf{r}_{G/B} \times m\bar{\mathbf{v}} = \mathbf{H}_G$ $\qquad \mathbf{H}_B = (1.563 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} - (0.938 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$ ◀

## PROBLEM 18.17



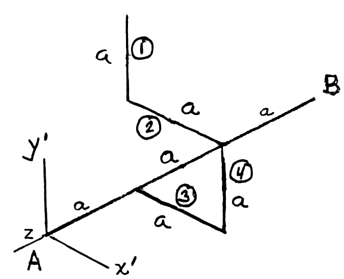

Two L-shaped arms, each weighing 4 lb, are welded at the third points of the 2-ft shaft $AB$. Knowing that shaft $AB$ rotates at the constant rate $\omega = 240$ rpm, determine (*a*) the angular momentum of the body about $A$, (*b*) the angle formed by the angular momentum and shaft $AB$.

## SOLUTION

$$W = 4 \text{ lb}, \quad m = \frac{4}{32.2} = 0.12422 \text{ lb}\cdot\text{s}^2/\text{ft}, \quad a = 8 \text{ in.} = 0.66667 \text{ ft}$$

$$\omega = \frac{(2\pi)(240)}{60} = 8\pi \text{ rad/s}, \quad \omega_x = 0, \quad \omega_y = 0, \quad \omega_z = 8\pi \text{ rad/s}$$

Use parallel axes $x', y', z'$ with origin at Point $A$ as shown.




$$(H_A)_{x'} = -I_{x'z'}\omega$$

$$(H_A)_{y'} = -I_{y'z'}\omega$$

$$(H_A)_{z'} = I_{z'}\omega$$

Segments 1, 2, 3, and 4, each of mass $m' = 0.06211 \text{ lb}\cdot\text{s}^2/\text{ft}$, contribute to $\bar{I}_{x'z'}$, $\bar{I}_{y'z'}$, and $\bar{I}_{z'}$.

| Part | $I_{x'z'}$ | $I_{y'z'}$ | $I_{z'}$ |
|---|---|---|---|
| 1 | $2m'a^2$ | $-m'a^2$ | $\left(\frac{1}{12}+\frac{1}{4}+1\right)m'a^2$ |
| 2 | $m'a^2$ | 0 | $\frac{1}{3}m'a^2$ |
| 3 | $-\frac{1}{2}m'a^2$ | 0 | $\frac{1}{3}m'a^2$ |
| 4 | $-m'a^2$ | $-\frac{1}{2}m'a^2$ | $\left(\frac{1}{12}+\frac{1}{4}+1\right)m'a^2$ |
| Σ | $\frac{3}{2}m'a^2$ | $-\frac{3}{2}m'a^2$ | $\frac{10}{3}m'a^2$ |

## PROBLEM 18.17 (Continued)

(*a*) Angular momentum about *A*.

$$(H_A)_{x'} = -I_{x'z'}\omega = -\frac{3}{2}m'a^2\omega$$

$$= -\frac{3}{2}(0.06211)(0.66667)^2(8\pi)$$

$$= -1.04067 \text{ lb}\cdot\text{s}\cdot\text{ft}$$

$$(H_A)_{y'} = -I_{y'z'}\omega = -\left(-\frac{3}{2}m'a^2\right)\omega$$

$$= \frac{3}{2}(0.06211)(0.66667)^2(8\pi)$$

$$= 1.04067 \text{ lb}\cdot\text{ft}\cdot\text{s}$$

$$(H_A)_{z'} = I_{z'}\omega = \frac{10}{3}m'a^2\omega$$

$$= \frac{10}{3}(0.06211)(0.66667)^2(8\pi)$$

$$= 2.3126 \text{ lb}\cdot\text{ft}\cdot\text{s}$$

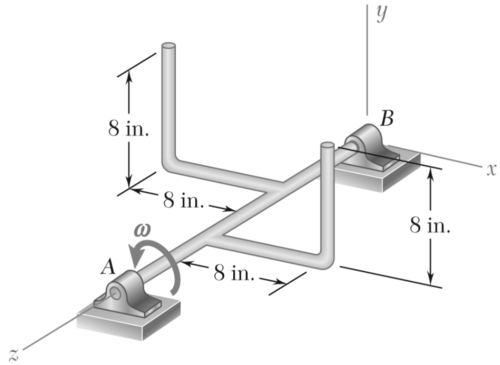
$$\mathbf{H}_A = -(1.041 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{i} + (1.041 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{j} + (2.31 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{k} \blacktriangleleft$$

$$H_A = \sqrt{(1.04067)^2 + (1.04067)^2 + (2.3126)^2} = 2.7412 \text{ lb}\cdot\text{ft}\cdot\text{s}$$

(*b*) Angle formed by $\mathbf{H}_A$ and shaft *AB*.

Unit vector along shaft *AB*: $\boldsymbol{\lambda} = -\mathbf{k}$

$$\cos\theta = \frac{\mathbf{H}_A\cdot\boldsymbol{\lambda}}{H_A} = \frac{-2.3126}{2.7412} = -0.84365 \qquad \theta = 147.5° \blacktriangleleft$$




## PROBLEM 18.18

For the body of Problem 18.17, determine (*a*) the angular momentum about *B*, (*b*) the angle formed by the angular momentum about shaft *BA*.

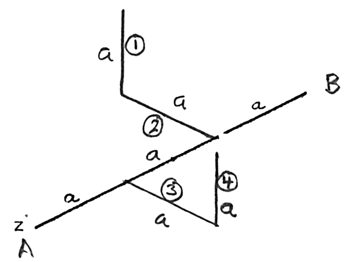
## SOLUTION

$$W = 4 \text{ lb.} \quad m = \frac{4}{32.2} = 0.12422 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$a = 8 \text{ in.} = 0.66667 \text{ ft}$$

$$\omega = \frac{(2\pi)(240)}{60} = 8\pi \text{ rad/s}, \quad \omega_x = 0, \quad \omega_y = 0, \quad \omega_z = 8\pi \text{ rad/s}$$

Use parallel axes $x'$, $y'$, $z$ with origin at Point *B* as shown.




$$(H_B)_x = -I_{xz}\omega$$

$$(H_B)_y = -I_{yz}\omega$$

$$(H_B)_z = I_z\omega$$

Segments 1, 2, 3, and 4, each of mass $m' = 0.06211 \text{ lb}\cdot\text{s}^2/\text{ft}$, contribute to $I_{xz}$, $I_{yz}$, and $I_z$.

| Part | $I_{xz}$ | $I_{yz}$ | $I_z$ |
|---|---|---|---|
| 1 | $-m'a^2$ | $\frac{1}{2}m'a^2$ | $\left(\frac{1}{12}+\frac{1}{4}+1\right)m'a^2$ |
| 2 | $-\frac{1}{2}m'a^2$ | 0 | $\frac{1}{3}m'a^2$ |
| 3 | $m'a^2$ | 0 | $\frac{1}{3}m'a^2$ |
| 4 | $2m'a^2$ | $m'a^2$ | $\left(\frac{1}{12}+\frac{1}{4}+1\right)m'a^2$ |
| $\Sigma$ | $\frac{3}{2}m'a^2$ | $\frac{3}{2}m'a^2$ | $\frac{10}{3}m'a^2$ |

(*a*) Angular momentum about $B$.

$$(H_B)_x = -I_{xz}\omega = -\frac{3}{2}m'a^2\omega$$
$$= -\frac{3}{2}(0.06211)(0.66667)^2(8\pi)$$
$$= -1.04067 \text{ lb}\cdot\text{s}\cdot\text{ft}$$

$$(H_B)_y = -I_{yz}\omega = -\frac{3}{2}m'a^2\omega$$
$$= -\frac{3}{2}(0.06211)(0.66667)^2(8\pi)$$
$$= -1.04067 \text{ lb}\cdot\text{s}\cdot\text{ft}$$

$$(H_B)_z = I_z\omega = \frac{10}{3}m'a^2\omega$$
$$= \frac{10}{3}(0.06211)(0.66667)^2(8\pi)$$
$$= 2.3126 \text{ lb}\cdot\text{s}\cdot\text{ft}$$

$$\mathbf{H}_B = -(1.041 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{i} - (1.041 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{j} + (2.31 \text{ lb}\cdot\text{ft}\cdot\text{s})\mathbf{k} \blacktriangleleft$$

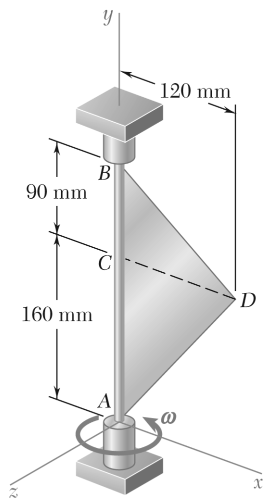
$$H_B = \sqrt{(1.04067)^2 + (1.04067)^2 + (2.3126)^2} = 2.7412 \text{ lb}\cdot\text{ft}\cdot\text{s}$$

(*b*) Angle formed by $\mathbf{H}_B$ and shaft $BA$.

Unit vector along shaft $BA$: $\boldsymbol{\lambda} = \mathbf{k}$

$$\cos\theta = \frac{\mathbf{H}_B\cdot\boldsymbol{\lambda}}{H_B} = \frac{2.3126}{2.7412} = 0.84365 \qquad \theta = 32.5° \blacktriangleleft$$

## PROBLEM 18.19



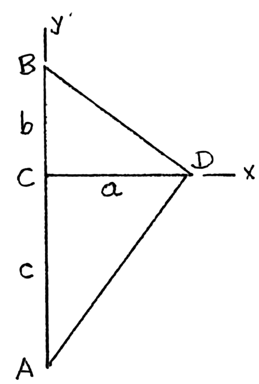

The triangular plate shown has a mass of 7.5 kg and is welded to a vertical shaft *AB*. Knowing that the plate rotates at the constant rate $\omega = 12$ rad/s, determine its angular momentum about (*a*) Point *C*, (*b*) Point *A*. (*Hint*: To solve part *b* find $\bar{\mathbf{v}}$ and use the property indicated in part *a* of Problem 18.14.)

## SOLUTION

$$\boldsymbol{\omega} = (12 \text{ rad/s})\mathbf{j}, \quad \omega_x = 0, \quad \omega_y = 12 \text{ rad/s}, \quad \omega_z = 0$$

(*a*) $$(H_C)_x = -I_{xy}\omega, \quad (H_C)_y = I_y\omega, \quad (H_C)_z = -I_{yz}\omega$$

Use axes with origin at *C* as shown. Divide the plate *ABD* into right triangles *ACD* and *CBD*.




For plate *ACD*, the product of inertia of the area is

$$(I_{xy})_{\text{area}} = -\frac{1}{24}a^2c^2$$

For plate *BCD*, it is

$$(I_{xy})_{\text{area}} = \frac{1}{24}a^2b^2$$

For both areas together,

$$(I_{xy})_{\text{area}} = -\frac{1}{24}(c^2 - b^2)a^2$$

Area: $$A = \frac{1}{2}(c+b)a$$

$$(I_{xy})_{\text{mass}} = \frac{m}{A}(I_{xy})_{\text{area}} = -\frac{m(c-b)a}{12}$$

For both areas together, $$(I_y)_{\text{area}} = \frac{1}{12}(c+b)a^3$$

$$(I_y)_{\text{mass}} = \frac{m}{A}(I_y)_{\text{area}} = \frac{1}{6}ma^2$$

$$(I_{xz})_{\text{mass}} \approx 0$$

## PROBLEM 18.19 (Continued)

Data: $m = 7.5 \text{ kg} \quad a = 120 \text{ mm} = 0.12 \text{ m}$

$$b = 90 \text{ mm} = 0.09 \text{ m} \quad c = 160 \text{ mm} = 0.16 \text{ m}$$

$$(I_{xy})_{\text{mass}} = -\frac{(7.5)(0.07)(0.12)}{12} = -0.00525 \text{ kg}\cdot\text{m}^2$$

$$(I_y)_{\text{mass}} = \frac{(7.5)(0.12)^2}{6} = 0.018 \text{ kg}\cdot\text{m}^2$$

$$(H_C)_x = -(-0.00525)(12) = 0.063 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_C)_y = (0.018)(12) = 0.216 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_C)_z = 0 \qquad \mathbf{H}_C = (0.063 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} + (0.216 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} \blacktriangleleft$$

Locate the mass center. $\mathbf{r}_{G/C} = \dfrac{a}{3}\mathbf{i} + \bar{y}\mathbf{j}$

Velocity of mass center: $\bar{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{r}_{G/C}$

$$\bar{\mathbf{v}} = \omega\mathbf{j} \times \left(\frac{a}{3}\mathbf{i} + \bar{y}\mathbf{j}\right) = -\frac{1}{3}\omega a\mathbf{k} = -\left(\frac{1}{3}\right)(12)(0.12)\mathbf{k} = -(0.48 \text{ m/s})\mathbf{k}$$

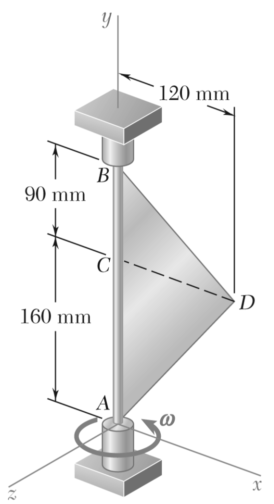
$$\mathbf{r}_{C/A} = c\mathbf{j} = (0.16 \text{ m})\mathbf{j}$$

$$\mathbf{r}_{C/A} \times m\bar{\mathbf{v}} = (0.16\mathbf{j}) \times [(7.5)(-0.48\mathbf{k})] = -(0.576 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i}$$

(*b*) $$\mathbf{H}_A = \mathbf{H}_C + \mathbf{r}_{C/A} \times m\bar{\mathbf{v}} = (0.063 - 0.576)\mathbf{i} + 0.216\mathbf{j}$$

$$\mathbf{H}_A = -(0.513 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} + (0.216 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} \blacktriangleleft$$

## PROBLEM 18.20




The triangular plate shown has a mass of 7.5 kg and is welded to a vertical shaft *AB*. Knowing that the plate rotates at the constant rate $\omega = 12$ rad/s, determine its angular momentum about (*a*) Point *C*, (*b*) Point *B*. (See hint of Problem 18.19.)

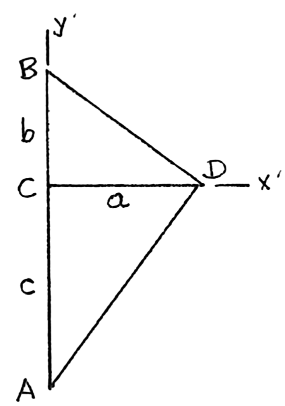
## SOLUTION

$$\boldsymbol{\omega} = (12 \text{ rad/s})\mathbf{j}, \quad \omega_x = 0, \quad \omega_y = 12 \text{ rad/s}, \quad \omega_z = 0$$

(*a*) $$(H_C)_x = -I_{xy}\omega, \quad (H_C)_y = I_y\omega, \quad (H_C)_z = -I_{yz}\omega$$

Use axes with origin at *C* as shown. Divide the plate *ABD* into right triangles *ACD* and *CBD*.

For plate *ACD*, the product of inertia of the area is




$$(I_{xy})_{\text{area}} = -\frac{1}{24}a^2c^2$$

For plate *BCD*, it is

$$(I_{xy})_{\text{area}} = \frac{1}{24}a^2b^2$$

For both areas together,

$$(I_{xy})_{\text{area}} = -\frac{1}{24}(c^2 - b^2)a^2$$

Area: $$A = \frac{1}{2}(c+b)a$$

$$(I_{xy})_{\text{mass}} = \frac{m}{A}(I_{xy})_{\text{area}} = -\frac{m(c-b)a}{12}$$

For both areas together,

$$(I_y)_{\text{area}} = \frac{1}{12}(c+b)a^3$$

$$(I_y)_{\text{mass}} = \frac{m}{A}(I_y)_{\text{area}} = \frac{1}{6}ma^2$$

$$(I_{xz})_{\text{mass}} \approx 0$$

Data: $m = 7.5 \text{ kg} \quad a = 120 \text{ mm} = 0.12 \text{ m}$

$$b = 90 \text{ mm} = 0.09 \text{ m} \quad c = 160 \text{ mm} = 0.16 \text{ m}$$

$$(I_{xy})_{\text{mass}} = -\frac{(7.5)(0.07)(0.12)}{12} = -0.00525 \text{ kg}\cdot\text{m}^2$$

$$(I_y)_{\text{mass}} = \frac{(7.5)(0.12)^2}{6} = 0.018 \text{ kg}\cdot\text{m}^2$$

$$(H_C)_x = -(-0.00525)(12) = 0.063 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_C)_y = (0.018)(12) = 0.216 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_C)_z = 0$$

$$\mathbf{H}_C = (0.063 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} + (0.216 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} \blacktriangleleft$$

Locate the mass center. $\mathbf{r}_{G/C} = \dfrac{a}{3}\mathbf{i} + \bar{y}\mathbf{j}$

Velocity of mass center: $\bar{\mathbf{v}} = \omega \times \mathbf{r}_{G/C}$

$$\bar{\mathbf{v}} = \omega\mathbf{j} \times \left(\frac{a}{3}\mathbf{i} + \bar{y}\mathbf{j}\right) = -\frac{1}{3}\omega a\mathbf{k} = -\left(\frac{1}{3}\right)(12)(0.12)\mathbf{k} = -(0.48 \text{ m/s})\mathbf{k}$$

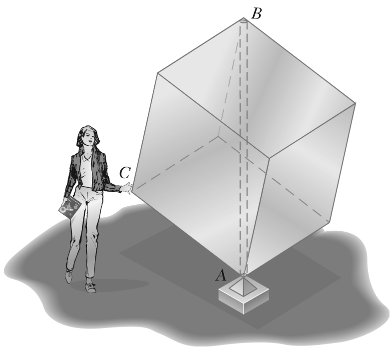
$$\mathbf{r}_{C/B} = -b\mathbf{j} = -(0.09 \text{ m})\mathbf{j}$$

$$\mathbf{r}_{C/B} \times m\bar{\mathbf{v}} = (-0.09\mathbf{j}) \times [(7.5)(-0.48\mathbf{k})] = (0.324 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i}$$

(*b*) $$\mathbf{H}_B = \mathbf{H}_C + \mathbf{r}_{C/B} \times m\bar{\mathbf{v}} = (0.063 + 0.324)\mathbf{i} + 0.216\mathbf{j}$$

$$\mathbf{H}_B = (0.387 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} + (0.216 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} \blacktriangleleft$$

## PROBLEM 18.21




One of the sculptures displayed on a university campus consists of a hollow cube made of six aluminum sheets, each $1.5 \times 1.5$ m, welded together and reinforced with internal braces of negligible weight. The cube is mounted on a fixed base at $A$ and can rotate freely about its vertical diagonal $AB$. As she passes by this display on the way to a class in mechanics, an engineering student grabs corner $C$ of the cube and pushes it for 1.2 s in a direction perpendicular to the plane $ABC$ with an average force of 50 N. Having observed that it takes 5 s for the cube to complete one full revolution, she flips out her calculator and proceeds to determine the mass of the cube. What is the result of her calculation? (*Hint:* The perpendicular distance from the diagonal joining two vertices of a cube to any of its other six vertices can be obtained by multiplying the side of the cube by $\sqrt{2/3}$.)

## SOLUTION

Let $m' = \frac{1}{6}m$ be the mass of one side of the cube. Choose $x$, $y$, and $z$ axes perpendicular to the face of the cube. Let $a$ be the side of the cube.

For a side perpendicular to the $x$ axis, $(I_x)_1 = \frac{1}{6}m'a^2$.

For a side perpendicular to the $y$ or $z$ axis, $$(I_x)_2 = \left(\frac{1}{12} + \frac{1}{4}\right)m'a^2 = \frac{1}{3}m'a^2$$

Total moment of inertia: $$I_x = 2(I_x)_1 + 4(I_x)_2 = \frac{5}{3}m'a^2 = \frac{5}{18}ma^2$$

By symmetry, $I_y = I_x$ and $I_z = I_x$.

Since all three moments of inertia are equal, the ellipsoid of inertia is a sphere. All centroidal axes are principal axes.

Moment of inertia about the vertical axis: $$I_v = \frac{5}{18}ma^2$$

Let $b = \sqrt{\frac{2}{3}}a$ be the moment arm of the impulse applied to the corner.

Using the impulse-momentum principle and taking moments about the vertical axis,

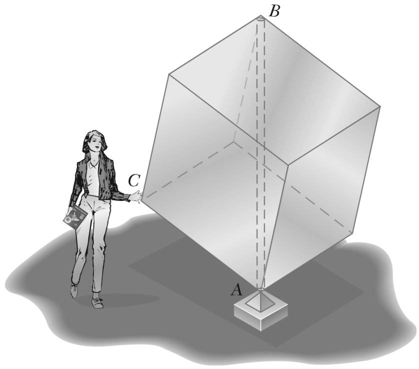
$$bF(\Delta t) = H_v = I_v\omega = \frac{5}{18}ma^2\omega \qquad (1)$$

Data: $$a = 1.5 \text{ m}, \qquad b = \sqrt{\frac{2}{3}}(1.5) = 1.22474 \text{ m}$$

$$\omega = \frac{2\pi}{5} = 1.25664 \text{ rad/s}, \qquad F = 50 \text{ N}, \qquad \Delta t = 1.2 \text{ s}.$$

Solving Equation (1) for $m$, $$m = \frac{18}{5}\frac{bF(\Delta t)}{a^2\omega} = \frac{18}{5}\frac{(1.22474)(50)(1.2)}{(1.5)^2(1.25664)} = 93.563 \text{ kg}$$

$m = 93.6$ kg ◀




## PROBLEM 18.22

If the aluminum cube of Problem 18.21 were replaced by a cube of the same size, made of six plywood sheets with mass 8 kg each, how long would it take for that cube to complete one full revolution if the student pushed its corner $C$ in the same way that she pushed the corner of the aluminum cube?

## SOLUTION

Let $m' = \frac{1}{6}m$ be the mass of one side of the cube. Choose $x$, $y$, and $z$ axes perpendicular to the face of the cube. Let $a$ be the side of the cube.

For a side perpendicular to the $x$ axis, $(I_x)_1 = \frac{1}{6}m'a^2$.

For a side perpendicular to the $y$ or $z$ axis, $$(I_x)_2 = \left(\frac{1}{12} + \frac{1}{4}\right)m'a^2 = \frac{1}{3}m'a^2$$

Total moment of inertia: $$I_x = 2(I_x)_1 + 4(I_x)_2 = \frac{5}{3}m'a^2 = \frac{5}{18}ma^2$$

By symmetry, $I_y = I_x$ and $I_z = I_x$.

Since all three moments of inertia are equal, the ellipsoid of inertia is a sphere. All centroidal axes are principal axes.

Moment of inertia about the vertical axis: $$I_v = \frac{5}{18}ma^2$$

Let $b = \sqrt{\frac{2}{3}}a$ be the moment arm of the impulse applied to the corner.

Using the impulse-momentum principle and taking moments about the vertical axis,

$$bF(\Delta t) = H_v = I_v\omega = \frac{5}{18}ma^2\omega \qquad (1)$$

Data: $$m' = 8 \text{ kg}, \qquad m = 6m' = 48 \text{ kg}, \qquad a = 1.5 \text{ m},$$

$$b = \sqrt{\frac{2}{3}}a = 1.22474 \text{ m}, \qquad F = 50 \text{ N}, \qquad \Delta t = 1.2 \text{ s}$$

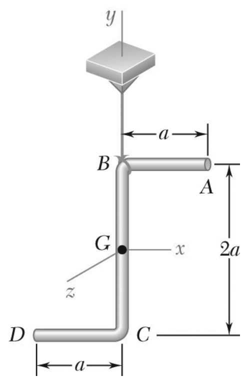
Solving (1) for $\omega$,

$$\omega = \frac{18}{5}\frac{bF(\Delta t)}{ma^2} = \frac{18}{5}\frac{(1.22474)(50)(1.2)}{(48)(1.5)^2} = 2.4495 \text{ rad/s}$$

$$t = \frac{2\pi}{\omega} = \frac{2\pi}{2.4495}$$

$t = 2.57$ s ◀

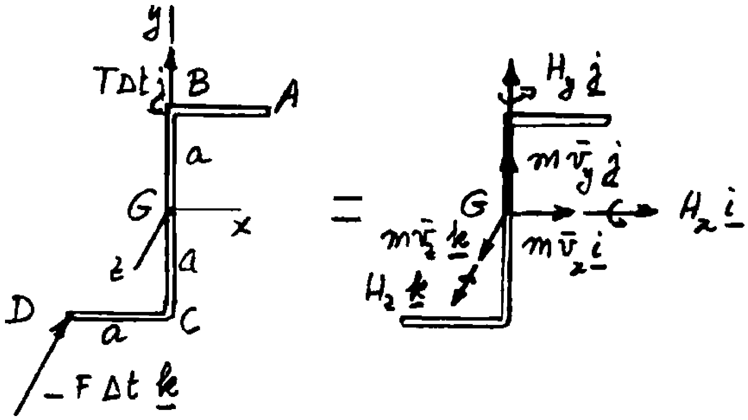
## PROBLEM 18.23




A uniform rod of total mass $m$ is bent into the shape shown and is suspended by a wire attached at $B$. The bent rod is hit at $D$ in a direction perpendicular to the plane containing the rod (in the negative $z$ direction). Denoting the corresponding impulse by $\mathbf{F}\Delta t$, determine (*a*) the velocity of the mass center of the rod, (*b*) the angular velocity of the rod.

## SOLUTION

We apply the principle of impulse and momentum, considering only the impulsive forces.




(*a*) *Velocity of mass center*

From constraints: $\bar{v}_y = 0$

$\xrightarrow{+}$ $x$ components: $0 = m\bar{v}_x$ $\qquad \bar{v}_x = 0$

$+\uparrow$ $y$ components: $T\Delta t = m\bar{v}_y = 0$ $\qquad T\Delta t = 0$

$+\swarrow$ $z$ components: $-F\Delta t = m\bar{v}_t$ $\qquad \bar{v}_z = -\dfrac{F\Delta t}{m}$

$$\bar{\mathbf{v}} = -\frac{F\Delta t}{m}\mathbf{k} \blacktriangleleft$$

(*b*) *Angular velocity*

Equating moments about $G$:

$$(-a\mathbf{i} - a\,\mathbf{j}) \times (-F\Delta t\,\mathbf{k}) = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

$$aF\Delta t\,\mathbf{i} - aF\Delta t\,\mathbf{j} = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

Thus: $$H_x = aF\Delta t, \quad H_y = -aF\Delta t, \quad H_z = 0 \qquad (1)$$

To determine angular velocity, we shall use Eqs. (18.7).

First, we determine the moments & products of inertia:

$$\bar{I}_x = \frac{1}{12}\left(\frac{m}{2}\right)(2a)^2 + \frac{m}{4}a^2 + \frac{m}{4}a^2 = \frac{2}{3}ma^2 \qquad (2)$$

$$\bar{I}_y = 2\left[\frac{1}{3}\left(\frac{m}{4}\right)a^2\right] = \frac{1}{6}ma^2 \qquad (3)$$

$$\bar{I}_{xy} = \frac{m}{4}\left(\frac{a}{2}\right)(a) + \frac{m}{4}\left(-\frac{a}{2}\right)(-a) = +\frac{1}{4}ma^2 \qquad (4)$$

$$\bar{I}_{xz} = 0 \qquad \bar{I}_{yz} = 0 \qquad (5)$$

We substitute the expressions (1) through (5) into Eqs. (18.7):

$$aF\Delta t = \frac{2}{3}ma^2\omega_x - \frac{1}{4}ma^2\omega_y + 0 \qquad (6)$$

$$-aF\Delta t = -\frac{1}{4}ma^2\omega_x + \frac{1}{6}ma^2\omega_y + 0 \qquad (7)$$

$$0 = 0 + 0 + \bar{I}_z\omega_z \qquad (8)$$

Multiplying Eq. (7) by 3/2 and adding to Eq. (6):

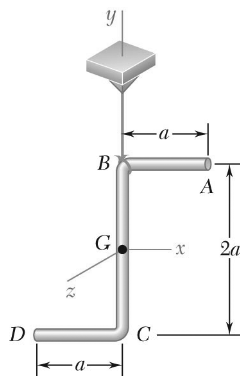
$$-\frac{1}{2}aF\Delta t = \frac{7}{24}ma^2\omega_x \qquad \omega_x = -\frac{12}{7}\frac{F\Delta t}{ma}$$

Substituting for $\omega$ into (7):

$$-aF\Delta t + \frac{1}{4}\left(-\frac{12^2}{7}\right)aF\Delta t = \frac{1}{6}ma^2\omega_y, \quad \omega_y = -\frac{60}{7}\frac{F\Delta t}{ma}$$

From Eq. (8): $\bar{I}_z\omega_z = 0 \quad \omega_z = 0$

Thus: $$\boldsymbol{\omega} = \frac{12}{7}\frac{F\Delta t}{ma}(-\mathbf{i} - 5\mathbf{j}) \blacktriangleleft$$



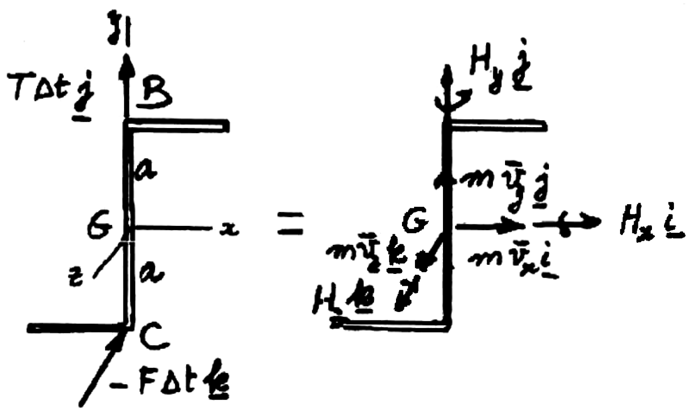

## PROBLEM 18.24

Solve Problem 18.23, assuming that the bent rod is hit at $C$.

**PROBLEM 18.23** A uniform rod of total mass $m$ is bent into the shape shown and is suspended by a wire attached at $B$. The bent rod is hit at $D$ in a direction perpendicular to the plane containing the rod (in the negative $z$ direction). Denoting the corresponding impulse by $\mathbf{F}\Delta t$, determine ($a$) the velocity of the mass center of the rod, ($b$) the angular velocity of the rod.

## SOLUTION

We apply the principle of impulse and momentum, consider only impulsive forces.




($a$) *Velocity of mass center*

From constraints: $\bar{v}_y = 0$

$\xrightarrow{+}$ $x$ components: $0 = m\bar{v}_x$ $\qquad \bar{v}_x = 0$

$+\uparrow$ $y$ components: $T\Delta t = m\bar{v}_y = 0$ $\qquad T\Delta t = 0$

$+\swarrow$ $z$ components: $-F\Delta t = m\bar{v}_z$ $\qquad \bar{v}_z = -\dfrac{F\Delta t}{m}$

$$\bar{\mathbf{v}} = \frac{F\Delta t}{m}\mathbf{k} \blacktriangleleft$$

($b$) *Angular velocity*

Equating moments about $G$:

$$-a\,\mathbf{j}\times(-F\Delta t\,\mathbf{k}) = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

$$aF\Delta t\,\mathbf{i} = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

Thus: $\qquad H_x = aF\Delta t, \quad H_y = 0, \quad H_z = 0 \qquad (1)$

To determine angular velocity, we shall use Eqs. (18.7) first, we determine the moments & products of inertia:

$$\bar{I}_x = \frac{1}{12}\left(\frac{m}{2}\right)(2a)^2 + \frac{m}{4}a^2 + \frac{m}{4}a^2 = \frac{2}{3}ma^2 \tag{2}$$

$$\bar{I}_y = 2\left[\frac{1}{3}\left(\frac{m}{4}\right)a^2\right] = \frac{1}{6}ma^2 \tag{3}$$

$$\bar{I}_{xy} = \frac{m}{4}\left(\frac{a}{2}\right)(a) + \frac{m}{4}\left(-\frac{a}{2}\right)(-a) = +\frac{1}{4}ma^2 \tag{4}$$

$$\bar{I}_{xz} = 0 \qquad \bar{I}_{yz} = 0 \tag{5}$$

$$aF\Delta t = \frac{2}{3}ma^2\omega_x - \frac{1}{4}ma^2\omega_y + 0 \tag{6}$$

$$0 = -\frac{1}{4}ma^2\omega_x + \frac{1}{6}ma^2\omega_y + 0 \tag{7}$$

$$0 = 0 + 0 + \bar{I}_z\omega_z \tag{8}$$

Multiplying Eq. (7) by 3/2 and adding to Eq. (6):

$$aF\Delta t = \left(\frac{2}{3} - \frac{3}{8}\right)ma^2\omega_{x,} \qquad \omega_x = \frac{24}{7}\frac{F\Delta t}{ma}$$

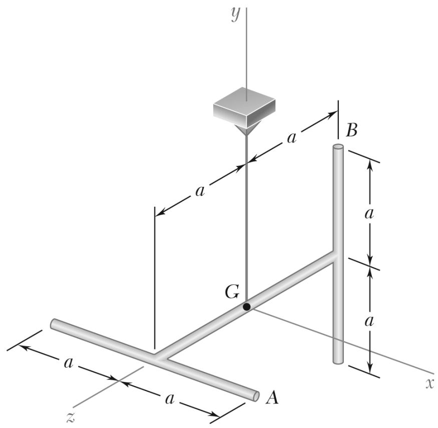
Substituting for $\omega_x$ into (7):

$$\omega_y = \frac{3}{2}\omega_x = \frac{3}{2}\frac{24}{7}aF\Delta t, \quad \omega_y = \frac{36}{7}\frac{F\Delta t}{ma}$$

From Eq. (8): $\bar{I}_z\omega_z = 0, \quad \omega_z = 0$

Thus: $\boldsymbol{\omega} = \frac{12}{7}\frac{F\Delta t}{ma}(2\mathbf{i} + 3\mathbf{j})$ ◀

Note that $\omega_y \neq 0$, even though Point $C$ where impulse is applied is on the $y$ axis.




## PROBLEM 18.25

Three slender rods, each of mass $m$ and length $2a$, are welded together to form the assembly shown. The assembly is hit at $A$ in a vertical downward direction. Denoting the corresponding impulse by $\mathbf{F}\,\Delta t$, determine immediately after the impact ($a$) the velocity of the mass center $G$, ($b$) the angular velocity of the rod.

## SOLUTION

Computation of moments and products of inertia.

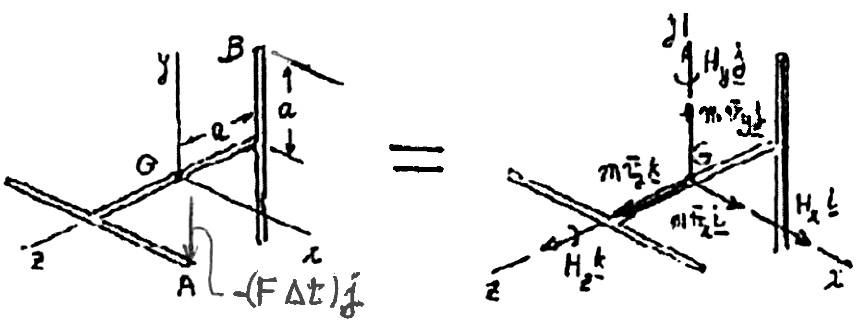
$$\bar{I}_x = (I_x)_1 + (I_x)_2 + (I_x)_3 = ma^2 + \frac{1}{3}ma^2 + m\left(a^2 + \frac{a^2}{3}\right) = \frac{8}{3}ma^2$$

$$\bar{I}_y = (I_y)_1 + (I_y)_2 + (I_y)_3 = m\left(a^2 + \frac{a^2}{3}\right) + \frac{1}{3}ma^2 + ma^2 = \frac{8}{3}ma^2 \qquad (1)$$

$$\bar{I}_z = (I_z)_1 + (I_z)_2 + (I_z)_3 = \frac{1}{3}ma^2 + 0 + \frac{1}{3}ma^2 = \frac{2}{3}ma^2$$

$$\bar{I}_{xy} = 0, \quad \bar{I}_{yz} = 0, \quad I_{zx} = 0$$

Impulse-momentum principle.




The impulses consist of $-(F\Delta t)\mathbf{j}$ applied at $A$ and $(T\Delta t)\mathbf{j}$ applied at $G$. Because of constraints, $\bar{v}_y = 0$.

($a$) Velocity of mass center.

Equate sums of vectors: $(T\Delta t)\mathbf{j} - (F\Delta t)\mathbf{j} = m\bar{v}_x\mathbf{i} + m\bar{v}_z\mathbf{k}$

Thus, $\quad T\Delta t = F\Delta t, \quad v_x = 0, \quad v_z = 0. \quad$ Since $\quad v_y = 0$ from above, $\qquad \bar{\mathbf{v}} = 0$ ◀

(*b*) Angular velocity.

Equate moments about $G$:

$$(a\mathbf{i} + a\mathbf{k}) \times (-F\Delta t)\mathbf{j} = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

$$-(aF\Delta t)\mathbf{k} + (aF\Delta t)\mathbf{i} = H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k}$$

Thus, $$H_x = aF\Delta t, \quad H_y = 0, \quad H_z = -aF\Delta t \tag{2}$$

Since the three products of inertia are zero, the $x$, $y$, and $z$ axes are principal centroidal axes and we can use Eqs. (18.10). Substituting from Eqs. (1) and (2) into these equations, we have

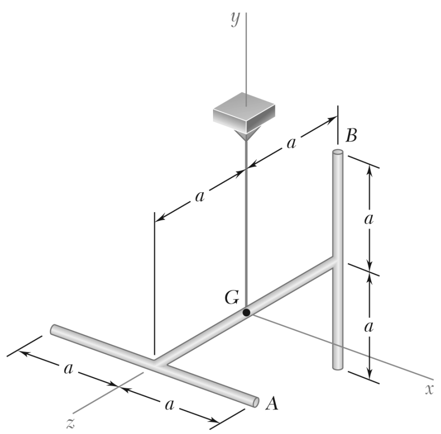
$$H_x = \bar{I}_x\omega_x: \quad aF\Delta t = \frac{8}{3}ma^2\omega_x \quad \omega_x = 3F\Delta t/8ma \tag{3}$$

$$H_y = \bar{I}_y\omega_y: \quad 0 = \frac{8}{3}ma^2\omega_y \quad \omega_y = 0 \tag{4}$$

$$H_z = \bar{I}_z\omega_z: \quad -aF\Delta t = \frac{2}{3}ma^2 \quad \omega_z = -3F\Delta t/2ma \tag{5}$$

Therefore,

$$\boldsymbol{\omega} = (3F\Delta t/8ma)(\mathbf{i} - 4\mathbf{k}) \blacktriangleleft$$




## PROBLEM 18.26

Solve Problem 18.25, assuming that the assembly is hit at $B$ in a direction opposite to that of the $x$ axis.

**PROBLEM 18.25** Three slender rods, each of mass $m$ and length $2a$, are welded together to form the assembly shown. The assembly is hit at $A$ in a vertical downward direction. Denoting the corresponding impulse by $\mathbf{F}\,\Delta t$, determine immediately after the impact (*a*) the velocity of the mass center $G$, (*b*) the angular velocity of the rod.

## SOLUTION

Computation of moments and products of inertia.

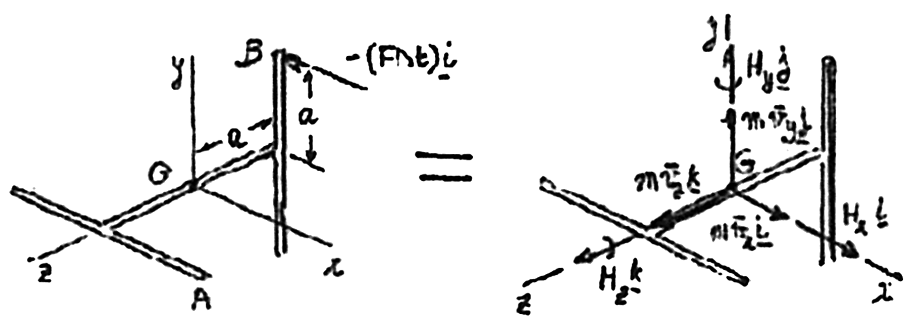
$$\bar{I}_x = (I_x)_1 + (I_x)_2 + (I_x)_3 = ma^2 + \frac{1}{3}ma^2 + m\left(a^2 + \frac{a^2}{3}\right) = \frac{8}{3}ma^2$$

$$\bar{I}_y = (I_y)_1 + (I_y)_2 + (I_y)_3 = m\left(a^2 + \frac{a^2}{3}\right) + \frac{1}{3}ma^2 + ma^2 = \frac{8}{3}ma^2 \qquad (1)$$

$$\bar{I}_z = (I_z)_1 + (I_z)_2 + (I_z)_3 = \frac{1}{3}ma^2 + 0 + \frac{1}{3}ma^2 = \frac{2}{3}ma^2$$

$$\bar{I}_{xy} = \bar{I}_{yz} = \bar{I}_{zx} = 0$$

Impulse-momentum principle.




The only impulse is $F\Delta t = -(F\Delta t)\mathbf{i}$.

(*a*) Velocity of mass center.

Equate sums of vectors: $\quad -(F\Delta t)\mathbf{i} = m\bar{v}_x\mathbf{i} + mv_y\mathbf{j} + mv_z\mathbf{k}$

Thus, $\quad v_x = -F\Delta t/m, \quad v_y = 0, \quad v_z = 0$

$$\bar{\mathbf{v}} = -(F\Delta t/m)\mathbf{i} \blacktriangleleft$$

## PROBLEM 18.26 (Continued)

(*b*) Angular velocity.

Equate moments about $G$:

$$(a\mathbf{j}-a\mathbf{k})\times(-F\Delta t)\mathbf{i}=H_x\mathbf{i}+H_y\mathbf{j}+H_z\mathbf{k}$$

$$(aF\Delta t)\mathbf{k}+(aF\Delta t)\mathbf{j}=H_x\mathbf{i}+H_y\mathbf{j}+H_z\mathbf{k}$$

Thus, $$H_x=0,\quad H_y=aF\Delta t,\quad H_z=aF\Delta t \qquad (2)$$

Since the three products of inertia are zero, the $x$, $y$, and $z$ axes are principal centroidal axes and we can use Eqs. (18.10). Substituting from Eqs. (1) and (2) into these equations, we have

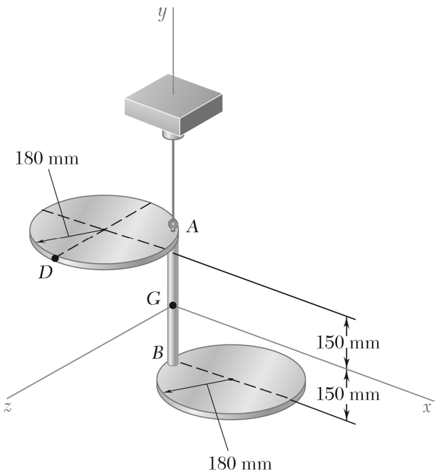
$$H_x=\bar{I}_x\omega_x:\qquad 0=\frac{8}{3}ma^2\omega_x\quad \omega_x=0 \qquad (3)$$

$$H_y=\bar{I}_y\omega_y:\qquad aF\Delta t=\frac{8}{3}ma^2\omega_y\quad \omega_y=3F\Delta t/8ma \qquad (4)$$

$$H_z=\bar{I}_z\omega_z:\qquad aF\Delta t=\frac{2}{3}ma^2\omega_z\quad \omega_z=3F\Delta t/2ma \qquad (5)$$

Therefore,

$$\boldsymbol{\omega}=(3F\Delta t/8ma)(\mathbf{j}+4\mathbf{k})\ \blacktriangleleft$$




## PROBLEM 18.27

Two circular plates, each of mass 4 kg, are rigidly connected by a rod $AB$ of negligible mass and are suspended from Point $A$ as shown. Knowing that an impulse $\mathbf{F}\,\Delta t = -(2.4\ \text{N}\cdot\text{s})\mathbf{k}$ is applied at Point $D$, determine ($a$) the velocity of the mass center $G$ of the assembly, ($b$) the angular velocity of the assembly.

## SOLUTION

Moments and products of inertia:

$$I_x = 2\left(\frac{1}{4}mr^2 + mb^2\right) = 2\left[\frac{1}{4}(4)(0.18)^2 + (4)(0.15)^2\right] = 0.2448\ \text{kg}\cdot\text{m}^2$$

$$I_y = 2\left(\frac{1}{2}mr^2 + mr^2\right) = 3mr^2 = (3)(4)(0.18)^2 = 0.3888\ \text{kg}\cdot\text{m}^2$$

$$I_z = 2\left[\frac{1}{4}mr^2 + m(b^2 + r^2)\right]$$

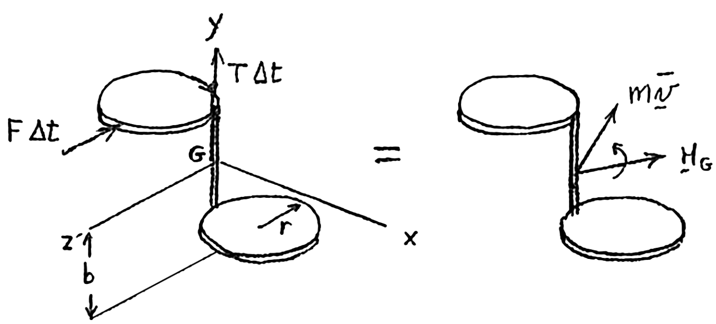
$$= 2\left[\frac{1}{4}(4)(0.18)^2 + (4)(0.15^2 + 0.18^2)\right] = 0.504\ \text{kg}\cdot\text{m}^2$$

$$I_{xy} = mb(-r) + m(-b)(r) = -2mbr = -(2)(4)(0.15)(0.18) = -0.216\ \text{kg}\cdot\text{m}^2$$

$$I_{xz} = 0, \quad I_{yz} = 0, \qquad \text{total mass} = 2m$$

Constraint of cable: $(v_G)_y = 0 \quad \bar{\mathbf{v}} = \mathbf{v}_G$

Principle of impulse-momentum: $2m\bar{\mathbf{v}} = 0,\ \ \mathbf{H}_G = 0$ initially.



## PROBLEM 18.27 (Continued)

(*a*) Direct components:

$$\mathbf{F}\Delta t = 2m\overline{\mathbf{v}}$$

$$0 = 2m\overline{v}_x \qquad \overline{v}_x = 0$$

$$T\Delta t = 2m\overline{v}_y = 0$$

$$-F\Delta t = 2m\overline{v}_z$$

$$\overline{v}_z = -\frac{F\Delta t}{2m} = -\frac{2.4}{(2)(4)} = -0.3 \text{ m/s} \qquad \overline{\mathbf{v}} = -(0.300 \text{ m/s})\mathbf{k} \blacktriangleleft$$

(*b*) Moments about *G*. $(b\mathbf{j} - r\mathbf{i}) \times (-F\Delta t\mathbf{k}) = \mathbf{H}_G$

$$-bF(\Delta t)\mathbf{i} - rF\Delta t\mathbf{j} = (I_x\omega_x - I_{xy}\omega_y)\mathbf{i} + (-I_{xy}\omega_x + I_y\omega_y)\mathbf{j} + I_z\omega_z\mathbf{k}$$

Resolve into components and apply the numerical data.

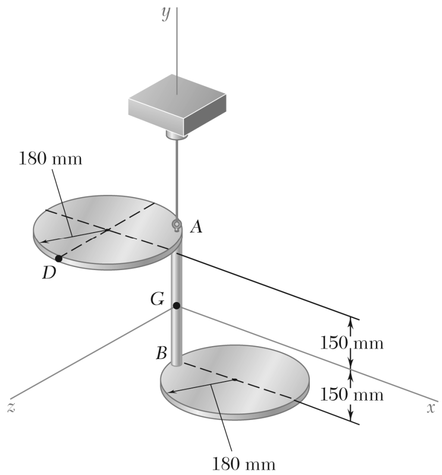
$$\mathbf{i}: \quad -(0.15)(2.4) = 0.2448\omega_x - (-0.216)\omega_y \qquad (1)$$

$$\mathbf{j}: \quad -(0.18)(2.4) = -(-0.216)\omega_x + 0.3888\omega_y \qquad (2)$$

$$\mathbf{k}: \quad 0 = 0.504\omega_z \qquad \omega_z = 0$$

Solving Eqs. (1) and (2), $\omega_x = -0.962$ rad/s, $\omega_y = -0.577$ rad/s

$$\boldsymbol{\omega} = -(0.962 \text{ rad/s})\mathbf{i} - (0.577 \text{ rad/s})\mathbf{j} \blacktriangleleft$$




## PROBLEM 18.28

Two circular plates, each of mass 4 kg, are rigidly connected by a rod $AB$ of negligible mass and are suspended from Point $A$ as shown. Knowing that an impulse $\mathbf{F}\Delta t = (2.4\ \text{N}\cdot\text{s})\mathbf{j}$ is applied at Point $D$, determine ($a$) the velocity of the mass center $G$ of the assembly, ($b$) the angular velocity of the assembly.

## SOLUTION

Moments and products of inertia:

$$I_x = 2\left(\frac{1}{4}mr^2 + mb^2\right) = 2\left[\frac{1}{4}(4)(0.18)^2 + (4)(0.15)^2\right] = 0.2448\ \text{kg}\cdot\text{m}^2$$

$$I_y = 2\left(\frac{1}{2}mr^2 + mr^2\right) = 3mr^2 = (3)(4)(0.18)^2 = 0.3888\ \text{kg}\cdot\text{m}^2$$

$$I_z = 2\left[\frac{1}{4}mr^2 + m(b^2 + r^2)\right]$$

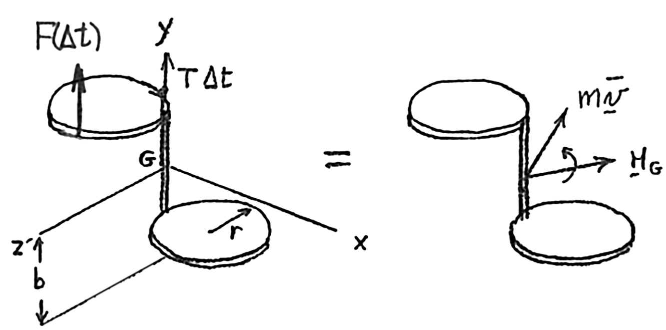
$$= 2\left[\frac{1}{4}(4)(0.18)^2 + (4)(0.15^2 + 0.18^2)\right] = 0.504\ \text{kg}\cdot\text{m}^2$$

$$I_{xy} = mb(-r) + m(-b)(r) = -2mbr = -(2)(4)(0.15)(0.18) = -0.216\ \text{kg}\cdot\text{m}^2$$

$$I_{xz} = 0, \quad I_{yz} = 0, \qquad \text{total mass} = 2m$$

Constraint of cable: $(v_G)_y = 0 \quad \overline{\mathbf{v}} = \mathbf{v}_G$

Principle of impulse-momentum: $2m\overline{\mathbf{v}} = 0, \quad \mathbf{H}_G = 0$ initially.



(*a*) Direct components:

$$\mathbf{F}\Delta t = 2m\overline{\mathbf{v}}$$

$$0 = 2m\overline{v}_x \qquad \overline{v}_x = 0$$

$$F(\Delta t) + T\Delta t = 2m\overline{v}_y = 0 \qquad \overline{v}_y = \frac{F(\Delta t) + T(\Delta t)}{2m}$$

$$0 = 2m\overline{v}_z \quad v_z = 0$$

Point $A$ moves upward. The cord becomes slack. $T(\Delta t) = 0$

$$v_y = \frac{2.4}{(2)(4)} = 0.300 \text{ m/s} \qquad \overline{\mathbf{v}} = (0.300 \text{ m/s})\mathbf{j} \blacktriangleleft$$

(*b*) <u>Moments about $G$</u>. $(b\mathbf{j} - r\mathbf{i}) \times (-F\Delta t\mathbf{j}) = \mathbf{H}_G$

$$-rF(\Delta t)\mathbf{i} - rF(\Delta t)\mathbf{k} = (I_x\omega_x - I_{xy}\omega_y)\mathbf{i} + (-I_{xy}\omega_x + I_y\omega_y)\mathbf{j} + I_z\omega_z\mathbf{k}$$

Resolve into components and apply the numerical data.

$$\mathbf{i}: \quad -(0.18)(2.4) = 0.2448\omega_x - (-0.216)\omega_y \qquad (1)$$

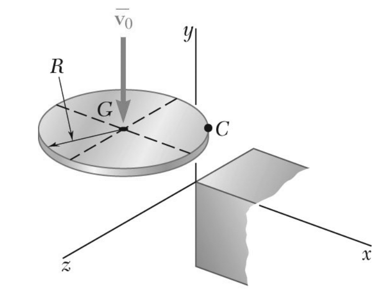
$$\mathbf{j}: \quad 0 = -(-0.216)\omega_x + 0.3888\omega_y \qquad (2)$$

$$\mathbf{k}: \quad -(0.18)(2.4) = 0.504\omega_z \qquad \omega_z = -0.857 \text{ rad/s}$$

Solving Eqs. (1) and (2), $\omega_x = -3.46$ rad/s, $\omega_y = -1.923$ rad/s

$$\boldsymbol{\omega} = -(3.46 \text{ rad/s})\mathbf{i} + (1.923 \text{ rad/s})\mathbf{j} - (0.857 \text{ rad/s})\mathbf{k} \blacktriangleleft$$

## PROBLEM 18.29

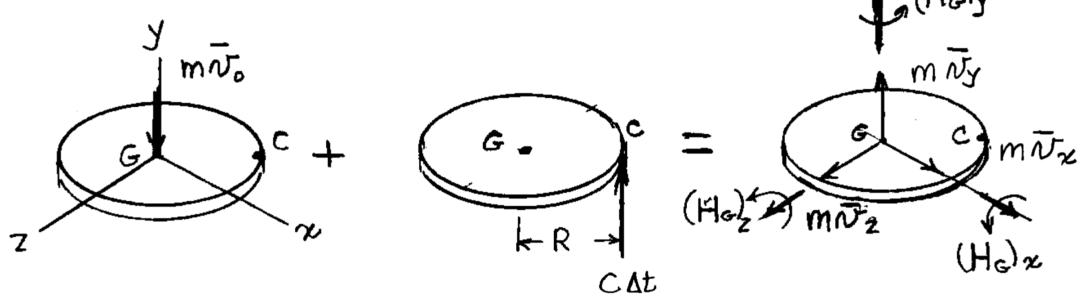



A circular plate of mass $m$ is falling with a velocity $\bar{\mathbf{v}}_0$ and no angular velocity when its edge $C$ strikes an obstruction. Assuming the impact to be perfectly plastic $(e = 0)$, determine the angular velocity of the plate immediately after the impact.

## SOLUTION

Principal moments of inertia. $\bar{I}_y = \frac{1}{2}mR^2, \quad \bar{I}_x = \bar{I}_z = \frac{1}{4}mR^2$

Principle of impulse and momentum.




**Syst. Momenta₁** + **Syst. Ext. Imp.₁→₂** = **Syst. Momenta₂**

Linear momentum: $-mv_0\mathbf{j} + C\Delta t\,\mathbf{j} = m\bar{v}_x\mathbf{i} + m\bar{v}_y\mathbf{j} + m\bar{v}_z\mathbf{k}$

**i**: $0 = m\bar{v}_x \qquad \bar{v}_x = 0$

**j**: $-m\bar{\mathbf{v}}_0 + C\Delta t = m\bar{v}_y \qquad C\Delta t = m(\bar{v}_0 + \bar{v}_y)$ (1)

**k**: $0 = m\bar{v}_z \qquad \bar{v}_z = 0$

Geometry: $\mathbf{r}_{C/G} = \frac{1}{\sqrt{2}}R(\mathbf{i} - \mathbf{k})$

Condition of impact: $(e = 0) \qquad (v_C)_y = 0$

Kinematics: $\bar{\mathbf{v}}_C = \bar{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{r}_{C/G}$

$$(v_C)_x\mathbf{i} + (v_C)_z\mathbf{k} = \bar{v}_y\mathbf{j} + (\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \times \frac{R}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$$

$$= \bar{v}_y\mathbf{j} + \frac{R}{\sqrt{2}}\omega_x\mathbf{j} - \frac{R}{\sqrt{2}}\omega_y(\mathbf{k} + \mathbf{i}) + \frac{R}{\sqrt{2}}\omega_z\mathbf{j}$$

**j**: $0 = \bar{v}_y + \frac{R}{\sqrt{2}}(\omega_x + \omega_z) \qquad \bar{v}_y = -\frac{R}{\sqrt{2}}(\omega_x + \omega_z)$

## PROBLEM 18.29 (Continued)

Moments about $G$:

$$0 + \mathbf{r}_{C/G} \times C\Delta t\,\mathbf{j} = (H_G)_x\mathbf{i} + (H_G)_y\mathbf{j} + (H_G)_z\mathbf{k}$$

$$\frac{R}{\sqrt{2}}(\mathbf{i} - \mathbf{k}) \times (C\Delta t\,\mathbf{j}) = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$\frac{R}{\sqrt{2}}(C\Delta t)(\mathbf{k} + \mathbf{i}) = \frac{1}{4}mR^2\omega_x\mathbf{i} + \frac{1}{2}mR^2\omega_y\mathbf{j} + \frac{1}{4}mR^2\omega_z\mathbf{k} \qquad (2)$$

$\mathbf{i}$: $$\frac{R}{\sqrt{2}}C\Delta t = \frac{1}{4}mR^2\omega_x \qquad (3)$$

$\mathbf{j}$: $$0 = \frac{1}{2}mR^2\omega_y \qquad \omega_y = 0$$

$\mathbf{k}$: $$\frac{R}{\sqrt{2}}C\Delta t = \frac{1}{4}mR^2\omega_z \qquad (4)$$

From Eqs. (1) and (2), $$C\Delta t = m\left[\bar{v}_0 - \frac{R}{\sqrt{2}}(\omega_x + \omega_z)\right]$$

$$\omega_x = 2\sqrt{2}\frac{\bar{v}_0}{R} - 2(\omega_x + \omega_z) \qquad 3\omega_x + 2\omega_z = 2\sqrt{2}\frac{\bar{v}_0}{R} \qquad (5)$$

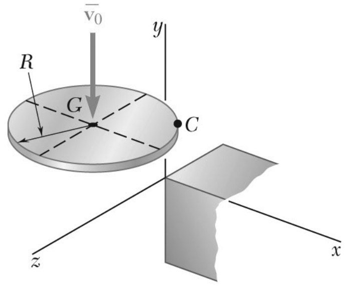
$$\omega_z = 2\sqrt{2}\frac{\bar{v}_0}{R} - 2(\omega_x + \omega_z) \qquad 2\omega_x + 3\omega_z = 2\sqrt{2}\frac{\bar{v}_0}{R} \qquad (6)$$

Solving Eqs. (5) and (6) simultaneously,

$$\omega_x = \omega_z = \frac{2\sqrt{2}}{5}\frac{\bar{v}_0}{R}$$

*Angular velocity.* $$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

$$\boldsymbol{\omega} = \frac{2\sqrt{2}}{5}\frac{v_0}{R}(\mathbf{i} + \mathbf{k}) \blacktriangleleft$$



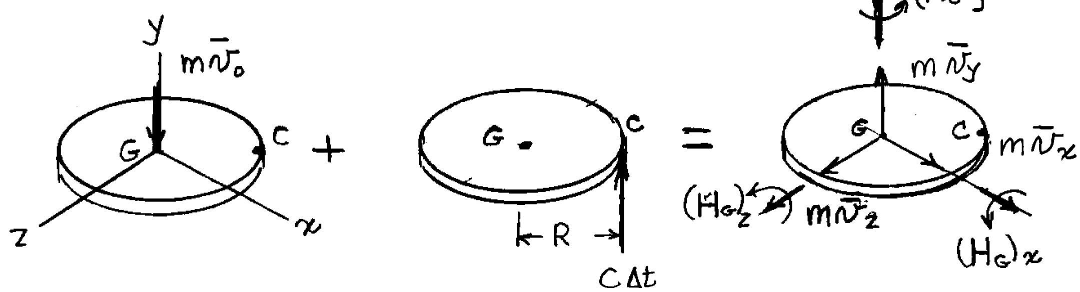

## PROBLEM 18.30

For the plate of Problem 18.29, determine (*a*) the velocity of its mass center *G* immediately after the impact, (*b*) the impulse exerted on the plate by the obstruction during the impact.

## SOLUTION

Principal moments of inertia. $\bar{I}_y = \frac{1}{2}mR^2, \quad \bar{I}_x = \bar{I}_z = \frac{1}{4}mR^2$

Principle of impulse and momentum.




**Syst. Momenta$_1$** + **Syst. Ext. Imp.$_{1\to2}$** = **Syst. Momenta$_2$**

Linear momentum: $-mv_0\mathbf{j} + C\Delta t\,\mathbf{j} = m\bar{v}_x\mathbf{i} + m\bar{v}_y\mathbf{j} + m\bar{v}_z\mathbf{k}$

$\mathbf{i}$: $0 = m\bar{v}_x \qquad \bar{v}_x = 0$

$\mathbf{j}$: $-m\bar{v}_0 + C\Delta t = m\bar{v}_y \qquad C\Delta t = m(\bar{v}_0 + \bar{v}_y)$ (1)

$\mathbf{k}$: $0 = m\bar{v}_z \qquad \bar{v}_z = 0$

Geometry: $\mathbf{r}_{C/G} = \frac{1}{\sqrt{2}}R(\mathbf{i} - \mathbf{k})$

Condition of impact: $(e = 0) \qquad (v_C)_y = 0$

Kinematics: $\bar{\mathbf{v}}_C = \bar{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{r}_{C/G}$

$$(v_C)_x\mathbf{i} + (v_C)_z\mathbf{k} = \bar{v}_y\mathbf{j} + (\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \times \frac{R}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$$

$$= \bar{v}_y\mathbf{j} + \frac{R}{\sqrt{2}}\omega_x\mathbf{j} - \frac{R}{\sqrt{2}}\omega_y(\mathbf{k} + \mathbf{i}) + \frac{R}{\sqrt{2}}\omega_z\mathbf{j}$$

$\mathbf{j}$: $0 = \bar{v}_y + \frac{R}{\sqrt{2}}(\omega_x + \omega_z) \qquad \bar{v}_y = -\frac{R}{\sqrt{2}}(\omega_x + \omega_z)$

Moments about $G$:

$$0+\mathbf{r}_{C/G}\times C\Delta t\,\mathbf{j}=(H_G)_x\mathbf{i}+(H_G)_y\mathbf{j}+(H_G)_z\mathbf{k}$$

$$\frac{R}{\sqrt{2}}(\mathbf{i}-\mathbf{k})\times(C\Delta t\,\mathbf{j})=\bar{I}_x\omega_x\mathbf{i}+\bar{I}_y\omega_y\mathbf{j}+I_z\omega_z\mathbf{k}$$

$$\frac{R}{\sqrt{2}}(C\Delta t)(\mathbf{k}+\mathbf{i})=\frac{1}{4}mR^2\omega_x\mathbf{i}+\frac{1}{2}mR^2\omega_y\mathbf{j}+\frac{1}{4}mR^2\omega_z\mathbf{k} \qquad (2)$$

$\mathbf{i}$: $$\frac{R}{\sqrt{2}}C\Delta t=\frac{1}{4}mR^2\omega_x \qquad (3)$$

$\mathbf{j}$: $$0=\frac{1}{2}mR^2\omega_y \qquad \omega_y=0$$

$\mathbf{k}$: $$\frac{R}{\sqrt{2}}C\Delta t=\frac{1}{4}mR^2\omega_z \qquad (4)$$

From Eqs. (1) and (2), $$C\Delta t=m\left[\bar{v}_0-\frac{R}{\sqrt{2}}(\omega_x+\omega_z)\right]$$

$$\omega_x=2\sqrt{2}\frac{\bar{v}_0}{R}-2(\omega_x+\omega_z) \qquad 3\omega_x+2\omega_z=2\sqrt{2}\frac{\bar{v}_0}{R} \qquad (5)$$

$$\omega_z=2\sqrt{2}\frac{\bar{v}_0}{R}-2(\omega_x+\omega_z) \qquad 2\omega_x+3\omega_z=2\sqrt{2}\frac{\bar{v}_0}{R} \qquad (6)$$

Solving Eqs. (5) and (6) simultaneously,

$$\omega_x=\omega_z=\frac{2\sqrt{2}}{5}\frac{\bar{v}_0}{R}$$

(*a*) *Velocity of the mass center.*

From Eq. (2), $$\bar{v}_y=-\frac{R}{\sqrt{2}}(\omega_x+\omega_z)=-\frac{4}{5}\bar{v}_0$$

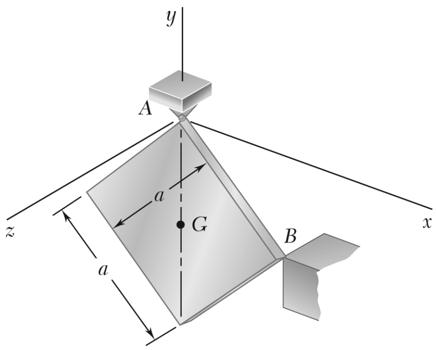
$$\bar{\mathbf{v}}=\bar{v}_x\mathbf{i}+\bar{v}_y\mathbf{j}+\bar{v}_z\mathbf{k}$$

$$\bar{\mathbf{v}}=-\frac{4}{5}\bar{v}_0\mathbf{j} \blacktriangleleft$$

(*b*) *Impulse at C.*

From Eq. (1), $$C\Delta t=m\left(\bar{v}_0-\frac{4}{5}\bar{v}_0\right)=\frac{1}{5}m\bar{v}_0$$

$$\mathbf{C}\Delta t=\frac{1}{5}m\bar{v}_0\mathbf{j} \blacktriangleleft$$

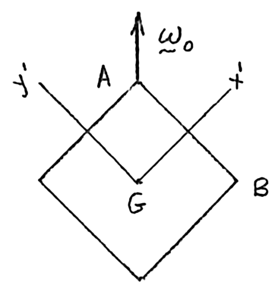



## PROBLEM 18.31

A square plate of side $a$ and mass $m$ supported by a ball-and-socket joint at $A$ is rotating about the $y$ axis with a constant angular velocity $\boldsymbol{\omega} = \omega_0 \mathbf{j}$ when an obstruction is suddenly introduced at $B$ in the $xy$ plane. Assuming the impact at $B$ to be perfectly plastic $(e = 0)$, determine immediately after impact ($a$) the angular velocity of the plate, ($b$) the velocity of its mass center $G$.

## SOLUTION

For the $x'$ and $y'$ axes shown, the initial angular velocity $\omega_0 \mathbf{j}$ has components




$$\omega_{x'} = \frac{\sqrt{2}}{2}\omega_0, \qquad \omega_{y'} = \frac{\sqrt{2}}{2}\omega_0,$$

Initial angular momentum about the mass center:

$$(\mathbf{H}_G)_0 = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}' = \frac{1}{12}ma^2 \frac{\sqrt{2}}{2}\omega_0(\mathbf{i}' + \mathbf{j}')$$

Initial velocity of the mass center: $\qquad \bar{\mathbf{v}}_0 = 0$

Let $\boldsymbol{\omega}$ be the angular velocity and $\bar{\mathbf{v}}$ be the velocity of the mass center immediately after impact.
Let $(F\Delta t)\mathbf{k}$ be the impulse at $B$.

Kinematics: $\qquad \mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A} = (\omega_{x'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times (-a\mathbf{j}')$

$$\mathbf{v}_B = a(\omega_{z'}\mathbf{i}' + \omega_{x'}\mathbf{k}')$$

Since the corner $B$ does not rebound, $\quad (v_B)_{z'} = 0 \quad$ or $\quad \omega_{x'} = 0$

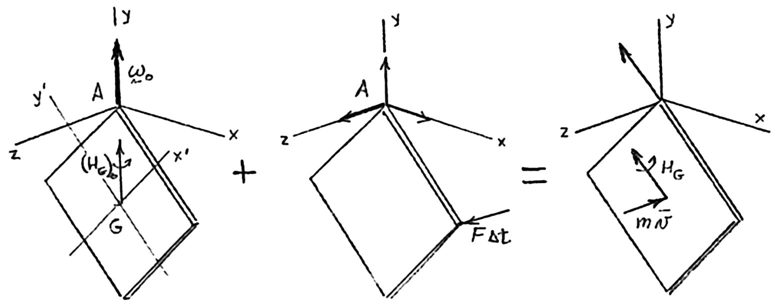
$$\bar{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{r}_{G/A} = (\omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times \left(\frac{1}{2}a\right)(-\mathbf{i}' - \mathbf{j}')$$

$$= \frac{1}{2}a(\omega_{z'}\mathbf{i}' - \omega_{z'}\mathbf{j} + \omega_{y}\mathbf{k}')$$

Also, $\qquad \mathbf{r}_{G/A} \times m\bar{\mathbf{v}} = \frac{1}{4}ma^2(-\omega_{y'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + 2\omega_{z'}\mathbf{k}')$

and $\qquad \mathbf{H}_G = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + \bar{I}_{z'}\omega_{z'}\mathbf{k}' = \frac{1}{12}ma^2\omega_{y'}\mathbf{j}' + \frac{1}{6}ma^2\omega_{z'}\mathbf{k}'$

Principle of impulse-momentum.




Moments about $A$:
$$(\mathbf{H}_A)_0 + (-a\mathbf{j}) \times (F\Delta t)\mathbf{k} = \mathbf{H}_A$$

$$(\mathbf{H}_G)_0 + \mathbf{r}_{G/A} \times m\overline{\mathbf{v}}_0 - (aF\Delta t)\mathbf{i} = \mathbf{H}_G + \mathbf{r}_{G/A} \times m\overline{\mathbf{v}}$$

Resolve into components.

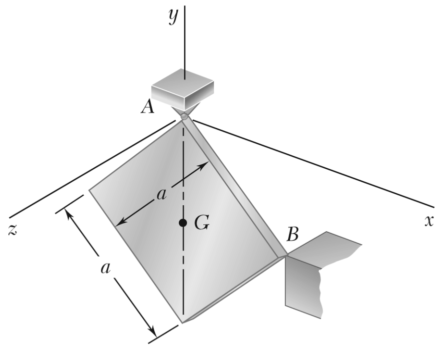
$$\mathbf{i}': \quad \frac{1}{24}\sqrt{2}ma^2\omega_0 - aF(\Delta t) = -\frac{1}{4}ma^2\omega_{y'}$$

$$\mathbf{j}': \quad \frac{1}{24}\sqrt{2}ma^2\omega_0 = \frac{1}{12}ma^2\omega_{y'} + \frac{1}{4}ma^2\omega_{y'} \qquad \omega_{y'} = \frac{\sqrt{2}}{8}\omega_0$$

$$\mathbf{k}': \quad 0 = \frac{1}{6}ma^2\omega_{z'} + \frac{1}{2}ma^2\omega_{z'} \qquad \omega_{z'} = 0$$

(*a*) $$\boldsymbol{\omega} = \frac{\sqrt{2}}{8}\omega_0\mathbf{j}' = \frac{1}{8}\sqrt{2}\omega_0\frac{\sqrt{2}}{2}(\mathbf{j} - \mathbf{i}) \qquad \boldsymbol{\omega} = \frac{1}{8}\omega_0(-\mathbf{i} + \mathbf{j}) \blacktriangleleft$$

(*b*) $$\overline{\mathbf{v}} = \frac{1}{2}a\omega_y\mathbf{k}' = \frac{\sqrt{2}}{16}a\omega_0\mathbf{k} \qquad \overline{\mathbf{v}} = 0.0884a\omega_0\mathbf{k} \blacktriangleleft$$

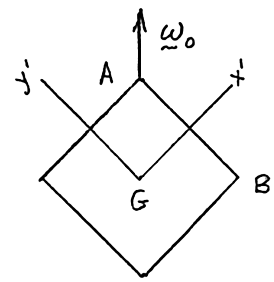



## PROBLEM 18.32

Determine the impulse exerted on the plate of Problem 18.31 during the impact by (*a*) the obstruction at *B*, (*b*) the support at *A*.

**PROBLEM 18.31** A square plate of side $a$ and mass $m$ supported by a ball-and-socket joint at $A$ is rotating about the $y$ axis with a constant angular velocity $\boldsymbol{\omega} = \omega_0 \mathbf{j}$ when an obstruction is suddenly introduced at $B$ in the $xy$ plane. Assuming the impact at $B$ to be perfectly plastic $(e = 0)$, determine immediately after impact (*a*) the angular velocity of the plate, (*b*) the velocity of its mass center $G$.

## SOLUTION




For the simpler $x'$ and $y'$ axes, the initial angular velocity $\omega_0 \mathbf{j}$ has components

$$\omega_{x'} = \frac{\sqrt{2}}{2}\omega_0, \qquad \omega_{y'} = \frac{\sqrt{2}}{2}\omega_0,$$

Initial angular momentum about the mass center:

$$(\mathbf{H}_G)_0 = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}' = \frac{1}{12}ma^2\frac{\sqrt{2}}{2}\omega_0(\mathbf{i}' + \mathbf{j}')$$

Initial velocity of the mass center: $\quad \bar{\mathbf{v}}_0 = 0$

Let $\boldsymbol{\omega}$ be the angular velocity and $\bar{\mathbf{v}}$ be the velocity of the mass center immediately after impact.
Let $(F\Delta t)\mathbf{k}$ be the impulse at $B$.

Kinematics:

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A} = (\omega_{x'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times (-a\mathbf{j}')$$

$$\mathbf{v}_B = a(\omega_{z'}\mathbf{i}' + \omega_{x'}\mathbf{k}')$$

Since the corner $B$ does not rebound, $\quad (v_B)_{z'} = 0 \quad$ or $\quad \omega_{x'} = 0$

$$\bar{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{r}_{G/A} = (\omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times \left(\frac{1}{2}a\right)(-\mathbf{i}' - \mathbf{j}')$$

$$= \frac{1}{2}a(\omega_{z'}\mathbf{i}' - \omega_{z'}\mathbf{j} + \omega_{y}\mathbf{k}')$$

Also,

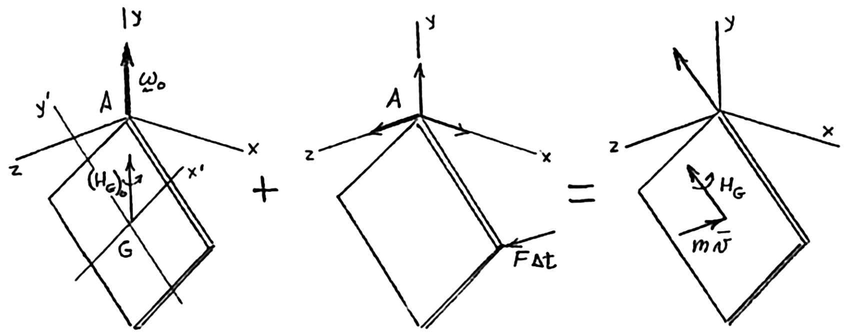
$$\mathbf{r}_{G/A} \times m\bar{\mathbf{v}} = \frac{1}{4}ma^2(-\omega_{y'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + 2\omega_{z'}\mathbf{k}')$$

and

$$\mathbf{H}_G = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + \bar{I}_{z'}\omega_{z'}\mathbf{k}'$$

$$= \frac{1}{12}ma^2\omega_{y'}\mathbf{j}' + \frac{1}{6}ma^2\omega_{z'}\mathbf{k}'$$

<u>Principle of impulse-momentum</u>.




Moments about $A$: $$(\mathbf{H}_A)_0 + (-a\mathbf{j}') \times (F\Delta t)\mathbf{k} = \mathbf{H}_A$$

$$(\mathbf{H}_G)_0 + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}}_0 + aF\Delta t\mathbf{i}' = \mathbf{H}_G + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}}$$

Resolve into components.

$$\mathbf{i}': \quad \frac{1}{24}\sqrt{2}ma^2\omega_0 - aF(\Delta t) = -\frac{1}{4}ma^2\omega_{y'} \tag{1}$$

$$\mathbf{j}': \quad \frac{1}{24}\sqrt{2}ma^2\omega_0 = \frac{1}{12}ma^2\omega_{y'} + \frac{1}{4}ma^2\omega_{y'} \qquad \omega_{y'} = \frac{1}{8}\sqrt{2}\omega_0$$

$$\mathbf{k}': \quad 0 = \frac{1}{6}ma^2\omega_{z'} + \frac{1}{2}ma^2\omega_{z'} \qquad \omega_{z'} = 0$$

(*a*) From Eq. (1), $$F\Delta t = \frac{1}{24}\sqrt{2}ma\omega_0 + \frac{1}{32}\sqrt{2}ma\omega_0 = \frac{7}{96}\sqrt{2}ma\omega_0$$

$$(F\Delta t)\mathbf{k} = 0.1031ma\omega_0\mathbf{k} \blacktriangleleft$$

$$\bar{\mathbf{v}} = \frac{1}{2}a\omega_y\mathbf{k}' = \frac{1}{16}\sqrt{2}a\omega_0\mathbf{k}'$$

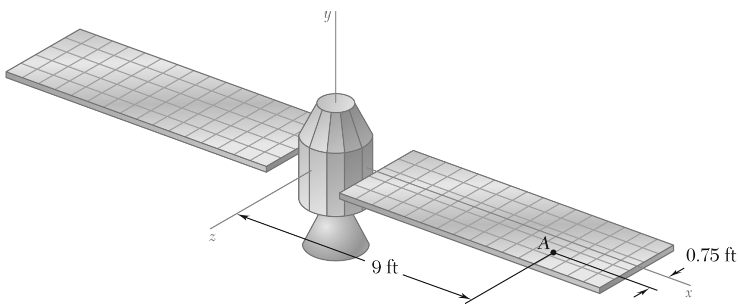
Linear momentum: $$m\bar{\mathbf{v}}_0 + \mathbf{A}\Delta t + F\Delta t\mathbf{k} = m\bar{\mathbf{v}}$$

$$0 + \mathbf{A}\Delta t + \frac{7}{96}\sqrt{2}ma\omega_0\mathbf{k}' = \frac{1}{16}\sqrt{2}ma\omega_0\mathbf{k}'$$

(*b*) $$\mathbf{A}\Delta t = -\frac{1}{96}\sqrt{2}ma\omega_0 \qquad \mathbf{A}\Delta t = -0.01473ma\omega_0\mathbf{k} \blacktriangleleft$$

## PROBLEM 18.33

The coordinate axes shown represent the principal centroidal axes of inertia of a 3000-lb space probe whose radii of gyration are $k_x = 1.375$ ft, $k_y = 1.425$ ft, and $k_z = 1.250$ ft. The probe has no angular velocity when a 5-oz meteorite strikes one of its solar panels at Point $A$ with a velocity $\mathbf{v}_0 = (2400 \text{ ft/s})\mathbf{i} - (3000 \text{ ft/s})\mathbf{j} + (3200 \text{ ft/s})\mathbf{k}$ relative to the probe. Knowing that the meteorite emerges on the other side of the panel with no change in the direction of its velocity, but with a speed reduced by 20 percent, determine the final angular velocity of the probe.




## SOLUTION

Masses: Space probe: $m' = \dfrac{3000}{32.2} = 93.17 \text{ lb} \cdot \text{s}^2/\text{ft}$

Meteorite: $m = \dfrac{5}{(16)(32.2)} = 0.009705 \text{ lb} \cdot \text{s}^2/\text{ft}$

Point of impact: $\text{r}_A = (9 \text{ ft})\mathbf{i} + (0.75 \text{ ft})\mathbf{k}$

Initial linear momentum of the meteorite, $(\text{lb} \cdot \text{s})$:

$$m\mathbf{v}_0 = (0.009705)(2400\mathbf{i} + 3000\mathbf{j} + 3200\mathbf{k}) = 23.292\mathbf{i} - 29.115\mathbf{j} + 31.056\mathbf{k}$$

Its moment about the origin, $(\text{lb} \cdot \text{ft} \cdot \text{s})$:

$$\mathbf{r}_A \times m\mathbf{v}_0 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 9 & 0 & 0.75 \\ 23.292 & -29.115 & 31.056 \end{vmatrix} = 21.836\mathbf{i} - 262.04\mathbf{j} - 262.04\mathbf{k}$$

Final linear momentum of meteorite and its moment about the origin, $(\text{lb} \cdot \text{s})$ and $(\text{lb} \cdot \text{s} \cdot \text{ft})$:

$$0.8m\mathbf{v}_0 = 18.634\mathbf{i} - 23.292\mathbf{j} + 24.845\mathbf{k}$$

$$\mathbf{r}_A \times (0.8m\mathbf{v}_0) = 17.469\mathbf{i} - 209.63\mathbf{j} - 209.63\mathbf{k}$$

## PROBLEM 18.33 (Continued)

Let $\mathbf{H}_A$ be the angular momentum of the probe and $m'$ be its mass. Conservation of angular momentum about the origin for a system of particles consisting of the probe plus the meteorite:

$$\mathbf{r}_A \times m\mathbf{v}_0 = \mathbf{H}_A + \mathbf{r}_A \times (0.8m\mathbf{v}_0)$$

$$\mathbf{H}_A = (4.367 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{i} - (52.41 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{j} - (52.41 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{k}$$

$$I_x\omega_x = (H_A)_x \qquad \omega_x = \frac{(H_A)_x}{m'k_x^2} = \frac{4.367}{(93.17)(1.375)^2} = 0.02479 \text{ rad/s}$$

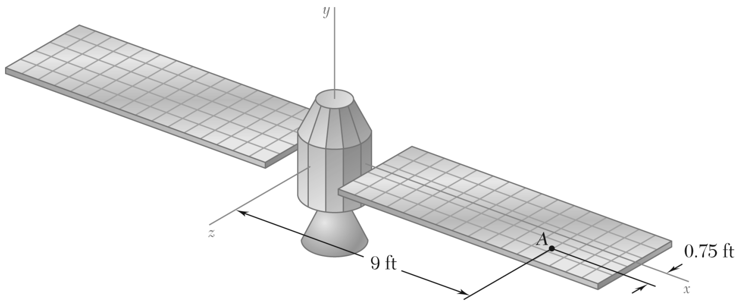
$$I_y\omega_y = (H_A)_y \qquad \omega_y = \frac{(H_A)_y}{m'k_y^2} = \frac{-52.41}{(93.17)(1.425)^2} = -0.2770 \text{ rad/s}$$

$$I_z\omega_z = (H_A)_z \qquad \omega_z = \frac{(H_A)_z}{m'k_z^2} = \frac{-52.41}{(93.17)(1.250)^2} = -0.3600 \text{ rad/s}$$

$$\boldsymbol{\omega} = (0.0248 \text{ rad/s})\mathbf{i} - (0.277 \text{ rad/s})\mathbf{j} - (0.360 \text{ rad/s})\mathbf{k} \blacktriangleleft$$

## PROBLEM 18.34

The coordinate axes shown represent the principal centroidal axes of inertia of a 3000-lb space probe whose radii of gyration are $k_x = 1.375$ ft, $k_y = 1.425$ ft, and $k_z = 1.250$ ft. The probe has no angular velocity when a 5-oz meteorite strikes one of its solar panels at Point $A$ and emerges on the other side of the panel with no change in the direction of its velocity, but with a speed reduced by 25 percent. Knowing that the final angular velocity of the probe is $\boldsymbol{\omega} = (0.05 \text{ rad/s})\mathbf{i} - (0.12 \text{ rad/s})\mathbf{j} + \omega_z\mathbf{k}$ and that the $x$ component of the resulting change in the velocity of the mass center of the probe is $-0.675$ in./s, determine ($a$) the component $\omega_z$ of the final angular velocity of the probe, ($b$) the relative velocity $\mathbf{v}_0$ with which the meteorite strikes the panel.




## SOLUTION

Masses: Space probe: $m' = \dfrac{3000}{32.2} = 93.17 \text{ lb}\cdot\text{s}^2/\text{ft}$

Meteorite: $m = \dfrac{5}{(16)(32.2)} = 0.009705 \text{ lb}\cdot\text{s}^2/\text{ft}$

Point of impact: $\mathbf{r}_A = (9 \text{ ft})\mathbf{i} + (0.75 \text{ ft})\mathbf{k}$

Initial linear momentum of the meteorite, $(\text{lb}\cdot\text{s})$:

$$m\mathbf{v}_0 = (0.009705)(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})$$

Its moment about the origin, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$(\mathbf{H}_A)_0 = \mathbf{r}_A \times m\mathbf{v}_0 = 0.009705\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 9 & 0 & 0.75 \\ v_x & v_y & v_z \end{vmatrix}$$

$$= 0.009705[-0.75v_y\mathbf{i} + (0.75v_x - 9v_z)\mathbf{j} + 9v_y\mathbf{k}]$$

Final linear momentum of the meteorite, $(\text{lb}\cdot\text{s})$:

$$0.75m\mathbf{v}_0 = 0.007279(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})$$

Its moment about the origin, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$\mathbf{r}_A \times (0.75m\mathbf{v}_0) = 0.007279[-0.75v_y\mathbf{i} + (0.75v_x - 9v_z)\mathbf{j} + 9v_y\mathbf{k}]$$

## PROBLEM 18.34 (Continued)

Initial linear momentum of the space probe, $(\text{lb}\cdot\text{s})$: $\quad m'\mathbf{v}_0' = 0$

Final linear momentum of the space probe, $(\text{lb}\cdot\text{s})$:

$$m'(v_x'\mathbf{i} + v_y'\mathbf{j} + v_z'\mathbf{k}) = 93.17\left(-\frac{0.675}{12}\mathbf{i} + v_y'\mathbf{j} + v_z'\mathbf{k}\right)$$

Final angular momentum of space probe, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$\mathbf{H}_A = m'\left(k_x^2\omega_x\mathbf{i} + k_y^2\omega_y\mathbf{j} + k_z^2\omega_z\mathbf{k}\right)$$

$$= 93.17[(1.375)^2(0.05)\mathbf{i} + (1.425)^2(-0.12)\mathbf{j} + (1.250)^2\omega_z\mathbf{k}]$$

$$= 8.8075\mathbf{i} - 22.703\mathbf{j} + 145.58\omega_z\mathbf{k}$$

Conservation of linear momentum of the probe plus the meteorite, $(\text{lb}\cdot\text{s})$:

$$0.009705(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) = 0.007279(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) + 93.17(-0.05625\mathbf{i} + v_y'\mathbf{j} + v_z'\mathbf{k})$$

$\mathbf{i}$: $\quad 0.002426v_x = -5.2408 \qquad v_x = -2160 \text{ ft/s}$

$\mathbf{j}$: $\quad 0.002426v_y = 93.17v_y'$

$\mathbf{k}$: $\quad 0.002426v_z = 93.17v_z'$

Conservation of angular momentum about the origin, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$(0.009705)[-0.75v_y\mathbf{i} + (0.75v_x - 9v_z)\mathbf{j} + 9v_y\mathbf{k}]$$

$$= (0.007279)[-0.75v_y\mathbf{i} + (0.75v_x - 9v_z)\mathbf{j} + 9v_y\mathbf{k}] + 8.8075\mathbf{i} - 22.703\mathbf{j} + 145.58\omega_z\mathbf{k}$$

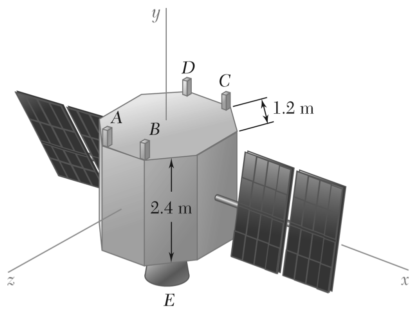
$\mathbf{i}$: $\quad -0.0018195v_y = 8.8075 \qquad v_y = 4840.5 \text{ ft/s}$

$\mathbf{j}$: $\quad -0.021834v_z + 0.0018195v_x = -22.703 \qquad v_z = 0.08333v_x + 1039.8 = 859.8 \text{ ft/s}$

$\mathbf{k}$: $\quad -0.021834v_y = 145.58\omega_z$

(*a*) $\quad \omega_z = -149.98\times 10^{-6}v_y \qquad \omega_z = -0.726 \text{ rad/s} \blacktriangleleft$

(*b*) $\quad \mathbf{v}_0 = -(2160 \text{ ft/s})\mathbf{i} - (4840 \text{ ft/s})\mathbf{j} + (860 \text{ ft/s})\mathbf{k} \blacktriangleleft$




## PROBLEM 18.35

A 2500-kg probe in orbit about the moon is 2.4 m high and has octagonal bases of sides 1.2 m. The coordinate axes shown are the principal centroidal axes of inertia of the probe, and its radii of gyration are $k_x = 0.98$ m, $k_y = 1.06$ m, and $k_z = 1.02$ m. The probe is equipped with a main 500-N thruster $E$ and with four 20-N thrusters $A$, $B$, $C$, and $D$ which can expel fuel in the positive $y$ direction. The probe has an angular velocity $\boldsymbol{\omega} = (0.040 \text{ rad/s})\mathbf{i} + (0.060 \text{ rad/s})\mathbf{k}$ when two of the 20-N thrusters are used to reduce the angular velocity to zero. Determine ($a$) which of the thrusters should be used, ($b$) the operating time of each of these thrusters, ($c$) for how long the main thruster $E$ should be activated if the velocity of the mass center of the probe is to remain unchanged.

## SOLUTION

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k} = m\left(k_x^2\omega_x\mathbf{i} + k_y^2\omega_y^2\mathbf{j} + k_2^2\omega_z\mathbf{k}\right)$$

$$= (2500)[(0.98)^2(0.040)\mathbf{i} + (1.06)^2(0)\mathbf{j} + (1.02)^2(0.060)\mathbf{k}]$$

$$= (96.04 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} + (156.06 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$$

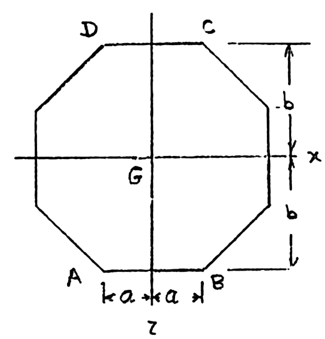
Let $-A\mathbf{j}$, $-B\mathbf{j}$, $-C\mathbf{j}$, and $-D\mathbf{j}$ be the impulses provided by the 20 N thrusters at $A$, $B$, $C$, and $D$, respectively. Let $E\mathbf{j}$ be that provided by the 500 N main thruster.

Position vectors for intersections of the lines of action of the thruster impulses with the $xz$ plane:

$$a = \frac{1}{2}(1.2) = 0.6 \text{ m}, \qquad b = 0.6 + 0.6\sqrt{2} = 1.4485 \text{ m}$$

$$\mathbf{r}_A = -a\mathbf{i} + b\mathbf{k}, \qquad \mathbf{r}_B = a\mathbf{i} + b\mathbf{k}$$

$$\mathbf{r}_C = a\mathbf{i} - b\mathbf{k}, \qquad \mathbf{r}_D = -a\mathbf{i} - b\mathbf{k}$$




The final linear and angular momenta are zero.

Principle of impulse-momentum. Moments about $G$:

$$\mathbf{H}_G + \mathbf{r}_A \times (-A\mathbf{j}) + \mathbf{r}_B \times (-B\mathbf{j}) + \mathbf{r}_C \times (-C\mathbf{j}) + \mathbf{r}_D \times (-D\mathbf{j}) = 0$$

$$\mathbf{H}_G + b(A + B - C - D)\mathbf{i} + a(A - B - C + D)\mathbf{k} = 0$$

Resolve into components.

$$\mathbf{i}: \quad A + B - C - D = -\frac{(H_G)_x}{b} = -\frac{96.04}{1.4485} = -66.30 \text{ N}\cdot\text{s} \qquad (1)$$

$$\mathbf{k}: \quad A - B - C + D = -\frac{(H_G)_z}{a} = -\frac{156.06}{0.6} = -260.1 \text{ N}\cdot\text{s} \qquad (2)$$

Of $A$, $B$, $C$, and $D$, two must be zero or positive, the other two zero.

## PROBLEM 18.35 (Continued)

Set $A = 0$ and $B - D = N$. Solve the simultaneous equations (1) and (2).

$C = 163.2\ \text{N}\cdot\text{s}$ and $N = 96.9.$ Set $D = 0$ and $B = 96.9\ \text{N}\cdot\text{s}$

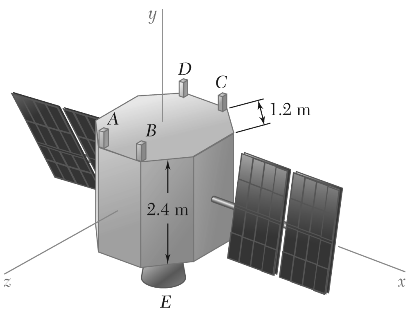
(*a*) Use thrusters $C$ and $B$. ◀

(*b*) $F_C(\Delta t_C) = C, \quad \Delta t_C = \dfrac{C}{F_C} = \dfrac{163.2}{20}$ $\Delta t_C = 8.16\ \text{s}$ ◀

$F_B(\Delta t_B) = B, \quad \Delta t_B = \dfrac{B}{F_B} = \dfrac{96.9}{20}$ $\Delta t_B = 4.84\ \text{s}$ ◀

(*c*) Linear momentum: $E\mathbf{j} - B\mathbf{j} - C\mathbf{j} = 0, \quad E = 30.291\ \text{lb}\cdot\text{s}$

$F_E(\Delta t_E) = E \quad \Delta t_E = \dfrac{E}{F_E} = \dfrac{260.1}{500}$ $\Delta t_E = 0.520\ \text{s}$ ◀




## PROBLEM 18.36

Solve Problem 18.35, assuming that the angular velocity of the probe is $\boldsymbol{\omega} = (0.060 \text{ rad/s})\mathbf{i} - (0.040 \text{ rad/s})\mathbf{k}$.

**PROBLEM 18.35** A 2500-kg probe in orbit about the moon is 2.4 m high and has octagonal bases of sides 1.2 m. The coordinate axes shown are the principal centroidal axes of inertia of the probe, and its radii of gyration are $k_x = 0.98$ m, $k_y = 1.06$ m, and $k_z = 1.02$ m. The probe is equipped with a main 500-N thruster $E$ and with four 20-N thrusters $A$, $B$, $C$, and $D$ which can expel fuel in the positive $y$ direction. The probe has an angular velocity $\boldsymbol{\omega} = (0.040 \text{ rad/s})\mathbf{i} + (0.060 \text{ rad/s})\mathbf{k}$ when two of the 20-N thrusters are used to reduce the angular velocity to zero. Determine ($a$) which of the thrusters should be used, ($b$) the operating time of each of these thrusters, ($c$) for how long the main thruster $E$ should be activated if the velocity of the mass center of the probe is to remain unchanged.

## SOLUTION

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= m\left(k_x^2\omega_x\mathbf{i} + k_y^2\omega_y^2\mathbf{j} + k_2^2\omega_z\mathbf{k}\right)$$

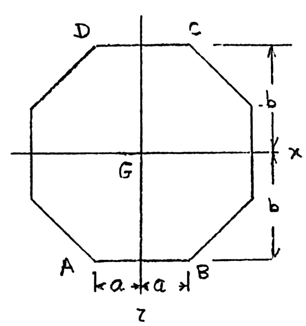
$$= (2500)[(0.98)^2(0.060) + (1.06)^2(0) + (1.02)^2(-0.040)\mathbf{k}]$$

$$= (144.06 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} - (104.04 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$$

Let $-A\mathbf{j}$, $-B\mathbf{j}$, $-C\mathbf{j}$, and $-D\mathbf{j}$ be the impulses provided by the 20 N thrusters at $A$, $B$, $C$, and $D$, respectively. Let $E\mathbf{j}$ be that provided by the 500 N main thruster.

Position vectors for intersections of the lines of action of the thruster impulses with the $xz$ plane:

$$a = \frac{1}{2}(1.2) = 0.6 \text{ m}, \qquad b = 0.6 + 0.6\sqrt{2} = 1.4485 \text{ m}$$




$$\mathbf{r}_A = -a\mathbf{i} + b\mathbf{k}, \qquad \mathbf{r}_B = a\mathbf{i} + b\mathbf{k}$$

$$\mathbf{r}_C = a\mathbf{i} - b\mathbf{k}, \qquad \mathbf{r}_D = -a\mathbf{i} - b\mathbf{k}$$

The final linear and angular momenta are zero.

<u>Principle of impulse-momentum</u>. Moments about $G$:

$$\mathbf{H}_G + \mathbf{r}_A \times (-A\mathbf{j}) + \mathbf{r}_B \times (-B\mathbf{j}) + \mathbf{r}_C \times (-C\mathbf{j}) + \mathbf{r}_D \times (-D\mathbf{j}) = 0$$

$$\mathbf{H}_G + b(A + B - C - D)\mathbf{i} + a(A - B - C + D)\mathbf{k} = 0$$

## PROBLEM 18.36 (Continued)

Resolve into components.

$$\mathbf{i}: \quad A + B - C - D = -\frac{(H_G)_x}{b} = -\frac{144.06}{1.4485} = -99.455 \text{ N}\cdot\text{s} \qquad (1)$$

$$\mathbf{k}: \quad A - B - C + D = -\frac{(H_G)_z}{a} = -\frac{-104.04}{0.6} = 173.4 \text{ N}\cdot\text{s} \qquad (2)$$

Of $A$, $B$, $C$, and $D$, two must be zero or positive, the other two zero.

Set $B = 0$ and $A - C = N$. Solve the simultaneous equations (1) and (2).

$D = 136.43 \text{ N}\cdot\text{s}$ and $N = 36.97 \text{ N}\cdot\text{s}$. Set $C = 0$ and $A = 36.97 \text{ N}\cdot\text{s}$

(*a*) Use thrusters $D$ and $A$. ◀

(*b*) $F_D(\Delta t_D) = D, \quad \Delta t_D = \dfrac{D}{F_D} = \dfrac{136.43}{20}$ $\quad \Delta t_D = 6.82 \text{ s}$ ◀

$F_A(\Delta t_A) = A, \quad \Delta t_A = \dfrac{A}{F_A} = \dfrac{36.97}{20}$ $\quad \Delta t_A = 1.848 \text{ s}$ ◀

(*c*) Linear momentum: $E\mathbf{j} - D\mathbf{j} - A\mathbf{j} = 0 \quad E = 173.4 \text{ N}\cdot\text{s}$

$F_E(\Delta t_E) = E \quad \Delta t_E = \dfrac{E}{F_E} = \dfrac{173.4}{500}$ $\quad \Delta t_E = 0.347 \text{ s}$ ◀

## PROBLEM 18.37

Denoting, respectively, by $\boldsymbol{\omega}$, $\mathbf{H}_O$, and $T$ the angular velocity, the angular momentum, and the kinetic energy of a rigid body with a fixed Point $O$, (*a*) prove that $\mathbf{H}_O \cdot \boldsymbol{\omega} = 2T$; (*b*) show that the angle $\theta$ between $\omega$ and $\mathbf{H}_O$ will always be acute.

## SOLUTION

(*a*)

$$\mathbf{H}_0 = (I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\mathbf{i} + (-I_{xy}\omega_x + I_y\omega_y - I_{yz}\omega_z)\mathbf{j} + (-I_{xz}\omega_x - I_{yz}\omega_y + I_z\omega_z)\mathbf{k}$$

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

$$\mathbf{H}_0 \cdot \boldsymbol{\omega} = I_x\omega_x^2 - I_{xy}\omega_y\omega_x - I_{xz}\omega_z\omega_x - I_{xy}\omega_x\omega_y + I_y\omega_y^2 - I_{yz}\omega_z\omega_y$$

$$-I_{xz}\omega_x\omega_z - I_{yz}\omega_y\omega_z + I_z\omega_z^2$$

$$= (2)\left(\frac{1}{2}\right)\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_x\omega_y - 2I_{yz}\omega_y\omega_z - 2I_{xz}\omega_x\omega_z\right)$$

$$= 2T$$

(*b*)

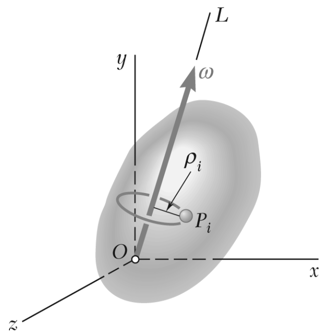
$$\mathbf{H}_0 \cdot \boldsymbol{\omega} = H_0\omega\cos\theta$$

$$2T = H_0\omega\cos\theta$$

$$\cos\theta = \frac{2T}{H_0\omega}$$

But $\quad T > 0,\ H_0 > 0,\ \omega > 0$

$$\cos\theta > 0 \qquad \theta < 90^\circ$$




## PROBLEM 18.38

Show that the kinetic energy of a rigid body with a fixed Point $O$ can be expressed as $T = \frac{1}{2} I_{OL} \omega^2$, where $\omega$ is the instantaneous angular velocity of the body and $I_{OL}$ is its moment of inertia about the line of action $OL$ of $\omega$. Derive this expression (*a*) from Eqs. (9.46) and (18.19), (*b*) by considering $T$ as the sum of the kinetic energies of particles $P_i$ describing circles of radius $\rho_i$ about line $OL$.

## SOLUTION

(*a*)

$$T = \frac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_x\omega_y - 2I_{yz}\omega_y\omega_z - 2I_{xz}\omega_x\omega_z\right)$$

Let

$$\omega_x = \omega\cos\theta_x = \omega\lambda_x$$

$$\omega_y = \omega\cos\theta_y = \omega\lambda_y$$

$$\omega_z = \omega\cos\theta_z = \omega\lambda_z$$

$$T = \frac{1}{2}\left(I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{xz}\lambda_x\lambda_z\right)\omega^2$$

$$= \frac{1}{2} I_{OL}\omega^2$$

$$T = \frac{1}{2} I_{OL}\omega^2 \blacktriangleleft$$

(*b*) Each particle of mass $(\Delta m)_i$ describes a circle of radius $\rho_i$.

The speed of the particle is $v_i = \rho_i\omega$.

Its kinetic energy is

$$(\Delta T)_i = \frac{1}{2}(\Delta m_i)v_i^2 = \frac{1}{2}(\Delta m_i)\rho_i^2\omega^2$$

The kinetic energy of the entire body is

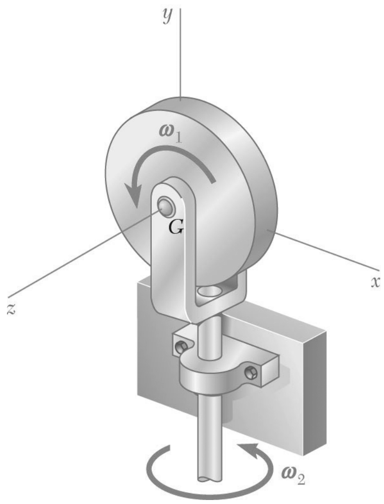
$$T = \Sigma(\Delta T)_i = \frac{1}{2}\Sigma(\Delta m_i)\rho_i^2\omega^2$$

but

$$I_{OL} = \Sigma(\Delta m_i)\rho_i^2$$

Hence,

$$T = \frac{1}{2} I_{OL}\omega^2 \blacktriangleleft$$




## PROBLEM 18.39

Determine the kinetic energy of the disk of Problem 18.1.

**PROBLEM 18.1** A thin, homogeneous disk of mass $m$ and radius $r$ spins at the constant rate $\omega_1$ about an axle held by a fork-ended vertical rod, which rotates at the constant rate $\omega_2$. Determine the angular momentum $\mathbf{H}_G$ of the disk about its mass center $G$.

## SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega_2 \mathbf{j} + \omega_1 \mathbf{k}$

Moments of inertia: $\bar{I}_x = \frac{1}{4}mr^2, \quad \bar{I}_y = \frac{1}{4}mr^2, \quad \bar{I}_z = \frac{1}{2}mr^2$

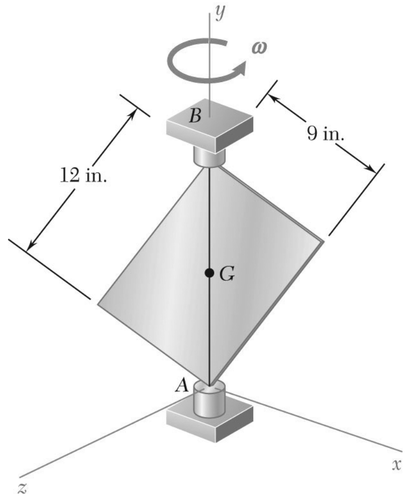
Products of inertia: by symmetry, $\bar{I}_{xy} = \bar{I}_{yz} = \bar{I}_{zx} = 0$

Kinetic energy:

$$T = \frac{1}{2}\left(\bar{I}_x\omega_x^2 + \bar{I}_y\omega_y^2 + \bar{I}_z\omega_z\right)$$

$$T = \frac{1}{2}\left[0 + \left(\frac{1}{4}mr^2\right)\omega_2^2 + \left(\frac{1}{2}mr^2\right)\omega_1^2\right]$$

$$T = \frac{1}{8}mr^2\left(\omega_2^2 + 2\omega_1^2\right) \blacktriangleleft$$



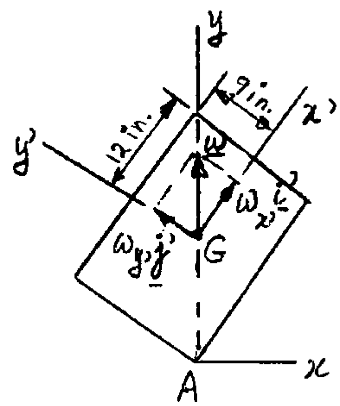

## PROBLEM 18.40

Determine the kinetic energy of the plate of Problem 18.2.

**PROBLEM 18.2** A thin rectangular plate of weight 15 lb rotates about its vertical diagonal *AB* with an angular velocity $\omega$. Knowing that the $z$ axis is perpendicular to the plate and that $\omega$ is constant and equal to 5 rad/s, determine the angular momentum of the plate about its mass center *G*.

## SOLUTION

$$h = \sqrt{(9\text{ in.})^2 + (12\text{ in.})^2} = 15\text{ in.}$$




Resolving $\omega$ along the principal axes $x'$, $y'$, $z$:

$$\omega_{x'} = \frac{12}{15}\omega = 0.8(5\text{ rad/s}) = 4\text{ rad/s}$$

$$\omega_{y'} = \frac{9}{15}\omega = 0.6(5\text{ rad/s}) = 3\text{ rad/s}$$

$$\omega_z = 0$$

Moments of inertia:

$$I_{x'} = \frac{1}{12}\left(\frac{15\text{ lb}}{32.2}\right)\left(\frac{9}{12}\text{ ft}\right) = 0.021836\text{ slug}\cdot\text{ft}^2$$

$$I_{y'} = \frac{1}{12}\left(\frac{15\text{ lb}}{32.2}\right)\left(\frac{12}{12}\text{ ft}\right)^2 = 0.038820\text{ slug}\cdot\text{ft}^2$$

From Eqs. (18.20):

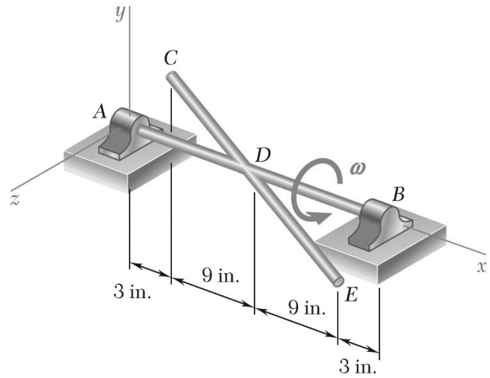
$$T = \frac{1}{2}(\bar{I}_{x'}\omega_{x'}^2 + \bar{I}_{y'}\omega_{y'}^2 + \bar{I}_{z'}\omega_{z'}^2)$$

$$= \frac{1}{2}[(0.021836)(4)^2 + (0.038820)(3)^2 + 0]$$

$$T = 0.34938\text{ ft}\cdot\text{lb}$$

$$T = 0.349\text{ ft}\cdot\text{lb} \blacktriangleleft$$

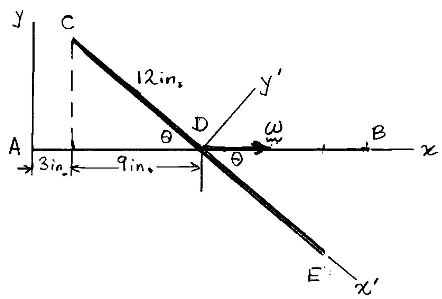
## PROBLEM 18.41




Determine the kinetic energy of the assembly of Problem 18.3.

**PROBLEM 18.3** Two uniform rods *AB* and *CE*, each of weight 3 lb and length 2 ft, are welded to each other at their midpoints. Knowing that this assembly has an angular velocity of constant magnitude $\omega = 12$ rad/s, determine the magnitude and direction of the angular momentum $\mathbf{H}_D$ of the assembly about *D*.

## SOLUTION




$$m = \frac{W}{g} = \frac{3}{32.2} = 0.093168 \text{ lb}\cdot\text{s}^2/\text{ft}, \qquad l = 24 \text{ in.} = 2 \text{ ft},$$

$$\boldsymbol{\omega} = (12 \text{ rad/s})\mathbf{i}$$

For rod *ADB*, $T = \frac{1}{2}I_x\omega^2 \approx 0$, since $I_x \approx 0$.

For rod *CDE*, use principal axes $x'$, $y'$ as shown.

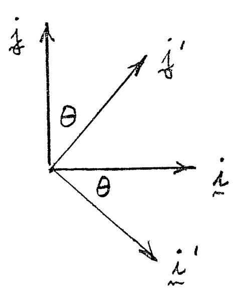
$$\cos\theta = \frac{9}{12}, \qquad \theta = 41.410°$$

$$\omega_{x'} = \omega\cos\theta = 9 \text{ rad/s}^2$$

$$\omega_{y'} = \omega\sin\theta = 7.93725 \text{ rad/s}^2$$

$$\omega_{z'} = 0$$

$$\bar{I}_{x'} \approx 0$$



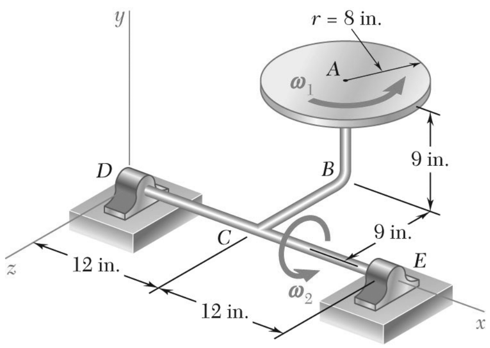

$$\bar{I}_{y'} = \frac{1}{12}ml^2 = \frac{1}{12}(0.093168)(2)^2$$

$$= 0.0310559 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}_{x'}\omega_{x'}^2 + \frac{1}{2}\bar{I}_{y'}\omega_{y'}^2 + \frac{1}{2}\bar{I}_{z'}\omega_{z'}^2$$

$$= 0 + 0 + \frac{1}{2}(0.0310559)(7.93725)^2 + 0$$

$$= 0.97826 \text{ ft}\cdot\text{lb} \qquad T = 0.978 \text{ ft}\cdot\text{lb} ◀$$




## PROBLEM 18.42

Determine the kinetic energy of the disk of Problem 18.4.

**PROBLEM 18.4** A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm $ABC$, which is welded to a shaft $DCE$ rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the angular momentum $\mathbf{H}_A$ of the disk about its center $A$.

## SOLUTION

$$m = \frac{W}{g} = \frac{6}{32.2} = 0.1863 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\boldsymbol{\omega} = \omega_2\mathbf{i} + \omega_1\mathbf{j} = (8 \text{ rad/s})\mathbf{i} + (16 \text{ rad/s})\mathbf{j}$$

For axes $x'$, $y'$, $z'$ parallel to $x$, $y$, $z$ with origin at $A$,

$$\bar{I}_{x'} = \frac{1}{4}mr^2 = \frac{1}{4}(0.1863)\left(\frac{8}{12}\right)^2 = 0.0207 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_{z'} = \bar{I}_{x'} = 0.0207 \text{ lb}\cdot\text{s}^2\cdot\text{ft}, \quad \bar{I}_{y'} = \bar{I}_{x'} + \bar{I}_{z'} = 0.0414 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Point $A$ is the mass center of the disk.

$$\mathbf{r}_{A/C} = (9 \text{ in.})\mathbf{i} - (9 \text{ in.})\mathbf{k} = (0.75 \text{ ft})\mathbf{i} - (0.75 \text{ ft})\mathbf{k}$$

$$\bar{\mathbf{v}} = \mathbf{v}_A = \omega_2\mathbf{i} \times \mathbf{r}_{A/C} = 8\mathbf{i} \times (0.75\mathbf{j} - 0.75\mathbf{k})$$

$$= (6 \text{ ft/s})\mathbf{j} + (6 \text{ ft/s})\mathbf{k}$$

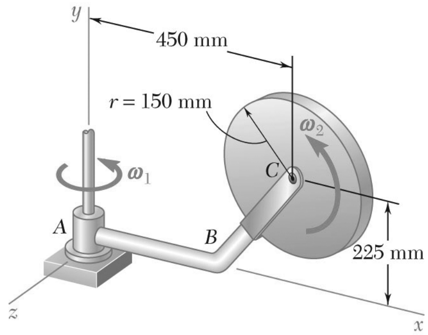
$$\bar{v} = \sqrt{(6)^2 + (6)^2} = 8.4853 \text{ ft/s}$$

Kinetic energy:

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}_{x'}\omega_x^2 + \frac{1}{2}\bar{I}_{y'}\omega_y^2 + \frac{1}{2}\bar{I}_{z'}\omega_z^2$$

$$T = \frac{1}{2}(0.1863)(8.4853)^2 + \frac{1}{2}(0.0207)(8)^2 + \frac{1}{2}(0.0414)(16)^2 + 0$$

$$= 6.7068 + 0.6624 + 5.2992 = 12.6684 \text{ ft}\cdot\text{lb} \qquad T = 12.67 \text{ ft}\cdot\text{lb} \blacktriangleleft$$



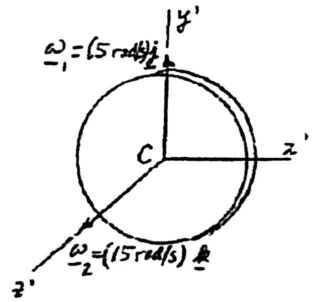

## PROBLEM 18.43

Determine the kinetic energy of the disk of Problem 18.5.

**PROBLEM 18.5** A thin disk of mass $m = 4$ kg rotates at the constant rate $\omega_2 = 15$ rad/s with respect to arm *ABC*, which itself rotates at the constant rate $\omega_1 = 5$ rad/s about the *y* axis. Determine the angular momentum of the disk about its center *C*.

## SOLUTION

$$r = 150 \text{ mm}$$




Angular velocity of disk:

$$\boldsymbol{\omega} = \omega_1 \mathbf{j} + \omega_2 \mathbf{k}$$
$$= (5 \text{ rad/s})\mathbf{j} + (15 \text{ rad/s})\mathbf{k}$$

Centroidal moments of inertia:

$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4} m r^2$$
$$= \frac{1}{4}(4)(0.150 \text{ m})^2 = 0.0225 \text{ kg}\cdot\text{m}^2$$
$$\bar{I}_{z'} = \frac{1}{2} m r^2 = 0.045 \text{ kg}\cdot\text{m}^2$$

Location of mass center. $\mathbf{r}_{C/A} = (0.450 \text{ m})\mathbf{i} + (0.225 \text{ m})\mathbf{j}$

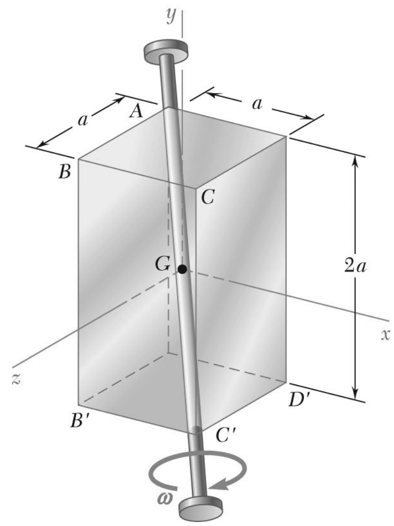
Velocity of mass center. $\bar{\mathbf{v}} = \boldsymbol{\omega}_1 \times \mathbf{r}_{C/A} = 5\mathbf{j} \times (0.45\mathbf{i} + 0.225\mathbf{j})$

$$= -(2.25 \text{ m/s})\mathbf{k}$$

Kinetic energy:

$$T = \frac{1}{2} m \bar{v}^2 + \frac{1}{2}\bar{I}_x \omega_x^2 + \frac{1}{2}\bar{I}_y \omega_y^2 + \frac{1}{2}\bar{I}_z \omega_z^2$$
$$= \frac{1}{2}(4)(2.25)^2 + 0 + \frac{1}{2}(0.0225)(5)^2 + \frac{1}{2}(0.0450)(15)^2$$
$$= 10.125 + 0 + 0.28125 + 5.0625$$

$T = 15.47$ J ◀




## PROBLEM 18.44

Determine the kinetic energy of the solid parallelepiped of Problem 18.6.

**PROBLEM 18.6** A solid rectangular parallelepiped of mass $m$ has a square base of side $a$ and a length $2a$. Knowing that it rotates at the constant rate $\omega$ about its diagonal $AC'$ and that its rotation is observed from $A$ as counterclockwise, determine ($a$) the magnitude of the angular momentum $\mathbf{H}_G$ of the parallelepiped about its mass center $G$, ($b$) the angle that $\mathbf{H}_G$ forms with the diagonal $AC'$.

## SOLUTION

Body diagonal: $$d = \sqrt{a^2 + (2a)^2 + a^2} = \sqrt{6}a$$

$$\boldsymbol{\omega} = \frac{\omega}{d}(-a\mathbf{i} + 2a\mathbf{j} - a\mathbf{k}) = -\frac{\omega}{\sqrt{6}}\mathbf{i} + \frac{2\omega}{\sqrt{6}}\mathbf{j} - \frac{\omega}{\sqrt{6}}\mathbf{k}$$

$$I_x = \frac{1}{12}m[(2a)^2 + a^2] = \frac{5}{12}ma^2$$

$$I_y = \frac{1}{12}m[a^2 + a^2] = \frac{1}{6}ma^2$$

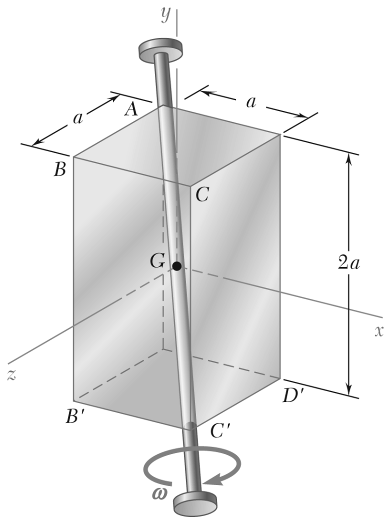
$$I_z = \frac{1}{12}m[a^2 + (2a)^2] = \frac{5}{12}ma^2$$

Axis of rotation passes through the mass center, hence $\overline{\mathbf{v}} = 0$.

Kinetic energy: $$T = \frac{1}{2}m\overline{v}^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

$$T = 0 + \frac{1}{2}\left(\frac{5}{12}ma^2\right)\left(\frac{\omega}{\sqrt{6}}\right)^2 + \frac{1}{2}\left(\frac{1}{6}ma^2\right)\left(\frac{2\omega}{\sqrt{6}}\right)^2 + \frac{1}{2}\left(\frac{5}{12}ma^2\right)\left(\frac{\omega}{\sqrt{6}}\right)^2 = \frac{1}{8}ma^2\omega^2$$

$$T = 0.1250\ ma^2\omega^2 \blacktriangleleft$$




## PROBLEM 18.45

Determine the kinetic energy of the hollow parallelepiped of Problem 18.7.

**PROBLEM 18.7** Solve Problem 18.6, assuming that the solid rectangular parallelepiped has been replaced by a hollow one consisting of six thin metal plates welded together.

## SOLUTION

Body diagonal: $$d = \sqrt{a^2 + (2a)^2 + a^2} = \sqrt{6}a$$

$$\boldsymbol{\omega} = \frac{\omega}{d}(-a\mathbf{i} + 2a\mathbf{j} - a\mathbf{k}) = -\frac{\omega}{\sqrt{6}}\mathbf{i} + \frac{2\omega}{\sqrt{6}}\mathbf{j} - \frac{\omega}{\sqrt{6}}\mathbf{k}$$

$$\text{Total area} = 2(a^2 + 2a^2 + 2a^2) = 10a^2$$

For each square plate, $$m' = \frac{1}{10}m$$

$$I_x = \frac{1}{12}m'a^2 + m'a^2 = \frac{13}{12}m'a^2 = \frac{13}{120}ma^2$$

$$I_y = \frac{1}{6}m'a^2 = \frac{1}{60}ma^2$$

$$I_z = I_x = \frac{13}{120}ma^2$$

For each plate parallel to the *yz* plane, $$m' = \frac{1}{5}m$$

$$I_x = \frac{1}{12}m'[a^2 + (2a)^2] = \frac{5}{12}m'a^2 = \frac{1}{12}ma^2$$

$$I_y = \frac{1}{12}m'a^2 + m'\left(\frac{a}{2}\right)^2 = \frac{1}{3}m'a^2 = \frac{1}{15}ma^2$$

$$I_z = \frac{1}{12}m'(2a)^2 + m'\left(\frac{a}{2}\right)^2 = \frac{7}{12}m'a^2 = \frac{7}{60}ma^2$$

## PROBLEM 18.45 (Continued)

For each plate parallel to the $xy$ plane, $m' = \frac{1}{5}m$

$$I_x = \frac{1}{12}m'(2a)^2 + m'\left(\frac{a}{2}\right)^2 = \frac{7}{12}m'a^2 = \frac{7}{60}ma^2$$

$$I_y = \frac{1}{12}m'a^2 + m'\left(\frac{a}{2}\right)^2 = \frac{1}{3}m'a^2 = \frac{1}{15}ma^2$$

$$I_z = \frac{1}{12}m'[a^2 + (2a)^2] = \frac{5}{12}m'a^2 = \frac{1}{12}ma^2$$

Total moments of inertia:

$$I_x = 2\left(\frac{13}{120} + \frac{1}{12} + \frac{7}{60}\right)ma^2 = \frac{37}{60}ma^2$$

$$I_y = 2\left(\frac{1}{60} + \frac{1}{15} + \frac{1}{15}\right)ma^2 = \frac{3}{10}ma^2$$

$$I_z = 2\left(\frac{13}{120} + \frac{7}{60} + \frac{1}{12}\right)ma^2 = \frac{37}{60}ma^2$$

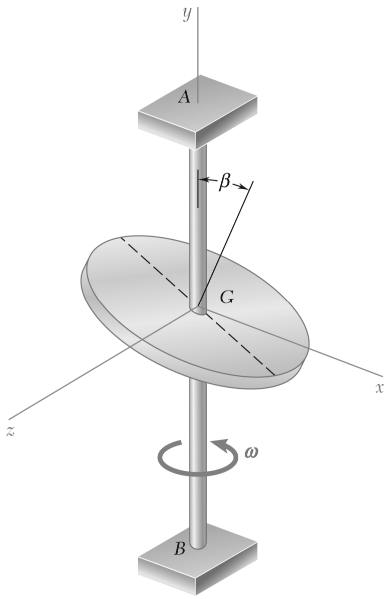
Axis of rotation passes through the mass center, hence $\overline{\mathbf{v}} = 0.$

Kinetic energy:

$$T = \frac{1}{2}m\overline{v}^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

$$T = 0 + \frac{1}{2}\left(\frac{37}{60}ma^2\right)\left(\frac{\omega}{\sqrt{6}}\right)^2 + \frac{1}{2}\left(\frac{3}{10}ma^2\right)\left(\frac{2\omega}{\sqrt{6}}\right)^2 + \frac{1}{2}\left(\frac{37}{60}ma^2\right)\left(\frac{\omega}{\sqrt{6}}\right)^2 = \frac{73}{360}\ ma^2\omega^2$$

$$T = 0.203\ ma^2\omega^2 \blacktriangleleft$$



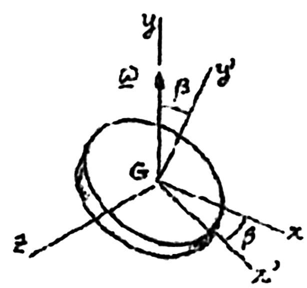

## PROBLEM 18.46

Determine the kinetic energy of the disk of Problem 18.8.

**PROBLEM 18.8** A homogeneous disk of mass $m$ and radius $r$ is mounted on the vertical shaft $AB$. The normal to the disk at $G$ forms an angle $\beta = 25°$ with the shaft. Knowing that the shaft has a constant angular velocity $\omega$, determine the angle $\theta$ formed by the shaft $AB$ and the angular momentum $\mathbf{H}_G$ of the disk about its mass center $G$.

## SOLUTION

Use the principal centroidal axes $Gx'y'z$.




Moments of inertia.

$$\bar{I}_{x'} = \bar{I}_z = \frac{1}{4}mr^2$$

$$\bar{I}_{y'} = \frac{1}{2}mr^2$$

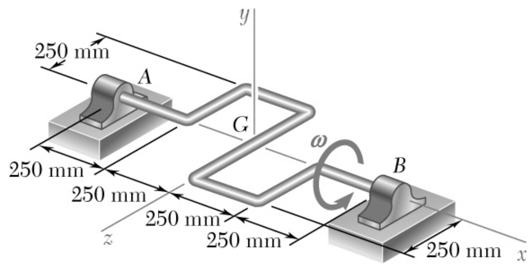
Angular velocities.

$$\omega_{x'} = -\omega \sin\beta$$
$$\omega_{y'} = \omega\cos\beta$$
$$\omega_z = 0$$

Kinetic energy:

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}_{x'}^2\omega_{x'}^2 + \frac{1}{2}\bar{I}_{y'}\omega_{y'}^2 + \frac{1}{2}\bar{I}_{z'}\omega_{z'}^2$$
$$= 0 + \frac{1}{2}\cdot\frac{1}{4}mr^2(-\omega\sin\beta)^2 + \frac{1}{2}\cdot\frac{1}{2}mr^2(\omega\cos\beta)^2 + 0$$
$$= \frac{1}{8}mr^2\omega^2(\sin^2\beta + 2\cos^2\beta)$$
$$= \frac{1}{8}mr^2\omega^2(\sin^2 25° + 2\cos^2 25°)$$

$$T = 0.228\, mr^2\omega^2 \blacktriangleleft$$

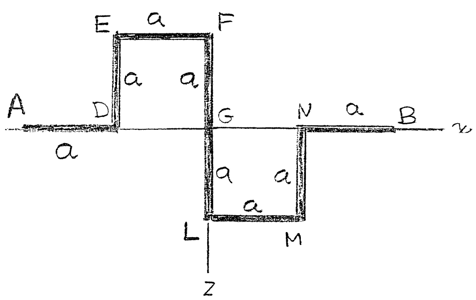



## PROBLEM 18.47

Determine the kinetic energy of the shaft of Problem 18.15.

**PROBLEM 18.15** A 5-kg rod of uniform cross section is used to form the shaft shown. Knowing that the shaft rotates with a constant angular velocity $\boldsymbol{\omega}$ of magnitude 12 rad/s, determine (*a*) the angular momentum $\mathbf{H}_G$ of the shaft about its mass center $G$, (*b*) the angle formed by $\mathbf{H}_G$ and the axis $AB$.

## SOLUTION




$$\omega = (12 \text{ rad/s})\mathbf{i}, \qquad \omega_y = \omega_z = 0$$

$$T = \frac{1}{2}\bar{I}_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2 - I_{xy}\omega_x\omega_y - I_{yz}\omega_y\omega_z - I_{xz}\omega_x\omega_z$$

$$= \frac{1}{2}I_x\omega^2$$

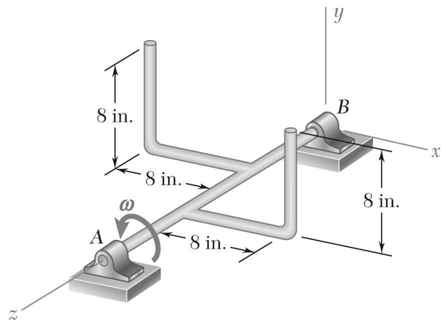
The shaft is comprised of eight sections, each of length

$a = 0.25$ m and of mass $m' = \dfrac{m}{8} = 0.625$ kg.

$$\bar{I}_x = (4)\left(\frac{1}{3}m'a^2\right) + (2)(m'a^2) = \frac{10}{3}m'a^2 = \frac{10}{3}(0.625)(0.25)^2 = 0.130208 \text{ kg}\cdot\text{m}^2$$

$$T = \frac{1}{2}(0.130208)(12)^2 = 9.38 \text{ kg}\cdot\text{m}^2/\text{s}^2 = 9.38 \text{ N}\cdot\text{m} = 9.38 \text{ J}$$

$T = 9.38$ J ◀




## PROBLEM 18.48

Determine the kinetic energy of the body of Problem 18.17.

**PROBLEM 18.17** Two L-shaped arms, each weighing 4 lb, are welded at the third points of the 2-ft shaft $AB$. Knowing that shaft $AB$ rotates at the constant rate $\omega = 240$ rpm, determine (*a*) the angular momentum of the body about $A$, (*b*) the angle formed by the angular momentum and shaft $AB$.

## SOLUTION

$$W = 4 \text{ lb} \quad m = \frac{4}{32.2} = 0.12422 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$a = 8 \text{ in.} = 0.66667 \text{ ft}$$

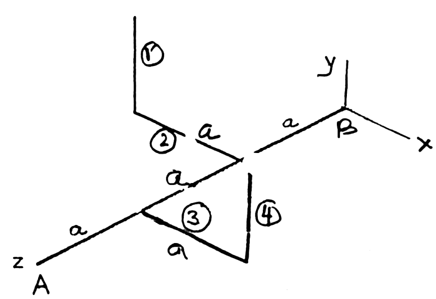
$$\omega = \frac{(2\pi)(240)}{60} = 8\pi \text{ rad/s}, \quad \omega_x = 0, \quad \omega_y = 0, \quad \omega_z = 8\pi \text{ rad/s}$$

For rotation about the fixed Point $B$, the kinetic energy is

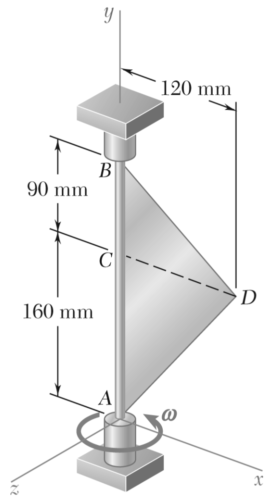
$$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2 - I_{xy}\omega_x\omega_y - I_{xz}\omega_x\omega_z - I_{yz}\omega_y\omega_z = \frac{1}{2}I_z\omega^2$$

Calculation of $I_z$: Segments 1, 2, 3, and 4, each of mass $m' = 0.06211 \text{ lb}\cdot\text{s}^2/\text{ft}$ contribute to

| Part | $I_z$ |
|---|---|
| 1 | $\left(\frac{1}{12} + \frac{1}{4} + 1\right)m'a^2$ |
| 2 | $\frac{1}{3}m'a^2$ |
| 3 | $\frac{1}{3}m'a^2$ |
| 4 | $\left(\frac{1}{12} + \frac{1}{4} + 1\right)m'a^2$ |
| Σ | $\frac{10}{3}m'a^2$ |




Kinetic energy: $T = \frac{1}{2}\left(\frac{10}{3}m'a^2\right)\omega^2 = \frac{1}{2}\left(\frac{10}{3}\right)(0.06211)(0.66667)^2(8\pi)^2$ $\quad T = 29.1 \text{ ft}\cdot\text{lb}$ ◀




## PROBLEM 18.49

Determine the kinetic energy of the triangular plate of Problem 18.19.

**PROBLEM 18.19** The triangular plate shown has a mass of 7.5 kg and is welded to a vertical shaft *AB*. Knowing that the plate rotates at the constant rate $\omega = 12$ rad/s, determine its angular momentum about (*a*) Point *C*, (*b*) Point *A*. (*Hint*: To solve part *b* find $\overline{\mathbf{v}}$ and use the property indicated in part *a* of Problem 18.13.)

## SOLUTION

$$\boldsymbol{\omega} = (12 \text{ rad/s})\mathbf{j}, \quad \omega_x = 0, \quad \omega_y = 12 \text{ rad/s}, \quad \omega_z = 0$$

Kinetic energy:

$$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2 - I_{yz}\omega_y\omega_z - I_{zx}\omega_z\omega_x - I_{xy}\omega_x\omega_y$$

$$= \frac{1}{2}I_y\omega^2$$

For the plate:

$$b = AB = 90 + 160 = 250 \text{ mm} = 0.25 \text{ m}$$

$$h = CD = 120 \text{ mm} = 0.12 \text{ m}$$

Area:

$$A = \frac{1}{2}bh = 15\times 10^{-3} \text{ m}^2$$

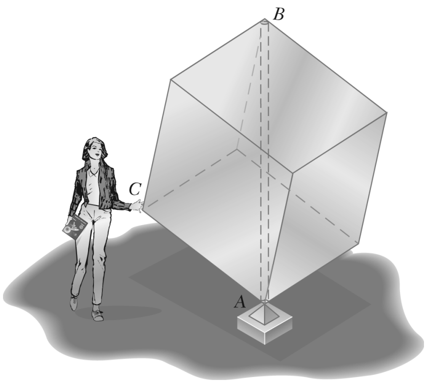
$$(I_y)_{\text{area}} = \frac{1}{12}bh^3 = 36\times 10^{-6} \text{ m}^4$$

$$m = 7.5 \text{ kg}$$

$$(I_y)_{\text{mass}} = \frac{m}{A}(I_y)_{\text{area}} = \frac{(7.5)(36\times 10^{-6})}{15\times 10^{-3}} = 18\times 10^{-3} \text{ kg}\cdot\text{m}^2$$

$$T = \frac{1}{2}(18\times 10^{-3})(12)^2 = 1.296 \text{ J}$$

$T = 1.296$ J ◀




## PROBLEM 18.50

Determine the kinetic energy imparted to the cube of Problem 18.21.

**PROBLEM 18.21** One of the sculptures displayed on a university campus consists of a hollow cube made of six aluminum sheets, each $1.5 \times 1.5$ m, welded together and reinforced with internal braces of negligible weight. The cube is mounted on a fixed base at *A* and can rotate freely about its vertical diagonal *AB*. As she passes by this display on the way to a class in mechanics, an engineering student grabs corner *C* of the cube and pushes it for 1.2 s in a direction perpendicular to the plane *ABC* with an average force of 50 N. Having observed that it takes 5 s for the cube to complete one full revolution, she flips out her calculator and proceeds to determine the mass of the cube. What is the result of her calculation? (*Hint:* The perpendicular distance from the diagonal joining two vertices of a cube to any of its other six vertices can be obtained by multiplying the side of the cube by $\sqrt{2/3}$.)

## SOLUTION

Let $m' = \frac{1}{6}m$ be the mass of one side of the cube. Choose *x*, *y*, and *z* axes perpendicular to the face of the cube. Let *a* be the side of the cube. For a side perpendicular to the *x* axis, $(I_x)_1 = \frac{1}{6}m'a^2$.

For a side perpendicular to the *y* or *z* axis, $(I_x)_2 = \left(\frac{1}{12} + \frac{1}{4}\right)m'a^2 = \frac{1}{3}m'a^2$

Total moment of inertia: $$I_x = 2(I_x)_1 + 4(I_x)_2 = \frac{5}{3}m'a^2 = \frac{5}{18}ma^2$$

By symmetry, $I_y = I_x$ and $I_z = I_x$.

Since all three moments of inertia are equal, the ellipsoid of inertia is a sphere. All centroidal axes are principal axes.

Moment of inertia about the vertical axis: $$I_v = \frac{5}{18}ma^2$$

Let $b = \sqrt{\frac{2}{3}}a$ be the moment arm of the impulse applied to the corner.

Using the impulse-momentum principle and taking moments about the vertical axis,

$$bF(\Delta t) = H_v = I_v\omega = \frac{5}{18}ma^2\omega \tag{1}$$

Data: $$a = 1.5 \text{ m}, \quad b = \sqrt{\frac{2}{3}}(1.5) = 1.22474 \text{ m}$$

$$\omega = \frac{2\pi}{5} = 1.25664 \text{ rad/s}, \quad F = 50 \text{ N}, \quad \Delta t = 1.2 \text{ s.}$$

## PROBLEM 18.50 (Continued)

Solving Equation (1) for $m$,

$$m = \frac{18}{5}\frac{bF(\Delta t)}{a^2\omega} = \frac{18}{5}\frac{(1.22474)(50)(1.2)}{(1.5)^2(1.25664)} = 93.563 \text{ kg}$$

$$m = 93.6 \text{ kg}$$

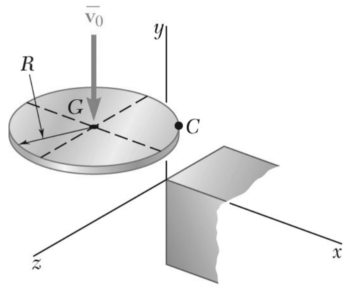
For principal axes,

$$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

$$T = \frac{1}{2}I_v\omega^2 = \frac{1}{2}\left(\frac{5}{18}ma^2\right)\omega^2$$

$$= \frac{1}{2}\frac{5}{18}(93.563)(1.5)^2(1.2566)^2$$

$$T = 46.2 \text{ J} \blacktriangleleft$$




## PROBLEM 18.51

Determine the kinetic energy lost when edge $C$ of the plate of Problem 18.29 hits the obstruction.

## SOLUTION

For principal moments of inertia, the kinetic energy is

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}_x\omega_x^2 + \frac{1}{2}\bar{I}_y\omega_y^2 + \frac{1}{2}\bar{I}_z\omega_z^2$$

Before impact: $\bar{v} = \bar{v}_0$

$$\omega_x = \omega_y = \omega_z = 0$$

$$T_0 = \frac{1}{2}m\bar{v}_0^2$$

After impact:

From Problem 18.30, $\bar{v}_x = \bar{v}_z = 0$

$$\bar{v}_y = -\frac{4}{5}\bar{v}_0$$

$$\bar{v} = \frac{4}{5}\bar{v}_0$$

From Problem 18.29, $\omega_x = \omega_z = \frac{2\sqrt{2}}{5}\frac{v_0}{R}$

$$\omega_y = 0$$

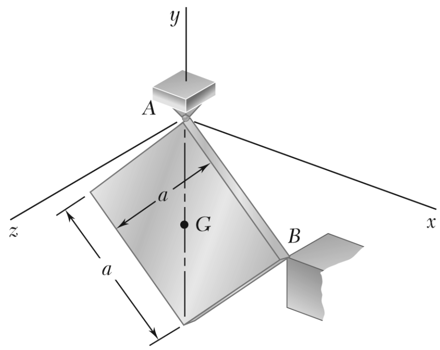
$$\bar{I}_x = \bar{I}_z = \frac{1}{4}mR^2$$

$$T = \frac{1}{2}m\left(\frac{4}{5}v_0\right)^2 + \left(\frac{1}{2}\right)\left(\frac{1}{4}mR^2\right)\left(\frac{2\sqrt{2}}{5}\frac{v_0}{R}\right)^2 + 0 + \left(\frac{1}{2}\right)\left(\frac{1}{4}mR^2\right)\left(\frac{2\sqrt{2}}{5}\frac{\bar{v}_0}{R}\right)^2$$

$$= \frac{1}{2}\left[\frac{16}{25} + \frac{2}{25} + 0 + \frac{2}{25}\right]m\bar{v}_0^2 = \frac{2}{5}m\bar{v}_0^2$$

Energy loss: $T_0 - T = \frac{1}{2}m\bar{v}_0^2 - \frac{2}{5}m\bar{v}_0^2$

$$T_0 - T = \frac{1}{10}m\bar{v}_0^2 \blacktriangleleft$$



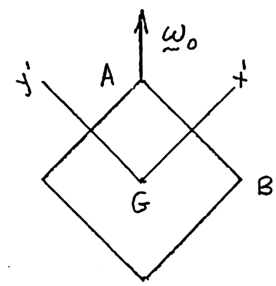

## PROBLEM 18.52

Determine the kinetic energy lost when the plate of Problem 18.31 hits the obstruction at $B$.

## SOLUTION

For the $x'$ and $y'$ axes shown, the initial angular velocity $\omega_0 \mathbf{j}$ has components

$$\omega_{x'} = \frac{\sqrt{2}}{2}\omega_0, \quad \omega_{y'} = \frac{\sqrt{2}}{2}\omega_0$$




Initial angular momentum about the mass center:

$$(\mathbf{H}_G)_0 = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}'$$

$$= \frac{1}{12}ma^2\frac{\sqrt{2}}{2}\omega_0(\mathbf{i}' + \mathbf{j}')$$

Initial velocity of the mass center: $\quad \bar{\mathbf{v}}_0 = 0$

Let $\boldsymbol{\omega}$ be the angular velocity and $\bar{\mathbf{v}}$ be the velocity of the mass center immediately after impact. Let $(F\Delta t)\mathbf{k}$ be the impulse at $B$.

Kinematics:

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A} = (\omega_{x'}\mathbf{i}' + \omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times (-a\mathbf{j}')$$

$$\mathbf{v}_B = a(\omega_{z'}\mathbf{i}' + \omega_{x'}\mathbf{k}')$$

Since the corner $B$ does not rebound, $\quad (v_B)_{z'} = 0 \quad$ or $\quad \omega_{x'} = 0$

$$\bar{\mathbf{v}} = \boldsymbol{\omega} \times \mathbf{r}_{G/A} = (\omega_{y'}\mathbf{j}' + \omega_{z'}\mathbf{k}') \times \left(\frac{1}{2}a\right)(-\mathbf{i}' - \mathbf{j}')$$

$$= \frac{1}{2}a(\omega_{z'}\mathbf{i}' - \omega_{z'}\mathbf{j} + \omega_y\mathbf{k}')$$

Also,

$$\mathbf{r}_{G/A} \times m\bar{\mathbf{v}} = \frac{1}{4}ma^2(-\omega_y\mathbf{i}' + \omega_{y'}\mathbf{j}' + 2\omega_{z'}\mathbf{k}')$$

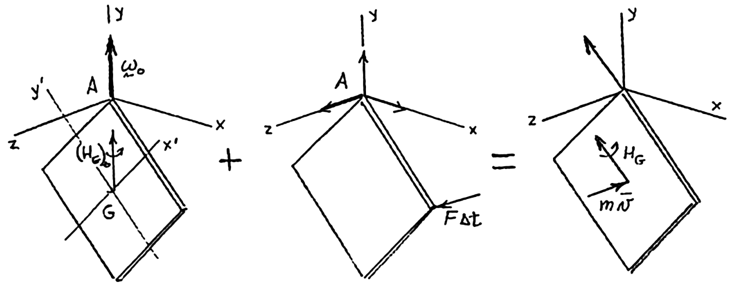
and

$$\mathbf{H}_G = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + \bar{I}_{z'}\omega_{z'}\mathbf{k}'$$

$$= \frac{1}{12}ma^2\omega_{y'}\mathbf{j}' + \frac{1}{6}ma^2\omega_{z'}\mathbf{k}'$$

## PROBLEM 18.52 (Continued)

Principle of impulse-momentum.




Moments about $A$:
$$(\mathbf{H}_A)_0 + (-a\mathbf{j}) \times (F\Delta t)\mathbf{k} = \mathbf{H}_A$$
$$(\mathbf{H}_G)_0 + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}}_0 - (aF\Delta t)\mathbf{i} = \mathbf{H}_G + \mathbf{r}_{G/A} \times m\bar{\mathbf{v}}$$

Resolve into components.

$$\mathbf{i}': \quad \frac{1}{24}\sqrt{2}ma^2\omega_0 - aF(\Delta t) = -\frac{1}{4}ma^2\omega_{y'}$$

$$\mathbf{j}': \quad \frac{1}{24}\sqrt{2}ma^2\omega_0 = \frac{1}{12}ma^2\omega_{y'} + \frac{1}{4}ma^2\omega_{y'} \quad \omega_{y'} = \frac{\sqrt{2}}{8}\omega_0$$

$$\mathbf{k}': \quad 0 = \frac{1}{6}ma^2\omega_{z'} + \frac{1}{2}ma^2\omega_{z'} \quad \omega_{z'} = 0$$

$$\boldsymbol{\omega} = \frac{\sqrt{2}}{8}\omega_0\mathbf{j}' = \frac{1}{8}\sqrt{2}\omega_0\frac{\sqrt{2}}{2}(\mathbf{j} - \mathbf{i})$$

$$\bar{\mathbf{v}} = \frac{1}{2}a\omega_y\mathbf{k}' = \frac{\sqrt{2}}{16}a\omega_0\mathbf{k}$$

Kinetic energy:
$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}_x\omega_{x'}^2 + \frac{1}{2}\bar{I}_y\omega_{y'}^2 + \frac{1}{2}\bar{I}_{z'}\omega_{z'}^2$$

Before impact:
$$\bar{v}_0 = 0, \quad \omega_{x'} = \frac{\sqrt{2}}{2}\omega_0, \quad \omega_{y'} = \frac{\sqrt{2}}{2}\omega_0, \quad \omega_{z'} = 0$$

$$T_0 = 0 + \frac{1}{2}\left(\frac{1}{12}ma^2\right)\left(\frac{\sqrt{2}}{2}\omega_0\right)^2 + \frac{1}{2}\left(\frac{1}{12}ma^2\right)\left(-\frac{\sqrt{2}}{2}\omega_0\right)^2 + 0 = \frac{1}{24}ma^2\omega_0^2$$

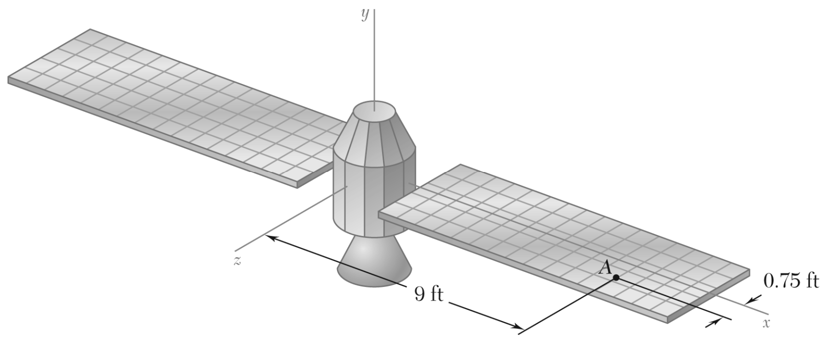
After impact:
$$\bar{v} = \frac{\sqrt{2}}{16}a\omega_0, \quad \omega_{x'} = 0, \quad \omega_{y'} = \frac{\sqrt{2}}{8}\omega_0, \quad \omega_{z'} = 0$$

$$T_1 = \frac{1}{2}m\left(\frac{\sqrt{2}}{16}a\omega_0\right)^2 + 0 + \frac{1}{2}\left(\frac{1}{12}ma^2\right)\left(\frac{\sqrt{2}}{8}\omega_0\right)^2 + 0 = \frac{1}{192}ma^2\omega_0^2$$

Kinetic energy lost.
$$T_0 - T_1 = \frac{7}{192}ma^2\omega_0^2 \blacktriangleleft$$

## PROBLEM 18.53

Determine the kinetic energy of the space probe of Problem 18.33 in its motion about its mass center after its collision with the meteorite.




## SOLUTION

Masses: Space probe: $m' = \dfrac{3000}{32.2} = 93.17 \text{ lb} \cdot \text{s}^2/\text{ft}$

Meteorite: $m = \dfrac{5}{(16)(32.2)} = 0.009705 \text{ lb} \cdot \text{s}^2/\text{ft}$

Point of impact: $\mathbf{r}_A = (9 \text{ ft})\mathbf{i} + (0.75 \text{ ft})\mathbf{k}$

Initial linear momentum of the meteorite, $(\text{lb} \cdot \text{s})$:

$$m\mathbf{v}_0 = (0.009705)(2400\mathbf{i} + 3000\mathbf{j} + 3200\mathbf{k}) = 23.292\mathbf{i} - 29.115\mathbf{j} + 31.056\mathbf{k}$$

Its moment about the origin, $(\text{lb} \cdot \text{ft} \cdot \text{s})$:

$$\mathbf{r}_A \times m\mathbf{v}_0 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 9 & 0 & 0.75 \\ 23.292 & -29.115 & 31.056 \end{vmatrix} = 21.836\mathbf{i} - 262.04\mathbf{j} - 262.04\mathbf{k}$$

Final linear momentum of meteorite and its moment about the origin, $(\text{lb} \cdot \text{s})$ and $(\text{lb} \cdot \text{s} \cdot \text{ft})$:

$$0.8m\mathbf{v}_0 = 18.634\mathbf{i} - 23.292\mathbf{j} + 24.845\mathbf{k}$$
$$\mathbf{r}_A \times (0.8m\mathbf{v}_0) = 17.469\mathbf{i} - 209.63\mathbf{j} - 209.63\mathbf{k}$$

## PROBLEM 18.53 (Continued)

Let $\mathbf{H}_A$ be the angular momentum of the probe and $m'$ be its mass. Conservation of angular momentum about the origin for a system of particles consisting of the probe plus the meteorite:

$$\mathbf{r}_A \times m\mathbf{v}_0 = \mathbf{H}_A + \mathbf{r}_A \times (0.8m\mathbf{v}_0)$$

$$\mathbf{H}_A = (4.367 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{i} - (52.41 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{j} - (52.41 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{k}$$

$$I_x\omega_x = (H_A)_x \qquad \omega_x = \frac{(H_A)_x}{m'k_x^2} = \frac{4.367}{(93.17)(1.375)^2} = 0.02479 \text{ rad/s}$$

$$I_y\omega_y = (H_A)_y \qquad \omega_y = \frac{(H_A)_y}{m'k_y^2} = \frac{-52.41}{(93.17)(1.425)^2} = -0.2770 \text{ rad/s}$$

$$I_z\omega_z = (H_A)_z \qquad \omega_z = \frac{(H_A)_z}{m'k_z^2} = \frac{-52.41}{(93.17)(1.250)^2} = -0.3600 \text{ rad/s}$$

Kinetic energy of motion of the probe about its mass center:

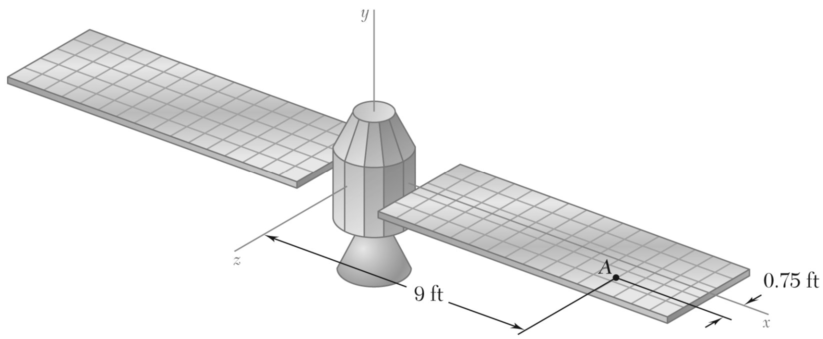
$$T' = \frac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right) = \frac{m}{2}\left(k_x^2\omega_x^2 + k_y^2\omega_y^2 + k_z^2\omega_z^2\right)$$

$$= \frac{93.17}{2}[(1.375)^2(0.02479)^2 + (1.425)^2(-0.2770)^2 + (1.250)^2(-0.3600)^2]$$

$$= 16.75 \text{ ft}\cdot\text{lb} \qquad T' = 16.75 \text{ ft}\cdot\text{lb} \blacktriangleleft$$

## PROBLEM 18.54

Determine the kinetic energy of the space probe of Problem 18.34 in its motion about its mass center after its collision with the meteorite.




## SOLUTION

Masses: Space probe: $m' = \dfrac{3000}{32.2} = 93.17 \text{ lb}\cdot\text{s}^2/\text{ft}$

Meteorite: $m = \dfrac{5}{(16)(32.2)} = 0.009705 \text{ lb}\cdot\text{s}^2/\text{ft}$

Point of impact: $\mathbf{r}_A = (9 \text{ ft})\mathbf{i} + (0.75 \text{ ft})\mathbf{k}$

Initial linear momentum of the meteorite, $(\text{lb}\cdot\text{s})$:

$$m\mathbf{v}_0 = (0.009705)(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})$$

Its moment about the origin, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$(\mathbf{H}_A)_0 = \mathbf{r}_A \times m\mathbf{v}_0 = 0.009705\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 9 & 0 & 0.75 \\ v_x & v_y & v_z \end{vmatrix}$$

$$= 0.009705[-0.75v_y\mathbf{i} + (0.75v_x - 9v_z)\mathbf{j} + 9v_y\mathbf{k}]$$

Final linear momentum of the meteorite, $(\text{lb}\cdot\text{s})$:

$$0.75m\mathbf{v}_0 = 0.007279(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})$$

Its moment about the origin, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$\mathbf{r}_A \times (0.75m\mathbf{v}_0) = 0.007279[-0.75v_y\mathbf{i} + (0.75v_x - 9v_z)\mathbf{j} + 9v_y\mathbf{k}]$$

Initial linear momentum of the space probe, $(\text{lb}\cdot\text{s})$: $m'\mathbf{v}'_0 = 0$

Final linear momentum of the space probe, $(\text{lb}\cdot\text{s})$:

$$m'(v'_x\mathbf{i}+v'_y\mathbf{j}+v'_z\mathbf{k})=93.17\left(-\frac{0.675}{12}\mathbf{i}+v'_y\mathbf{j}+v'_z\mathbf{k}\right)$$

Final angular momentum of space probe, $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$\begin{aligned}\mathbf{H}_A&=m'(k_x^2\omega_x\mathbf{i}+k_y^2\omega_y\mathbf{j}+k_z^2\omega_z\mathbf{k})\\&=93.17[(1.375)^2(0.05)\mathbf{i}+(1.425)^2(-0.12)\mathbf{j}+(1.250)^2\omega_z\mathbf{k}]\\&=8.8075\mathbf{i}-22.703\mathbf{j}+145.58\omega_z\mathbf{k}\end{aligned}$$

Conservation of linear momentum of the probe plus the meteorite $(\text{lb}\cdot\text{s})$:

$$0.009705(v_x\mathbf{i}+v_y\mathbf{j}+v_z\mathbf{k})=0.007279(v_x\mathbf{i}+v_y\mathbf{j}+v_z\mathbf{k})+93.17(-0.05625\mathbf{i}+v'_y\mathbf{j}+v'_z\mathbf{k})$$

$$\mathbf{i}:\quad 0.002426v_x=-5.2408\qquad v_x=-2160\text{ ft/s}$$

$$\mathbf{j}:\quad 0.002426v_y=93.17v'_y$$

$$\mathbf{k}:\quad 0.002426v_z=93.17v'_z$$

Conservation of angular momentum about the origin $(\text{lb}\cdot\text{ft}\cdot\text{s})$:

$$\begin{aligned}(0.009705)[-0.75v_y\mathbf{i}+(0.75v_x-9v_z)\mathbf{j}+9v_y\mathbf{k}]&=(0.007279)[-0.75v_y\mathbf{i}+(0.75v_x-9v_z)\mathbf{j}+9v_y\mathbf{k}]\\&\quad+8.8075\mathbf{i}-22.703\mathbf{j}+145.58\omega_z\mathbf{k}\end{aligned}$$

$$\mathbf{i}:\quad -0.0018195v_y=8.8075\qquad v_y=-4840.5\text{ ft/s}$$

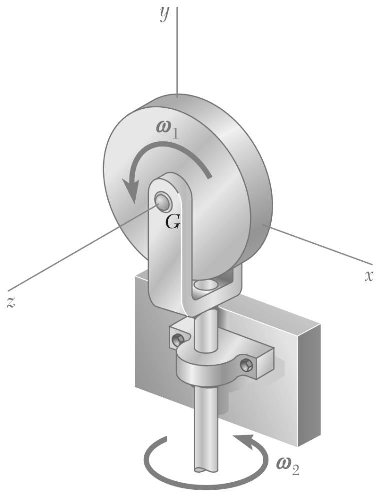
$$\mathbf{k}:\quad -0.021834v_y=145.58\omega_z$$

$$\omega_z=-149.98\times10^{-6}v_y\qquad \omega_z=-0.726\text{ rad/s}$$

Kinetic energy of motion of probe relative to its mass center:

$$\begin{aligned}T'&=\frac{1}{2}\left(I_x\omega_x^2+I_y\omega_y^2+I_z\omega_z^2\right)=\frac{1}{2}m\left(k_x^2\omega_x^2+k_y^2\omega_z^2\omega_z^2\right)\\&=\frac{1}{2}(93.17)[(1.375)^2(0.05)^2+(1.425)^2(-0.12)^2+(1.250)^2(-0.726)^2]\\&=39.9\text{ ft}\cdot\text{lb}\end{aligned}$$

$T'=39.9\text{ ft}\cdot\text{lb}$ ◀




## PROBLEM 18.55

Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Problem 18.1.

**PROBLEM 18.1** A thin, homogeneous disk of mass $m$ and radius $r$ spins at the constant rate $\omega_1$ about an axle held by a fork-ended vertical rod, which rotates at the constant rate $\omega_2$. Determine the angular momentum $\mathbf{H}_G$ of the disk about its mass center $G$.

## SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega_2 \mathbf{j} + \omega_1 \mathbf{k}$

Moments of inertia: $\bar{I}_x = \frac{1}{4}mr^2, \quad \bar{I}_y = \frac{1}{4}mr^2, \quad \bar{I}_z = \frac{1}{2}mr^2$

Products of inertia: by symmetry, $\bar{I}_{xy} = \bar{I}_{yz} = \bar{I}_{zx} = 0$

Angular momentum:

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

$$\mathbf{H}_G = 0 + \left(\frac{1}{4}mr^2\right)\omega_2\mathbf{j} + \frac{1}{2}mr^2\omega_1\mathbf{k}$$

$$\mathbf{H}_G = \frac{1}{4}mr^2\omega_2\mathbf{j} + \frac{1}{2}mr^2\omega_1\mathbf{k}$$

Rate of change of angular momentum. Let the frame of reference $Gxyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega_1\mathbf{j}$$

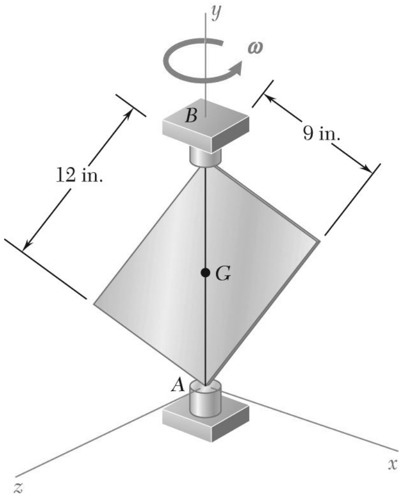
Then

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$

$$= 0 + \omega_2\mathbf{j} \times \left(\frac{1}{4}mr^2\omega_2\mathbf{j} + \frac{1}{2}mr^2\omega_1\mathbf{k}\right)$$

$$= \frac{1}{2}mr^2\omega_1\omega_2\mathbf{i}$$

$\dot{\mathbf{H}}_G = \frac{1}{2}mr^2\omega_1\omega_2\mathbf{i}$ ◀



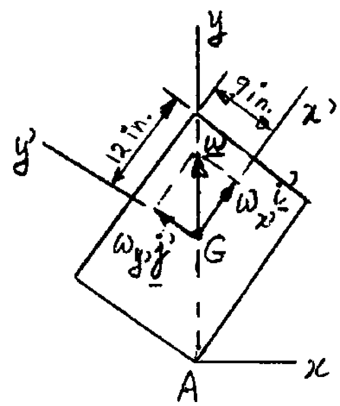

## PROBLEM 18.56

Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the plate of Problem 18.2.

**PROBLEM 18.2** A thin rectangular plate of weight 15 lb rotates about its vertical diagonal $AB$ with an angular velocity $\omega$. Knowing that the $z$ axis is perpendicular to the plate and that $\omega$ is constant and equal to 5 rad/s, determine the angular momentum of the plate about its mass center $G$.

## SOLUTION

$$h = \sqrt{(9\text{ in.})^2 + (12\text{ in.})^2} = 15\text{ in.}$$




Resolving $\omega$ along the principal axes $x'$, $y'$, $z$:

$$\omega_{x'} = \frac{12}{15}\omega = 0.8(5\text{ rad/s}) = 4\text{ rad/s}$$

$$\omega_{y'} = \frac{9}{15}\omega = 0.6(5\text{ rad/s}) = 3\text{ rad/s}$$

$$\omega_z = 0$$

Moments of inertia:

$$I_{x'} = \frac{1}{12}\left(\frac{15\text{ lb}}{32.2}\right)\left(\frac{9}{12}\text{ft}\right)^2 = 0.021836\text{ slug}\cdot\text{ft}^2$$

$$I_{y'} = \frac{1}{12}\left(\frac{15\text{ lb}}{32.2}\right)\left(\frac{12}{12}\text{ft}\right)^2 = 0.038820\text{ slug}\cdot\text{ft}^2$$

From Eqs. (18.10):

$$H_{x'} = I_{x'}\omega_{x'} = (0.021836)(4) = 0.087345\text{ slug ft}^2/\text{s}$$

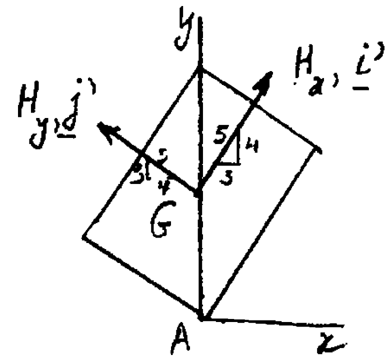
$$H_{y'} = I_{y'}\omega_{y'} = (0.038820)(3) = 0.11646\text{ slug ft}^2/\text{s}$$

$$H_z = I_z\omega_z = 0$$

$$\mathbf{H}_G = (0.087345\text{ slug ft}^2/\text{s})\mathbf{i}' + (0.11646\text{ slug ft}^2/\text{s})\mathbf{j}'$$

## PROBLEM 18.56 (Continued)

Components along $x$ and $y$ axes:




$$H_x = \frac{3}{5}H_{x'} - \frac{4}{5}H_{y'}$$
$$= \frac{3}{5}(0.087345) - \frac{4}{5}(0.11646)$$
$$= -0.040761$$
$$H_y = \frac{4}{5}H_{x'} + \frac{3}{5}H_{y'}$$
$$= \frac{4}{5}(0.087345) + \frac{3}{5}(0.11646)$$
$$= 0.13975$$

$$\mathbf{H}_G = (-0.040761\ \text{slug}\cdot\text{ft}^2/\text{s})\mathbf{i} + (0.13975\ \text{slug}\cdot\text{ft}^2/\text{s})\mathbf{j}$$

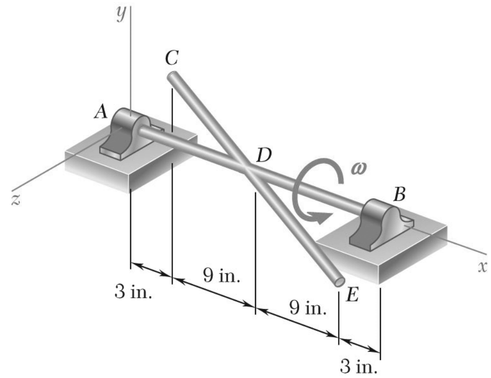
where the frame $Axyz$ rotates with the plate with the angular velocity.

$$\boldsymbol{\Omega} = \boldsymbol{\omega} = (5\ \text{rod/s})\mathbf{j}$$

We have $(\dot{\mathbf{H}}_G)_{Axyz} = 0.$ Substituting into Eq. (18.22):

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_G = 0 + 5\mathbf{j}\times(-0.040761\mathbf{i} + 0.13975\mathbf{j})$$
$$= (0.20380\ \text{ft/lb})\mathbf{k}$$

$$\dot{\mathbf{H}}_G = (0.204\ \text{ft/lb})\mathbf{k} \blacktriangleleft$$




## PROBLEM 18.57

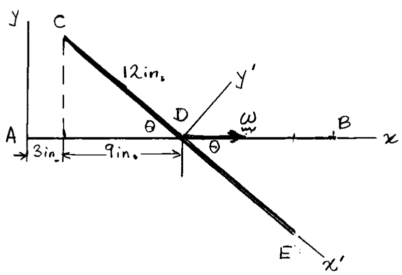
Determine the rate of change $\dot{\mathbf{H}}_D$ of the angular momentum $\mathbf{H}_D$ of the assembly of Problem 18.3.

**PROBLEM 18.3** Two uniform rods *AB* and *CE*, each of weight 3 lb and length 2 ft, are welded to each other at their midpoints. Knowing that this assembly has an angular velocity of constant magnitude $\omega = 12$ rad/s, determine the magnitude and direction of the angular momentum $\mathbf{H}_D$ of the assembly about *D*.

## SOLUTION

$$m = \frac{W}{g} = \frac{3}{32.2} = 0.093168 \text{ lb}\cdot\text{s}^2/\text{ft}, \qquad l = 24 \text{ in.} = 2 \text{ ft},$$

$$\boldsymbol{\omega} = (12 \text{ rad/s})\mathbf{i}$$




For rod *ADB*, $\quad \mathbf{H}_D = I_x \omega \mathbf{i} \approx 0, \quad$ since $I_x \approx 0.$

For rod *CDE*, use principal axes $x'$, $y'$ as shown.

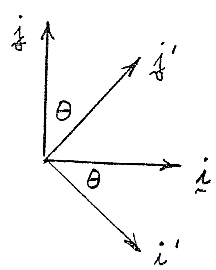
$$\cos\theta = \frac{9}{12}, \qquad \theta = 41.410°$$

$$\omega_{x'} = \omega\cos\theta = 9 \text{ rad/s}^2$$

$$\omega_{y'} = \omega\sin\theta = 7.93725 \text{ rad/s}^2$$

$$\omega_{z'} = 0$$

$$I_{x'} \approx 0$$




$$I_{y'} = \frac{1}{12}ml^2 = \frac{1}{12}(0.093168)(2)^2$$

$$= 0.0310559 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\mathbf{H}_D = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + I_{z'}\omega_{z'}\mathbf{k}'$$

$$= 0 + (0.0310559)(7.93725)\mathbf{j}' + 0$$

$$= 0.246498\mathbf{j}'$$

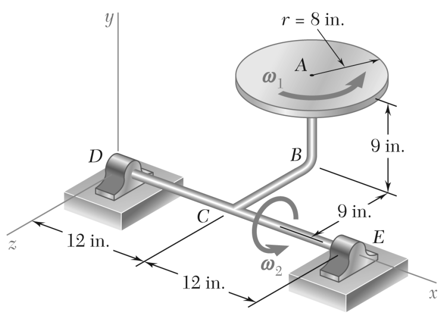
$$\mathbf{H}_D = 0.246498(\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = 0.163045\mathbf{i} + 0.184874\mathbf{j}$$

Let the frame of reference *Dxyz* be rotating with angular velocity

$$\boldsymbol{\Omega} = \boldsymbol{\omega} = (12 \text{ rad/s})\mathbf{i}$$

Then, $\quad \dot{\mathbf{H}}_D = (\dot{\mathbf{H}}_D)_{Dxyz} + \boldsymbol{\Omega}\times\mathbf{H}_D = 0 + \boldsymbol{\omega}\times\mathbf{H}_D$

$$\dot{\mathbf{H}}_D = 12\mathbf{i}\times(0.163045\mathbf{i} + 0.184874\mathbf{j}) \qquad \dot{\mathbf{H}}_D = (2.22 \text{ lb}\cdot\text{ft})\mathbf{k} \blacktriangleleft$$




## PROBLEM 18.58

Determine the rate of change $\dot{\mathbf{H}}_A$ of the angular momentum $\mathbf{H}_A$ of the disk of Problem 18.4.

**PROBLEM 18.4** A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm *ABC*, which is welded to a shaft *DCE* rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the angular momentum $\mathbf{H}_A$ of the disk about its center *A*.

## SOLUTION

$$\boldsymbol{\omega} = \omega_2\mathbf{i} + \omega_1\mathbf{j} = (8\text{ rad/s})\mathbf{i} + (16\text{ rad/s})\mathbf{j}$$

For axes $x'$, $y'$, $z'$ parallel to $x$, $y$, $z$ with origin at $A$,

$$m = \frac{W}{g} = \frac{6}{32.2} = 0.186335\text{ lb}\cdot\text{s}^2/\text{ft}$$

$$\bar{I}_{x'} = \frac{1}{4}mr^2 = \frac{1}{4}(0.186335)\left(\frac{8}{12}\right)^2 = 0.020704\text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_{z'} = \bar{I}_{x'} = 0.020704\text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_{y'} = \bar{I}_{x'} + \bar{I}_{z'} = 0.041408\text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\begin{aligned}\mathbf{H}_A &= \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\,\omega_{y'}\mathbf{j} + I_{z'}\,\omega_{z'}\mathbf{k} \\ &= (0.020704)(8)\mathbf{i} + (0.041408)(16)\mathbf{j} \\ &= (0.1656\text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{i} + (0.6625\text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{j}\end{aligned}$$

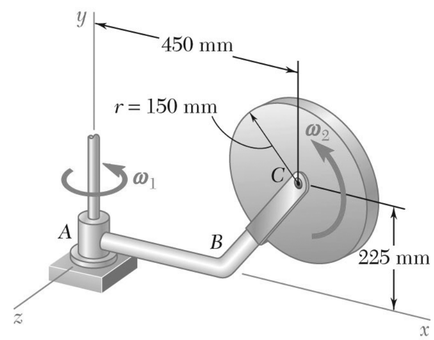
Let the frame of reference *Axyz* be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega_2\mathbf{i} = (8\text{ rad/s})\mathbf{i}$$

Then
$$\dot{\mathbf{H}}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_A = 0 + \omega_2\mathbf{i}\times\mathbf{H}_A$$

$$\dot{\mathbf{H}}_A = 8\mathbf{i}\times(0.1656\mathbf{i} + 0.6625\mathbf{j}) = (5.30\text{ lb}\cdot\text{ft})\mathbf{k}$$

$\dot{\mathbf{H}}_A = (5.30\text{ lb}\cdot\text{ft})\mathbf{k}$ ◀




## PROBLEM 18.59

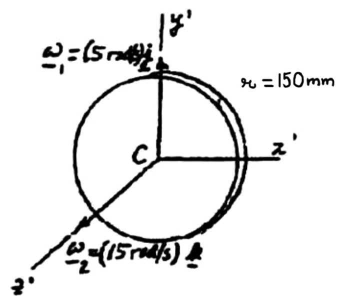
Determine the rate of change $\dot{\mathbf{H}}_C$ of the angular momentum $\mathbf{H}_C$ of the disk of Problem 18.5.

**PROBLEM 18.5** A thin disk of mass $m = 4\text{ kg}$ rotates at the constant rate $\omega_2 = 15\text{ rad/s}$ with respect to arm *ABC*, which itself rotates at the constant rate $\omega_1 = 5\text{ rad/s}$ about the *y* axis. Determine the angular momentum of the disk about its center *C*.

## SOLUTION

$$r = 150\text{ mm}$$

Angular velocity of disk:




$$\boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{k}$$
$$= (5\text{ rad/s})\mathbf{j} + (15\text{ rad/s})\mathbf{k}$$

Centroidal moments of inertia:

$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4}mr^2$$
$$= \frac{1}{4}(4)(0.150\text{ m})^2 = 0.0225\text{ kg}\cdot\text{m}^2$$
$$\bar{I}_{z'} = \frac{1}{2}mr^2 = 0.045\text{ kg}\cdot\text{m}^2$$

Angular momentum about *C*.

$$\mathbf{H}_C = \bar{I}_{x'}\,\omega_{x'}\mathbf{i} + \bar{I}_{y'}\,\omega_{y'}\mathbf{j} + I_{z'}\,\omega_{z'}\mathbf{k}$$
$$= 0 + (0.0225)(5)\mathbf{j} + (0.045)(15)\mathbf{k}$$
$$= (0.1125\text{ kg}\cdot\text{m}^2\text{/s})\mathbf{j} + (0.6750\text{ kg}\cdot\text{m}^2\text{/s})\mathbf{k}$$

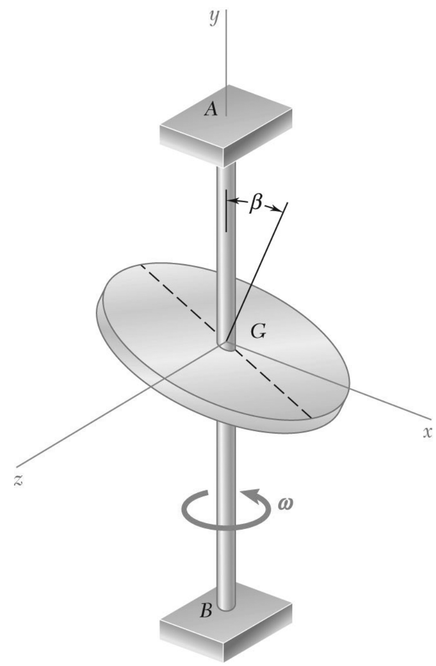
Rate of change of angular momentum. Let the reference frame *Oxyz* be rotating with angular velocity.

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_1 = (5\text{ rad/s})\mathbf{j}$$

Then

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Oxyz} + \boldsymbol{\Omega}\times\mathbf{H}_C$$
$$= 0 + 5\mathbf{j}\times(0.1125\mathbf{j} + 0.6750\mathbf{k})$$
$$= (3.3750\text{ N}\cdot\text{m})\mathbf{i}$$

$\dot{\mathbf{H}}_C = (3.38\text{ N}\cdot\text{m})\mathbf{i}$ ◀



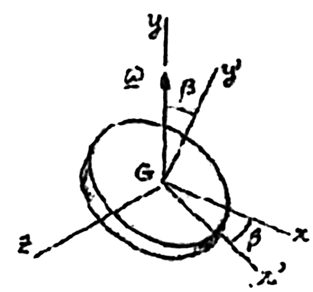

## PROBLEM 18.60

Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Problem 18.8.

**PROBLEM 18.8** A homogeneous disk of mass $m$ and radius $r$ is mounted on the vertical shaft $AB$. The normal to the disk at $G$ forms an angle $\beta = 25°$ with the shaft. Knowing that the shaft has a constant angular velocity $\omega$, determine the angle $\theta$ formed by the shaft $AB$ and the angular momentum $\mathbf{H}_G$ of the disk about its mass center $G$.

## SOLUTION

Use the principal centroidal axes $Gx'y'z'$,




Moments of inertia.

$$\bar{I}_{x'} = \bar{I}_z = \frac{1}{4}mr^2$$

$$\bar{I}_{y'} = \frac{1}{2}mr^2$$

Angular velocity.

$$\omega_{x'} = -\omega \sin\beta$$

$$\omega_{y'} = \omega \cos\beta$$

$$\omega_z = 0$$

Angular momentum about $G$.

Using Eqs. (18.10),

$$H_{x'} = \bar{I}_{x'}\omega_{x'} = -\frac{1}{4}mr^2\omega\sin\beta$$

$$H_{y'} = \bar{I}_{y'}\omega_{y'} = \frac{1}{2}mr^2\omega\cos\beta$$

$$H_z = \bar{I}_z\omega_z = 0$$

$$\mathbf{H}_G = H_{x'}\mathbf{i}' + H_{y'}\mathbf{j}' + H_z\mathbf{k}$$

## PROBLEM 18.60 (Continued)

where $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}$ are the unit vectors along the $x'y'z$ axes.

$$\mathbf{H}_G = -\frac{1}{4}mr^2\omega\sin\beta\mathbf{i}' + \frac{1}{2}mr^2\omega\cos\beta\mathbf{j}'$$

<u>Rate of change of angular momentum</u>. Let the frame of reference $Gx'y'z'$ be rotating with angular velocity

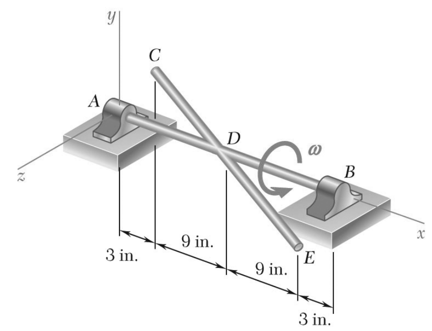
$$\mathbf{\Omega} = \omega\mathbf{j} = \omega(-\sin\beta\mathbf{i}' + \cos\beta\mathbf{j}')$$

Then

$$\begin{aligned}\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega}\times\mathbf{H}_G \\ &= 0 + \omega(-\sin\beta\mathbf{i}' + \cos\beta\mathbf{j}')\times\left(-\frac{1}{4}mr^2\omega\sin\beta\mathbf{i}' + \frac{1}{2}mr^2\omega\cos\beta\mathbf{j}'\right) \\ &= \frac{1}{4}mr^2\omega^2\cos\beta\sin\beta\mathbf{k} - \frac{1}{2}mr^2\omega^2\sin\beta\cos\beta\mathbf{k} \\ &= -\frac{1}{4}mr^2\omega^2\sin\beta\cos\beta\mathbf{k}\end{aligned}$$

With $\beta = 25°$,

$$\dot{\mathbf{H}}_G = -\frac{1}{4}mr^2\omega^2\sin 25°\cos 25°\mathbf{k} \qquad \dot{\mathbf{H}}_G = -0.0958mr^2\omega^2\mathbf{k} \blacktriangleleft$$




## PROBLEM 18.61

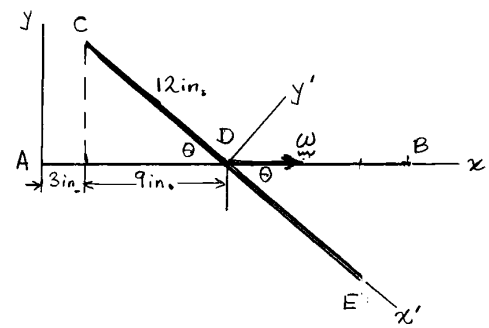
Determine the rate of change $\dot{\mathbf{H}}_D$ of the angular momentum $\mathbf{H}_D$ of the assembly of Problem 18.3, assuming that at the instant considered the assembly has an angular velocity $\boldsymbol{\omega} = (12\text{ rad/s})\mathbf{i}$ and an angular acceleration $\alpha = -(96\text{ rad/s}^2)\mathbf{i}$.

## SOLUTION

$$m = \frac{W}{g} = \frac{3}{32.2} = 0.093168\text{ lb}\cdot\text{s}^2/\text{ft}^2, \quad l = 24\text{ in.} = 2\text{ ft}, \quad \boldsymbol{\omega} = (12\text{ rad/s})\mathbf{i}$$

For rod *ADB*, $\mathbf{H}_D = I_x\omega\mathbf{i} \approx 0,$ since $I_x \approx 0.$

For rod *CDE*, use principal axes $x', y'$ as shown.




$$\cos\theta = \frac{9}{12}, \quad \theta = 41.410°$$

$$\omega_{x'} = \omega\cos\theta = 9\text{ rad/s}^2$$

$$\omega_{y'} = \omega\sin\theta = 7.93725\text{ rad/s}^2$$

$$\omega_{z'} = 0$$

$$\alpha = -(96\text{ rad/s}^2)\mathbf{i}$$

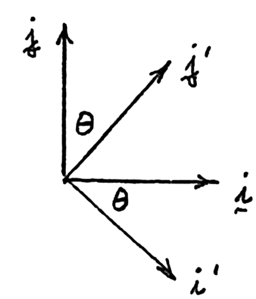
$$\alpha_{x'} = \alpha\cos\theta = -72\text{ rad/s}$$

$$\alpha_{y'} = \alpha\sin\theta = -63.498\text{ rad/s}^2$$

$$I_{x'} \approx 0$$

$$I_{y'} = \frac{1}{12}ml^2 = \frac{1}{12}(0.93168)(2)^2 = 0.0310559\text{ kg}\cdot\text{m}^2$$

$$I_{z'} = I_{y'}$$




$$\mathbf{H}_D = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + I_{z'}\omega_{z'}\mathbf{k}'$$

$$= 0 + (0.0310559)(7.93725)\mathbf{j}' + 0 = (0.24698\text{ lb}\cdot\text{s/ft})\mathbf{j}'$$

Let the reference frame $Dx'y'z'$ be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega\mathbf{i} = \omega_{x'}\mathbf{i}' + \omega_{y'}\mathbf{j}' = (9\text{ rad/s})\mathbf{i}' + (7.93725\text{ rad/s})\mathbf{j}'$$

$$\dot{\mathbf{H}}_D = (\dot{\mathbf{H}}_D)_{Dx'y'z'} + \boldsymbol{\Omega}\times\mathbf{H}_D$$

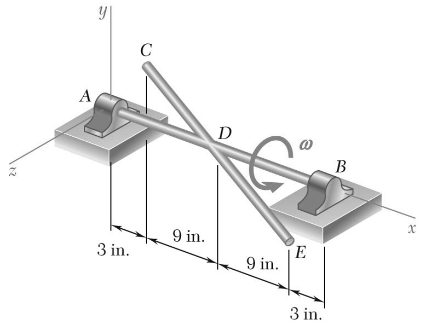
$$= I_{x'}\alpha_{x'}\mathbf{i}' + I_{y'}\alpha_{y'}\mathbf{j}' + I_{z'}\alpha_{z'}\mathbf{k}' + \boldsymbol{\Omega}\times\mathbf{H}_D$$

$$= 0 + (0.0310559)(-63.498)\mathbf{j} + 0 + (9\mathbf{i}' + 7.93725\mathbf{j}')\times(0.246498\mathbf{j}')$$

$$= -1.97199\mathbf{j}' + 2.21848\mathbf{k}$$

$$= -(1.97199\text{ kg}\cdot\text{m}^2/\text{s}^2)(\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) + (2.21848\text{ kg}\cdot\text{m}^2/\text{s}^2)\mathbf{k}$$

$$\dot{\mathbf{H}}_D = -(1.304\text{ N}\cdot\text{m})\mathbf{i} - (1.479\text{ N}\cdot\text{m})\mathbf{j} + (2.22\text{ N}\cdot\text{m})\mathbf{k} \blacktriangleleft$$




## PROBLEM 18.62

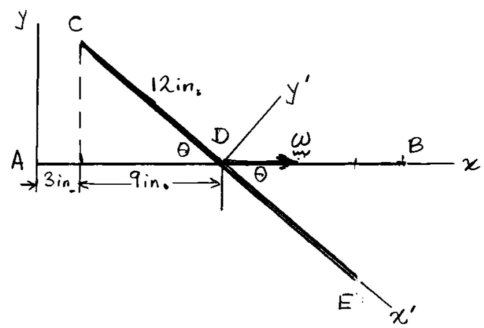
Determine the rate of change $\dot{\mathbf{H}}_D$ of the angular momentum $\mathbf{H}_D$ of the assembly of Problem 18.3, assuming that at the instant considered the assembly has an angular velocity $\boldsymbol{\omega} = (12 \text{ rad/s})\mathbf{i}$ and an angular acceleration $\boldsymbol{\alpha} = (96 \text{ rad/s}^2)\mathbf{i}$.

## SOLUTION

$$m = \frac{W}{g} = \frac{3}{32.2} = 0.093168 \text{ lb}\cdot\text{s}^2/\text{ft}, \quad l = 24 \text{ in.} = 2 \text{ ft}, \quad \boldsymbol{\omega} = (12 \text{ rad/s})\mathbf{i}$$

For rod $ADB$, $\mathbf{H}_D = I_x\omega\mathbf{i} \approx 0$, since $I_x \approx 0$.

For rod $CDE$, use principal axes $x'$, $y'$ as shown.




$$\cos\theta = \frac{9}{12}, \quad \theta = 41.410°$$

$$\omega_{x'} = \omega\cos\theta = 9 \text{ rad/s}^2$$

$$\omega_{y'} = \omega\sin\theta = 7.93725 \text{ rad/s}^2$$

$$\omega_{z'} = 0$$

$$\alpha = (96 \text{ rad/s}^2)\mathbf{i}$$

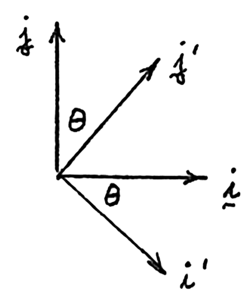
$$\alpha_{x'} = \alpha\cos\theta = 72 \text{ rad/s}$$

$$\alpha_{y'} = \alpha\sin\theta = 63.498 \text{ rad/s}^2$$

$$I_{x'} \approx 0$$

$$I_{y'} = \frac{1}{12}ml^2 = \frac{1}{12}(0.93168)(2)^2 = 0.0310559 \text{ kg}\cdot\text{m}^2$$

$$I_{z'} = I_{y'}$$




$$\mathbf{H}_D = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + I_{z'}\omega_{z'}\mathbf{k}'$$
$$= 0 + (0.0310559)(7.93725)\mathbf{j} + 0 = (0.246498 \text{ lb}\cdot\text{s/ft})\mathbf{j}'$$

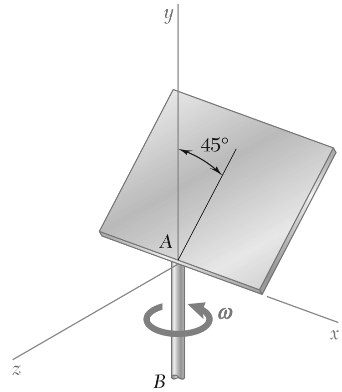
Let the reference frame $Dx'y'z'$ be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega\mathbf{i} = \omega_{x'}\mathbf{i}' + \omega_{y'}\mathbf{j}' = (9 \text{ rad/s})\mathbf{i}' + (7.93725 \text{ rad/s})\mathbf{j}'$$

$$\dot{\mathbf{H}}_D = (\dot{\mathbf{H}}_D)_{Dx'y'z'} + \boldsymbol{\Omega}\times\mathbf{H}_D$$
$$= I_{x'}\alpha_{x'}\mathbf{i}' + I_{y'}\alpha_{y'}\mathbf{j}' + I_{z'}\alpha_{z'}\mathbf{k}' + \boldsymbol{\Omega}\times\mathbf{H}_D$$
$$= 0 + (0.310559)(63.498)\mathbf{j} + 0 + (9\mathbf{i}' + 7.93725\mathbf{j}')\times(0.246498\mathbf{j}')$$
$$= 1.97199\mathbf{j}' + 2.21848\mathbf{k}$$
$$= (1.97199 \text{ kg}\cdot\text{m}^2/\text{s}^2)(\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) + (2.21848 \text{ kg}\cdot\text{m}^2/\text{s}^2)\mathbf{k}$$

$$\dot{\mathbf{H}}_D = (1.304 \text{ N}\cdot\text{m})\mathbf{i} + (1.479 \text{ N}\cdot\text{m})\mathbf{j} + (2.22 \text{ N}\cdot\text{m})\mathbf{k} \blacktriangleleft$$

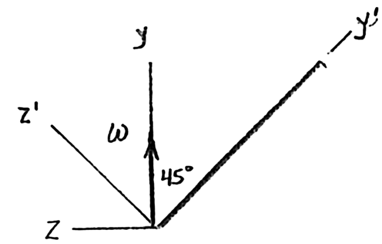
## PROBLEM 18.63




A thin homogeneous square of mass $m$ and side $a$ is welded to a vertical shaft $AB$ with which it forms an angle of 45°. Knowing that the shaft rotates with an angular velocity $\boldsymbol{\omega} = \omega \mathbf{j}$ and an angular acceleration $\boldsymbol{\alpha} = \alpha \mathbf{j}$, determine the rate of change $\dot{\mathbf{H}}_A$ of the angular momentum $\mathbf{H}_A$ of the plate assembly.

## SOLUTION

Use principal axes $y'$, $z'$ as shown.

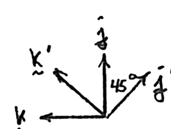



$$\omega_{y'} = \omega \cos 45°, \quad \omega_{z'} = \omega \sin 45°$$

$$\omega_{x'} = 0$$

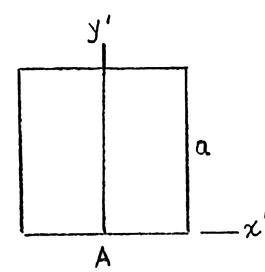
$$I_{x'} = \frac{1}{3} ma^2, \quad I_{y'} = \frac{1}{12} ma^2$$

$$I_{z'} = I_{x'} + I_{y'} = \frac{5}{12} ma^2$$




$$\mathbf{H}_A = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}' + I_{z'}\omega_{z'}\mathbf{k}'$$

$$= 0 + \left(\frac{1}{12} ma^2\right)(\omega \cos 45°)\mathbf{j}' + \left(\frac{5}{12} ma^2\right)(\omega \sin 45°)\mathbf{k}'$$




$$\mathbf{H}_A = \left(\frac{1}{12} ma^2\right)(\omega \cos 45°)(\cos 45° \mathbf{j} - \sin 45° \mathbf{k})$$

$$+ \left(\frac{5}{12} ma^2\right)(\omega \sin 45°)(\sin 45° \mathbf{j} + \cos 45° \mathbf{k})$$

$$\mathbf{H}_A = \frac{ma^2\omega}{12}(3\mathbf{j} + 2\mathbf{k})$$

<u>Angular acceleration</u>. $\quad \boldsymbol{\alpha} = \alpha \mathbf{j}; \quad \dot{\omega}_{y'} = \alpha \cos 45°, \quad \dot{\omega}_{z'} = \alpha \sin 45°$

Let the reference frame $Axyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega \mathbf{j}$$

With respect to this frame,

$$(\dot{\mathbf{H}}_A)_{Axyz} = I_{x'}\dot{\omega}_{x'}\mathbf{i}' + I_{y'}\dot{\omega}_{y'}\mathbf{j}' + I_{z'}\dot{\omega}_{z}\mathbf{k}'$$

$$= 0 + \left(\frac{1}{12} ma^2\right)(\alpha \cos 45°)\mathbf{i}' + \left(\frac{5}{12} ma^2\right)(\alpha \sin 45°)\mathbf{k}'$$

$$= \frac{ma^2\alpha}{12}(3\mathbf{j} + 2\mathbf{k})$$

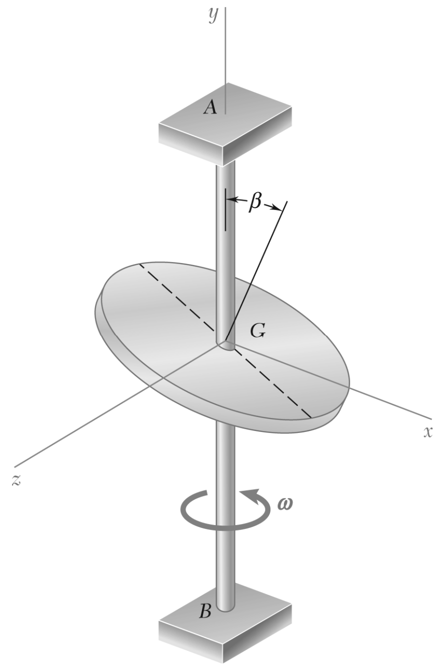
## PROBLEM 18.63 (Continued)

With respect to the fixed reference frame,

$$\dot{\mathbf{H}}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega} \times \mathbf{H}_A$$

$$= \frac{ma^2\alpha}{12}(3\mathbf{j} + 2\mathbf{k}) + \omega\mathbf{j} \times \left[\frac{ma^2\omega}{12}(3\mathbf{j} + 2\mathbf{k})\right]$$

$$\dot{\mathbf{H}}_A = \frac{ma^2}{12}(2\omega^2\mathbf{i} + 3\alpha\mathbf{j} + 2\alpha\mathbf{k}) \blacktriangleleft$$



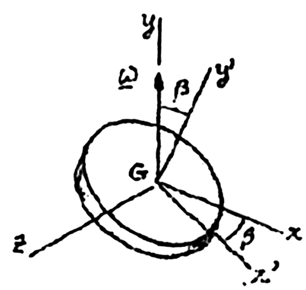

## PROBLEM 18.64

Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Problem 18.8, assuming that at the instant considered the assembly has an angular velocity $\boldsymbol{\omega} = \omega\mathbf{j}$ and an angular acceleration $\boldsymbol{\alpha} = \alpha\mathbf{j}$.

**PROBLEM 18.8** A homogeneous disk of mass $m$ and radius $r$ is mounted on the vertical shaft $AB$. The normal to the disk at $G$ forms an angle $\beta = 25°$ with the shaft. Knowing that the shaft has a constant angular velocity $\omega$, determine the angle $\theta$ formed by the shaft $AB$ and the angular momentum $\mathbf{H}_G$ of the disk about its mass center $G$.

## SOLUTION

Use the principal centroidal axes $Gx'y'z'$




Moments of inertia.

$$\bar{I}_{x'} = \bar{I}_{z'} = \frac{1}{4}mr^2$$

$$\bar{I}_{y'} = \frac{1}{2}mr^2$$

Angular velocity.

$$\omega_{x'} = -\omega \sin\beta$$

$$\omega_{y'} = \omega\cos\beta$$

$$\omega_z = 0$$

Angular momentum about $G$.

Using Eqs. (18.10),

$$H_{x'} = \bar{I}_{x'}\omega_{x'} = -\frac{1}{4}mr^2\omega\sin\beta$$

$$H_{y'} = \bar{I}_{y'}\omega_{y'} = \frac{1}{2}mr^2\omega\cos\beta$$

$$H_z = \bar{I}_z\omega_z = 0$$

$$\mathbf{H}_G = H_{x'}\mathbf{i}' + H_{y'}\mathbf{j} + H_z\mathbf{k}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along the $x', y', z'$ axes.

$$\mathbf{H}_G = -\frac{1}{4}mr^2\omega\sin\beta\mathbf{i}' + \frac{1}{2}mr^2\omega\cos\beta\mathbf{j}'$$

Angular acceleration. $\boldsymbol{\alpha} = \alpha\mathbf{j}, \quad \dot{\omega}_{x'} = -\alpha\sin\beta, \quad \dot{\omega}_{y'} = \alpha\cos\beta, \quad \dot{\omega}_{z'} = 0$

Rate of change of angular momentum. Let the reference frame $Gxyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega\mathbf{j} = -(\omega\sin\beta)\mathbf{i}' + (\omega\cos\beta)\mathbf{j}'$$

With respect to this frame,

$$\begin{aligned}
(\dot{\mathbf{H}}_G)_{Gxyz} &= \bar{I}_{x'}\dot{\omega}_{x'}\mathbf{i}' + \bar{I}_{y'}\dot{\omega}_{y'}\mathbf{j}' + \bar{I}_{z'}\dot{\omega}_{z'}\mathbf{k}' \\
&= \left(\frac{1}{4}mr^2\right)(-\alpha\sin\beta)\mathbf{i}' + \left(\frac{1}{2}mr^2\right)(\alpha\cos\beta)\mathbf{j}' + 0 \\
&= -\frac{1}{4}mr^2\alpha\sin\beta\mathbf{i}' + \frac{1}{2}mr^2\alpha\cos\beta\mathbf{j}' \\
&= -\frac{1}{4}mr^2\alpha\sin\beta(\mathbf{i}\cos\beta - \mathbf{j}\sin\beta) + \frac{1}{2}mr^2\alpha\cos\beta(\mathbf{i}\sin\beta + \mathbf{j}\cos\beta) \\
&= \frac{1}{4}mr^2\alpha[\mathbf{i}\sin\beta\cos\beta + \mathbf{j}(2\cos^2\beta + \sin^2\beta)] \\
&= \frac{1}{4}mr^2\alpha[\mathbf{i}\sin 25^\circ\cos 25^\circ + \mathbf{j}(2\cos^2 25^\circ + \sin^2 25^\circ)] \\
&= mr^2\alpha(0.0957556\mathbf{i} + 0.45535\mathbf{j})
\end{aligned}$$

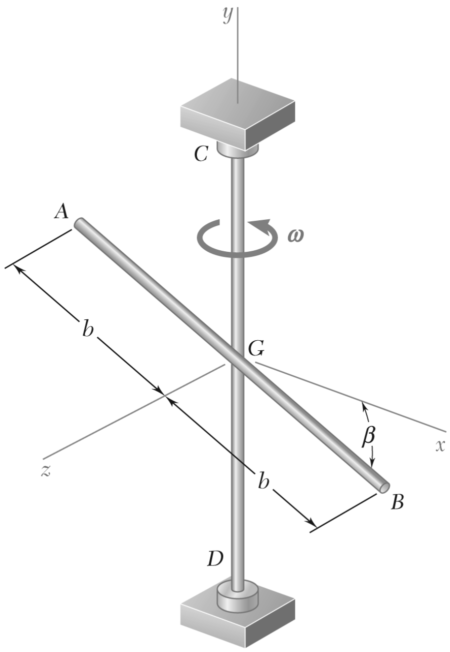
With respect to the fixed reference frame,

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega}\times\mathbf{H}_G \quad \text{where}$$

$$\begin{aligned}
\boldsymbol{\Omega}\times\mathbf{H}_G &= +(-\omega\sin\beta\mathbf{i}' + \omega\cos\beta\mathbf{j}') \\
&\quad \times\left(-\frac{1}{4}mr^2\omega\sin\beta + \frac{1}{2}mr^2\omega\cos\beta\right) \\
&= \frac{1}{4}mr^2\omega^2\cos\beta\sin\beta\mathbf{k} - \frac{1}{2}mr^2\omega^2\sin\beta\cos\beta\mathbf{k} \\
&= -\frac{1}{4}mr^2\omega^2\sin\beta\cos\beta\mathbf{k} \\
&= -\frac{1}{4}mr^2\omega^2\sin 25^\circ\cos 25^\circ\mathbf{k} = -0.095756mr^2\omega^2\mathbf{k}
\end{aligned}$$

Then $\dot{\mathbf{H}}_G = mr^2(0.0958\alpha\mathbf{i} + 0.455\alpha\mathbf{j} - 0.0958\omega^2\mathbf{k})$ ◀

## PROBLEM 18.65




A slender, uniform rod $AB$ of mass $m$ and a vertical shaft $CD$, each of length $2b$, are welded together at their midpoints $G$. Knowing that the shaft rotates at the constant rate $\omega$, determine the dynamic reactions at $C$ and $D$.

## SOLUTION

Using the principal axes $Gx'y'z$:

$$\bar{I}_{x'} = 0, \quad \bar{I}_{y'} = \bar{I}_z = \frac{1}{3}mb^2$$

$$\omega_{x'} = -\omega \sin\beta, \quad \omega_{y'} = \omega\cos\beta, \quad \omega_z = 0$$

Angular momentum about $G$.
$$\mathbf{H}_G = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}' + \bar{I}_z\omega_z\mathbf{k}$$

$$\mathbf{H}_G = \frac{1}{3}mb^2\omega\cos\beta\,\mathbf{j}'$$

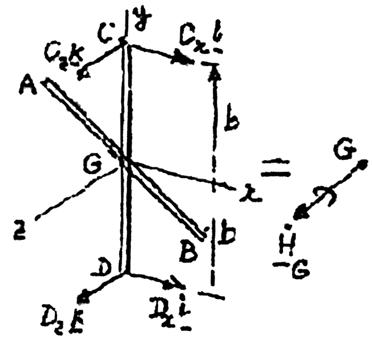
or, since $\mathbf{j}' = \mathbf{i}\sin\beta + \mathbf{j}\cos\beta$:
$$\mathbf{H}_G = \frac{1}{3}mb^2\omega\cos\beta(\sin\beta\,\mathbf{i} + \cos\beta\,\mathbf{j}) \qquad (1)$$

Rate of change of angular momentum.

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega}\times\mathbf{H}_G = 0 + \boldsymbol{\omega}\times\mathbf{H}_B$$

$$= \omega\mathbf{j}\times\left[\frac{1}{3}mb^2\omega\cos\beta(\sin\beta\,\mathbf{i} + \cos\beta\,\mathbf{j})\right]$$

$$= -\frac{1}{3}mb^2\omega^2\sin\beta\cos\beta\,\mathbf{k}$$




<u>Equations of motion</u>.

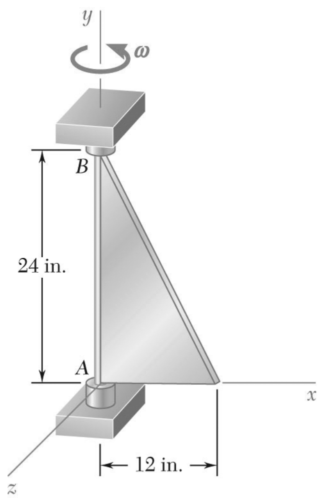
We equate the systems of external and effective forces

$$\Sigma\mathbf{M}_D = \Sigma(\mathbf{M}_D)_{\text{eff}}: \quad 2b\mathbf{j}\times(C_x\mathbf{i}+C_z\mathbf{k}) = \dot{\mathbf{H}}_G$$

$$-2bC_x\mathbf{k} + 2bC_z\mathbf{i} = -\frac{1}{3}mb^2\omega^2\sin\beta\cos\beta\,\mathbf{k}$$

Thus, $\quad C_x = \dfrac{1}{6}mb\omega^2\sin\beta\cos\beta, \quad C_z = 0$ $\quad \mathbf{C} = \dfrac{1}{6}mb\omega^2\sin\beta\cos\beta\,\mathbf{i}$ ◀

$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}: \quad \mathbf{C}+\mathbf{D} = 0$ $\quad \mathbf{D} = -\dfrac{1}{6}mb\omega^2\sin\beta\cos\beta\,\mathbf{i}$ ◀




## PROBLEM 18.66

A thin homogeneous triangular plate of weight 10 pounds is welded to a light vertical axle supported by bearings at $A$ and $B$. Knowing that the plate rotates at the constant rate $\omega = 8$ rad/s, determine the dynamic reactions at $A$ and $B$.

## SOLUTION

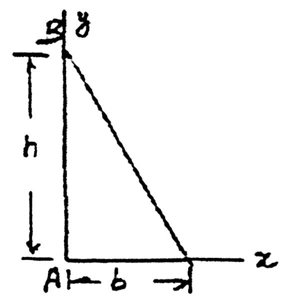
We shall use Eqs. (18.1) and (18.28):

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \quad \Sigma \mathbf{M}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega} \times \mathbf{H}_A$$

Computation of $\mathbf{H}_A$:

$$\boldsymbol{\omega} = \omega \mathbf{j}$$

$$\mathbf{H}_A = -I_{xy}\omega\mathbf{i} + I_y\omega\mathbf{j} - I_{yz}\omega\mathbf{k}$$




The moment of inertia of the triangle is

$$(I_y)_{\text{area}} = \frac{1}{12}b^3h, \quad A = \frac{1}{2}bh$$

$$(I_y)_{\text{mass}} = \frac{m}{A}(I_y)_{\text{area}} = \frac{1}{6}mb^2$$

The product of inertia of the triangle is

$$(I_{xy})_{\text{area}} = \frac{1}{24}b^2h^2$$

$$(I_{xy})_{\text{mass}} = \frac{m}{A}(I_{xy})_{\text{area}} = \frac{1}{12}mbh$$

Since the $z$-coordinate is negligible,

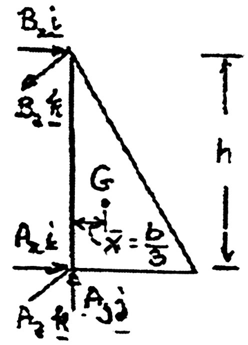
$$(I_{yz})_{\text{mass}} = 0$$

Thus,

$$\mathbf{H}_A = -\frac{1}{12}mbh\omega\mathbf{i} + \frac{1}{6}mb^2\omega\mathbf{j}. \tag{1}$$

where the frame of reference $Axyz$ rotates with the plate with the angular velocity

$$\mathbf{\Omega} = \boldsymbol{\omega} = \omega\mathbf{j}$$




Equations of motion. (Weight is omitted for dynamic reactions.)

Eq. (18.28),

$$\Sigma\mathbf{M}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega}\times\mathbf{H}_A.$$

Since $\omega = \text{constant}$, it follows from Eq. (1) that

$$(\dot{\mathbf{H}}_A)_{Axyz} = 0$$

Thus,

$$h\mathbf{j}\times(B_x\mathbf{i}+B_z\mathbf{k}) = 0 + \omega\mathbf{j}\times\left(-\frac{mbh}{12}\omega\mathbf{i} + \frac{mb^2}{6}\omega\mathbf{j}\right) - hB_x\mathbf{k} + h\mathbf{B}_z\mathbf{i}$$

$$= +\frac{1}{12}mbh\omega^2\mathbf{k}$$

Equating coefficients of unit vectors: $B_x = -\frac{1}{12}mb\omega^2, \quad B_z = 0$

$$\mathbf{B} = -\frac{1}{12}mb\omega^2\mathbf{i}$$

The dynamic reactions must also satisfy Eq. (18.1):

$$\Sigma\mathbf{F} = m\bar{\mathbf{a}}: \quad \mathbf{A}+\mathbf{B} = -m\bar{x}\omega^2\mathbf{i} = -m\left(\frac{b}{3}\right)\omega^2\mathbf{i}$$

$$\mathbf{A} = -\frac{1}{3}mb\omega^2\mathbf{i} - \left(-\frac{1}{12}mb\omega^2\mathbf{i}\right), \quad \mathbf{A} = -\frac{1}{4}mb\omega^2\mathbf{i}$$

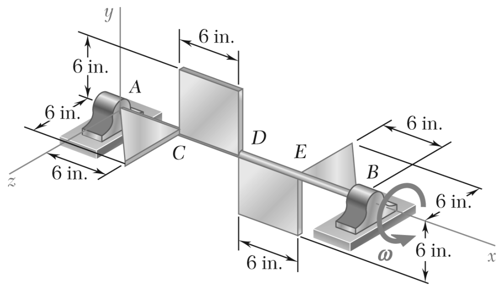
Given data: $W = 10\text{ lb}, \quad b = 12\text{ in.} = 1\text{ ft}, \quad h = 24\text{ in.} = 2\text{ ft}, \quad \omega = 8\text{ rad/s}$

$$\mathbf{A} = -\frac{1}{4}\left(\frac{10\text{ lbs}}{32.2}\right)(1\text{ ft})(8\text{ rad/s})^2\mathbf{i}$$

$\mathbf{A} = -(4.97\text{ lb})\mathbf{i}$ ◀

$$\mathbf{B} = -\frac{1}{12}\left(\frac{10\text{ lbs}}{32.2}\right)(1\text{ ft})(8\text{ rad/s})^2\mathbf{i}$$

$\mathbf{B} = -(1.656\text{ lb})\mathbf{i}$ ◀




## PROBLEM 18.67

The assembly shown consists of pieces of sheet aluminum of uniform thickness and of total weight 2.7 lb welded to a light axle supported by bearings at $A$ and $B$. Knowing that the assembly rotates at the constant rate $\omega = 240$ rpm, determine the dynamic reactions at $A$ and $B$.

## SOLUTION

Mass of sheet metal: $m = \dfrac{2.7}{32.2} = 0.08385 \text{ lb}\cdot\text{s}^2/\text{ft}$

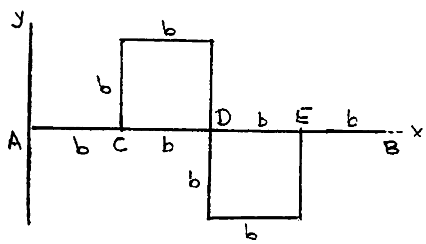
Sheet metal dimension: $b = 6 \text{ in.} = 0.5 \text{ ft}$

Area of sheet metal: $A = \dfrac{1}{2}b^2 + b^2 + b^2 + \dfrac{1}{2}b^2 = 3b^2 = 0.75 \text{ ft}^2$

Let $\rho = \dfrac{m}{A} = \dfrac{0.08385}{0.75} = 0.1118 \text{ lb}\cdot\text{s}^2/\text{ft}^3 = \text{mass per unit area.}$

Moments and products of inertia: $I_{\text{mass}} = \rho I_{\text{area}}$

*xy plane* (rectangles)




$$I_x = \frac{1}{3}b^4 + \frac{1}{3}b^4 = \frac{2}{3}b^4$$

$$I_x = \frac{2}{3}\rho b^4$$

$$= \frac{2}{3}(0.1118)(0.5)^4$$

$$= 4.658 \times 10^{-3} \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

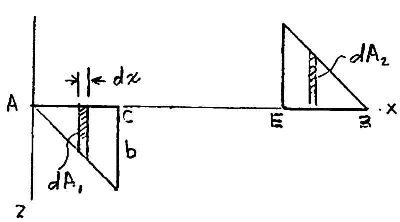
$$I_{xy} = (b^2)\left(\frac{3}{2}b\right)\left(\frac{1}{2}b\right) + (b^2)\left(\frac{5}{2}b\right)\left(-\frac{1}{2}b\right)$$

$$= -\frac{1}{2}b^4$$

$$I_{xy} = -\frac{1}{2}\rho b^4 = -\frac{1}{2}(0.1118)(0.5)^4$$

$$= -3.4938 \times 10^{-3} \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

*xz plane* (triangles)




$$I_x = \frac{1}{12}b^4 + \frac{1}{12}b^4 = \frac{1}{6}b^4$$

$$I_x = \frac{1}{6}\rho b^4 = \frac{1}{6}(0.1118)(0.5)^4$$

$$= 1.1646 \times 10^{-3} \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

For calculation of $I_{xz}$, use pairs of elements $dA_1$ and $dA_2$:

$$dA_2 = dA_1.$$

$$I_{xz} = \int x\frac{z}{2}dA_1 + \int(4b-x)\left(-\frac{z}{2}\right)dA_2 = -\int(2b-x)z\,dA_1 = -\int_0^b(2b-x)z^2dx$$

but $\quad z = x.$

Hence, $\quad I_{xz} = -\int_0^a(2bx^2 - x^3)dx = -\left(\frac{2}{3}b^4 - \frac{1}{4}b^4\right) = -\frac{5}{12}b^4$

$$I_{xz} = -\frac{5}{12}\rho b^4 = -\frac{5}{12}(0.1118)(0.5)^4 = -2.9115\times10^{-3}\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

Total for $I_x$: $\quad I_x = 4.658\times10^{-3} + 1.1646\times10^{-3} = 5.823\times10^{-3}\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$

The mass center lies on the rotation axis, therefore

$$\bar{\mathbf{a}} = 0$$

$$\Sigma\mathbf{F} = \mathbf{A} + \mathbf{B} = m\bar{\mathbf{a}} = 0 \quad \mathbf{A} = -\mathbf{B}$$

$$\mathbf{H}_A = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k} \quad \boldsymbol{\omega} = \omega\mathbf{i}, \quad \boldsymbol{\alpha} = \alpha\mathbf{i}$$

Let the frame of reference $Axyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = \boldsymbol{\omega} = \omega\mathbf{i}$$

$$\Sigma\mathbf{M}_A = \dot{\mathbf{H}}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_A$$

$$M_0\mathbf{i} + 4b\mathbf{i}\times(B_y\mathbf{j} + B_z\mathbf{k}) = I_x\alpha\mathbf{i} - I_{xy}\alpha\mathbf{j} - I_{xz}\alpha\mathbf{k} + \omega\mathbf{i}\times(I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k})$$

$$M_0\mathbf{i} - 4bB_z\mathbf{j} + 4bB_y\mathbf{k} = I_x\alpha\mathbf{i} - (I_{xy}\alpha - I_{xz}\omega^2)\mathbf{j} - (I_{xz}\alpha + I_{xy}\omega^2)\mathbf{k}$$

Resolve into components and solve for $B_y$ and $B_z$.

$$\mathbf{i}: \quad M_0 = I_x\alpha$$

$$\mathbf{j}: \quad B_z = \frac{(I_{xy}\alpha - I_{xz}\omega^2)}{4b}$$

$$\mathbf{k}: \quad B_y = -\frac{(I_{xz}\alpha + I_{xy}\omega^2)}{4b}$$

<u>Data</u>: $\quad \alpha = 0, \quad \omega = \frac{2\pi(240)}{60} = 25.133\ \text{rad/s}, \quad b = 0.5\ \text{ft} \quad M_0 = 0$

$$B_z = \frac{0 - (-2.9115\times10^{-3})(25.133)^2}{(4)(0.5)} = 0.91955\ \text{lb.}$$

## PROBLEM 18.67 (Continued)

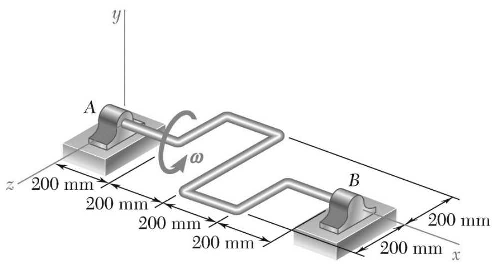
$$B_y = \frac{0-(-3.4938\times 10^{-3})(25.133)^2}{(4)(0.5)} = 1.10346 \text{ lb.}$$

$$A_y = -B_y = -1.10346 \text{ lb.}$$

$$A_z = -B_z = -0.91955 \text{ lb.}$$

$$\mathbf{A} = -(1.103 \text{ lb})\mathbf{j} - (0.920 \text{ lb})\mathbf{k} \blacktriangleleft$$

$$\mathbf{B} = (1.103 \text{ lb})\mathbf{j} + (0.920 \text{ lb})\mathbf{k} \blacktriangleleft$$




## PROBLEM 18.68

The 8-kg shaft shown has a uniform cross section. Knowing that the shaft rotates at the constant rate $\omega = 12$ rad/s, determine the dynamic reactions at $A$ and $B$.

## SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega \mathbf{i}$

Angular momentum about the mass center $G$:

$$\mathbf{H}_G = \bar{I}_x \omega \mathbf{i} - \bar{I}_{xy} \omega \mathbf{j} - \bar{I}_{xz} \omega \mathbf{k}$$

Let the reference frame $Axyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = \omega \mathbf{i}.$$

$$\begin{aligned}
\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Axyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \\
&= 0 + \omega \mathbf{i} \times (\bar{I}_x \omega \mathbf{i} - \bar{I}_{xy} \omega \mathbf{j} - \bar{I}_{xz} \omega \mathbf{k}) \\
&= \bar{I}_{xz} \omega^2 \mathbf{j} - \bar{I}_{xy} \omega^2 \mathbf{k}
\end{aligned}$$

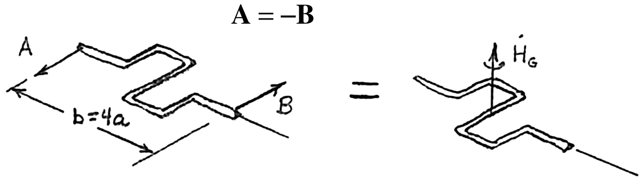
Since the shaft lies in the $xz$ plane, $\bar{I}_{xy} = 0$.

By symmetry, the mass center lies on line $AB$.

$$m\bar{\mathbf{a}} = 0$$

$$\Sigma \mathbf{F} = \Sigma F_{\text{eff}}: \quad \mathbf{A} + \mathbf{B} = m\bar{\mathbf{a}} = 0 \quad \mathbf{A} \text{ and } \mathbf{B} \text{ form a couple.}$$

$$\mathbf{A} = -\mathbf{B}$$




$$b\mathbf{i} \times (B_y \mathbf{j} + B_z \mathbf{k}) = \dot{\mathbf{H}}_G$$

$$-bB_z \mathbf{j} + bB_y \mathbf{k} = \bar{I}_{xz} \omega \mathbf{j} \quad B_z = -\frac{\bar{I}_{xz} \omega^2}{b}, \quad B_y = 0$$

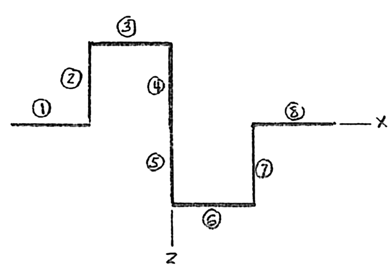
Calculation of $\bar{I}_{xz}$.

Divide the shaft into eight segments, each of length

$$a = 200 \text{ mm} = 0.2 \text{ m}$$

Let $m'$ be the mass of one segment.

$$m' = \frac{8 \text{ kg}}{8} = 1 \text{ kg}$$




For 1, 4, 5, and 8, $I_{xz} = 0$

## PROBLEM 18.68 (Continued)

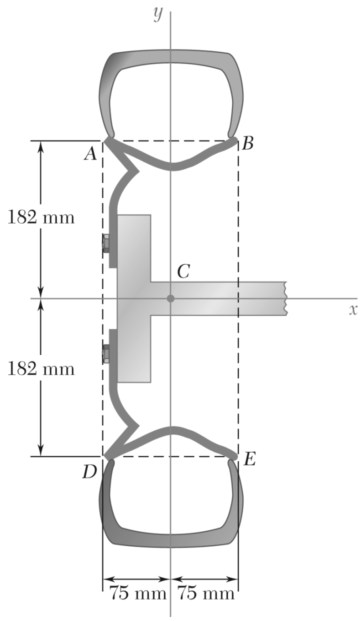
Then $$I_{xz} = m'(-a)\left(-\frac{a}{2}\right) + m'\left(-\frac{a}{2}\right)(-a) + m'\left(\frac{a}{2}\right)(a) + m'(a)\left(\frac{a}{2}\right) = 2m'a^2$$

$$B_z = -\frac{2m'a^2\omega^2}{4a} = -\frac{(2)(1)(0.2)^2(12)^2}{(4)(0.2)} = -14.4 \text{ N}$$

$$A_y = 0, \qquad A_z = -B_z = 14.4 \text{ N}$$

$\mathbf{A} = (14.4 \text{ N})\mathbf{k}$ ◀

$\mathbf{B} = -(14.4 \text{ N})\mathbf{k}$ ◀




## PROBLEM 18.69

After attaching the 18-kg wheel shown to a balancing machine and making it spin at the rate of 15 rev/s, a mechanic has found that to balance the wheel both statically and dynamically, he should use two corrective masses, a 170-g mass placed at $B$ and a 56-g mass placed at $D$. Using a right-handed frame of reference rotating with the wheel (with the $z$ axis perpendicular to the plane of the figure), determine before the corrective masses have been attached ($a$) the distance from the axis of rotation to the mass center of the wheel and the products of inertia $I_{xy}$ and $I_{zx}$, ($b$) the force-couple system at $C$ equivalent to the forces exerted by the wheel on the machine.

## SOLUTION

$$m = 18 \text{ kg}, \quad \boldsymbol{\omega} = 2\pi(15)\mathbf{i} = (94.248 \text{ rad/s})\mathbf{i}$$

$$m_B = 170 \text{ g} = 0.17 \text{ kg}$$

$$m_D = 56 \text{ g} = 0.056 \text{ kg}$$

$$x_B = 75 \text{ mm} = 0.075 \text{ m}, \quad y_B = 182 \text{ mm} = 0.182 \text{ m}$$

$$x_D = -0.075 \text{ m}, \quad y_D = -0.182 \text{ m}$$

($a$) Balance masses are added to move the mass center to Point $C$ and to reduce the products of inertia to zero.

$$m_B y_B + m_D y_D + m\bar{y} = 0$$

$$(0.17)(0.182) + (0.056)(-0.182) + 18\bar{y} = 0$$

$$\bar{y} = -1.15267 \times 10^{-3} \text{ m} \qquad \bar{y} = -1.153 \text{ mm} \blacktriangleleft$$

$$\bar{z} = 0 \qquad \bar{z} = 0 \blacktriangleleft$$

$$m_B x_B y_B + m_D x_D y_D + I_{xy} = 0$$

$$(0.17)(0.075)(0.182) + (0.056)(-0.075)(-0.182) + I_{xy} = 0$$

$$I_{xy} = -3.0848 \times 10^{-3} \qquad I_{xy} = -3.08 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \blacktriangleleft$$

$$I_{zx} = 0 \blacktriangleleft$$

## PROBLEM 18.69 (Continued)

(*b*) Force-couple system exerted on the wheel:

$$\mathbf{F}' = m\overline{\mathbf{a}}_n = -m\overline{\mathbf{r}}\omega^2 = -(18)(-1.15267\times 10^{-3}\,\mathbf{j})(94.248)^2$$
$$= (184.3\ \text{N})\mathbf{j}$$

$$\mathbf{H}_C = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k}$$

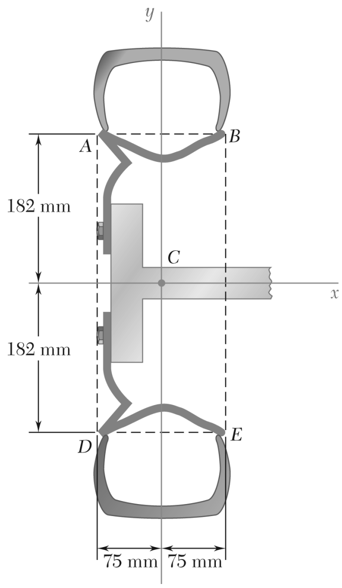
$$\mathbf{M}'_C = \dot{\mathbf{H}}_C = \boldsymbol{\omega}\times\mathbf{H}_C = I_{xz}\omega^2\mathbf{j} - I_{xy}\omega^2\mathbf{k}$$
$$= 0 - (-3.0848\times 10^{-3})(94.248)^2\mathbf{k} = (27.4\ \text{N}\cdot\text{m})\mathbf{k}$$

Force-couple system exerted by the wheel on the machine:

$$\mathbf{F} = -\mathbf{F}' \qquad \mathbf{F} = -(184.3\ \text{N})\mathbf{j}\ \blacktriangleleft$$

$$\mathbf{M}_C = -\mathbf{M}'_C \qquad \mathbf{M}_C = -(27.4\ \text{N}\cdot\text{m})\mathbf{k}\ \blacktriangleleft$$

## PROBLEM 18.70




When the 18-kg wheel shown is attached to a balancing machine and made to spin at a rate of 12.5 rev/s, it is found that the forces exerted by the wheel on the machine are equivalent to a force-couple system consisting of a force $\mathbf{F} = (160\text{ N})\mathbf{j}$ applied at $C$ and a couple $\mathbf{M}_C = (14.7\text{ N}\cdot\text{m})\mathbf{k}$, where the unit vectors form a triad which rotates with the wheel. (*a*) Determine the distance from the axis of rotation to the mass center of the wheel and the products of inertia $I_{xy}$ and $I_{zx}$. (*b*) If only two corrective masses are to be used to balance the wheel statically and dynamically, what should these masses be and at which of the Points $A$, $B$, $D$, or $E$ should they be placed?

## SOLUTION

$$m = 18\text{ kg}, \quad \omega = 2\pi(12.5)\mathbf{i} = (78.54\text{ rad/s})\mathbf{i}$$

(*a*) The force-couple system acting on the wheel is

$$\mathbf{F}' = -(160\text{ N})\mathbf{j}, \quad \mathbf{M}'_C = -(14.7\text{ N}\cdot\text{m})\mathbf{k}$$

$$\mathbf{F}' = m\bar{\mathbf{a}}_n = -m\bar{\mathbf{r}}\omega^2 \qquad \bar{\mathbf{r}} = \bar{y}\mathbf{j} + \bar{z}\mathbf{k} = \frac{-\mathbf{F}'}{m\omega^2}$$

$$\bar{y} = \frac{-F'_y}{m\omega^2} = \frac{160}{(18)(78.54)^2} = 1.441\times10^{-3}\text{ m} \qquad \bar{y} = 1.441\text{ mm}$$

$$\bar{z} = \frac{-F'_z}{m\omega^2} = 0 \qquad \bar{r} = 1.441\text{ mm} \blacktriangleleft$$

$$\mathbf{H}_C = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k}$$

$$\mathbf{M}'_C = \dot{\mathbf{H}}_C = \boldsymbol{\omega}\times\mathbf{H}_C = \omega\mathbf{i}\times(I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k})$$

$$(M'_C)_y\mathbf{j} + (M'_C)_z\mathbf{k} = I_{xz}\omega^2\mathbf{j} - I_{xy}\omega^2\mathbf{k}$$

$$I_{xy} = -\frac{(M'_C)_z}{\omega^2} = \frac{-(-14.7)}{(78.54)^2} = 2.3831\times10^{-3}\text{ kg}\cdot\text{m}^2$$

$$I_{xy} = 2.38\times10^{-3}\text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

$$I_{xz} = \frac{(M'_C)_y}{\omega^2} = 0 \qquad I_{xz} = 0 \blacktriangleleft$$

(*b*) Positions for the balance masses.

$$y_A = y_B = -y_E = -y_D = 182 \text{ mm} = 0.182 \text{ m}$$

$$-x_A = x_B = x_E = -x_D = 75 \text{ mm} = 0.075 \text{ m}$$

Balance masses must be added to move the mass center to Point *C* and to reduce the product of inertia to zero.

$$m_A y_A + m_B y_B + m_E y_E + m_D y_D + m\bar{y} = 0$$

$$(0.182)(m_A + m_B - m_E - m_D) + (18)(1.441\times 10^{-3})$$

$$m_A + m_B - m_E - m_D = -0.1425 \text{ kg} \qquad (1)$$

$$m_A x_A y_A + m_B x_B y_B + m_E x_E y_E + m_D x_D y_D + I_{xy} = 0$$

$$(0.075)(0.182)(-m_A + m_B - m_E + m_D) + 2.3831\times 10^{-3} = 0$$

$$-m_A + m_B - m_E + m_D = -0.17459 \text{ kg} \qquad (2)$$

To solve Eqs. (1) and (2), set $m_B = 0$ and let $m_{AD} = m_A - m_D$.

Then $$m_{AD} - m_E = -0.1425$$

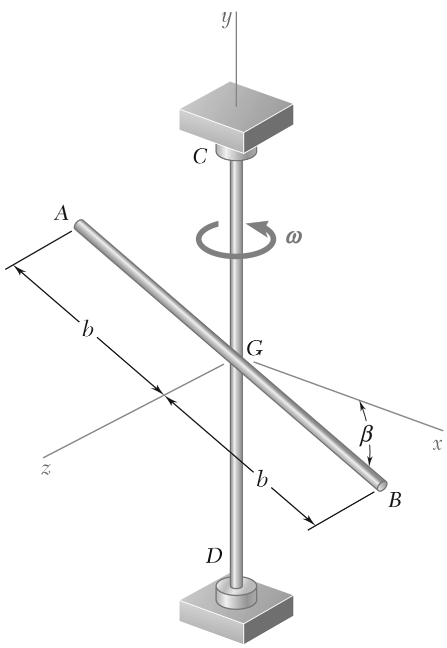
and $$-m_{AD} - m_E = -0.17459$$

Solving, $$m_{AD} = 0.01605 \text{ kg}, \quad m_E = 0.1585 \text{ kg}$$

Set $m_D = 0$, so that $m_A = 0.01605$ kg

$m_E = 158.5$ g ◀

$m_A = 16.05$ g ◀




## PROBLEM 18.71

Knowing that the assembly of Problem 18.65 is initially at rest $(\omega = 0)$ when a couple of moment $\mathbf{M}_0 = M_0\mathbf{j}$ is applied to shaft $CD$, determine (*a*) the resulting angular acceleration of the assembly, (*b*) the dynamic reactions at $C$ and $D$ immediately after the couple is applied.

## SOLUTION

Using the principal axes $Gx'y'z$:

$$\bar{I}_{x'} = 0, \quad \bar{I}_{y'} = \bar{I}_z = \frac{1}{3}mb^2$$

$$\omega_{x'} = -\omega\sin\beta, \quad \omega_{y'} = \omega\cos\beta, \quad \omega_z = 0$$

Angular momentum about $G$.

$$\mathbf{H}_G = \bar{I}_{x'}\omega_{x'}\mathbf{i}' + \bar{I}_{y'}\omega_{y'}\mathbf{j}' + I_z\omega_z\mathbf{k}$$

$$\mathbf{H}_G = \frac{1}{3}mb^2\omega\cos\beta\,\mathbf{j}'$$

or, since $\quad \mathbf{j}' = \mathbf{i}\sin\beta + \mathbf{j}\cos\beta: \quad \mathbf{H}_G = \frac{1}{3}mb^2\omega\cos\beta(\sin\beta\,\mathbf{i} + \cos\beta\,\mathbf{j}) \qquad (1)$

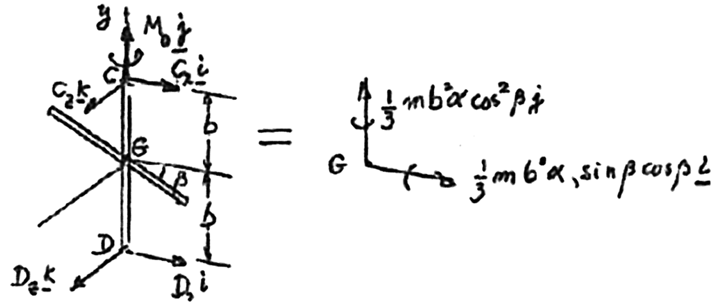
Rate of change of angular momentum.

Eq. (18.22): $\quad \dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega}\times\mathbf{H}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + 0$

Since $\boldsymbol{\Omega} = \boldsymbol{\omega} = 0$ when couple is applied, thus,

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} = \frac{1}{3}mb^2\alpha\cos\beta(\sin\beta\,\mathbf{i} + \cos\beta\,\mathbf{j}) \qquad (2)$$

Equations of motion: Equivalence of applied and effective forces.




$$\Sigma\mathbf{M}_D = \Sigma(\mathbf{M}_D)_{\text{eff}}: \quad 2b\mathbf{j}\times(C_x\mathbf{i}+C_z\mathbf{k})+M_0\mathbf{j} = \frac{1}{3}mb^2\alpha\sin\beta\cos\beta\mathbf{i}+\frac{1}{3}mb^2\alpha\cos^2\beta\mathbf{j}$$

$$-2bC_x\mathbf{k}+2bC_z\mathbf{i}+M_0\mathbf{j} = \frac{1}{3}mb^2\alpha\sin\beta\cos\beta\mathbf{i}+\frac{1}{3}mb^2\alpha\cos^2\beta\mathbf{j}$$

Equating the coefficients of **i**, **j**, **k**:

$$\mathbf{i}: \quad 2bC_z = \frac{1}{3}mb^2\alpha\sin\beta\cos\beta \tag{3}$$

$$\mathbf{j}: \quad M_0 = \frac{1}{3}mb^2\alpha\cos^2\beta \tag{4}$$

$$\mathbf{k}: \quad C_x = 0 \tag{5}$$

(*a*) Angular acceleration.

From Eq. (4),

$$\alpha = \frac{3M_0}{mb^2\cos^2\beta} \blacktriangleleft$$

(*b*) Initial dynamic reactions.

From Eq. (3),

$$C_z = \frac{1}{6}mb\alpha\sin\beta\cos\beta = \frac{1}{6}mb\sin\beta\cos\beta\left(\frac{3M_0}{mb^2\cos^2\beta}\right)$$

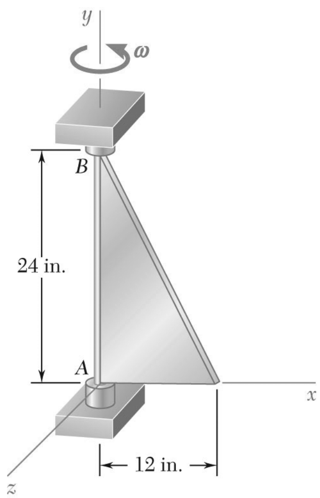
$$C_z = \left(\frac{M_0}{2b}\right)\tan\beta$$

Recalling Eq. (5), $\quad C_x = 0$

$$\mathbf{C} = \left(\frac{M_0}{2b}\right)\tan\beta\mathbf{k} \blacktriangleleft$$

$\Sigma\mathbf{F} = \Sigma(\mathbf{F})_{\text{eff}}: \quad \mathbf{C}+\mathbf{D} = 0, \quad \mathbf{D} = -\mathbf{C}$

$$\mathbf{D} = -\left(\frac{M_0}{2b}\right)\tan\beta\mathbf{k} \blacktriangleleft$$



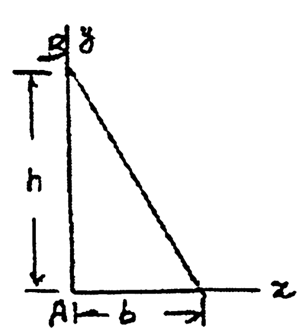

## PROBLEM 18.72

Knowing that the plate of Problem 18.66 is initially at rest $(\omega = 0)$ when a couple of moment $\mathbf{M}_0 = (0.75 \text{ ft} \cdot \text{lb})\mathbf{j}$ is applied to it, determine (*a*) the resulting angular acceleration of the plate, (*b*) the dynamic reactions $A$ and $B$ immediately after the couple has been applied.

**PROBLEM 18.66** A thin homogeneous triangular plate of weight 10 pounds is welded to a light vertical axle supported by bearings at $A$ and $B$. Knowing that the plate rotates at the constant rate $\omega = 8 \text{ rad/s}$, determine the dynamic reactions at $A$ and $B$.

## SOLUTION

We shall use Eqs. (18.1) and (18.28):




$$\Sigma\mathbf{F} = m\bar{\mathbf{a}} \quad \Sigma\mathbf{M}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega} \times \mathbf{H}_A$$

Computation of $\mathbf{H}_A$:

$$\boldsymbol{\omega} = \omega\mathbf{j}$$

$$\mathbf{H}_A = -I_{xy}\omega\mathbf{i} + I_y\omega\mathbf{j} - I_{yz}\omega\mathbf{k}$$

The moment of inertia of the triangle is

$$(I_y)_{\text{area}} = \frac{1}{12}b^3h, \quad A = \frac{1}{2}bh$$

$$(I_y)_{\text{mass}} = \frac{m}{A}(I_y)_{\text{area}} = \frac{1}{6}mb^2$$

The product of inertia of the triangle is

$$(I_{xy})_{\text{area}} = \frac{1}{24}b^2h^2$$

$$(I_{xy})_{\text{mass}} = \frac{m}{A}(I_{xy})_{\text{area}} = \frac{1}{12}mbh$$

Since the $z$-coordinate is negligible,

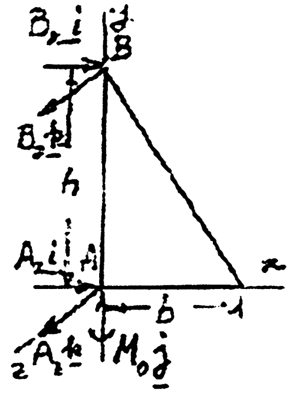
$$(I_{yz})_{\text{mass}} = 0$$

Thus,

$$\mathbf{H}_A = -\frac{1}{12}mbh\omega\mathbf{i} + \frac{1}{6}mb^2\omega\mathbf{j} \qquad (1)$$

where the frame of reference $Axyz$ rotates with the plate with the angular velocity

$$\mathbf{\Omega} = \boldsymbol{\omega} = \omega\mathbf{j}$$




Equations of motion. We first use Eq. (18.28):

$$\Sigma\mathbf{M}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega}\times\mathbf{H}_A$$

Where $\mathbf{H}_A$ was obtained in Eq. (1) of Problem 18.66:

$$\mathbf{H}_A = -\frac{1}{12}mbh\omega\mathbf{i} + \frac{1}{6}mb^2\omega\mathbf{j}$$

Differentiating with respect to the rotating frame:

$$(\dot{\mathbf{H}}_A)_{Axyz} = -\frac{1}{12}mbh\alpha\mathbf{i} + \frac{1}{6}mb^2\alpha\mathbf{j}$$

Substituting for $\mathbf{H}_A$ and $(\dot{\mathbf{H}}_A)_{Axyz}$ into Eq. (18.28), noting that $\omega = 0$, and computing $\Sigma\mathbf{M}_A$ from diagram:

$$M_0\mathbf{j} + h\mathbf{j}\times(B_x\mathbf{i} + B_z\mathbf{k}) = -\frac{1}{12}mbh\alpha\mathbf{i} + \frac{1}{6}mb^2\alpha\mathbf{j} + 0$$

$$M_0\mathbf{j} - hB_x\mathbf{k} + hB_z\mathbf{i} = -\frac{1}{12}mbh\alpha\mathbf{i} + \frac{1}{6}mb^2\alpha\mathbf{j}$$

Equating the coefficients of the unit vectors:

$$\mathbf{j}: \qquad M_0 = \frac{1}{6}mb^2\alpha \qquad\qquad \alpha = \frac{6M_0}{mb^2}$$

(*a*) $$\alpha = \frac{6(0.75\text{ lb}\cdot\text{ft})}{(\frac{10\text{ lb}}{32.2})(1\text{ ft})^2} = 14.49\text{ rad/s}^2$$ $\boldsymbol{\alpha} = (14.49\text{ rad/s}^2)\mathbf{j}$ ◀

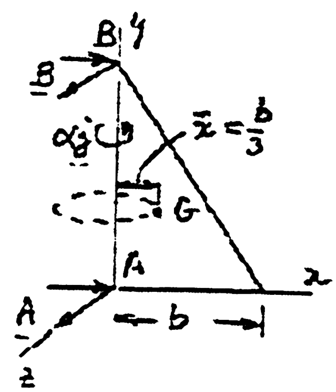
$$\mathbf{k}: \qquad -hB_x = 0 \qquad\qquad B_x = 0$$

$$\mathbf{i}: \qquad hB_z = -\frac{1}{12}mbh\alpha, \quad B_z = -\frac{1}{12}mb\alpha = -\frac{1}{12}\left(\frac{10\text{ lb}}{32.2}\right)(1)(14.49)$$

(*b*) $$B_z = -0.375\text{ lb}$$ $\mathbf{B} = -(0.375\text{ lb})\mathbf{k}$ ◀

We shall now apply Eq. (18.1): $\Sigma\mathbf{F} = m\bar{\mathbf{a}}$:

Since $\omega = 0$: $$\bar{\mathbf{a}} = \bar{\mathbf{a}}_t = \alpha\mathbf{j}\times\frac{b}{3}\mathbf{i} = -\frac{1}{3}b\alpha\mathbf{k}$$



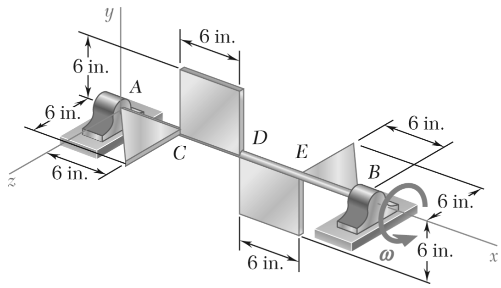

Substituting in Eq. (12.1):

$$\mathbf{A} + \mathbf{B} = -\frac{1}{3}mb\alpha\mathbf{k}$$

$$\mathbf{A} = -\frac{1}{3}mb\alpha\mathbf{k} - B$$

$$= -\frac{1}{3}mb\alpha\mathbf{k} + \frac{1}{12}mb\alpha\mathbf{k}$$

$$\mathbf{A} = -\frac{1}{4}mb\alpha\mathbf{k} = -\frac{1}{4}\left(\frac{10\text{ lb}}{32.2}\right)(1\text{ ft})(14.49\text{ rad/s}^2)\mathbf{k}$$ $\mathbf{A} = -(1.125\text{ lb})\mathbf{k}$ ◀




## PROBLEM 18.73

The assembly of Problem 18.67 is initially at rest ($\omega = 0$) when a couple $\mathbf{M}_0$ is applied to axle $AB$. Knowing that the resulting angular acceleration of the assembly is $\boldsymbol{\alpha} = (150 \text{ rad/s}^2)\mathbf{i}$, determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at *A* and *B* immediately after the couple is applied.

## SOLUTION

Mass of sheet metal: $m = \dfrac{2.7}{32.2} = 0.08385 \text{ lb} \cdot \text{s}^2/\text{ft}$

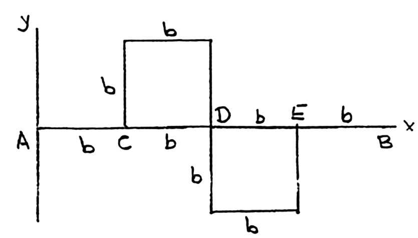
Sheet metal dimension: $b = 6 \text{ in.} = 0.5 \text{ ft}$

Area of sheet metal: $A = \dfrac{1}{2}b^2 + b^2 + b^2 + \dfrac{1}{2}b^2 = 3b^2 = 0.75 \text{ ft}^2$

Let $\rho = \dfrac{m}{A} = \dfrac{0.08385}{0.75} = 0.1118 \text{ lb} \cdot \text{s}^2/\text{ft}^3 = \text{mass per unit area}$

Moments and products of inertia: $(I)_{\text{mass}} = \rho I_{(\text{area})}$

*xy plane* (rectangles)




$$I_x = \frac{1}{3}b^4 + \frac{1}{3}b^4 = \frac{2}{3}b^4$$

$$I_x = \frac{2}{3}\rho b^4 = \frac{2}{3}(0.1118)(0.5)^4$$

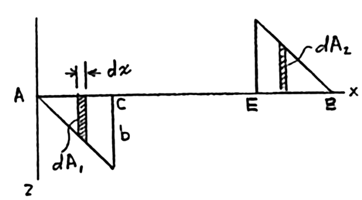
$$= 4.658 \times 10^{-3} \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

$$I_{xy} = (b^2)\left(\frac{3}{2}b\right)\left(\frac{1}{2}b\right) + (b^2)\left(\frac{5}{2}b\right)\left(-\frac{1}{2}b\right) = -\frac{1}{2}b^4$$

$$I_{xy} = -\frac{1}{4}\rho b^4 = -\frac{1}{4}(0.1118)(0.5)^4$$

$$= -3.4938 \times 10^{-3} \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

*xz plane* (triangles)




$$I_x = \frac{1}{12}b^4 + \frac{1}{12}b^4 = \frac{1}{6}b^4$$

$$I_x = \frac{1}{6}\rho b^4 = \frac{1}{6}(0.1118)(0.5)^4$$

$$= 1.1646 \times 10^{-3} \text{ lb} \cdot \text{s}^2 \cdot \text{ft}$$

For calculation of $I_{xz}$, use pairs of elements $dA_1$ and $dA_2$: $\quad dA_2 = dA_1$

$$I_{xz} = \int x\frac{z}{2}dA_1 + \int(4b-x)\left(-\frac{z}{2}\right)dA_2 = -\int(2b-x)z\,dA_1 = -\int_0^b(2b-x)z^2dx$$

But $z = x$.

Hence,
$$I_{xz} = -\int_0^a(2bx^2 - x^3)dx = -\left(\frac{2}{3}b^4 - \frac{1}{4}b^4\right) = -\frac{5}{12}b^4$$

$$I_{xz} = -\frac{5}{12}\rho b^4 = -\frac{5}{12}(0.1118)(0.5)^4 = -2.9115\times10^{-3}\ \text{lb}\cdot\text{s}^2\cdot\text{ft}$$

Total for $I_x$: $\quad I_x = 4.658\times10^{-3} + 1.1646\times10^{-3} = 5.823\times10^{-3}\ \text{kg}\cdot\text{m}^2$

The mass center lies on the rotation axis, therefore, $\bar{\mathbf{a}} = 0$

$$\Sigma\mathbf{F} = \mathbf{A} + \mathbf{B} = m\bar{\mathbf{a}} = 0 \qquad \mathbf{A} = -\mathbf{B}$$

$$\mathbf{H}_A = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k} \qquad \boldsymbol{\omega} = \omega\mathbf{i}, \qquad \boldsymbol{\alpha} = \alpha\mathbf{i}$$

Let the frame of reference $Axyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega} = \omega\mathbf{i}$.

$$\Sigma\mathbf{M}_A = \dot{\mathbf{H}}_A = (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_A$$

$$M_0\mathbf{i} + 4b\mathbf{i}\times(B_y\mathbf{j} + B_z\mathbf{k}) = I_x\alpha\mathbf{i} - I_{xy}\alpha\mathbf{j} - I_{xz}\alpha\mathbf{k} + \omega\mathbf{i}\times(I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k})$$

$$M_0\mathbf{i} - 4bB_z\mathbf{j} + 4bB_y\mathbf{k} = I_x\alpha\mathbf{i} - (I_{xy}\alpha - I_{xz}\omega^2)\mathbf{j} - (I_{xz}\alpha + I_{xy}\omega^2)\mathbf{k}$$

Resolve into components and solve for $B_y$ and $B_z$.

$\mathbf{i}$: $\quad M_0 = I_x\alpha$

$\mathbf{j}$: $\quad B_z = \dfrac{(I_{xy}\alpha - I_{xz}\omega^2)}{4b}$

$\mathbf{k}$: $\quad B_y = -\dfrac{(I_{xz}\alpha + I_{xy}\omega^2)}{4b}$

<u>Data:</u> $\quad \alpha = 150\ \text{rad/s}^2, \qquad \omega = 0, \qquad b = 0.5\ \text{ft}$

(*a*) $\quad M_0 = (5.823\times10^{-3})(150) = 0.87345\ \text{lb}\cdot\text{ft} \qquad M_0 = (0.873\ \text{lb}\cdot\text{ft})\mathbf{i}$ ◀

(*b*)
$$B_z = \frac{(-3.4938\times10^{-3})(150) - 0}{(4)(0.5)} = -0.262\ \text{lb}$$

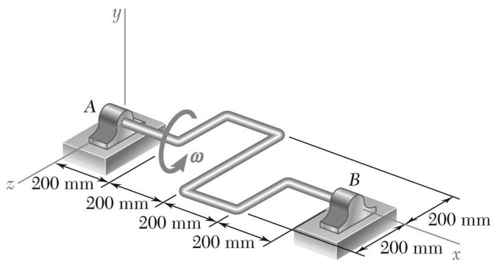
$$B_y = -\frac{(-2.9115\times10^{-3})(150) + 0}{(4)(0.5)} = 0.218\ \text{lb}$$

$$A_y = -B_y = -0.218\ \text{lb}$$

$$A_z = -B_z = 0.262\ \text{lb}$$

$\mathbf{A} = -(0.218\ \text{lb})\mathbf{j} + (0.262\ \text{lb})\mathbf{k}$ ◀

$\mathbf{B} = (0.218\ \text{lb})\mathbf{j} - (0.262\ \text{lb})\mathbf{k}$ ◀




## PROBLEM 18.74

The shaft of Problem 18.68 is initially at rest $(\omega = 0)$ when a couple $\mathbf{M}_0$ is applied to it. Knowing that the resulting angular acceleration of the shaft is $\boldsymbol{\alpha} = (20\ \text{rad/s}^2)\mathbf{i}$, determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at *A* and *B* immediately after the couple is applied.

**PROBLEM 18.68** The 8-kg shaft shown has a uniform cross section. Knowing that the shaft rotates at the constant rate $\omega = 12$ rad/s, determine the dynamic reactions at *A* and *B*.

## SOLUTION

Angular velocity and angular acceleration: $\dot{\boldsymbol{\omega}} = \dot{\omega}\mathbf{i} = \alpha\mathbf{i}$

Angular momentum about the mass center *G*: $\mathbf{H}_G = \bar{I}_x\omega\mathbf{i} - \bar{I}_{xy}\omega\mathbf{j} - \bar{I}_{xz}\omega\mathbf{k}$

Let the reference frame *Axyz* be rotating with angular velocity $\Omega = \omega\mathbf{i} = 0$

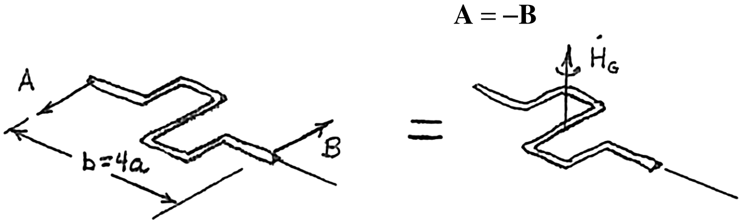
$$\begin{aligned}\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_G \\ &= \bar{I}_x\alpha\mathbf{i} - \bar{I}_{xy}\alpha\mathbf{j} - \bar{I}_{xz}\alpha\mathbf{k} + 0 \\ &= \bar{I}_x\alpha\mathbf{i} - \bar{I}_{xy}\alpha\mathbf{j} - \bar{I}_{xz}\alpha\mathbf{k}\end{aligned}$$

Since the shaft lies in the *xz* plane, $\bar{I}_{xy} = 0$.

By symmetry, the mass center lies on line *AB*. $m\bar{\mathbf{a}} = 0$

$$\Sigma\mathbf{F} = \Sigma F_{\text{eff}}:\quad \mathbf{A} + \mathbf{B} = m\bar{\mathbf{a}} = 0 \qquad \mathbf{A} \text{ and } \mathbf{B} \text{ form a couple.}$$

$$\mathbf{A} = -\mathbf{B}$$



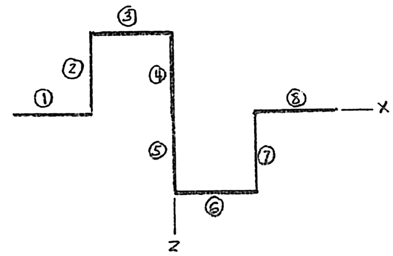

$$M_0\mathbf{i} + b\mathbf{i}\times(B_y\mathbf{j} + B_z\mathbf{k}) = \dot{\mathbf{H}}_G$$

$$M_0\mathbf{i} + bB_y\mathbf{k} - bB_z\mathbf{j} = \bar{I}_x\alpha\mathbf{i} - \bar{I}_{xz}\alpha\mathbf{k}$$

$$M_0 = \bar{I}_x\alpha, \quad B_y = -\frac{I_{xz}\alpha}{b}, \quad B_z = 0$$

# PROBLEM 18.74 (Continued)

Calculation of $\bar{I}_x$ and $I_{xz}$.




Divide the shaft into eight segments, each of length

$$a = 200 \text{ mm} = 0.2 \text{ m}$$

Let $m'$ be the mass of one segment.

$$m' = \frac{8 \text{ kg}}{8} = 1 \text{ kg}$$

For ① and ⑧, $\quad I_x = 0$

For ②, ④, ⑤, and ⑦, $\quad I_x = \frac{1}{3} ma^2$

For ③ and ⑥, $\quad I_x = m'a^2$

Total: $\quad \bar{I}_x = \frac{10}{3} m'a^2$

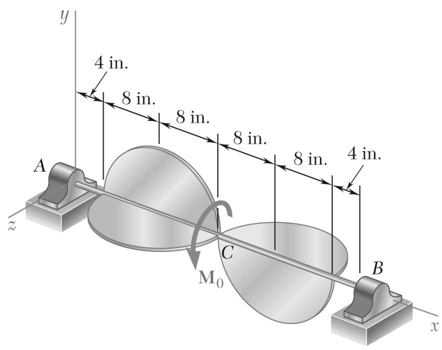
For ①, ④, ⑤, and ⑧, $\quad I_{xz} = 0$

For ②, ③, ⑥, and ⑦, $\quad I_{xz} = m'(-a)\left(-\frac{a}{2}\right) + m'\left(-\frac{a}{2}\right)(-a) + m'\left(\frac{a}{2}\right)(a) + m'(a)\left(\frac{a}{2}\right) = 2m'a^2$

(*a*) $\quad M_0 = \frac{10}{3} m'a^2 \alpha = \frac{10}{3}(1)(0.2)^2(20)$ $\quad \mathbf{M}_0 = (2.67 \text{ N} \cdot \text{m})\mathbf{i}$ ◀

(*b*) $\quad B_y = -\frac{2m'a^2\alpha}{b} = -\frac{(2)(1)(0.2)^2(20)}{(4)(0.2)}$ $\quad \mathbf{B} = -(2.00 \text{ N})\mathbf{j}$ ◀

$A_z = 0, \quad A_y = -B_y = 2 \text{ N}$ $\quad \mathbf{A} = (2.00 \text{ N})\mathbf{j}$ ◀




## PROBLEM 18.75

The assembly shown weighs 12 lb and consists of 4 thin 16-in.-diameter semicircular aluminum plates welded to a light 40-in.-long shaft $AB$. The assembly is at rest $(\omega = 0)$ at time $t = 0$ when a couple $\mathbf{M}_0$ is applied to it as shown, causing the assembly to complete one full revolution in 2 s. Determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at $A$ and $B$ at $t = 0$.

## SOLUTION

Fixed axis rotation with constant angular acceleration. $\quad \alpha = \alpha\mathbf{i}, \quad \omega = \omega\mathbf{i}$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2 \qquad \alpha = \frac{2\theta}{t^2} = \frac{2(2\pi)}{2^2} = 3.1416 \text{ rad/s}^2$$

$$\omega = 0$$

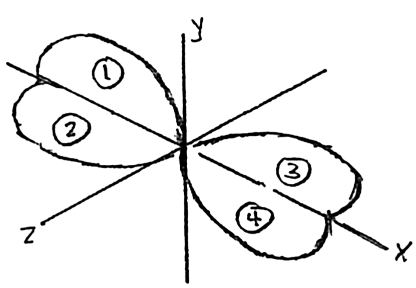
Use centroidal axes $x$, $y$, $z$ with origin at $C$. $\quad \mathbf{H}_C = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k}$

Let the reference frame $Cxyz$ rotate with angular velocity $\quad \mathbf{\Omega} = \boldsymbol{\omega} = \omega\mathbf{i}$

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Cxyz} + \mathbf{\Omega} \times \mathbf{H}_C = I_x\alpha\mathbf{i} - I_{xy}\alpha\mathbf{j} - I_{xz}\alpha\mathbf{k} + I_{xz}\omega^2\mathbf{j} - I_{xy}\omega^2\mathbf{k}$$

Required moments and products of inertia. Let $\rho = \dfrac{m}{A}$ = mass per unit area.

$$I_{\text{mass}} = \rho I_{\text{area}} \qquad r = 8 \text{ in.} = 0.66667 \text{ ft} \qquad m = \frac{12}{32.2} = 0.37267 \text{ lb}\cdot\text{s}^2/\text{ft}$$




| Part | $A$ | $I_x$ | $I_{xy}$ | $I_{xz}$ |
|---|---|---|---|---|
| 1 | $\frac{1}{2}\pi r^2$ | $\frac{1}{8}\pi r^4$ | $-\frac{2}{3}r^4$ | 0 |
| 2 | $\frac{1}{2}\pi r^2$ | $\frac{1}{8}\pi r^4$ | 0 | $-\frac{2}{3}r^4$ |
| 3 | $\frac{1}{2}\pi r^2$ | $\frac{1}{8}\pi r^4$ | 0 | $-\frac{2}{3}r^4$ |
| 4 | $\frac{1}{2}\pi r^2$ | $\frac{1}{8}\pi r^4$ | $-\frac{2}{3}r^4$ | 0 |
| $\Sigma$ | $2\pi r^2$ | $\frac{1}{2}\pi r^4$ | $-\frac{4}{3}r^4$ | $-\frac{4}{3}r^4$ |

$$\rho = \frac{m}{2\pi r^2} = \frac{0.37267}{2\pi(0.66667)^2} = 0.13345 \text{ lb}\cdot\text{s}^2/\text{ft}^3$$

$$I_x = (0.13345)\left(\frac{1}{2}\pi\right)(0.66667)^4 = 0.041407 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$I_{xy} = (0.13345)\left(-\frac{4}{3}\right)(0.66667)^4 = -0.035147 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$I_{xz} = (0.13345)\left(-\frac{4}{3}\right)(0.66667)^4 = -0.035147 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Since the mass center lies on the rotation axis, $\bar{\mathbf{a}} = 0$

$$\Sigma\mathbf{F} = \mathbf{A} + \mathbf{B} = \Sigma\mathbf{F}_{\text{eff}} = m\bar{\mathbf{a}} = 0 \qquad \mathbf{A} = -\mathbf{B}$$

$$\Sigma\mathbf{M}_C = M_0\mathbf{i} + (-b\mathbf{i})\times\mathbf{A} + b\mathbf{i}\times\mathbf{B} = M_0\mathbf{i} + 2b\mathbf{i}\times(B_y\mathbf{j} + B_z\mathbf{k})$$

$$= M_0\mathbf{i} - 2bB_z\mathbf{j} + 2bB_y\mathbf{k} \qquad \text{where} \qquad 2b = 40 \text{ in.} = 3.3333 \text{ ft}$$

$$\Sigma M_C = \dot{\mathbf{H}}_C \qquad \text{Resolve into components.}$$

(*a*) **i**: $M_0 = I_x\alpha = (0.041407)(3.1416)$ $\qquad \mathbf{M}_0 = (0.1301 \text{ lb}\cdot\text{ft})\mathbf{i}$ ◀

(*b*) **j**: $-2bB_z = -I_{xy}\alpha + I_{xz}\omega^2$

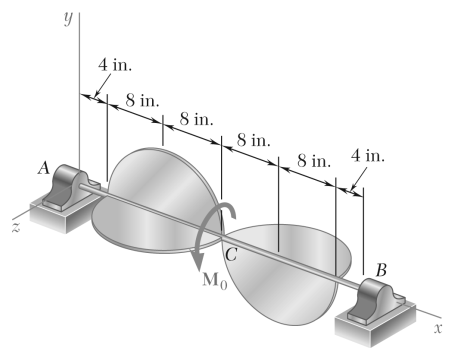
$$B_z = -\frac{-(-0.035147)(3.1416) + 0}{3.3333} = -0.0331 \text{ lb} \qquad A_z = 0.0331 \text{ lb}$$

**k**: $2bB_y = -I_{xz}\alpha - I_{xy}\omega^2$

$$B_y = -\frac{(-0.035147)(3.1416) + 0}{3.3333} = 0.0331 \text{ lb} \qquad A_y = -0.0331 \text{ lb}$$

$\mathbf{A} = -(0.0331 \text{ lb})\mathbf{j} + (0.0331 \text{ lb})\mathbf{k}$ ◀

$\mathbf{B} = (0.0331 \text{ lb})\mathbf{j} - (0.0331 \text{ lb})\mathbf{k}$ ◀




## PROBLEM 18.76

For the assembly of Problem 18.75, determine the dynamic reactions at $A$ and $B$ at $t = 2$ s.

**PROBLEM 18.75** The assembly shown weighs 12 lb and consists of four thin 16-in.-diameter semicircular aluminum plates welded to a light 40-in.-long shaft $AB$. The assembly is at rest $(\omega = 0)$ at time $t = 0$ when a couple $\mathbf{M}_0$ is applied to it as shown, causing the assembly to complete one full revolution in 2 s. Determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at $A$ and $B$ at $t = 0$.

## SOLUTION

Fixed axis rotation with constant angular acceleration. $\quad \boldsymbol{\alpha} = \alpha\mathbf{i}, \quad \boldsymbol{\omega} = \omega\mathbf{i}$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2 \qquad \alpha = \frac{2\theta}{t^2} = \frac{2(2\pi)}{2^2} = 3.1416 \text{ rad/s}^2$$

$$\omega = \omega_0 + \alpha t = \alpha t = (3.1416)(2) = 6.2832 \text{ rad/s}$$

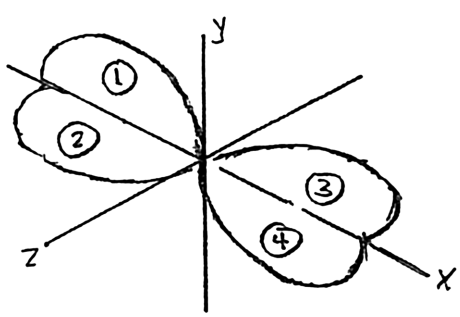
Use centroidal axes $x$, $y$, $z$ with origin at $C$. $\quad \mathbf{H}_C = I_x\omega\mathbf{i} - I_{xy}\omega\mathbf{j} - I_{xz}\omega\mathbf{k}$

Let the reference frame $Cxyz$ rotate with angular velocity $\quad \boldsymbol{\Omega} = \boldsymbol{\omega} = \omega\mathbf{i}$

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Cxyz} + \boldsymbol{\Omega}\times\mathbf{H}_C = I_x\alpha\mathbf{i} - I_{xy}\alpha\mathbf{j} - I_{xz}\alpha\mathbf{k} + I_{xz}\omega^2\mathbf{j} - I_{xy}\omega^2\mathbf{k}$$

Required moments and products of inertia. Let $\rho = \dfrac{m}{A} =$ mass per unit area.

$$I_{\text{mass}} = \rho I_{\text{area}} \qquad r = 8 \text{ in.} = 0.66667 \text{ ft} \quad m = \frac{12}{32.2} = 0.37267 \text{ lb}\cdot\text{s}^2/\text{ft}$$




| Part | $A$ | $I_x$ | $I_{xy}$ | $I_{xz}$ |
|---|---|---|---|---|
| 1 | $\frac{1}{2}\pi r^2$ | $\frac{1}{8}\pi r^4$ | $-\frac{2}{3}r^4$ | 0 |
| 2 | $\frac{1}{2}\pi r^2$ | $\frac{1}{8}\pi r^4$ | 0 | $-\frac{2}{3}r^4$ |
| 3 | $\frac{1}{2}\pi r^2$ | $\frac{1}{8}\pi r^4$ | 0 | $-\frac{2}{3}r^4$ |
| 4 | $\frac{1}{2}\pi r^2$ | $\frac{1}{8}\pi r^4$ | $-\frac{2}{3}r^4$ | 0 |
| $\Sigma$ | $2\pi r^2$ | $\frac{1}{2}\pi r^4$ | $-\frac{4}{3}r^4$ | $-\frac{4}{3}r^4$ |

$$\rho = \frac{m}{2\pi r^2} = \frac{0.37267}{2\pi(0.66667)^2} = 0.13345 \text{ lb}\cdot\text{s}^2/\text{ft}^3$$

$$I_x = (0.13345)\left(\frac{1}{2}\pi\right)(0.66667)^4 = 0.041407 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$I_{xy} = (0.13345)\left(-\frac{4}{3}\right)(0.66667)^4 = -0.035147 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$I_{xz} = (0.13345)\left(-\frac{4}{3}\right)(0.66667)^4 = -0.035147 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Since the mass center lies on the rotation axis, $\quad \bar{\mathbf{a}} = 0$

$$\Sigma\mathbf{F} = \mathbf{A} + \mathbf{B} = \Sigma\mathbf{F}_{\text{eff}} = m\bar{\mathbf{a}} = 0 \qquad \mathbf{A} = -\mathbf{B}$$

$$\Sigma\mathbf{M}_C = M_0\mathbf{i} + (-b\mathbf{i})\times\mathbf{A} + b\mathbf{i}\times\mathbf{B} = M_0\mathbf{i} + 2b\mathbf{i}\times(B_y\mathbf{j} + B_z\mathbf{k})$$

$$= M_0\mathbf{i} - 2bB_z\mathbf{j} + 2bB_y\mathbf{k} \qquad \text{where} \qquad 2b = 40 \text{ in.} = 3.3333 \text{ ft}$$

$$\Sigma M_C = \dot{\mathbf{H}}_C \qquad \text{Resolve into components.}$$

**i**: $\quad M_0 = I_x\alpha = (0.041407)(3.1416) \qquad M_0 = 0.1301 \text{ lb}\cdot\text{ft}$

**j**: $\quad -2b\,B_z = -I_{xy}\alpha + I_{xz}\omega^2$

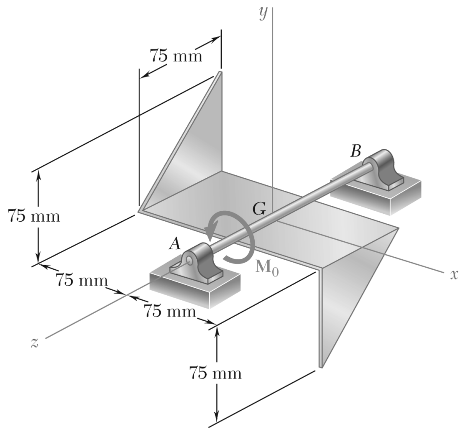
$$B_z = -\frac{-(-0.035147)(3.1416) + (-0.035147)(6.2832)^2}{3.3333} = 0.383 \text{ lb}, \qquad A_z = -0.383 \text{ lb}$$

**k**: $\quad 2bB_y = -I_{xz}\alpha - I_{xy}\omega^2$

$$B_y = -\frac{(-0.035147)(3.1416) + (-0.035147)(6.2832)^2}{3.3333} = 0.449 \text{ lb}, \qquad A_y = -0.449 \text{ lb}$$

$\mathbf{A} = -(0.449 \text{ lb})\mathbf{j} - (0.383 \text{ lb})\mathbf{k}$ ◀

$\mathbf{B} = (0.449 \text{ lb})\mathbf{j} + (0.383 \text{ lb})\mathbf{k}$ ◀




## PROBLEM 18.77

The sheet-metal component shown is of uniform thickness and has a mass of 600 g. It is attached to a light axle supported by bearings at $A$ and $B$ located 150 mm apart. The component is at rest when it is subjected to a couple $\mathbf{M}_0$ as shown. If the resulting angular acceleration is $\boldsymbol{\alpha} = (12\text{ rad/s}^2)\mathbf{k}$, determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at $A$ and $B$ immediately after the couple has been applied.

## SOLUTION

The sheet metal component rotates about the fixed $z$ axis, so that Equations (18.29) of the textbook apply. These are relisted below as equations (1), (2), and (3).

$$\Sigma M_x = -I_{xz}\alpha + I_{yz}\omega^2 \qquad (1)$$

$$\Sigma M_y = -I_{yz}\alpha - I_{xz}\omega^2 \qquad (2)$$

$$\Sigma M_z = I_z\alpha \qquad (3)$$

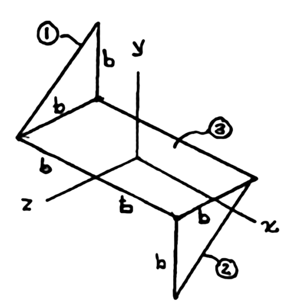
*Calculation of the required moment and products of inertia.*

Total mass: $m = 600\text{ g} = 0.6\text{ kg}$

Total area: $A = \frac{1}{2}(0.075)^2 + (0.150)(0.075) + \frac{1}{2}(0.075)^2 = 16.875\times10^{-3}\text{ m}^2$

Let $\rho$ be the mass per unit area. $\rho = \frac{m}{A} = 35.556\text{ kg/m}^2$

The component is comprised of 3 parts: triangle ①, triangle ②, and rectangle ③ as shown. Let the lengths of 75 mm be labeled $b$.




Triangle ①. The equation of the upper edge is

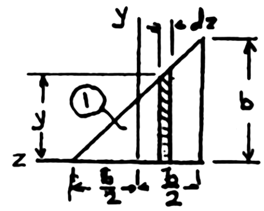
$$y = \frac{b}{2} - z$$

Use elements of width $dz$ and height $y$.

$$dm = \rho y dz \qquad (d\bar{I}_z)_{\text{el}} = \frac{1}{12}y^2 dm$$

$$(d\bar{I}_{xz})_{\text{el}} = 0 \qquad (d\bar{I}_{yz})_{\text{el}} = 0$$

Coordinates of the element mass center:



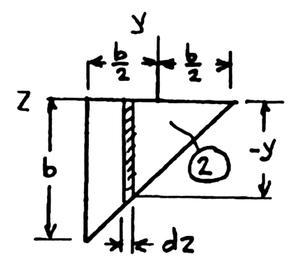

$$\bar{x}_{\text{el}} = -b, \qquad \bar{y}_{\text{el}} = \frac{1}{2}y, \qquad \bar{z}_{\text{el}} = z$$

$$dI_{xz} = (d\bar{I}_{xz})_{\text{el}} + \bar{x}_{\text{el}}\bar{z}_{\text{el}}dm$$

$$= 0 + (-b)z(\rho y d_z) = -\rho bz\left(\frac{b}{2} - z\right)dz$$

$$I_{xz} = -\rho b\int_{-b/2}^{b/2} z\left(\frac{b}{2} - z\right)dz = \frac{1}{12}\rho b^4$$

$$dI_{yz} = (d\bar{I}_{yz})_{\text{el}} + \bar{y}_{\text{el}}\bar{z}_{\text{el}}dm$$




$$= 0 + \left(\frac{1}{2}y\right)z\left(\rho y dz\right) = \frac{1}{2}\rho\left(\frac{b}{2} - z\right)^2 zdz$$

$$I_{yz} = \frac{1}{2}\rho\int_{-b/2}^{b/2}\left(\frac{b}{2} - z\right)^2 zdz = -\frac{1}{24}\rho b^4$$

$$dI_z = (d\bar{I}_z)_{\text{el}} + (\bar{x}_{\text{el}}^2 + \bar{y}_{\text{el}}^2)dm$$

$$= \left(\frac{1}{12}y^2 + b^2 + \frac{1}{4}y^2\right)\rho y dz = \rho\left(b^2 y + \frac{1}{3}y^3\right)dz$$

$$= \rho\left(\frac{13}{24}b^3 - \frac{5}{4}b^2 z + \frac{1}{2}bz^2 - \frac{1}{3}z^3\right)dz$$

$$I_z = \rho\int_{-b/2}^{b/2}\left(\frac{13}{24}b^3 - \frac{5}{4}b^2 z + \frac{1}{2}bz^2 - \frac{1}{3}z^3\right)dz = \frac{7}{12}\rho b^4$$

Applying the data,

$$I_{xz} = \frac{1}{12}(35.556)(0.075)^4 = 93.75\times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

$$I_{yz} = -\frac{1}{24}(35.556)(0.075)^4 = -46.875\times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

$$I_z = \frac{7}{12}(35.556)(0.075)^4 = 656.25\times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

*Triangle* ②. The equation of the lower edge is $y = -\left(\frac{b}{2} - z\right)$.

Use elements of width $dz$ and height $-y$.

$$dm = -\rho y dz, \qquad (d\bar{I}_z)_{\text{el}} = \frac{1}{12} y^2 dm, \qquad (d\bar{I}_{zx})_{\text{el}} = 0, \qquad (d\bar{I}_{yz})_{\text{el}} = 0$$

Coordinates of the element mass center:

$$\bar{x}_{\text{el}} = b, \qquad \bar{y}_{\text{el}} = \frac{1}{2} y, \qquad \bar{z}_{\text{el}} = z$$

The integrals for $I_{xz}, I_{yz}$, and $I_z$ turn out to be the same as those of triangle ①.

$$I_{xz} = 93.75 \times 10^{-6} \text{ kg}\cdot\text{m}^2, \quad I_{yz} = -46.875 \times 10^{-6} \text{ kg}\cdot\text{m}^2, \quad I_z = 656.25 \times 10^{-6} \text{ kg}\cdot\text{m}^2$$

*Rectangle* ③. Area: $A = (0.150)(0.075) = 11.25 \times 10^{-3} \text{ m}^2$

Mass: $m = \rho A = 400 \times 10^{-3}$ kg

$$I_{xz} = 0, \qquad I_{yz} = 0$$

$$I_z = \frac{1}{12} m(2b)^2 = \frac{1}{12}(400 \times 10^{-3})(0.150)^2 = 750 \times 10^{-6} \text{ kg}\cdot\text{m}^2$$

*Totals.*

$$I_{xz} = \Sigma I_{xz} = 187.5 \times 10^{-6} \text{ kg}\cdot\text{m}^2$$

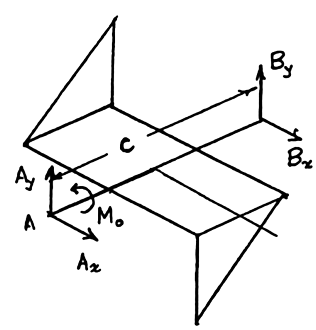
$$I_{yz} = \Sigma I_{yz} = -93.75 \times 10^{-6} \text{ kg}\cdot\text{m}^2$$

$$I_z = \Sigma I_z = 2062.5 \times 10^{-6} \text{ kg}\cdot\text{m}^2$$

Since the mass center lies on the fixed axis, the acceleration $\bar{\mathbf{a}}$ of the mass center is zero.

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} = 0$$

The reactions at $A$ and $B$ form a couple.




$$\mathbf{B} = -\mathbf{A}$$

Let $\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j}$

Resultant couple acting on the body:

$$\mathbf{M} = c\mathbf{k} \times (A_x \mathbf{i} + A_y \mathbf{j}) + M_0 \mathbf{k}$$

$$= -cA_y \mathbf{i} + cA_x \mathbf{j} + M_0 \mathbf{k}$$

(*a*) Moment $M_0$. Using Equation (3),

$$\Sigma M_z = M_0 = I_z \alpha = (2062.5 \times 10^{-6})(12)$$

$M_0 = 24.8 \times 10^{-3}$ N·m ◀

## PROBLEM 18.77 (Continued)

(*b*) Reactions at $A$ and $B$ for the case $\omega = 0$.

$$\Sigma M_x = -cA_y = -I_{xz}\alpha - I_{yz}\omega^2 = -I_{xz}\alpha$$

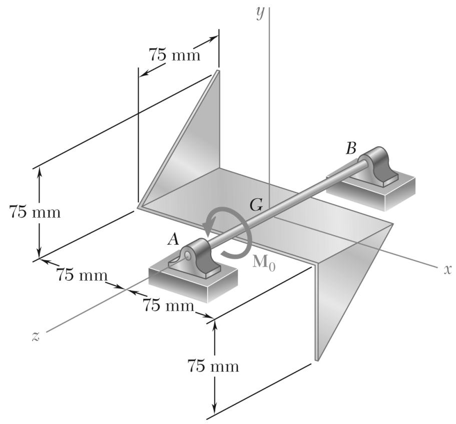
$$A_y = \frac{I_{xz}\alpha}{c} = \frac{(187.5\times10^{-6})(12)}{0.150} = 15\times10^{-3}\text{ N}$$

$$\Sigma M_y = cA_x = -I_{yz}\alpha - I_{xz}\omega^2 = -I_{yz}\alpha$$

$$A_x = -\frac{I_{yz}\alpha}{c} = -\frac{(-93.75\times10^{-6})(12)}{0.150} = 7.5\times10^{-3}\text{ N}$$

$$\mathbf{A} = (7.50\times10^{-3}\text{ N})\mathbf{i} + (15.00\times10^{-3}\text{ N})\mathbf{j} \blacktriangleleft$$

$$\mathbf{B} = -(7.50\times10^{-3}\text{ N})\mathbf{i} - (15.00\times10^{-3}\text{ N})\mathbf{j} \blacktriangleleft$$




## PROBLEM 18.78

For the sheet-metal component of Problem 18.77, determine (*a*) the angular velocity of the component 0.6 s after the couple $\mathbf{M}_0$ has been applied to it, (*b*) the magnitude of the dynamic reactions at *A* and *B* at that time.

## SOLUTION

The sheet metal component rotates about the fixed *z* axis with angular acceleration $\boldsymbol{\alpha} = (12 \text{ rad/s}^2)\mathbf{k}$.

(*a*) *Angular velocity at* $t = 0.6$ s.

$$\omega = \omega_0 + \alpha t = 0 + (12)(0.6) = 7.2 \text{ rad/s} \qquad \boldsymbol{\omega} = (7.20 \text{ rad/s})\mathbf{k} \blacktriangleleft$$

(*b*) *Dynamic reactions.*

Equations (18.29) of the textbook apply. These are relisted below as equations (1), (2), and (3).

$$\Sigma M_x = -I_{xz}\alpha + I_{yz}\omega^2 \qquad (1)$$

$$\Sigma M_y = -I_{yz}\alpha - I_{xz}\omega^2 \qquad (2)$$

$$\Sigma M_z = I_z\alpha \qquad (3)$$

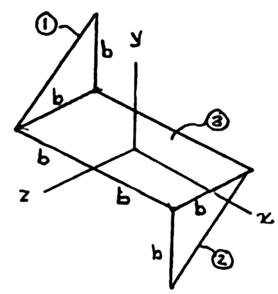
*Calculation of the required moment and products of inertia.*

Total mass: $m = 600 \text{ g} = 0.6 \text{ kg}$

Total area: $A = \frac{1}{2}(0.075)^2 + (0.150)(0.075) + \frac{1}{2}(0.075)^2 = 16.875 \times 10^{-3} \text{ m}^2$

Let $\rho$ be the mass per unit area. $\rho = \frac{m}{A} = 35.556 \text{ kg/m}^2$

The component is comprised of 3 parts: triangle ①, triangle ②, and rectangle ③ as shown. Let the lengths of 75 mm be labeled *b*.




Triangle ①. The equation of the upper edge is

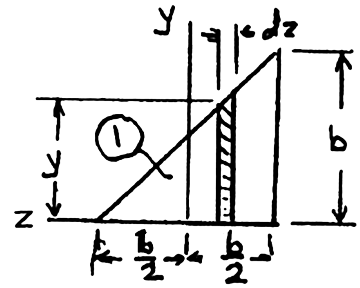
$$y = \frac{b}{2} - z$$

Use elements of width $dz$ and height $y$.

$$dm = \rho y dz \qquad (d\bar{I}_z)_{\text{el}} = \frac{1}{12}y^2 dm$$

$$(d\bar{I}_{xz})_{\text{el}} = 0 \qquad (d\bar{I}_{yz})_{\text{el}} = 0$$

Coordinates of the element mass center:

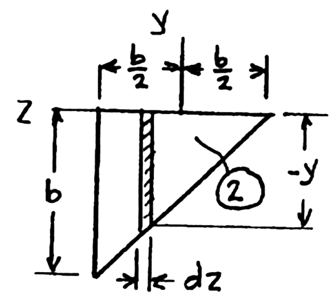



$$\bar{x}_{\text{el}} = -b, \ \bar{y}_{\text{el}} = \frac{1}{2}y, \quad \bar{z}_{\text{el}} = z$$

$$dI_{xz} = (d\bar{I}_{xz})_{\text{el}} + \bar{x}_{\text{el}}\bar{z}_{\text{el}}dm$$

$$= 0 + (-b)z(\rho y dz) = -\rho bz\left(\frac{b}{2} - z\right)dz$$

$$I_{xz} = -\rho b\int_{-b/2}^{b/2} z\left(\frac{b}{2} - z\right)dz = \frac{1}{12}\rho b^4$$




$$dI_{yz} = (d\bar{I}_{yz})_{\text{el}} + \bar{y}_{\text{el}}\bar{z}_{\text{el}}dm$$

$$= 0 + \left(\frac{1}{2}y\right)z(\rho y dz) = \frac{1}{2}\rho\left(\frac{b}{2} - z\right)^2 zdz$$

$$I_{yz} = \frac{1}{2}\rho\int_{-b/2}^{b/2}\left(\frac{b}{2} - z\right)^2 zdz = -\frac{1}{24}\rho b^4$$

$$dI_z = (d\bar{I}_z)_{\text{el}} + \left(\bar{x}_{\text{el}}^2 + \bar{y}_{\text{el}}^2\right)dm$$

$$= \left(\frac{1}{12}y^2 + b^2 + \frac{1}{4}y^2\right)\rho y_i dz = \rho\left(b^2 y + \frac{1}{3}y^3\right)dz$$

$$= \rho\left(\frac{13}{24}b^3 - \frac{5}{4}b^2 z + \frac{1}{2}bz^2 - \frac{1}{3}z^3\right)dz$$

$$I_z = \rho\int_{-b/2}^{b/2}\left(\frac{13}{24}b^3 - \frac{5}{4}b^2 z + \frac{1}{2}bz^2 - \frac{1}{3}z^3\right)dz = \frac{7}{12}\rho b^4$$

Applying the data,

$$I_{xz} = \frac{1}{12}(35.556)(0.075)^4 = 93.75 \times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

$$I_{yz} = -\frac{1}{24}(35.556)(0.075)^4 = -46.875 \times 10^{-6}\ \text{kg}\cdot\text{m}^2$$

$$I_z = \frac{7}{12}(35.556)(0.075)^4 = 656.25 \times 10^{6}\ \text{kg}\cdot\text{m}^2$$

Triangle ②. The equation of the lower edge is $\quad y = -\left(\frac{b}{2} - z\right).$

Use elements of width $dz$ and height $-y$.

$$dm = -\rho y dz, \quad (d\bar{I}_z)_{\text{el}} = \frac{1}{12}y^2 dm, \quad (d\bar{I}_{zx})_{\text{el}} = 0, \quad (d\bar{I}_{yz})_{\text{el}} = 0$$

## PROBLEM 18.78 (Continued)

Coordinates of the element mass center:

$$\bar{x}_{\text{el}} = b, \qquad \bar{y}_{\text{el}} = \frac{1}{2}y, \qquad \bar{z}_{\text{el}} = z$$

The integrals for $I_{xz}, I_{yz},$ and $I_z$ turn out to be the same as those of triangle ①.

$$I_{xz} = 93.75\times10^{-3}\ \text{kg}\cdot\text{m}^2, \quad I_{yz} = -46.875\times10^{-6}\ \text{kg}\cdot\text{m}^2, \quad I_z = 656.25\times10^{-6}\ \text{kg}\cdot\text{m}^2$$

Rectangle ③. Area: $A = (0.150)(0.075) = 11.25\times10^{-3}\ \text{m}^2$

Mass: $m = \rho A = 400\times10^{-3}\ \text{kg}$

$$I_{xz} = 0, \qquad I_{yz} = 0$$

$$I_z = \frac{1}{12}m(2b)^2 = \frac{1}{12}(400\times10^{-3})(0.150)^2 = 750\times10^{-6}\ \text{kg}\cdot\text{m}^2$$

*Totals.* $I_{xz} = \Sigma I_{xz} = 187.5\times10^{-6}\ \text{kg}\cdot\text{m}^2$

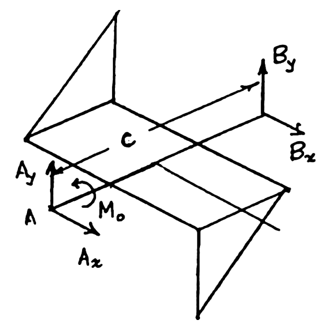
$$I_{yz} = \Sigma I_{yz} = -93.75\times10^{-6}\ \text{kg}\cdot\text{m}^2$$

$$I_z = \Sigma I_z = 2062.5\times10^{-6}\ \text{kg}\cdot\text{m}^2$$

Since the mass center lies on the fixed axis, the acceleration $\bar{\mathbf{a}}$ of the mass center is zero.

$$\Sigma\mathbf{F} = m\bar{\mathbf{a}} = 0$$

The reactions at $A$ and $B$ form a couple.




$$\mathbf{B} = -\mathbf{A}$$

Let $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$

Resultant couple acting on the body:

$$\mathbf{M} = c\mathbf{k}\times(A_x\mathbf{i} + A_y\mathbf{j}) + M_0\mathbf{k}$$

$$= -cA_y\mathbf{i} + cA_x\mathbf{j} + M_0\mathbf{k}$$

From Eq. (1), $-cA_y = -I_{xz}\alpha + I_{yz}\omega^2$

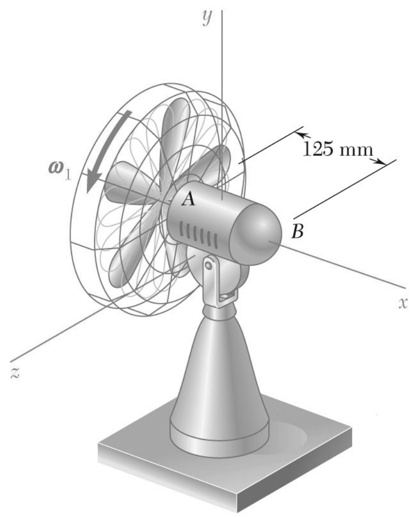
$$A_y = \frac{I_{xz}\alpha}{c} - \frac{I_{yz}\omega^2}{c} = \frac{(187.5\times10^{-6})(12)}{0.150} - \frac{(-93.75\times10^{-6})(7.2)^2}{0.150} = 47.4\times10^{-3}\ \text{N}$$

From Eq. (2), $cA_x = -I_{yz}\alpha - I_{xz}\omega^2$

$$A_x = -\frac{I_{yz}\alpha}{c} - \frac{I_{xz}\omega^2}{c} = -\frac{(-93.75\times10^{-6})(12)}{0.150} - \frac{(187.5\times10^{-6})(7.2)^2}{0.150} = -57.3\times10^{-3}\ \text{N}$$

$$\mathbf{A} = -(57.3\times10^{-3}\ \text{N})\mathbf{i} + (47.3\times10^{-3}\ \text{N})\mathbf{j} \ ◀$$

$$\mathbf{B} = (57.3\times10^{-3}\ \text{N})\mathbf{i} - (47.3\times10^{-3}\ \text{N})\mathbf{j} \ ◀$$




## PROBLEM 18.79

The blade of an oscillating fan and the rotor of its motor have a total weight of 300 g and a combined radius of gyration of 75 mm. They are supported by bearings at $A$ and $B$, 125 mm apart, and rotate at the rate $\omega_1 = 1800$ rpm. Determine the dynamic reactions at $A$ and $B$ when the motor casing has an angular velocity $\omega_2 = (0.6 \text{ rad/s})\mathbf{j}$.

## SOLUTION

$$\boldsymbol{\omega}_1 = \omega_1 \mathbf{i}$$

$$\omega_1 = \frac{2\pi(1800)}{60}$$

$$= 188.5 \text{ rad/s}$$

Angular velocity: $\boldsymbol{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j}$

Angular momentum: $\mathbf{H}_G = I_x \omega_1 \mathbf{i} + I_y \omega_2 \mathbf{j}$

Let the reference frame be rotating with angular velocity $\boldsymbol{\Omega} = \omega_2 \mathbf{j}$.

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$

$$= 0 + \omega_2 \mathbf{j} \times (I_x \omega_1 \mathbf{i} + I_y \omega_2 \mathbf{j})$$

$$= -I_x \omega_1 \omega_2 \mathbf{k}$$

Assume that the acceleration of the mass center is negligible. Then the dynamic reactions at $A$ and $B$ reduce to a couple.

$$\mathbf{A} = -\mathbf{B}$$

$$\mathbf{M} = b\mathbf{i} \times \mathbf{B}$$

$$= b\mathbf{i} \times (B_y \mathbf{j} + B_z \mathbf{k})$$

$$= -b B_z \mathbf{j} + b B_y \mathbf{k}$$

$\mathbf{M} = \dot{\mathbf{H}}_G$ Resolve into components.

$\mathbf{j}$: $-b B_z = 0 \qquad B_z = 0, \qquad A_z = 0$

$\mathbf{k}$: $b B_y = -I_x \omega_1 \omega_2 \qquad B_y = -\dfrac{I_x \omega_1 \omega_2}{b} = -\dfrac{m k_x^2 \omega_1 \omega_2}{b}$

Data:

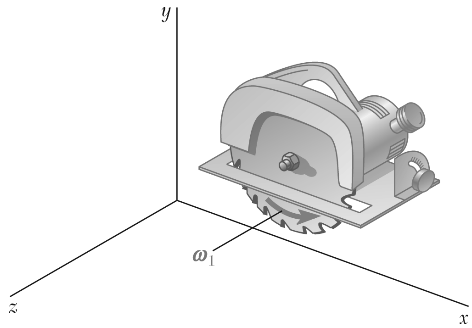
$$m = 0.300 \text{ kg}$$
$$k_x = 0.075 \text{ m}$$
$$b = 0.125 \text{ m}$$

$$B_y = -\frac{(0.300)(75\times 10^{-3})^2(188.5)(0.6)}{0.125} = -1.527 \text{ N}$$

$$A_y = 1.527 \text{ N}$$

$\mathbf{A} = (1.527 \text{ N})\mathbf{j}$ ◀

$\mathbf{B} = -(1.527 \text{ N})\mathbf{j}$ ◀




## PROBLEM 18.80

The blade of a portable saw and the rotor of its motor have a total weight of 2.5 lb and a combined radius of gyration of 1.5 in. Knowing that the blade rotates as shown at the rate $\omega_1 = 1500$ rpm, determine the magnitude and direction of the couple $\mathbf{M}$ that a worker must exert on the handle of the saw to rotate it with a constant angular velocity $\boldsymbol{\omega}_2 = -(2.4\ \text{rad/s})\mathbf{j}$.

## SOLUTION

$$\boldsymbol{\omega}_1 = \frac{(2\pi)(1500)}{60}$$
$$= 157.08 \text{ rad/s}$$
$$\boldsymbol{\omega}_1 = \omega_1\mathbf{k}$$

Angular velocity: $\boldsymbol{\omega} = \omega_2\mathbf{j} + \omega_1\mathbf{k}$

Angular momentum of rotor: $\mathbf{H}_G = I_y\omega_2\mathbf{j} + I_z\omega_1\mathbf{k}$

Let the reference frame $Gxyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \omega_2\mathbf{j}$.

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega}\times\mathbf{H}_G$$
$$= 0 + \omega_2\mathbf{j}\times(I_y\omega_2\mathbf{j} + I_z\omega_1\mathbf{k})$$
$$= I_z\omega_1\omega_2\mathbf{i}$$

Couple exerted on the saw: $\mathbf{M} = \dot{\mathbf{H}}_G$

$$= I_z\omega_1\omega_2\mathbf{i}$$
$$= mk_z^2\omega_1\omega_2\mathbf{i}$$

Data: $W = 2.5$ lb

$$m = \frac{2.5}{32.2}$$
$$= 0.07764 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$k_z = 1.5 \text{ in.}$$
$$= 0.125 \text{ ft}$$

$$\mathbf{M} = (0.07764)(0.125)^2(157.08)(-2.4)\mathbf{i} \qquad \mathbf{M} = -(0.457 \text{ lb}\cdot\text{ft})\mathbf{i} \blacktriangleleft$$

## PROBLEM 18.81

The flywheel of an automobile engine, which is rigidly attached to the crankshaft, is equivalent to a 400-mm-diameter, 15-mm-thick steel plate. Determine the magnitude of the couple exerted by the flywheel on the horizontal crankshaft as the automobile travels around an unbanked curve of 200-m radius at a speed of 90 km/h, with the flywheel rotating at 2700 rpm. Assume the automobile to have (*a*) a rear-wheel drive with the engine mounted longitudinally, (*b*) a front-wheel drive with the engine mounted transversely. (Density of steel $= 7860\ \text{kg/m}^3$.)

## SOLUTION

Let the $x$ axis be a horizontal axis directed along the engine mounting, i.e., longitudinally for rear-wheel drive and transversely for front-wheel drive.

Let the $y$ axis be vertical.

The angular velocity of the automobile, $\boldsymbol{\omega}_2$, is equal to $(v/\rho)\mathbf{j}$, where

$$v = 90\ \text{km/h} = 25\ \text{m/s} \quad \text{and} \quad \rho = 200\ \text{m}.$$

$$\boldsymbol{\omega}_2 = \frac{25}{200}\mathbf{j} = (0.125\ \text{rad/s})\mathbf{j}$$

Angular velocity of the fly wheel relative to the automobile: $\boldsymbol{\omega}_1 = \omega_1 \mathbf{i}$

where $\omega_1 = \dfrac{2\pi(2700)}{60} = 282.74\ \text{rad/s}$

Angular momentum of fly wheel: $\mathbf{H}_G = I_x \omega_1 \mathbf{i} + I_y \omega_y \mathbf{j}$

Let the reference frame $Gxyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \omega_2 \mathbf{j}$

$$\begin{aligned}\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \\ &= 0 + \omega_2 \mathbf{j} \times (I_x \omega_1 \mathbf{i} + I_y \omega_2 \mathbf{j}) \\ &= -I_x \omega_1 \omega_2 \mathbf{k}\end{aligned}$$

Couple exerted by the shaft on the fly wheel: $\mathbf{M} = -I_x \omega_1 \omega_2 \mathbf{k}$

Couple exerted by the fly wheel on the shaft: $\mathbf{M}' = -\mathbf{M} = I_x \omega_1 \omega_2 \mathbf{k} \qquad (1)$

Data for fly wheel: $m = \rho\left(\dfrac{\pi}{4}d^2\right)t = (7860)\dfrac{\pi}{4}(0.4)^2(0.015) = 14.816\ \text{kg}$

For a circular plate, $I_x = \dfrac{1}{2}mr^2 = \dfrac{1}{2}(14.816)(0.2)^2 = 0.29632\ \text{kg}\cdot\text{m}^2$

Using Equation (1), $\mathbf{M}' = (0.29632)(282.74)(0.125)\mathbf{k} = (10.47\ \text{N}\cdot\text{m})\mathbf{k}$

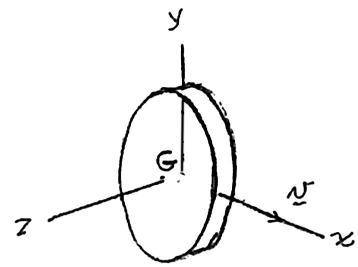
(*a*) Magnitude of couple for rear-wheel drive: $M' = 10.47\ \text{N}\cdot\text{m}$ ◀

(*b*) For front-wheel drive: $M' = 10.47\ \text{N}\cdot\text{m}$ ◀

## PROBLEM 18.82

Each wheel of an automobile has a mass of 22 kg, a diameter of 575 mm, and a radius of gyration of 225 mm. The automobile travels around an unbanked curve of radius 150 m at a speed of 95 km/h. Knowing that the transverse distance between the wheels is 1.5 m, determine the additional normal force exerted by the ground on each outside wheel due to the motion of the car.

## SOLUTION




For each wheel, $v = 95 \text{ km/h} = 26.389 \text{ m/s}$

$$\omega_x = 0$$

$$\omega_y = \frac{v}{\rho} = \frac{26.389}{150} = 0.17593 \text{ rad/s}$$

$$r = \frac{d}{2} = \frac{575}{2} = 287.5 \text{ mm} = 0.2875 \text{ m}$$

$$\bar{I}_z = mk^2 = (22)(0.225)^2 = 1.11375 \text{ kg}\cdot\text{m}^2$$

$$\omega_z = -\frac{v}{r} = -\frac{26.389}{0.2875} = -91.787 \text{ rad/s}$$

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$
$$= \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

Let reference frame $Gxyz$ be rotating with angular velocity $\mathbf{\Omega} = \omega_y\mathbf{j}$.

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega}\times\mathbf{H}_G$$
$$= 0 + \omega_y\mathbf{j}\times(\bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k})$$
$$= \bar{I}_z\omega_z\omega_y\mathbf{i}$$
$$\dot{\mathbf{H}}_G = (1.11375)(-91.787)(0.17593)\mathbf{i}$$
$$= -(17.985 \text{ N}\cdot\text{m})\mathbf{i}$$

Let $O$ be the point at the center of the axle. For the two wheels plus the axle,

$$\dot{\mathbf{H}}_O = \dot{\mathbf{H}}_G + \dot{\mathbf{H}}_G = -(35.97 \text{ N}\cdot\text{m})\mathbf{i}$$

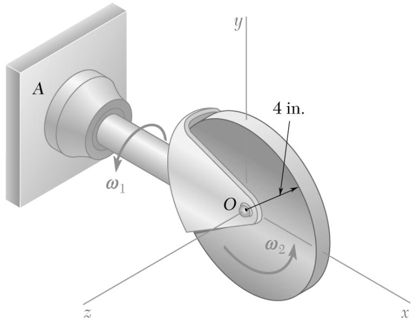
The distance between the wheels is 1.5 m.

$$\mathbf{M}_O = 1.5\mathbf{k}\times F_y\mathbf{j} = -1.5F_y\mathbf{i}$$

Set $\mathbf{M}_O = \dot{\mathbf{H}}_O$ and solve for $F_y$.

$$F_y = \frac{-35.97}{-1.5} = 23.98 \text{ N}$$

$\mathbf{F}_y = 24.0 \text{ N}\uparrow$ ◀




## PROBLEM 18.83

The uniform thin 5-lb disk spins at a constant rate $\omega_2 = 6$ rad/s about an axis held by a housing attached to a horizontal rod that rotates at the constant rate $\omega_1 = 3$ rad/s. Determine the couple which represents the dynamic reaction at the support $A$.

## SOLUTION

Angular velocity: $\omega_x = \omega_1, \quad \omega_y = 0, \quad \omega_z = \omega_2.$

$$\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{k}$$

Angular momentum:

$$\mathbf{H}_O = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

$$= \bar{I}_x\omega_1\mathbf{i} + \bar{I}_z\omega_2\mathbf{k}$$

Let frame $Oxyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \omega_1\,\mathbf{i}$.

Rate of change of angular momentum.

$$\dot{\mathbf{H}}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \boldsymbol{\Omega}\times\mathbf{H}_O$$

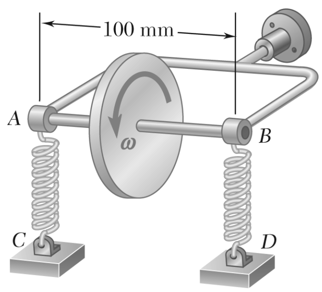
$$= \bar{I}_x\dot{\omega}_1\mathbf{i} + \bar{I}_z\dot{\omega}_2\mathbf{k} + \omega_1\mathbf{i}\times(\bar{I}_x\omega_1\mathbf{i} + \bar{I}_z\omega_2\mathbf{k})$$

$$= 0 + 0 + 0 - \bar{I}_z\omega_1\omega_2\mathbf{j} = -\frac{1}{2}mr^2\omega_1\omega_2\mathbf{j}$$

Dynamic reaction couple: $\mathbf{M} = \dot{\mathbf{H}}_O$

$$\mathbf{M} = -\frac{1}{2}mr^2\omega_1\omega_2\mathbf{j} = -\frac{1}{2}\left(\frac{5}{32.2}\right)\left(\frac{4}{12}\right)^2(3)(6)\mathbf{j}$$

$$\mathbf{M} = -(0.1553\ \text{lb}\cdot\text{ft})\mathbf{j} \blacktriangleleft$$

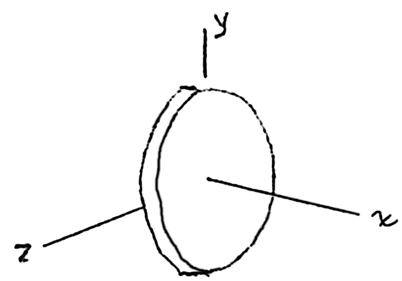



## PROBLEM 18.84

The essential structure of a certain type of aircraft turn indicator is shown. Each spring has a constant of 500 N/m, and the 200-g uniform disk of 40-mm radius spins at the rate of 10,000 rpm. The springs are stretched and exert equal vertical forces on yoke *AB* when the airplane is traveling in a straight path. Determine the angle through which the yoke will rotate when the pilot executes a horizontal turn of 750-m radius to the right at a speed of 800 km/h. Indicate whether Point *A* will move up or down.

## SOLUTION

Let the *x* axis lie along the axle *AB* and the *y* axis be vertical.




$$\omega_x = \frac{2\pi(10{,}000)}{60} = 1047.2 \text{ rad/s}$$

$$v = 800 \text{ km/h} = 222.22 \text{ m/s}$$

$$\rho = 750 \text{ m}$$

$$\omega_y = -\frac{v}{\rho} = -\frac{22.222}{750} = -0.2963 \text{ rad/s}$$

$$\omega_z = 0$$

Angular momentum:

$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k}$$
$$= I_x \omega_x \mathbf{i} + I_y \omega_y \mathbf{j}$$

Let the reference frame *Gxyz* be turning about the *y* axis with angular velocity $\mathbf{\Omega} = \omega_y \mathbf{j}$.

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G$$
$$= \omega_y \mathbf{j} \times (\bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j})$$
$$= -I_x \omega_x \omega_y \mathbf{k}$$

Data for the disk: $m = 200 \text{ g} = 0.2 \text{ kg}$

$$I_x = \frac{1}{2} m r^2$$
$$= \frac{1}{2}(0.2)(0.040)^2$$
$$= 160 \times 10^{-6} \text{ kg} \cdot \text{m}^2$$

$$\mathbf{M}_G = \dot{\mathbf{H}}_G$$
$$= -(160 \times 10^{-6})(1047.2)(-0.2963)\mathbf{k}$$
$$= (0.049646 \text{ N} \cdot \text{m})\mathbf{k}$$

The spring forces $\mathbf{F}_A$ and $\mathbf{F}_B$ exerted on the yoke provide the couple $\mathbf{M}_G$. The force exerted by spring $B$ is upward.

Let $\quad \mathbf{F}_B = F\mathbf{j}$

Then $\quad \mathbf{F}_A = -F\mathbf{j}$

$$\mathbf{M}_G = \mathbf{r}_{B/A}\mathbf{i} \times F\mathbf{j}$$
$$= 0.100\mathbf{i} \times F\mathbf{j}$$
$$= 0.1F\mathbf{k}$$

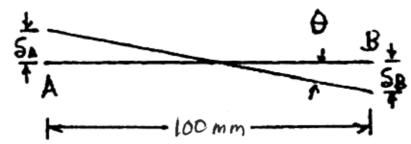
From $\quad \mathbf{M}_G = \dot{\mathbf{H}}_G,$

$$0.1F = 0.049646$$
$$F = 0.49646 \text{ N.}$$

Compression of spring $B$: $\quad \delta_B = \dfrac{F}{k}$

$$= \frac{0.49646}{500}$$
$$= 0.99291 \times 10^{-3} \text{ m}$$
$$= 0.9929 \text{ mm}$$

Point $B$ moves 0.9929 mm *down*. Point $A$ moves 0.9929 mm *up*.

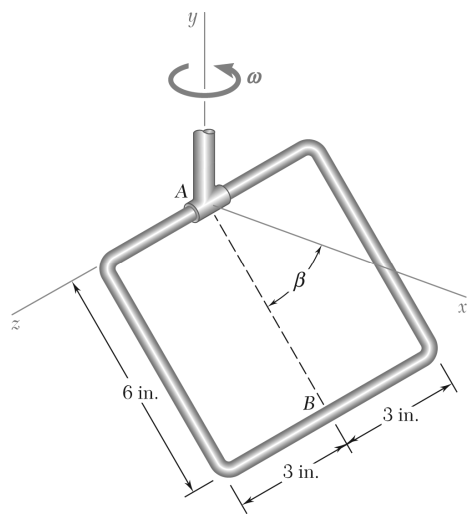



Turning angle for yoke: $\quad \theta = \dfrac{0.9929 + 0.9929}{100}$

$$= 0.019858 \text{ rad}$$

$\boldsymbol{\theta} = 1.138°$ ◀

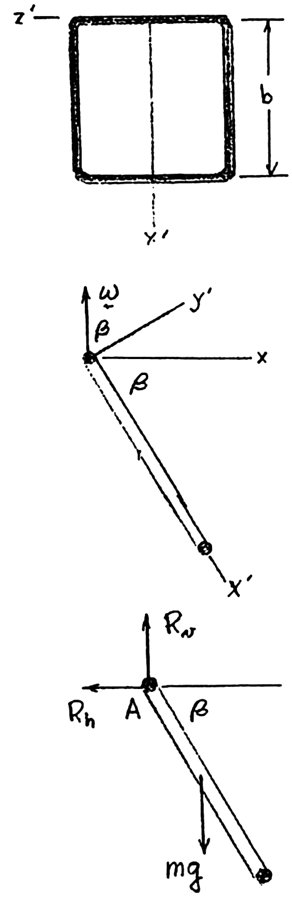
Point $A$ moves *up*. ◀




## PROBLEM 18.85

A slender rod is bent to form a square frame of side 6 in. The frame is attached by a collar at $A$ to a vertical shaft which rotates with a constant angular velocity $\boldsymbol{\omega}$. Determine the value of $\omega$ for which line $AB$ forms an angle $\beta = 48°$ with the horizontal $x$ axis.

## SOLUTION




Choose principal axes $x'$, $y'$, $z'$ with origin at the fixed Point $A$.

$$I_{x'} = 2\left(\frac{m}{4}\right)\left(\frac{1}{12}a^2\right) + 2\left(\frac{m}{4}\right)\left(\frac{a}{2}\right)^2 = \frac{1}{6}ma^2$$

$$I_{z'} = 2\left(\frac{m}{4}\right)\left(\frac{1}{3}a^2\right) + \left(\frac{m}{4}\right)(a)^2 = \frac{5}{12}ma^2$$

$$I_{y'} = I_{x'} + I_{z'} = \frac{7}{12}ma^2$$

Angular velocity: $\boldsymbol{\omega} = -\omega \sin\beta \mathbf{i}' + \omega \cos\beta \mathbf{j}'$

Angular momentum about $A$: $\mathbf{H}_A = -I_{x'}\omega \sin\beta \mathbf{i}' + I_{y'}\omega\cos\beta \mathbf{j}'$

Let the reference from $Axyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

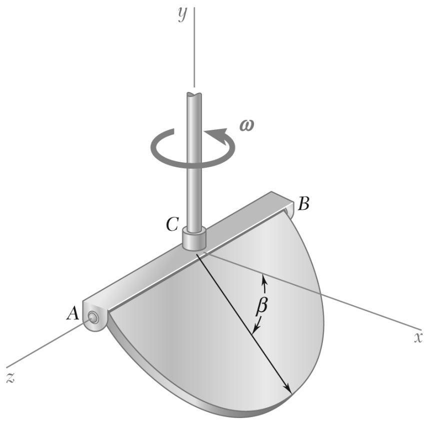
$$\begin{aligned}\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Axyz} + \boldsymbol{\Omega}\times\mathbf{H}_A = 0 + \boldsymbol{\omega}\times\mathbf{H}_A \\ &= (-\omega\sin\beta\mathbf{i}' + \omega\cos\beta\mathbf{j}') \\ &\quad \times(-I_{x'}\,\omega\sin\beta\mathbf{i}' + I_{y'}\,\omega\cos\beta\mathbf{j}') \\ &= -(I_y - I_z)\,\omega^2\sin\beta\cos\beta\mathbf{k} \\ &= -\frac{5}{12}ma^2\omega^2\sin\beta\cos\beta\mathbf{k}\end{aligned}$$

$$\Sigma\mathbf{M}_A = -mg\frac{a}{2}\cos\beta\mathbf{k} = \dot{\mathbf{H}}_A$$

$$-mg\frac{a}{2}\cos\beta = -\frac{5}{12}ma^2\omega^2\sin\beta\cos\beta$$

$$\omega^2 = \frac{6}{5}\frac{g}{a\sin\beta} = \frac{(6)(32.2)}{(5)(0.5)\sin 48°} = 103.99\,(\text{rad/s})^3$$

$\omega = 10.20$ rad/s ◀




## PROBLEM 18.86

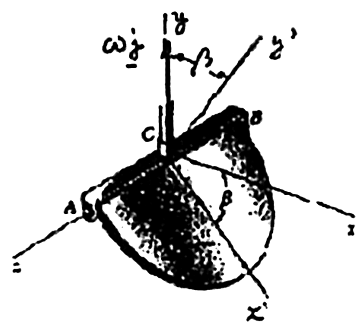
A uniform semicircular plate of radius 120 mm is hinged at $A$ and $B$ to a clevis which rotates with a constant angular velocity $\boldsymbol{\omega}$ about a vertical axis. Determine ($a$) the angle $\beta$ that the plate forms with the horizontal $x$ axis when $\omega = 15$ rad/s, ($b$) the largest value of $\omega$ for which the plate remains vertical ($\beta = 90°$).

## SOLUTION

Moments and products of inertia.

We use the axes $Cx'y'z$ shown.

We note that $I_{x'}$ and $I_{y'}$ are half those for a circular plate, and so is the mass $m$. Thus,




$$I_{x'} = \frac{1}{4}mr^2$$

$$I_{y'} = \frac{1}{2}mr^2$$

Because of symmetry, all products of inertia are equal to zero:

$$I_{x'y'} = I_{y'z} = I_{zx'} = 0$$

Angular momentum about $C$.

$$\mathbf{H}_C = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}'$$

$$= \frac{1}{4}mx^2(-\omega\sin\beta)\mathbf{i}' + \frac{1}{2}mr^2(\omega\cos\beta)\mathbf{j}'$$

$$= \frac{1}{4}mx^2\omega(-\sin\beta\mathbf{i}' + 2\cos\beta\mathbf{j}')$$

Since $C$ is a fixed point, we can use Equation (18.28):

$$\Sigma\mathbf{M}_C = (\dot{\mathbf{H}}_C)_{Cx'y'z} + \boldsymbol{\Omega}\times\mathbf{H}_C = 0 + \omega\mathbf{j}\times\mathbf{H}_C$$

Or, since $\mathbf{j} = -\mathbf{i}'\sin\beta + \mathbf{j}'\cos\beta$: $\quad \Sigma\mathbf{M}_C = \omega(-\mathbf{i}'\sin\beta + \mathbf{j}'\cos\beta)\times\frac{1}{4}mx^2\omega(-\sin\beta\mathbf{i}' + 2\cos\beta\mathbf{j}')$

$$= \frac{1}{4}mr^2\omega^2(-2\sin\beta\cos\beta\mathbf{k} + \cos\beta\sin\beta\mathbf{k})$$

$$\Sigma\mathbf{M}_C = -\frac{1}{4}mr^2\omega^2\sin\beta\cos\beta\mathbf{k} \qquad (1)$$

## PROBLEM 18.86 (Continued)

But $$\Sigma \mathbf{M}_C = -mg\overline{x}' \cos\beta \mathbf{k}$$

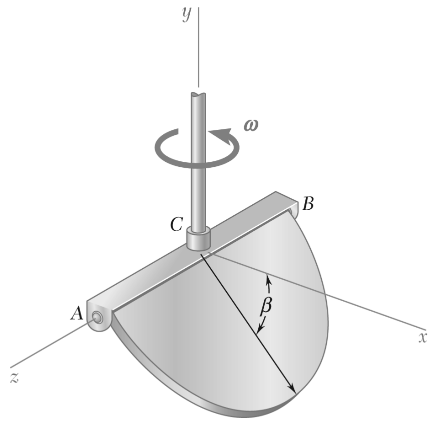

$$= -mg\frac{4r}{3\pi}\cos\beta \mathbf{k} \qquad (2)$$

Equating (1) and (2): $$\frac{1}{4}mr^2\omega^2 \sin\beta\cos\beta = \frac{4mgr}{3\pi}\cos\beta$$

$$\omega^2 \sin\beta = \frac{16}{3\pi}\frac{g}{r} = \frac{16}{3\pi}\frac{9.81 \text{ m/s}^2}{0.12 \text{ m}} \qquad \omega^2\sin\beta = 138.78 \text{ s}^{-2} \quad (3)$$

(*a*) Let $\omega = 15$ rad/s in Eq. (3): $$\sin\beta = \frac{138.78}{(15)^2} = 0.61681 \qquad \beta = 38.1° \blacktriangleleft$$

(*b*) Let $\beta = 90°$ in Eq. (3): $$\omega^2 = 138.78 \text{ s}^{-2} \qquad \omega = 11.78 \text{ rad/s} \blacktriangleleft$$




## PROBLEM 18.87

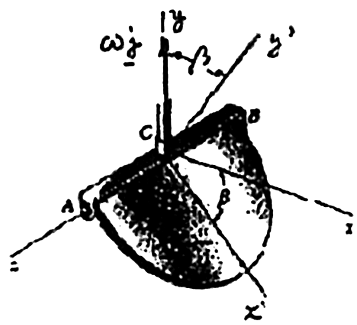
A uniform semicircular plate of radius 120 mm is hinged at $A$ and $B$ to a clevis which rotates with a constant angular velocity $\boldsymbol{\omega}$ about a vertical axis. Determine the value of $\omega$ for which the plate forms an angle $\beta = 50°$ with the horizontal $x$ axis.

## SOLUTION

<u>Moments and products of inertia</u>.

We use the axes $Cx'y'z$ shown.

We note that $I_{x'}$ and $I_{y'}$ are half those for a circular plate, and so is the mass $m$. Thus,




$$I_{x'} = \frac{1}{4}mr^2$$

$$I_{y'} = \frac{1}{2}mr^2$$

Because of symmetry, all products of inertia are equal to zero:

$$I_{x'y'} = I_{y'z} = I_{zx'} = 0$$

<u>Angular momentum about $C$</u>.

$$\mathbf{H}_C = I_{x'}\omega_{x'}\mathbf{i}' + I_{y'}\omega_{y'}\mathbf{j}'$$

$$= \frac{1}{4}mr^2(-\omega\sin\beta)\mathbf{i}' + \frac{1}{2}mr^2(\omega\cos\beta)\mathbf{j}'$$

$$= \frac{1}{4}mr^2\omega(-\sin\beta\mathbf{i}' + 2\cos\beta\mathbf{j}')$$

Since $C$ is a fixed point, we can use Equation (18.28):

$$\Sigma\mathbf{M}_C = (\dot{\mathbf{H}}_C)_{Cx'y'z} + \boldsymbol{\Omega}\times\mathbf{H}_C$$

$$= 0 + \omega\mathbf{j}\times\mathbf{H}_C$$

Or, since $\mathbf{j} = -\mathbf{i}'\sin\beta + \mathbf{j}'\cos\beta$:

$$\Sigma\mathbf{M}_C = \omega(-\mathbf{i}'\sin\beta + \mathbf{j}'\cos\beta)\times\frac{1}{4}mr^2\omega(-\sin\beta\mathbf{i}' + 2\cos\beta\mathbf{j}')$$

$$= \frac{1}{4}mr^2\omega^2(-2\sin\beta\cos\beta\mathbf{k} + \cos\beta\sin\beta\mathbf{k})$$

$$\Sigma\mathbf{M}_C = -\frac{1}{4}mr^2\omega^2\sin\beta\cos\beta\mathbf{k} \qquad (1)$$

## PROBLEM 18.87 (Continued)

But

$$\Sigma \mathbf{M}_C = -mg\overline{x}' \cos\beta\,\mathbf{k}$$

$$= -mg\frac{4r}{3\pi}\cos\beta\,\mathbf{k} \qquad (2)$$

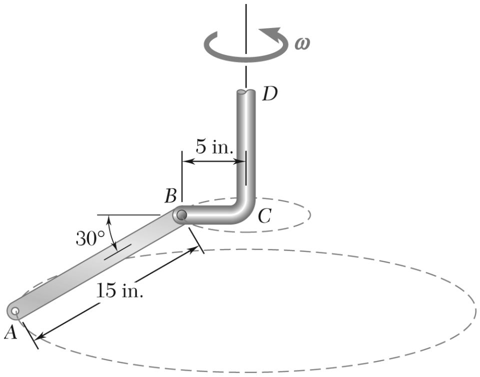
Equating (1) and (2):

$$\frac{1}{4}mr^2\omega^2 \sin\beta\cos\beta = \frac{4mgr}{3\pi}\cos\beta$$

$$\omega^2 \sin\beta = \frac{16}{3\pi}\frac{g}{r} = \frac{16}{3\pi}\frac{9.81\ \text{m/s}^2}{0.12\ \text{m}} \qquad \omega^2\sin\beta = 138.78\ \text{s}^{-2} \quad (3)$$

Let $\beta = 50°$ in Equation (3):

$$\omega^2 = \frac{138.78\ \text{s}^{-2}}{\sin 50°} = 181.17\ \text{s}^{-2} \qquad \omega = 13.46\ \text{rad/s} \blacktriangleleft$$



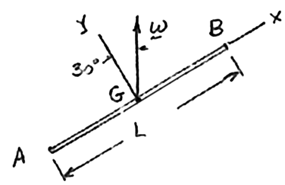

## PROBLEM 18.88

The slender rod *AB* is attached by a clevis to arm *BCD* which rotates with a constant angular velocity **ω** about the centerline of its vertical portion *CD*. Determine the magnitude of the angular velocity **ω**.

## SOLUTION

Let $AB = L = 15 \text{ in.} = 1.25 \text{ ft}, \quad \text{and} \quad BC = b = 5 \text{ in.} = 0.41667 \text{ ft}$

Choose $x, y, z$ axes as shown. $\bar{I}_x \approx 0, \quad \bar{I}_y = \bar{I}_z = \dfrac{1}{12}mL^2$




Angular velocity: $\boldsymbol{\omega} = \omega \sin 30°\mathbf{i} + \omega \cos 30°\mathbf{j}$

Angular momentum of rod *AB* about its mass center *G*:

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k} = \bar{I}_y\omega\cos 30°\mathbf{j}$$

Let the reference frame *Gxyz* be rotating with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$

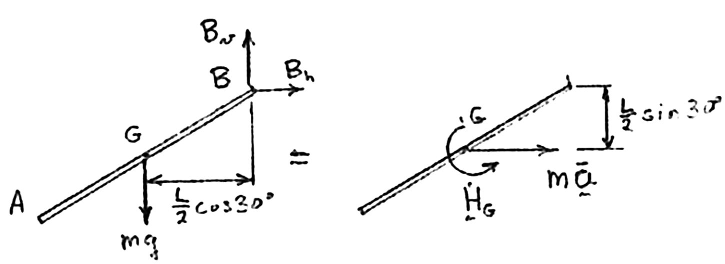
$$= 0 + (\omega \sin 30°\mathbf{i} + \omega\cos 30°\mathbf{j}) \times \bar{I}_y\omega\cos 30°\mathbf{j}$$

$$= \bar{I}_y\omega^2 \sin 30° \cos 30°\mathbf{k} = \frac{\sqrt{3}}{48}mL^2\omega^2\mathbf{k}$$

Radius of circular path of Point *G*: $r = \dfrac{L}{2}\cos 30° + b = 0.95793 \text{ ft}$

Acceleration of the mass center: $\bar{\mathbf{a}} = r\omega^2 \longrightarrow$

Equations of motion:



## PROBLEM 18.88 (Continued)

$$+\circlearrowleft \Sigma \mathbf{M}_B = mg\frac{L}{2}\cos 30°\mathbf{k} = \left(\frac{L}{2}\sin 30°\right)mr\omega^2\mathbf{k} + \dot{\mathbf{H}}_G$$

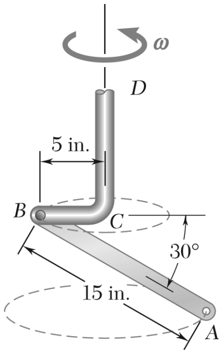
$$\frac{\sqrt{3}}{4}mgL\mathbf{k} = \left(\frac{1}{4}mLr + \frac{\sqrt{3}}{48}mL^2\right)\omega^2\mathbf{k}$$

$$\frac{\sqrt{3}}{4}g = \left(\frac{1}{4}r + \frac{\sqrt{3}}{48}L\right)\omega^2$$

$$\frac{\sqrt{3}}{4}(32.2) = \left[\frac{1}{4}(0.95793) + \frac{\sqrt{3}}{48}(1.25)\right]\omega^2$$

$$\omega^2 = 48.994$$

$\omega = 7.00$ rad/s ◀

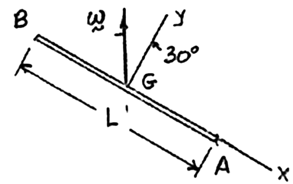



## PROBLEM 18.89

The slender rod $AB$ is attached by a clevis to arm $BCD$, which rotates with a constant angular velocity $\omega$ about the centerline of its vertical portion $CD$. Determine the magnitude of the angular velocity $\omega$.

## SOLUTION

Let $AB = L = 15 \text{ in.} = 1.25 \text{ ft}$, and $BC = b = 5 \text{ in.} = 0.41667 \text{ ft}$




Choose $x, y, z$ axes as shown. $\bar{I}_x \approx 0, \quad \bar{I}_y = \bar{I}_z = \dfrac{1}{12}mL^2$

Angular velocity: $\boldsymbol{\omega} = -\omega \sin 30°\mathbf{i} + \omega \cos 30°\mathbf{j}$

Angular momentum of rod $AB$ about its mass center $G$:

$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} = \bar{I}_y \omega \cos 30°\mathbf{j}$$

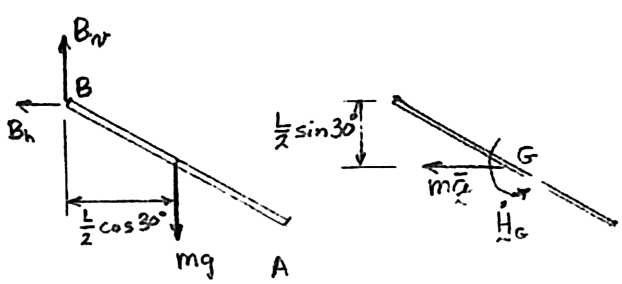
Let the reference frame $Gxyz$ be rotating with angular velocity $\mathbf{\Omega} = \boldsymbol{\omega}$.

$$\begin{aligned} \dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G \\ &= 0 + (-\omega \sin 30°\mathbf{i} + \omega \cos 30°\mathbf{j}) \times \bar{I}_y \omega \cos 30°\mathbf{j} \\ &= -\bar{I}_y \omega^2 \sin 30° \cos 30°\mathbf{k} = -\frac{\sqrt{3}}{48} mL^2 \omega^2 \mathbf{k} \end{aligned}$$

Radius of circular path of Point $G$: $r = \dfrac{L}{2}\cos 30° - b = 0.1246 \text{ ft}$

Acceleration of the mass center: $\bar{\mathbf{a}} = r\omega^2 \longleftarrow$

Equations of motion:



## PROBLEM 18.89 (Continued)

$$+\circlearrowleft \Sigma \mathbf{M}_B = -mg\frac{L}{2}\cos 30°\mathbf{k} = -\left(\frac{L}{2}\sin 30°\right)mr\omega^2\mathbf{k} + \dot{\mathbf{H}}_G$$

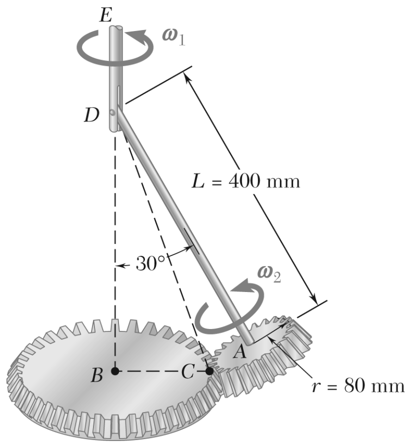
$$-\frac{\sqrt{3}}{4}mgL\mathbf{k} = -\left(\frac{1}{4}mLr + \frac{\sqrt{3}}{48}mL^2\right)\omega^2\mathbf{k}$$

$$\frac{\sqrt{3}}{4}g = \left(\frac{1}{4}r + \frac{\sqrt{3}}{48}L\right)\omega^2$$

$$\frac{\sqrt{3}}{4}(32.3) = \left[\frac{1}{4}(0.1246) + \frac{\sqrt{3}}{48}(1.25)\right]\omega^2$$

$$\omega^2 = 182.85$$

$\omega = 13.52$ rad/s ◀



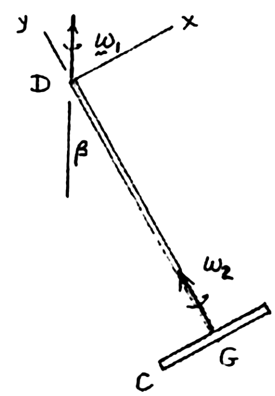

## PROBLEM 18.90

The 950-g gear $A$ is constrained to roll on the fixed gear $B$, but is free to rotate about axle $AD$. Axle $AD$, of length 400 mm and negligible mass, is connected by a clevis to the vertical shaft $DE$, which rotates as shown with a constant angular velocity $\boldsymbol{\omega}_1$. Assuming that gear $A$ can be approximated by a thin disk of radius 80 mm, determine the largest allowable value of $\omega_1$ if gear $A$ is not to lose contact with gear $B$.

## SOLUTION

$$\beta = 30°$$
$$L = 400 \text{ mm} = 0.4 \text{ m}$$
$$r = 80 \text{ mm} = 0.08 \text{ m}$$

Choose principal axes $x$, $y$, $z$ as shown.



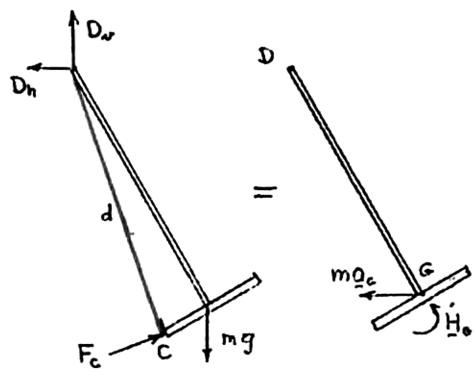

Kinematics:
$$\boldsymbol{\omega}_1 = \omega_1 \sin\beta\mathbf{i} + \omega_1 \cos\beta\mathbf{j}$$
$$\boldsymbol{\omega}_2 = \omega_2\mathbf{j}$$
$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$$
$$= \omega_1 \sin\beta\mathbf{i} + (\omega_1 \cos\beta + \omega_2)\mathbf{j}$$
$$\omega_x = \omega_1 \sin\beta$$
$$\omega_y = \omega_1 \cos\beta + \omega_2$$
$$\mathbf{v}_G = \boldsymbol{\omega}_1 \times \mathbf{r}_{G/D} = \boldsymbol{\omega}_1\mathbf{j} \times \mathbf{r}_{G/D}$$
$$= -\omega_1 L \sin\beta\mathbf{k}$$
$$\mathbf{a}_G = \boldsymbol{\omega}_1 \times \mathbf{v}_G = \omega_1^2 L \sin\beta \longleftarrow$$
$$\mathbf{v}_C = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}_{C/G}$$
$$0 = -\omega_1 L \sin\beta\mathbf{k} + (\omega_x\mathbf{i} + \omega_y\mathbf{j}) \times (-r\mathbf{i})$$
$$= -\omega_1 L \sin\beta\mathbf{k} + \omega_y r\mathbf{k}$$
$$\omega_y = \omega_1 \frac{L}{r} \sin\beta$$

Angular momentum:
$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j}$$

## PROBLEM 18.90 (Continued)

Let the reference frame $Dxyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}_1$.

$$\begin{aligned}\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \\ &= 0 + (\omega_1 \sin\beta\mathbf{i} + \omega_1 \cos\beta\mathbf{j}) \times (\bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j}) \\ &= (\bar{I}_y\omega_y\omega_1 \sin\beta - \bar{I}_x\omega_x\omega_1 \cos\beta)\mathbf{k} \\ &= \left(\frac{1}{2}mr^2\omega_y \sin\beta - \frac{1}{4}mr^2\omega_x \cos\beta\right)\omega_1\mathbf{k}\end{aligned}$$




Moments about $D$: $\quad \mathbf{M}_D = Fd\mathbf{k} - mgL\sin\beta\mathbf{k}$

where $\quad d = (DC) = \sqrt{L^2 + r^2}$

$$\begin{aligned}(\mathbf{M}_D)_{\text{eff}} &= \dot{\mathbf{H}}_G + \mathbf{r}_{G/D} \times m\mathbf{a}_G \\ &= \dot{\mathbf{H}}_G + m\omega_1^2L^2 \sin\beta\cos\beta\mathbf{k}\end{aligned}$$

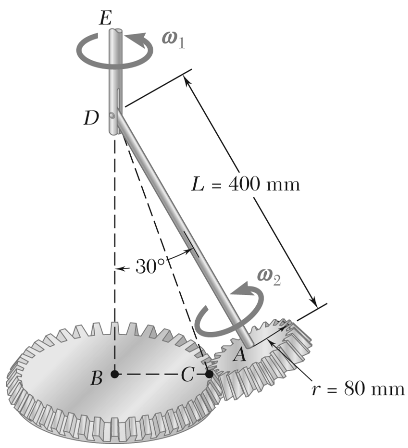
Equating $\mathbf{M}_D = (\mathbf{M}_D)_{\text{eff}}$ and taking the $z$ component,

$$\begin{aligned}F_Cd - mgL\sin\beta &= \left(\frac{1}{2}mr^2\frac{L}{r}\sin^2\beta - \frac{1}{4}mr^2 \sin\beta\cos\beta\right)\omega_1^2 + m\omega_1^2L^2 \sin\beta\cos\beta \\ &= m\omega_1^2 \sin\beta\left(\frac{1}{2}rL\sin\beta - \frac{1}{4}r^2 \cos\beta - L^2 \cos\beta\right)\end{aligned}$$

Set $F_C = 0$ and solve for $\omega_1^2$: $\quad \omega_1^2 = \dfrac{gL}{L^2\cos\beta + \frac{1}{4}r^2\cos\beta - \frac{1}{2}rL\sin\beta}$

$$= \frac{(9.81)(0.4)}{(0.4)^2\cos 30° + \frac{1}{4}(0.08)^2\cos 30° - \frac{1}{2}(0.08)(0.4)\sin 30°}$$

$$= 29.739 \qquad \omega_2 = 5.45 \text{ rad/s} ◀$$

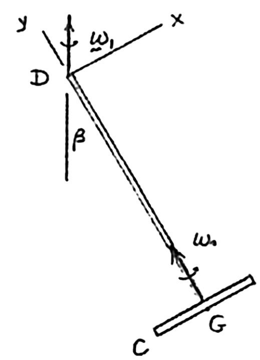



## PROBLEM 18.91

Determine the force **F** exerted by gear $B$ on gear $A$ of Problem 18.90 when shaft $DE$ rotates with the constant angular velocity $\omega_1 = 4$ rad/s. (*Hint:* The force **F** must be perpendicular to the line drawn from $D$ to $C$.)

## SOLUTION

$$\beta = 30°$$
$$L = 400 \text{ mm} = 0.4 \text{ m}$$
$$r = 80 \text{ mm} = 0.08 \text{ m}$$



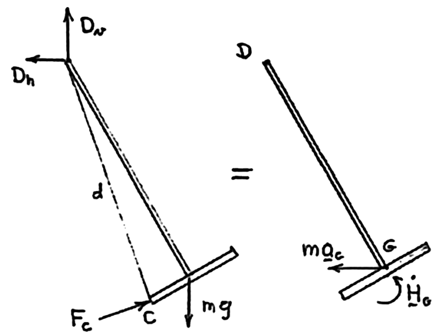

Choose principal axes $x$, $y$, $z$ as shown.

Kinematics:
$$\boldsymbol{\omega}_1 = \omega_1 \sin\beta\mathbf{i} + \omega_1 \cos\beta\mathbf{j}$$
$$\boldsymbol{\omega}_2 = \omega_2\mathbf{j}$$
$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$$
$$= \omega_1 \sin\beta\mathbf{i} + (\omega_1 \cos\beta + \omega_2)\mathbf{j}$$
$$\omega_x = \omega_1 \sin\beta$$
$$\omega_y = \omega_1 \cos\beta + \omega_2$$
$$\mathbf{v}_G = \boldsymbol{\omega}_1 \times \mathbf{r}_{G/D}$$
$$= \boldsymbol{\omega}_1\mathbf{j} \times \mathbf{r}_{G/D}$$
$$= -\omega_1 L \sin\beta\mathbf{k}$$
$$\mathbf{a}_G = \boldsymbol{\omega}_1 \times \mathbf{v}_G = \omega_1^2 L \sin\beta \longleftarrow$$
$$\mathbf{v}_C = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}_{C/G}$$
$$0 = -\omega_1 L \sin\beta\mathbf{k} + (\omega_x\mathbf{i} + \omega_y\mathbf{j}) \times (-r\mathbf{i})$$
$$= -\omega_1 L \sin\beta\mathbf{k} + \omega_y r\mathbf{k}$$
$$\omega_y = \omega_1 \frac{L}{r} \sin\beta$$

Angular momentum:
$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j}$$

Let the reference frame $Dxyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}_1$.

$$\begin{aligned}\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \\ &= 0 + (\omega_1 \sin\beta \mathbf{i} + \omega_1 \cos\beta \mathbf{j}) \times (\bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j}) \\ &= (\bar{I}_y \omega_y \omega_1 \sin\beta - \bar{I}_x \omega_x \omega_1 \cos\beta)\mathbf{k} \\ &= \left(\frac{1}{2}mr^2\omega_y \sin\beta - \frac{1}{4}mr^2\omega_x \cos\beta\right)\omega_1\mathbf{k}\end{aligned}$$




Moments about $D$: $\mathbf{M}_D = Fd\mathbf{k} - mgL\sin\beta\mathbf{k}$

where $d = (DC) = \sqrt{L^2 + r^2}$

$$(\mathbf{M}_D)_{\text{eff}} = \dot{\mathbf{H}}_G + \mathbf{r}_{G/D} \times m\mathbf{a}_G = \dot{\mathbf{H}}_G + m\omega_1^2 L^2 \sin\beta\cos\beta\mathbf{k}$$

Equating $\mathbf{M}_D = (\mathbf{M}_D)_{\text{eff}}$ and taking the $z$ component,

$$\begin{aligned}F_C d - mgL\sin\beta &= \left(\frac{1}{2}mr^2 \frac{L}{r}\sin^2\beta - \frac{1}{4}mr^2 \sin\beta\cos\beta\right)\omega_1^2 + m\omega_1^2 L^2 \sin\beta\cos\beta \\ &= m\omega_1^2 \sin\beta\left(\frac{1}{2}rL\sin\beta - \frac{1}{4}r^2\cos\beta - L^2\cos\beta\right)\end{aligned}$$

Solving for $F_C$, $$F_C = \frac{m\sin\beta}{d}\left[gL - \omega_1^2\left(L^2\cos\beta + \frac{1}{4}r^2\cos\beta - \frac{1}{2}rL\sin\beta\right)\right]$$

Additional data: $m = 950\,g = 0.95 \text{ kg}, \quad \omega_1 = 4 \text{ rad/s}$

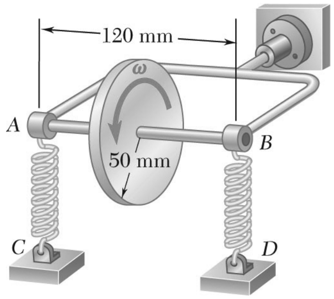
$$d = \sqrt{(0.4)^2 + (0.08)^2} = 0.40792 \text{ m}$$

$$F_C = \frac{(0.95)\sin 30°}{0.40792}\left\{(9.81)(0.4) - (4)^2\left[(0.4)^2\cos 30° + \frac{1}{4}(0.08)^2\cos 30° - \frac{1}{2}(0.4)(0.08)\sin 30°\right]\right\}$$
$$= 2.11 \text{ N}$$

$$\alpha = \beta - \tan^{-1}\frac{r}{L} = 30° - \tan^{-1}\left(\frac{4}{20}\right)$$

$\alpha = 18.7°$ $\qquad$ $2.11 \text{ N} \measuredangle 18.7°$ ◀

## PROBLEM 18.92




The essential structure of a certain type of aircraft turn indicator is shown. Springs *AC* and *BD* are initially stretched and exert equal vertical forces at *A* and *B* when the airplane is traveling in a straight path. Each spring has a constant of 600 N/m and the uniform disk has a mass of 250 g and spins at the rate of 12,000 rpm. Determine the angle through which the yoke will rotate when the pilot executes a horizontal turn of 800-m radius to the right at a speed of 720 km/h. Indicate whether point *A* will move up or down.

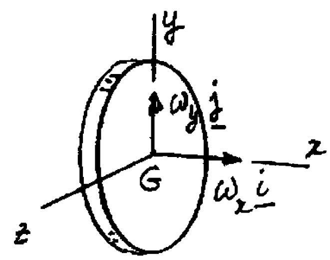
## SOLUTION

Aircraft speed: $v = 720 \text{ km/h} = 200 \text{ m/s}$

Radius of turn: $\rho = 800 \text{ m}$

Angular velocity: $\omega_x = 12000 \text{ rpm} = 1256.6 \text{ rad/s}$

$\omega_y$ is negative since the aircraft is turning to the right.




$$\omega_y = -\frac{v}{\rho} = -\frac{200 \text{ m/s}}{800 \text{ m}} = -0.25 \text{ rad/s}$$

$$\omega_z = 0$$

Angular momentum: $\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j}$

where the reference frame *Gxyz* is turning with the aircraft with an angular velocity

$$\mathbf{\Omega} = \omega_y \mathbf{j}$$

Rate of change of angular momentum:

Since $\omega_x$ and $\omega_y$ are constant, $(\dot{\mathbf{H}}_G)_{Gxyz} = 0$ and Eq. (18.22) yields

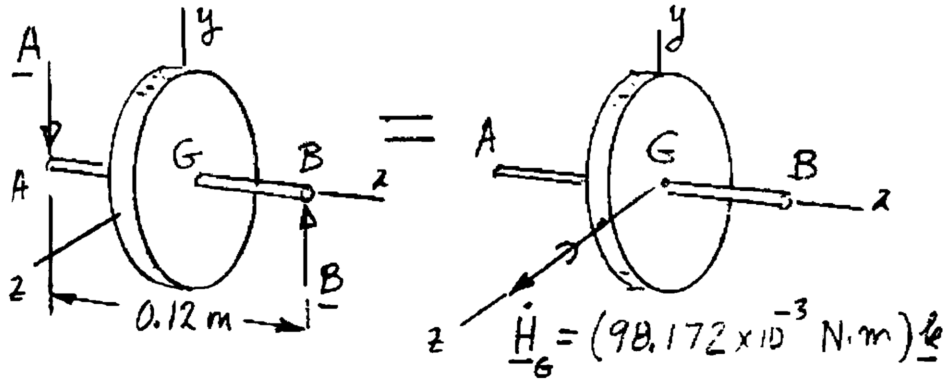
$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G = 0 + \omega_y \mathbf{j} \times (\bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j})$$

$$= -\bar{I}_x \omega_x \omega_y \mathbf{k} = -\left(\frac{1}{2} m r^2\right) \omega_x \omega_y \mathbf{k}$$

$$= -\frac{1}{2}(0.25 \text{ kg})(0.05 \text{ m})^2 (1256.6 \text{ rad/s})(-0.250 \text{ rad/s})\mathbf{k}$$

$$\dot{\mathbf{H}}_G = +(98.172 \times 10^{-3} \text{ N} \cdot \text{m})\mathbf{k}$$

**Free Body and kinetic diagrams**



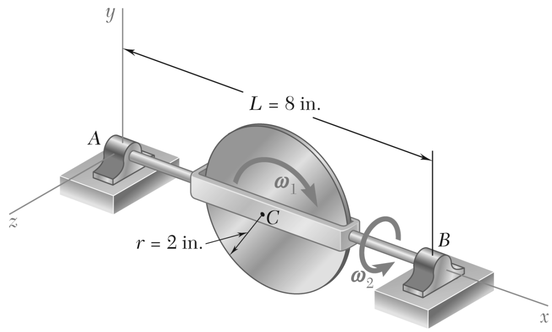

The forces $A$ and $B$ exerted by the springs must be equivalent to the couple $\dot{\mathbf{H}}_G$. They must therefore be directed as shown, which means that the spring at $A$ will stretch and <u>A will move up</u>. ◀

We have $A = B$, $\quad (0.12\text{ m})A = 98.172\times10^{-3}\text{ N}\cdot\text{m}$

$$F = A = 0.81810\text{ N}$$

$$\text{Deflection of spring } = x = \frac{F}{k} = \frac{0.81810\text{ N}}{600\text{ N/m}} = 1.3635\times10^{-3}\text{ m}$$

$$\text{Angle of rotation } = \frac{x}{GA} = \frac{1.3635\times10^{-3}\text{ m}}{0.06\text{ m}} = 0.022725\text{ rad} = 1.30° \quad ◀$$




## PROBLEM 18.93

The 10-oz disk shown spins at the rate $\omega_1 = 750$ rpm, while axle $AB$ rotates as shown with an angular velocity $\omega_2$ of 6 rad/s. Determine the dynamic reactions at $A$ and $B$.

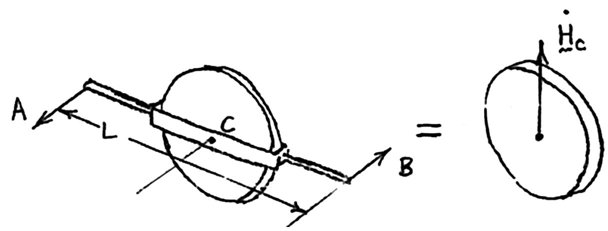
## SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega_2 \mathbf{i} - \omega_1 \mathbf{k}$

Angular momentum: $\mathbf{H}_C = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} = \bar{I}_x \omega_2 \mathbf{i} - \bar{I}_z \omega_1 \mathbf{k}$

Let the reference frame $Cxyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \omega_2 \mathbf{i}$.

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Cxyz} + \boldsymbol{\Omega} \times \mathbf{H}_C = 0 + \omega_2 \mathbf{i} \times (I_x \omega_2 \mathrm{i} - \bar{I}_z \omega_1 \mathbf{k}) = \bar{I}_z \omega_2 \omega_1 \mathbf{j}$$




Acceleration of mass center:

$$\bar{\mathbf{a}} = 0$$

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$

$$A\mathbf{k} - B\mathbf{k} = 0$$

$$A = B$$

$$\mathbf{M}_C = \dot{\mathbf{H}}_C$$

$$LB\mathbf{j} = \bar{I}_z \omega_2 \omega_1 \mathbf{j} \qquad B = \frac{\bar{I}_z \omega_2 \omega_1}{L}$$

Data: $$m = \frac{W}{g} = \frac{10}{(16)(32.2)} = 0.01941 \text{ lb}\cdot\text{s}^2/\text{ft} \qquad r = 2 \text{ in.} = 0.16667 \text{ ft}$$

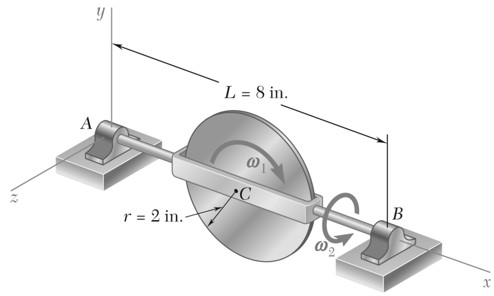
$$\bar{I}_Z = \frac{1}{2} m r^2 = \frac{1}{2}(0.01941)(0.16667)^2 = 269.6 \times 10^{-6} \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\omega_1 = \frac{2\pi(750)}{60} = 25\pi \text{ rad/s}, \quad \omega_2 = 6 \text{ rad/s}, \quad L = 8 \text{ in.} = 0.66667 \text{ ft}$$

$$A = B = \frac{(269.6 \times 10^{-6})(6)(25\pi)}{0.66667} = 0.1906 \text{ lb} \qquad \mathbf{A} = (0.1906 \text{ lb})\mathbf{k} \blacktriangleleft$$

$$\mathbf{B} = -(0.1906 \text{ lb})\mathbf{k} \blacktriangleleft$$

## PROBLEM 18.94




The 10-oz disk shown spins at the rate $\omega_1 = 750$ rpm, while axle $AB$ rotates as shown with an angular velocity $\omega_2$. Determine the maximum allowable magnitude of $\omega_2$ if the dynamic reactions at $A$ and $B$ are not to exceed 0.25 lb each.

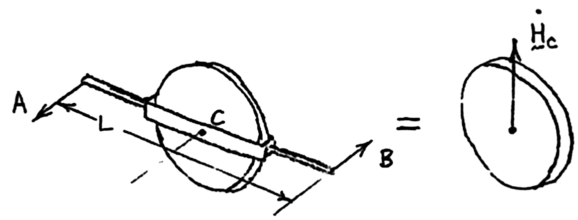
## SOLUTION

Angular velocity: $\boldsymbol{\omega} = \omega_2 \mathbf{i} - \omega_1 \mathbf{k}$

Angular momentum: $\mathbf{H}_C = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} = \bar{I}_x \omega_2 \mathbf{i} - \bar{I}_z \omega_1 \mathbf{k}$

Let the reference frame $Cxyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \omega_2 \mathbf{i}$.

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Cxyz} + \boldsymbol{\Omega} \times \mathbf{H}_C = 0 + \omega_2 \mathbf{i} \times (I_x \omega_2 \mathbf{i} - \bar{I}_z \omega_1 \mathbf{k}) = \bar{I}_z \omega_2 \omega_1 \mathbf{j}$$




Acceleration of mass center:

$$\bar{\mathbf{a}} = 0$$

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$

$$A\mathbf{k} - B\mathbf{k} = 0$$

$$A = B$$

$$\mathbf{M}_C = \dot{\mathbf{H}}_C$$

$$LB\mathbf{j} = \bar{I}_z \omega_2 \omega_1 \mathbf{j} \quad B = \frac{\bar{I}_z \omega_2 \omega_1}{L}$$

<u>Data</u>: $m = \dfrac{W}{g} = \dfrac{10}{(16)(32.2)} = 0.01941 \text{ lb}\cdot\text{s}^2/\text{ft} \quad r = 2 \text{ in.} = 0.16667 \text{ ft}$

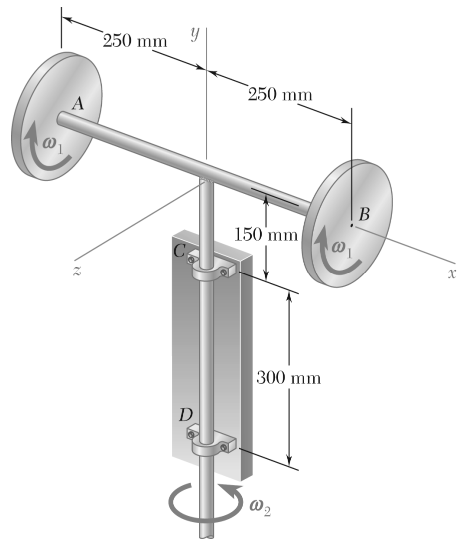
$$\bar{I}_z = \frac{1}{2} mr^2 = \frac{1}{2}(0.01941)(0.16667)^2 = 269.6 \times 10^{-6} \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\omega_1 = \frac{2\pi(750)}{60} = 25\pi \text{ rad/s}, \quad L = 8 \text{ in.} = 0.66667 \text{ ft}$$

$$A = B = 0.25 \text{ lb}$$

$$\omega_2 = \frac{LB}{\bar{I}_z \omega_1} = \frac{(0.66667)(0.25)}{(269.6 \times 10^{-6})(25\pi)}$$

$\omega_2 = 7.87$ rad/s ◀

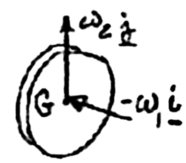



## PROBLEM 18.95

Two disks, each of mass 5 kg and radius 100 mm, spin as shown at the rate $\omega_1 = 1500$ rpm about a rod $AB$ of negligible mass which rotates about a vertical axis at the rate $\omega_2 = 45$ rpm. (*a*) Determine the dynamic reactions at $C$ and $D$. (*b*) Solve part *a*, assuming that the direction of spin of disk $B$ is reversed.

## SOLUTION

Angular momentum of each disk about its mass center.




$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} = -\frac{1}{2} m r^2 \omega_1 \mathbf{i} + \frac{1}{4} m r^2 \omega_2 \mathbf{j}$$

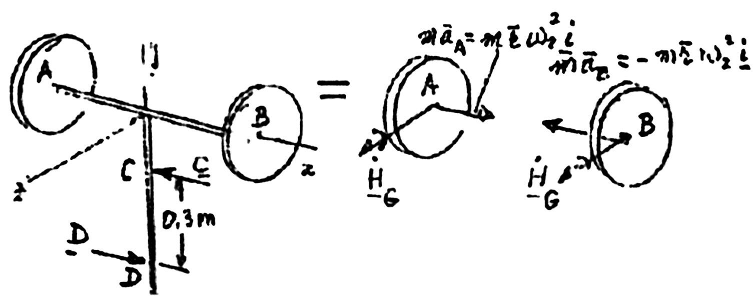
$$\mathbf{H}_G = \frac{1}{4} m r^2 (-2\omega_1 \mathbf{i} + \omega_2 \mathbf{j}) \qquad (1)$$

Eq. (18.22):

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G = 0 + \omega_2 \mathbf{j} \times \frac{1}{4} m r^2 (-2\omega_1 \mathbf{i} + \omega_2 \mathbf{j})$$

$$\dot{\mathbf{H}}_B = +\frac{1}{2} m r^2 \omega_1 \omega_2 \mathbf{k} \qquad (2)$$

Equations of motion.




Since $m\bar{\mathbf{a}}_A$ and $m\bar{\mathbf{a}}_B$ cancel out, effective forces reduce to couple $2\dot{\mathbf{H}}_G = m r^2 \omega_1 \omega_2 \mathbf{k}$.

It follows that the reactions form an equivalent couple with

$$-\mathbf{C} = \mathbf{D} = \left( \frac{m r^2 \omega_1 \omega_2}{0.3 \text{ m}} \right) \mathbf{i} \qquad (3)$$

## PROBLEM 18.95 (Continued)

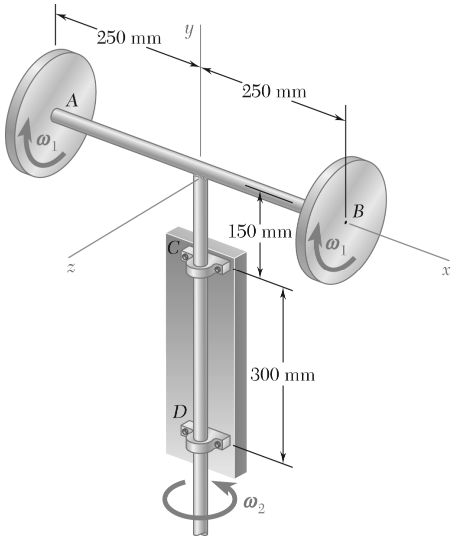
(*a*) With $m = 5$ kg, $r = 0.1$ m, $\omega_1 = 1500 \text{ rpm } = 50\pi \text{ rad/s}$, and $\omega_2 = 45 \text{ rpm} = 1.5\pi \text{ rad/s}$, Eq. (3) yields

$$C = D = (5 \text{ kg})(0.1 \text{ m})^2 (50\pi \text{ rad/s})\left(\frac{1.5\pi \text{ rad/s}}{0.3 \text{ m}}\right) = 123.37 \text{ N}$$

$\mathbf{C} = -(123.4 \text{ N})\mathbf{i}; \quad \mathbf{D} = (123.4 \text{ N})\mathbf{i}$ ◀

(*b*) With direction of spin of $B$ reversed, its angular momentum will also be reversed and the effective forces (and thus, the applied forces) reduce to zero:

$\mathbf{C} = \mathbf{D} = 0$ ◀

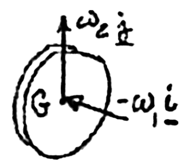



## PROBLEM 18.96

Two disks, each of mass 5 kg and radius 100 mm, spin as shown at the rate $\omega_1 = 1500$ rpm about a rod $AB$ of negligible mass which rotates about a vertical axis at a rate $\omega_2$. Determine the maximum allowable value of $\omega_2$ if the dynamic reactions at $C$ and $D$ are not to exceed 250 N each.

## SOLUTION

Angular momentum of each disk about its mass center.



$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} = -\frac{1}{2} m r^2 \omega_1 i + \frac{1}{4} m r^2 \omega_2 \mathbf{j}$$

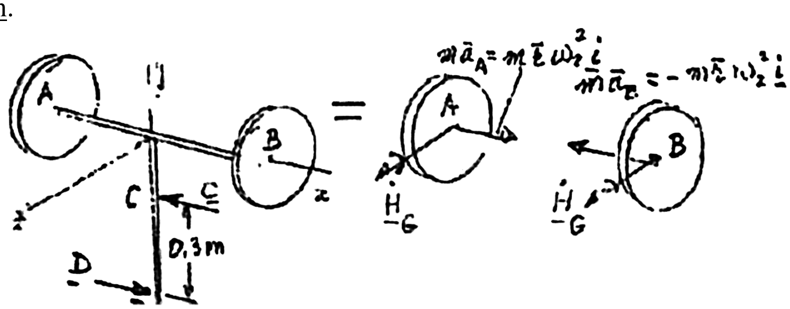
$$\mathbf{H}_G = \frac{1}{4} m r^2 (-2\omega_1 \mathbf{i} + \omega_2 \mathbf{j}) \tag{1}$$

Eq. (18.22):

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G = 0 + \omega_2 \mathbf{j} \times \frac{1}{4} m r^2 (-2\omega_1 \mathbf{i} + \omega_2 \mathbf{j})$$

$$\dot{\mathbf{H}}_B = +\frac{1}{2} m r^2 \omega_1 \omega_2 \mathbf{k} \tag{2}$$

Equations of motion.



Since $m\bar{\mathbf{a}}_A$ and $m\bar{\mathbf{a}}_B$ cancel out, effective forces reduce to couple $2\dot{\mathbf{H}}_G = m r^2 \omega_1 \omega_2 \mathbf{k}$. It follows that the reactions form an equivalent couple with

$$-\mathbf{C} = \mathbf{D} = \left( \frac{m r^2 \omega_1 \omega_2}{0.3 \text{ m}} \right) \mathbf{i} \tag{3}$$

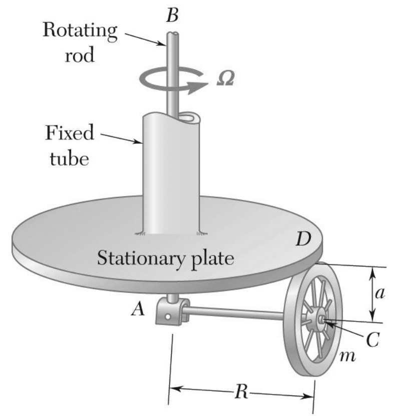
## PROBLEM 18.96 (Continued)

Making $C = D - 250$ N in Eq. (3) yields

$$\frac{mr^2\omega_1\omega_2}{0.3\text{ m}} = 250\text{ N}$$

With $m = 5$ kg, $r = 0.1$ m, $\omega_1 = 1500$ rpm $= 50\pi$ rad/s

We have $$\omega_2 = \frac{(250\text{ N})(0.3\text{ m})}{(5\text{ kg})(0.1\text{ m})^2(50\pi\text{ rad/s})} = 9.5493\text{ rad/s}$$ $\omega_2 = 91.2$ rpm ◀

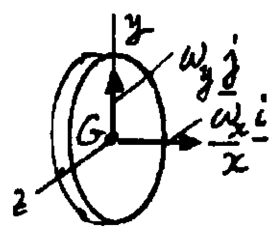



## PROBLEM 18.97

A stationary horizontal plate is attached to the ceiling by means of a fixed vertical tube. A wheel of radius $a$ and mass $m$ is mounted on a light axle $AC$ which is attached by means of a clevis at $A$ to a rod $AB$ fitted inside the vertical tube. The rod $AB$ is made to rotate with a constant angular velocity $\mathbf{\Omega}$ causing the wheel to roll on the lower face of the stationary plate. Determine the minimum angular velocity $\mathbf{\Omega}$ for which contact is maintained between the wheel and the plate. Consider the particular cases ($a$) when the mass of the wheel is concentrated in the rim, ($b$) when the wheel is equivalent to a thin disk of radius $a$.

## SOLUTION

<u>Angular momentum $\mathbf{H}_G$ of wheel:</u> $\omega_y = \Omega, \quad \omega_x = \frac{R}{a}\Omega, \quad \omega_z = 0$




$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k}$$

$$\mathbf{H}_G = \bar{I}_x \frac{R}{a}\Omega \mathbf{i} + \bar{I}_y \Omega \mathbf{j}$$

<u>Rate of change of angular momentum</u>:

Since $\omega_x$ and $\omega_y$ are constant, and observing that the frame $Gxyz$ rotates with the angular velocity $\mathbf{\Omega} = \Omega \mathbf{i}$:

$$(\dot{\mathbf{H}}_G)_{Gxyz} = 0$$

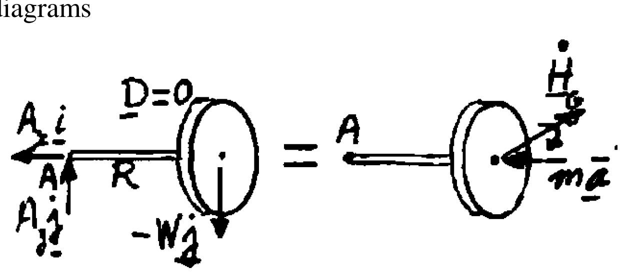
Eq. (18.22):

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G$$

$$\dot{\mathbf{H}}_G = 0 + \Omega \mathbf{j} \times \left( \bar{I}_x \frac{R}{a} a \mathbf{i} + \bar{I}_y \Omega \mathbf{j} \right)$$

$$\dot{\mathbf{H}}_G = -\bar{I}_z \frac{R}{a} \Omega^2 \mathbf{k} \qquad (1)$$

The free body and kinetic diagrams



## PROBLEM 18.97 (Continued)

Equating moments about $A$:

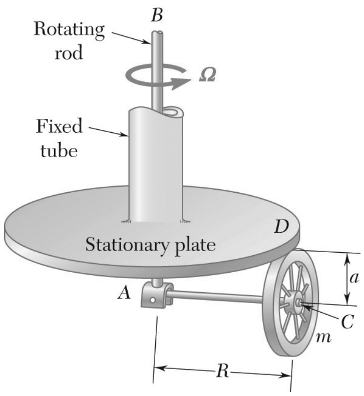
$$R\mathbf{i}\times(-W\,\mathbf{j}) = \dot{\mathbf{H}}_G$$

$$-Rmg\,\mathbf{k} = -\bar{I}_z\frac{R}{a}\Omega^2\mathbf{k}$$

$$\Omega = \sqrt{\frac{mga}{\bar{I}_x}}$$

(a) mass in rim: $\bar{I}_x = ma^2$ $\qquad \Omega = \sqrt{g/a}$ ◀

($b$) thin disk: $\bar{I}_x = \frac{1}{2}ma^2$ $\qquad \Omega = \sqrt{2g/a}$ ◀




## PROBLEM 18.98

Assuming that the wheel of Problem 18.97 weights 8 lb, has a radius $a = 4$ in. and a radius of gyration of 3 in., and that $R = 20$ in., determine the force exerted by the plate on the wheel when $\Omega = 25$ rad/s.

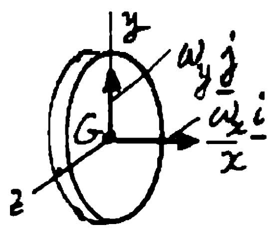
**PROBLEM 18.97** A stationary horizontal plate is attached to the ceiling by means of a fixed vertical tube. A wheel of radius $a$ and mass $m$ is mounted on a light axle $AC$ which is attached by means of a clevis at $A$ to a rod $AB$ fitted inside the vertical tube. The rod $AB$ is made to rotate with a constant angular velocity $\mathbf{\Omega}$ causing the wheel to roll on the lower face of the stationary plate. Determine the minimum angular velocity $\mathbf{\Omega}$ for which contact is maintained between the wheel and the plate. Consider the particular cases ($a$) when the mass of the wheel is concentrated in the rim, ($b$) when the wheel is equivalent to a thin disk of radius $a$.

## SOLUTION

Angular momentum $\mathbf{H}_G$ of wheel: $\omega_y = \Omega, \quad \omega_x = \dfrac{R}{a}\Omega, \quad \omega_z = 0$

$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k}$$

$$\mathbf{H}_G = \bar{I}_x \frac{R}{a}\Omega \mathbf{i} + \bar{I}_y \Omega \mathbf{j}$$




Rate of change of angular momentum:

Since $\omega_x$ and $\omega_y$ are constant, and observing that the frame $Gxyz$ rotates with the angular velocity $\mathbf{\Omega} = \Omega \mathbf{j}$:

$$(\dot{\mathbf{H}}_G)_{Gxyz} = 0$$

Eq. (18.22):

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G$$

$$\dot{\mathbf{H}}_G = 0 + \Omega \mathbf{j} \times \left( \bar{I}_x \frac{R}{a} a \mathbf{i} + \bar{I}_y \Omega \mathbf{j} \right)$$

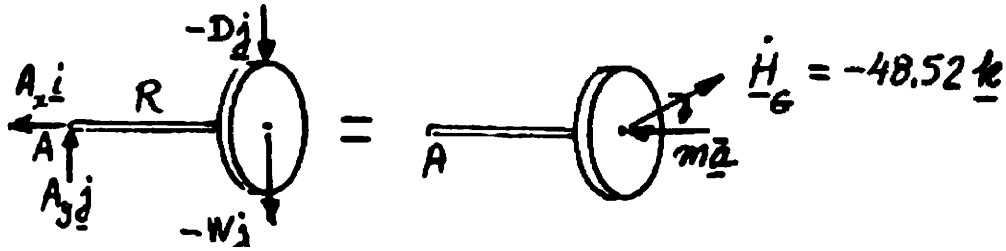
$$\dot{\mathbf{H}}_G = -\bar{I}_z \frac{R}{a} \Omega^2 \mathbf{k} \qquad (1)$$

With $\quad R = 20 \text{ in.}, \quad a = 4 \text{ in.}, \quad \Omega = 25 \text{ rad/s}, \quad m = \dfrac{8}{g}$

and $\quad \bar{I}_x = m\bar{k}_x^2 = \dfrac{8}{g}\left(\dfrac{3}{12}\text{ft}\right)^2 = 15.53 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$

$$\dot{\mathbf{H}}_G = -15.53 \times 10^{-3} \left(\frac{20}{4}\right)(25)^2 \mathbf{k} = 48.52 \mathbf{k}$$

## PROBLEM 18.98 (Continued)




Taking moments about $A$:

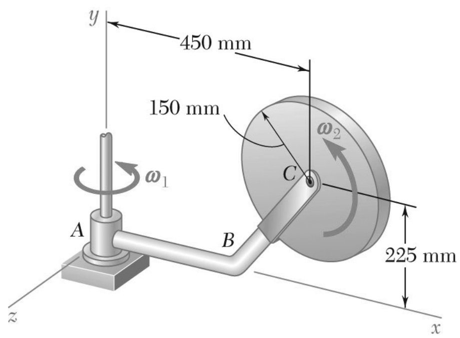
$$R\mathbf{i} \times (-D\mathbf{j} - W\mathbf{j}) = \dot{\mathbf{H}}_G$$

$$\frac{20}{12}\mathbf{i} \times (-D\mathbf{j} - 8\mathbf{j}) = -48.52\mathbf{k}$$

$$-\frac{5}{3}(D+8)\mathbf{k} = -48.52\mathbf{k}$$

$$D = \frac{3}{5}(48.52) - 8 = 21.1 \text{ lb}$$

$\mathbf{D} = 21.1 \text{ lb} \downarrow$ ◀




## PROBLEM 18.99

A thin disk of mass $m = 4$ kg rotates with an angular velocity $\omega_2$ with respect to arm $ABC$, which itself rotates with an angular velocity $\omega_1$ about the $y$ axis. Knowing that $\omega_1 = 5$ rad/s and $\omega_2 = 15$ rad/s and that both are constant, determine the force-couple system representing the dynamic reaction at the support at $A$.

## SOLUTION

*Angular velocity of the disk.* $\quad \boldsymbol{\omega} = \omega_1 \mathbf{j} + \omega_2 \mathbf{k} = (5 \text{ rad/s})\mathbf{j} + (15 \text{ rad/s})\mathbf{k}$

*Moments of inertia about principal axes passing through the mass center.*

$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4} m r^2$$

$$= \frac{1}{4}(4)(0.150 \text{ m})^2 = 0.0225 \text{ kg}\cdot\text{m}^2$$

$$\bar{I}_{z'} = \frac{1}{2} m r^2 = 0.045 \text{ kg}\cdot\text{m}^2$$

*Angular momentum about mass center C.*

$$\mathbf{H}_C = \bar{I}_{x'}\omega_{x'}\mathbf{i} + \bar{I}_{y'}\omega_{y'}\mathbf{j} + \bar{I}_{z'}\omega_{z'}\mathbf{k}$$

$$= 0 + (0.0225)5\mathbf{j} + (0.045)15\mathbf{k}$$

$$\mathbf{H}_C = (0.1125 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (0.6750 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$$

*Rate of change of* $\mathbf{H}_C$. Let the frame $Axyz$ be turning with angular velocity $\boldsymbol{\Omega} = \omega_1 \mathbf{j}$.

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Axyz} + \boldsymbol{\Omega} \times \mathbf{H}_C = 0 + \boldsymbol{\Omega} \times \mathbf{H}_C$$

$$= 5\mathbf{j} \times (0.1125\mathbf{j} + 0.675\mathbf{k}) = (3.375 \text{ N}\cdot\text{m})\mathbf{i}$$

*Position vector of Point C.*

$$\mathbf{r}_{C/A} = (0.450 \text{ m})\mathbf{i} + (0.225 \text{ m})\mathbf{j}$$

*Velocity of Point C, the mass center of the disk.*

$$\mathbf{v}_C = \omega_1 \times \mathbf{r}_{C/A} = 5\mathbf{j} \times (0.45\mathbf{i} + 0.225\mathbf{j})$$

$$= -(2.25 \text{ m/s})\mathbf{k}$$

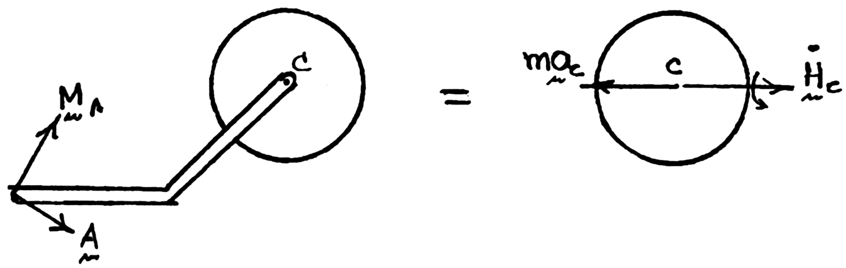
*Acceleration of Point C.*

$$\mathbf{a}_C = \alpha_1 \mathbf{j} \times \mathbf{r}_{C/A} + \omega_1 \mathbf{j} \times \mathbf{v}_C = 0 + 5\mathbf{j} \times (-2.25\mathbf{k}) = -(11.25 \text{ m/s}^2)\mathbf{i}$$

## PROBLEM 18.99 (Continued)

$$m\mathbf{a}_C = (4)(-11.25\mathbf{i}) = -(45\text{ N})\mathbf{i}$$

Free body and kinetic diagrams



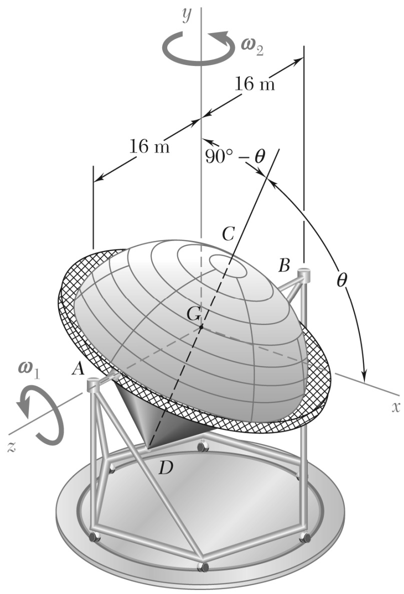

Linear components: $\mathbf{A} = m\mathbf{a}_C$ $\qquad$ $\mathbf{A} = -(45\text{ N})\mathbf{i}$ ◀

Moments about $A$.

$$\mathbf{M}_A = \mathbf{r}_{C/A} \times m\mathbf{a}_C + \dot{\mathbf{H}}_C$$

$$\mathbf{M}_A = (0.450\mathbf{i} + 0.225\mathbf{j}) \times (-45\mathbf{i}) + 3.375\mathbf{i}$$

$$\mathbf{M}_A = (3.38\text{ N}\cdot\text{m})\mathbf{i} + (10.13\text{ N}\cdot\text{m})\mathbf{k}$$ ◀



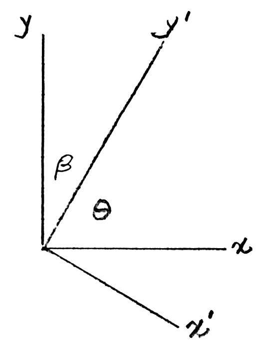

## PROBLEM 18.100

An experimental Fresnel-lens solar-energy concentrator can rotate about the horizontal axis $AB$, which passes through its mass center $G$. It is supported at $A$ and $B$ by a steel framework, which can rotate about the vertical $y$ axis. The concentrator has a mass of 30 Mg, a radius of gyration of 12 m about its axis of symmetry $CD$, and a radius of gyration of 10 m about any transverse axis through $G$. Knowing that the angular velocities $\omega_1$ and $\omega_2$ have constant magnitudes equal to 0.20 rad/s and 0.25 rad/s, respectively, determine for the position $\theta = 60°$ (*a*) the forces exerted on the concentrator at $A$ and $B$, (*b*) the couple $M_2\mathbf{k}$ applied to the concentrator at that instant.

## SOLUTION

Let the $y$ axis be vertical and the $y'$ axis be the symmetry axis.

Let the $z$ axis be directed along axle $BA$ as shown and the $x'$ axis be the transverse axis perpendicular to $BA$.




Unit vectors.

$$\beta = 90° - \theta$$
$$\mathbf{i}' = \mathbf{i}\cos\beta - \mathbf{j}\sin\beta$$
$$\mathbf{j}' = \mathbf{i}\sin\beta + \mathbf{j}\cos\beta$$
$$\mathbf{i} = \mathbf{i}'\cos\beta + \mathbf{j}'\sin\beta$$
$$\mathbf{j} = -\mathbf{i}'\sin\beta + \mathbf{j}\cos\beta$$

Angular velocity.

$$\boldsymbol{\omega} = \omega_2\mathbf{j} + \omega_1\mathbf{k}$$
$$\omega = -(\omega_2\sin\beta)\mathbf{i}' + (\omega_2\cos\beta)\mathbf{j}' + \omega_1\mathbf{k}$$
$$= -\omega_2\cos\theta\mathbf{i} + \omega_2\sin\theta\mathbf{j} + \omega_1\mathbf{k}$$

$$\omega_{x'} = \omega_2\cos\theta \qquad \omega_{y'} = \omega_2\sin\theta \qquad \omega_z = \omega_1$$

$$\dot{\theta} = \omega_1 \qquad \omega_1 \text{ and } \omega_2 \text{ are constant.}$$
$$\dot{\omega}_{x'} = \omega_2\omega_1\sin\theta$$
$$\dot{\omega}_{y'} = \omega_2\omega_1\cos\theta, \quad \dot{\omega}_2 = 0$$

Radii of gyration:

$$k_{x'} = 10 \text{ m}$$
$$k_{y'} = 12 \text{ m}$$
$$k_z = 10 \text{ m}$$

Moments of inertia: $I = mk^2$

## PROBLEM 18.100 (Continued)

Angular momentum about Point $G$. $\quad \mathbf{H}_G = I_{x'}\omega_{x'}\mathbf{i}' + I_y\omega_{y'}\mathbf{j} + I_z\omega_z\mathbf{k}$

$$\mathbf{H}_G = m[(10)^2\omega_{x'}\mathbf{i}' + (12)^2\omega_{y'}\mathbf{j}' + (10)^2\omega_z\mathbf{k}]$$
$$= m(100\omega_z\cos\theta\mathbf{i}' + 144\omega_2\sin\theta\mathbf{j}' + 100\omega_1\mathbf{k})$$

where $m$ is the mass. $\quad m = 30{,}000\text{ kg}$

Acceleration of the mass center.

Since the mass center lies at the center of the axle $BA$, $\quad \bar{\mathbf{a}} = 0$

Rate of change of angular momentum.

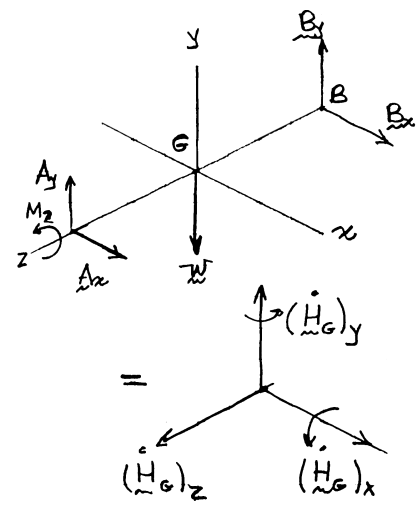
Let the reference frame $G_{x'y'z}$ be turning with angular velocity $\mathbf{\Omega} = \boldsymbol{\omega}$

$$\dot{\mathbf{H}}_G = \dot{\mathbf{H}}_{Gx'y'z} + \mathbf{\Omega}\times\mathbf{H}_G$$

where
$$\dot{\mathbf{H}}_{Gx'y'z} = I_{x'}\dot{\omega}_{x'}\mathbf{i}' + I_{y'}\dot{\omega}_y\mathbf{j}' + I_z\dot{\omega}_z\mathbf{k}$$
$$= m[(10)^2\omega_2\omega_1\sin\theta\mathbf{i}' + (12)^2\omega_2\omega_1\cos\theta\mathbf{j}' + 0]$$
$$= m[(100)(0.25)(0.20)\sin 60°\mathbf{i}' + (144)(0.25)(0.20)\cos 60°\mathbf{j}']$$
$$= m(4.3301\mathbf{i}' + 3.6\mathbf{j}')$$

and
$$\mathbf{\Omega}\times\mathbf{H}_G = m\begin{vmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k} \\ -\omega_2\cos\theta & \omega_2\sin\theta & \omega_1 \\ -100\omega_2\cos\theta & 144\omega_2\sin\theta & 100\omega_1 \end{vmatrix}$$
$$= m[-44\omega_1\omega_2\sin\theta\mathbf{i}' - 44\omega_2^2\sin\theta\cos\theta\mathbf{k}]$$
$$= m[-(44)(0.20)(0.25)\sin 60°\mathbf{i}'$$
$$+ (44)(0.25)^2\sin 60°\cos 60°\mathbf{k}]$$
$$= m(-1.9053\mathbf{i}' + 1.1908\mathbf{k})$$

$$\dot{\mathbf{H}}_G = m[2.4248\mathbf{i}' + 3.6\mathbf{j}' - 1.1908\mathbf{k}]$$
$$= (30{,}000)[2.4248(\mathbf{i}\cos 30° - \mathbf{j}\sin 30°)$$
$$+ 3.6(\mathbf{i}\sin 30° + \mathbf{j}\cos 30°) - 1.1908\mathbf{k}]$$
$$= (117\times 10^3\text{ N}\cdot\text{m})\mathbf{i} + (57.158\times 10^3\text{ N}\cdot\text{m})\mathbf{j}$$
$$-(35.724\times 10^3\text{ N}\cdot\text{m})\mathbf{k}$$




Weight: $\quad \mathbf{W} = -mg\mathbf{j} = -(30\times 10^3)(9.81)\mathbf{j}$

$$= -(294.3\times 10^3\text{ N})\mathbf{j}$$

Equations of motion.

$$\Sigma\mathbf{M}_B = \Sigma(\mathbf{M}_B)_{\text{eff}}: \quad \mathbf{r}_{A/B}\times(A_x\mathbf{i} + B_y\mathbf{j}) + \mathbf{r}_{G/B}\times\mathbf{W}) + M_2\mathbf{k} = \dot{\mathbf{H}}_G$$
$$32\mathbf{k}\times(A_x\mathbf{i} + A_y\mathbf{j}) + 16\mathbf{k}\times(-294.3\times 10^3)\mathbf{j} + M_2\mathbf{k} = \dot{\mathbf{H}}_G$$
$$(-32A_y + 4.7088\times 10^6)\mathbf{i} + 32A_x\mathbf{j} + M_2\mathbf{k} = \dot{\mathbf{H}}_G$$

## PROBLEM 18.100 (Continued)

Equate like components:

$$\mathbf{i}: \quad -32A_y + 4.7088\times10^6 = 117\times10^3 \qquad A_y = 143.5\times10^3\ \text{N}$$

$$\mathbf{j}: \quad 32A_x = 57.158\times10^3 \qquad A_x = 1.786\times10^3\ \text{N}$$

$$\mathbf{k}: \quad M_2 = -35.724\times10^3$$

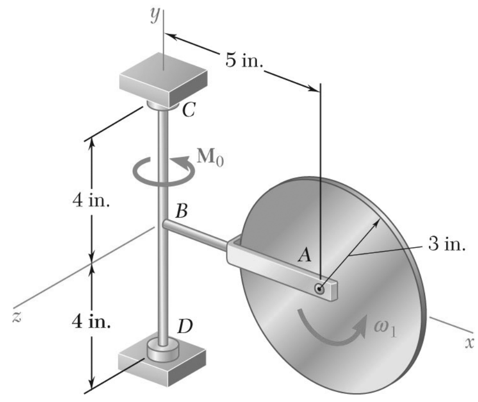
(*a*) Reading at $A$. — Reaction at $A$. $\quad \mathbf{A} = (1.786\ \text{kN})\mathbf{i} + (143.5\ \text{kN})\mathbf{j}$ ◀

$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}: \quad \mathbf{B} + \mathbf{A} + \mathbf{W} = 0$$

$$\mathbf{B} = -\mathbf{W} - \mathbf{A} = 294.3\times10^3\mathbf{j} - \mathbf{A}$$

Reaction at $B$. $\quad \mathbf{B} = -(1.786\ \text{kN})\mathbf{i} + (150.8\ \text{kN})\mathbf{j}$ ◀

(*b*) Couple $M_2\mathbf{k}$: $\quad M_2\mathbf{k} = -(35.7\ \text{kN}\cdot\text{m})\mathbf{k}$ ◀




## PROBLEM 18.101

A 6-lb homogeneous disk of radius 3 in. spins as shown at the constant rate $\omega_1 = 60$ rad/s. The disk is supported by the fork-ended rod $AB$, which is welded to the vertical shaft $CBD$. The system is at rest when a couple $\mathbf{M}_0 = (0.25 \text{ ft}\cdot\text{lb})\mathbf{j}$ is applied to the shaft for 2 s and then removed. Determine the dynamic reactions at $C$ and $D$ after the couple has been removed.

## SOLUTION

Angular velocity of shaft *CBD* and arm *AB*: $\quad \mathbf{\Omega} = \omega_2 \mathbf{j}$

Angular velocity of disk *A*: $\quad \boldsymbol{\omega} = \omega_2 \mathbf{j} + \omega_1 \mathbf{k}$

Its angular momentum about *A*: $\quad \mathbf{H}_A = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} = \bar{I}_y \omega_2 \mathbf{j} + \bar{I}_z \omega_1 \mathbf{k}$

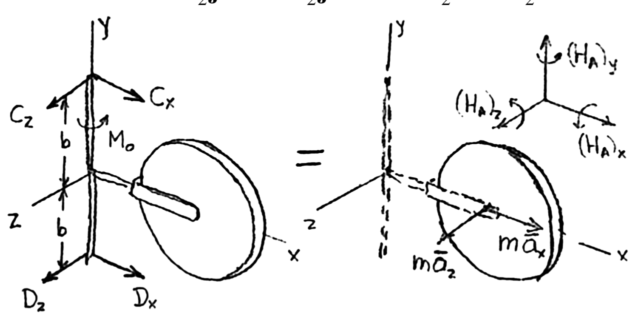
Let the reference frame *Bxyz* be rotating with angular velocity $\mathbf{\Omega}$.

$$\begin{aligned}
\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega} \times \mathbf{H}_A \\
&= \bar{I}_y \dot{\omega}_2 \mathbf{j} + \bar{I}_z \dot{\omega}_1 \mathbf{k} + \omega_2 \mathbf{j} \times (\bar{I}_y \omega_2 \mathbf{j} + \bar{I}_z \omega_1 \mathbf{k}) \\
&= \bar{I}_z \omega_1 \omega_2 \mathbf{i} + \bar{I}_y \dot{\omega}_2 \mathbf{j} + \bar{I}_z \dot{\omega}_1 \mathbf{k} \\
&= \frac{1}{2} m r^2 \omega_1 \omega_2 \mathbf{i} + \frac{1}{4} m r^2 \dot{\omega}_2 \mathbf{j} + \frac{1}{2} m r^2 \dot{\omega}_1 \mathbf{k}
\end{aligned}$$

Velocity and acceleration of the mass center *A* of the disk:

$$\bar{\mathbf{v}} = \omega_2 \mathbf{j} \times c\mathbf{i} = -c\omega_2 \mathbf{k},$$

$$\bar{\mathbf{a}} = \dot{\omega}_2 \mathbf{j} \times c\mathbf{i} + \omega_2 \mathbf{j} \times \bar{\mathbf{v}} = -c\dot{\omega}_2 \mathbf{k} - c\omega_2^2 \mathbf{i}$$




$$\Sigma \mathbf{F} = \Sigma \mathbf{F}_{\text{eff}} = m\bar{\mathbf{a}}$$

$$C_x \mathbf{i} + C_y \mathbf{k} + D_x \mathbf{i} + D_z \mathbf{k} = m\bar{\mathbf{a}}$$

Resolve into components.

$$C_x + D_x = -mc\omega_2^2$$

$$C_z + D_z = -mc\dot{\omega}_2$$

$$\Sigma \mathbf{M}_D = \dot{\mathbf{H}}_D = \dot{\mathbf{H}}_A + \mathbf{r}_{A/D} \times m\bar{\mathbf{a}}$$

$$= \dot{\mathbf{H}}_A + (c\mathbf{i} + b\mathbf{j}) \times (-c\dot{\omega}_2 \mathbf{k} - c\omega_2^2 \mathbf{i})$$

$$M_0 \mathbf{j} + 2b\mathbf{j} \times (C_x \mathbf{i} + C_z \mathbf{k}) = \dot{\mathbf{H}}_A - mbc\dot{\omega}_2 \mathbf{i} + mc^2 \dot{\omega}_2 \mathbf{j} + mbc\omega_2^2 \mathbf{k}$$

$$2bC_z \mathbf{i} + M_0 \mathbf{j} - 2bC_x \mathbf{k} = m\left(\frac{1}{2}r^2\omega_1\omega_2 - bc\dot{\omega}_2\right)\mathbf{i} + m\left(\frac{1}{4}r^2 + c^2\right)\dot{\omega}_2 \mathbf{j} + m\left(\frac{1}{2}r^2\dot{\omega}_1 + bc\omega_2^2\right)\mathbf{k}$$

$$\mathbf{j}: \quad M_0 = m\left(\frac{1}{4}r^2 + c^2\right)\dot{\omega}_2 \tag{1}$$

$$\mathbf{k}: \quad C_x = -\frac{m}{2b}\left(\frac{1}{2}r^2\dot{\omega}_1 + bc\omega_2^2\right) \quad D_x = -\frac{m}{2b}\left(-\frac{1}{2}r^2\dot{\omega}_1 + bc\omega_2^2\right) \tag{2}$$

$$\mathbf{i}: \quad C_z = \frac{m}{2b}\left(\frac{1}{2}r^2\omega_1\omega_2 - bc\dot{\omega}_2\right) \quad D_z = -\frac{m}{2b}\left(\frac{1}{2}r^2\omega_1\omega_2 + bc\dot{\omega}_2\right) \tag{3}$$

<u>Data</u>: $W = 6 \text{ lb}, \quad r = 3 \text{ in.} = 0.25 \text{ ft}, \quad b = 4 \text{ in.} = 0.33333 \text{ ft},$
$c = 5 \text{ in.} = 0.41667 \text{ ft}, \quad \omega_1 = 60 \text{ rad/s}, \quad \dot{\omega}_1 = 0$

While the couple is applied, $M_0 = 0.25 \text{ ft} \cdot \text{lb}$

Rearranging Equation (1) $$\dot{\omega}_2 = \frac{M_0}{m\left(\frac{1}{4}r^2 + c^2\right)} = \frac{0.25}{\left(\frac{6}{32.2}\right)\left[\frac{1}{4}(0.25)^2 + (0.41667)^2\right]} = 7.0899 \text{ rad/s}^2$$

At $t = 2\text{s}$, $\omega_2 = (\omega_2)_0 + \dot{\omega}_2 t = 0 + (7.0899)(2) = 14.18 \text{ rad/s}$

For $t > 2\text{s}$, $M_0 = 0, \quad \dot{\omega}_2 = 0$

From Equations (2), (3) $$C_x = \frac{\frac{6}{32.2}}{(2)(0.33333)}[0 + (0.33333)(0.41667)(14.1798)^2] = -7.8054 \text{ lb}$$

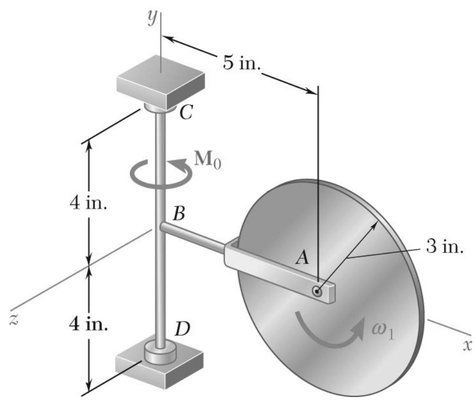
$$D_x = \frac{\frac{6}{32.2}}{(2)(0.33333)}[0 + (0.33333)(0.41667)(14.1798)^2] = -7.8054 \text{ lb}$$

$$C_z = \frac{\frac{6}{32.2}}{(2)(0.33333)}\left[\frac{1}{2}(0.25)^2(60)(14.1798) - 0\right] = 7.4312 \text{ lb}$$

$$D_z = \frac{\frac{6}{32.2}}{(2)(0.33333)}\left[-\frac{1}{2}(0.25)^2(60)(14.1798) - 0\right] = -7.4312 \text{ lb}$$

$\mathbf{C} = -(7.81 \text{ lb})\mathbf{i} + (7.43 \text{ lb})\mathbf{k}$ ◀

$\mathbf{D} = -(7.81 \text{ lb})\mathbf{i} - (7.43 \text{ lb})\mathbf{k}$ ◀




## PROBLEM 18.102

A 6-lb homogeneous disk of radius 3 in. spins as shown at the constant rate $\omega_1 = 60$ rad/s. The disk is supported by the fork-ended rod $AB$, which is welded to the vertical shaft $CBD$. The system is at rest when a couple $\mathbf{M}_0$ is applied as shown to the shaft for 3 s and then removed. Knowing that the maximum angular velocity reached by the shaft is 18 rad/s, determine (*a*) the couple $\mathbf{M}_0$, (*b*) the dynamic reactions at $C$ and $D$ after the couple has been removed.

## SOLUTION

Angular velocity of shaft *CBD* and arm *AB*: $\mathbf{\Omega} = \omega_2 \mathbf{j}$

Angular velocity of disk *A*: $\boldsymbol{\omega} = \omega_2 \mathbf{j} + \omega_1 \mathbf{k}$

Its angular momentum about *A*: $\mathbf{H}_A = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} = \bar{I}_y \omega_2 \mathbf{j} + \bar{I}_z \omega_1 \mathbf{k}$

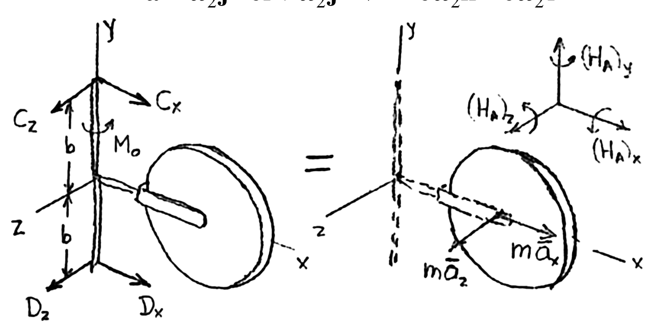
Let the reference frame *Bxyz* be rotating with angular velocity $\mathbf{\Omega}$.

$$\begin{aligned}
\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega} \times \mathbf{H}_A \\
&= \bar{I}_y \dot{\omega}_2 \mathbf{j} + \bar{I}_z \dot{\omega}_1 \mathbf{k} + \omega_2 \mathbf{j} \times (\bar{I}_y \omega_2 \mathbf{j} + \bar{I}_z \omega_1 \mathbf{k}) \\
&= \bar{I}_z \omega_1 \omega_2 \mathbf{i} + \bar{I}_y \dot{\omega}_2 \mathbf{j} + \bar{I}_z \dot{\omega}_1 \mathbf{k} \\
&= \frac{1}{2} m r^2 \omega_1 \omega_2 \mathbf{i} + \frac{1}{4} m r^2 \dot{\omega}_2 \mathbf{j} + \frac{1}{2} m r^2 \dot{\omega}_1 \mathbf{k}
\end{aligned}$$

Velocity and acceleration of the mass center *A* of the disk:

$$\bar{\mathbf{v}} = \omega_2 \mathbf{j} \times c\mathbf{i} = -c\omega_2 \mathbf{k},$$

$$\bar{\mathbf{a}} = \dot{\omega}_2 \mathbf{j} \times c\mathbf{i} + \omega_2 \mathbf{j} \times \bar{\mathbf{v}} = -c\dot{\omega}_2 \mathbf{k} - c\omega_2^2 \mathbf{i}$$




$$\Sigma \mathbf{F} = \Sigma \mathbf{F}_{\text{eff}} = m\bar{\mathbf{a}}$$

$$C_x \mathbf{i} + C_y \mathbf{k} + D_x \mathbf{i} + D_z \mathbf{k} = m\bar{\mathbf{a}}$$

Resolve into components.

$$C_x + D_x = -mc\omega_2^2$$

$$C_z + D_z = -mc\dot{\omega}_2$$

$$\Sigma \mathbf{M}_D = \dot{\mathbf{H}}_D = \dot{\mathbf{H}}_A + \mathbf{r}_{A/D} \times m\bar{\mathbf{a}}$$

$$= \dot{\mathbf{H}}_A + (c\mathbf{i} + b\mathbf{j}) \times (-c\dot{\omega}_2\mathbf{k} - c\omega_2^2\mathbf{i})$$

$$M_0\mathbf{j} + 2b\mathbf{j} \times (C_x\mathbf{i} + C_z\mathbf{k}) = \dot{\mathbf{H}}_A - mbc\dot{\omega}_2\mathbf{i} + mc^2\dot{\omega}_2\mathbf{j} + mbc\omega_2^2\mathbf{k}$$

$$2bC_z\mathbf{i} + M_0\mathbf{j} - 2bC_x\mathbf{k} = m\left(\frac{1}{2}r^2\omega_1\omega_2 - bc\dot{\omega}_2\right)\mathbf{i} + m\left(\frac{1}{4}r^2 + c^2\right)\dot{\omega}_2\mathbf{j} + m\left(\frac{1}{2}r^2\dot{\omega}_1 + bc\omega_2^2\right)\mathbf{k}$$

$$\mathbf{j}: \quad M_0 = m\left(\frac{1}{4}r^2 + c^2\right)\dot{\omega}_2 \tag{1}$$

$$\mathbf{k}: \quad C_x = -\frac{m}{2b}\left(\frac{1}{2}r^2\dot{\omega}_1 + bc\omega_2^2\right) \quad D_x = -\frac{m}{2b}\left(-\frac{1}{2}r^2\dot{\omega}_1 + bc\omega_2^2\right) \tag{2}$$

$$\mathbf{i}: \quad C_z = \frac{m}{2b}\left(\frac{1}{2}r^2\omega_1\omega_2 - bc\dot{\omega}_2\right) \quad D_z = -\frac{m}{2b}\left(\frac{1}{2}r^2\omega_1\omega_2 + bc\dot{\omega}_2\right) \tag{3}$$

Data: $W = 6$ lb, $r = 3$ in. $= 0.25$ ft, $b = 4$ in. $= 0.33333$ ft,
$c = 5$ in. $= 0.41667$ ft, $\omega_1 = 60$ rad/s, $\dot{\omega}_1 = 0$

(*a*) While the couple is applied $\dot{\omega}_2 = \dfrac{\omega_2}{t} = \dfrac{18}{3} = 6 \text{ rad/s}^2$

From Equation (1) $M_0 = m\left(\frac{1}{4}r^2 + c^2\right)\dot{\omega}_2$

$$= \left(\frac{6}{32.2}\right)\left[\frac{1}{4}(0.25)^2 + (0.41667)^2\right](6) \qquad M_0 = (0.212 \text{ ft}\cdot\text{lb})\mathbf{j} \blacktriangleleft$$

(*b*) For $t > 3$ s, $M_0 = 0, \quad \dot{\omega}_2 = 0$

From Equations (2), (3)

$$C_x = \frac{\frac{6}{32.2}}{(2)(0.33333)}[0 + (0.33333)(0.41667)(18)^2] = -12.578 \text{ lb}$$

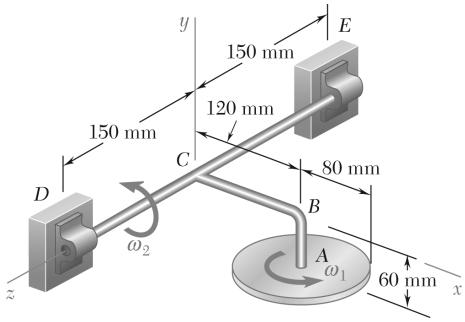
$$D_x = \frac{\frac{6}{32.2}}{(2)(0.33333)}[0 + (0.33333)(0.41667)(18)^2] = -12.578 \text{ lb}$$

$$C_z = \frac{\frac{6}{32.2}}{(2)(0.33333)}\left[\frac{1}{2}(0.25)^2(60)(18) - 0\right] = 9.4332 \text{ lb}$$

$$D_z = \frac{\frac{6}{32.2}}{(2)(0.33333)}\left[-\frac{1}{2}(0.25)^2(60)(18) - 0\right] = -9.4332 \text{ lb}$$

$$\mathbf{C} = -(12.58 \text{ lb})\mathbf{i} + (9.43 \text{ lb})\mathbf{k} \blacktriangleleft$$

$$\mathbf{D} = -(12.58 \text{ lb})\mathbf{i} - (9.43 \text{ lb})\mathbf{k} \blacktriangleleft$$




## PROBLEM 18.103

A 2.5 kg homogeneous disk of radius 80 mm rotates with an angular velocity $\boldsymbol{\omega}_1$ with respect to arm $ABC$, which is welded to a shaft $DCE$ rotating as shown at the constant rate $\omega_2 = 12$ rad/s. Friction in the bearing at $A$ causes $\omega_1$ to decrease at the rate of 15 rad/s$^2$. Determine the dynamic reactions at $D$ and $E$ at a time when $\omega_1$ has decreased to 50 rad/s.

## SOLUTION

Angular velocity of shaft $DCE$ and arm $CBA$: $\quad \mathbf{\Omega} = \omega_2 \mathbf{k}$

Angular velocity of disk $A$: $\quad \boldsymbol{\omega} = \omega_1 \mathbf{j} + \omega_2 \mathbf{k}$

Its angular momentum about $A$: $\quad \mathbf{H}_A = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} = \bar{I}_y \omega_1 \mathbf{j} + \bar{I}_z \omega_2 \mathbf{k}$

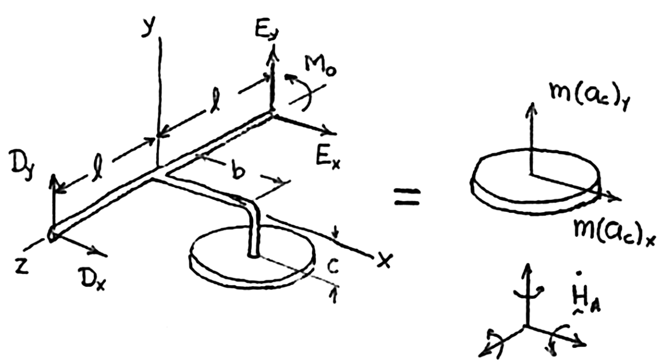
Let the reference frame $Cxyz$ be rotating with angular velocity $\mathbf{\Omega}$.

$$
\begin{aligned}
\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega} \times \mathbf{H}_A \\
&= \bar{I}_y \dot{\omega}_1 \mathbf{j} + \bar{I}_z \dot{\omega}_2 \mathbf{k} + \omega_2 \mathbf{k} \times (\bar{I}_y \omega_1 \mathbf{j} + \bar{I}_z \omega_2 \mathbf{k}) \\
&= -\bar{I}_y \omega_2 \omega_1 \mathbf{i} + \bar{I}_y \dot{\omega}_1 \mathbf{j} + \bar{I}_z \dot{\omega}_2 \mathbf{k} \\
&= -\frac{1}{2} m r^2 \omega_2 \omega_1 \mathbf{i} + \frac{1}{2} m r^2 \dot{\omega}_1 \mathbf{j} + \frac{1}{4} m r^2 \dot{\omega}_2 \mathbf{k}
\end{aligned}
$$

Velocity and acceleration of the mass center $A$ of the disk:

$$\mathbf{v}_A = \boldsymbol{\omega}_2 \mathbf{k} \times \mathbf{r}_{A/C} = \omega_2 \mathbf{k} \times (b\mathbf{i} - c\mathbf{j}) = c\omega_2 \mathbf{i} + b\omega_2 \mathbf{j}$$

$$\mathbf{a}_A = \dot{\omega}_2 \mathbf{k} \times \mathbf{r}_{A/C} + \omega_2 \mathbf{k} \times \mathbf{v}_A = (c\dot{\omega}_2 - b\omega_2^2)\mathbf{i} + (b\dot{\omega}_2 + c\omega_2^2)\mathbf{j}$$




$$\Sigma \mathbf{F} = \Sigma \mathbf{F}_{\text{eff}}$$

$$D_x \mathbf{i} + D_y \mathbf{j} + E_x \mathbf{i} + E_y \mathbf{j} = m\mathbf{a}_A$$

## PROBLEM 18.103 (Continued)

Resolve into components.

$$D_x + E_x = m(c\dot{\omega}_2 - b\omega_2^2)$$
$$D_y + E_y = m(b\dot{\omega}_2 + c\omega_2^2)$$
$$\Sigma\mathbf{M}_E = \dot{\mathbf{H}}_E = \dot{\mathbf{H}}_A + \mathbf{r}_{A/E} \times m\mathbf{a}_A = \dot{\mathbf{H}}_A + (b\mathbf{i} - c\mathbf{j} + l\mathbf{k}) \times m\mathbf{a}_A$$
$$2l\mathbf{k} \times (D_x\mathbf{i} + D_y\mathbf{j}) + M_0\mathbf{k} = \dot{\mathbf{H}}_A + m(bl\dot{\omega}_2 - cl\omega_2^2)\mathbf{i} + m(cl\dot{\omega}_2 + bl\omega_2^2)\mathbf{j} + m(b^2 + c^2)\dot{\omega}_2\mathbf{k}$$
$$-2lD_y\mathbf{i} + 2lD_x\mathbf{j} + M_0\mathbf{k} = m\left(-\frac{1}{2}r^2\omega_2\omega_1 + bl\dot{\omega}_2 - cl\omega_2^2\right)\mathbf{i}$$
$$+ m\left(\frac{1}{2}r^2\dot{\omega}_1 + cl\dot{\omega}_2 - bl\omega_2^2\right)\mathbf{j} + m\left(\frac{1}{4}r^2 + b^2 + c^2\right)\dot{\omega}_2\mathbf{k}$$

$$\mathbf{k}: \quad M_0 = m\left(\frac{1}{4}r^2 + b^2 + c^2\right)\dot{\omega}_2$$
$$\mathbf{j}: \quad D_x = \frac{m}{2l}\left(\frac{1}{2}r^2\dot{\omega}_1 + cl\dot{\omega}_2 - bl\omega_2^2\right) \quad E_x = \frac{m}{2l}\left(-\frac{1}{2}r^2\dot{\omega}_1 + cl\dot{\omega}_2 - bl\omega_2^2\right)$$
$$\mathbf{i}: \quad D_y = \frac{m}{2l}\left(\frac{1}{2}r^2\omega_2\omega_1 + bl\dot{\omega}_2 + cl\omega_2^2\right) \quad E_y = \frac{m}{2l}\left(-\frac{1}{2}r^2\omega_2\omega_1 + bl\dot{\omega}_2 + cl\omega_2^2\right)$$

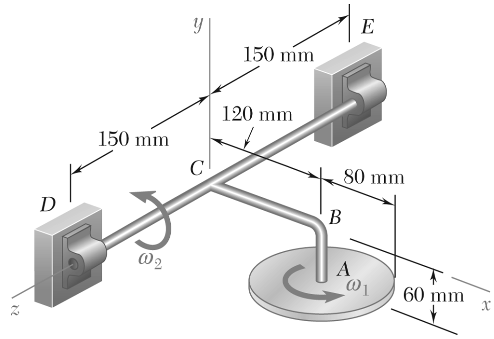
<u>Data</u>:
$$m = 2.5 \text{ kg}, \quad r = 80 \text{ mm} = 0.08 \text{ m}$$
$$b = 120 \text{ mm} = 0.12 \text{ m}, \quad c = 60 \text{ mm} = 0.06 \text{ m}, \quad l = 150 \text{ mm} = 0.15 \text{ m}$$
$$\omega_1 = 50 \text{ rad/s}, \quad \dot{\omega}_1 = -15 \text{ rad/s}^2, \quad \omega_2 = 12 \text{ rad/s}, \quad \dot{\omega}_2 = 0$$

$$D_x = \frac{2.5}{(2)(0.15)}\left[\frac{1}{2}(0.08)^2(-15) + 0 - (0.12)(0.15)(12)^2\right] = -22.0 \text{ N}$$
$$D_y = \frac{2.5}{(2)(0.15)}\left[\frac{1}{2}(0.08)^2(12)(50) + 0 + (0.06)(0.15)(12)^2\right] = 26.8 \text{ N}$$

$$\mathbf{D} = -(22.0 \text{ N})\mathbf{i} + (26.8 \text{ N})\mathbf{j} \blacktriangleleft$$

$$E_x = \frac{2.5}{(2)(0.15)}\left[-\frac{1}{2}(0.08)^2(-15) + 0 - (0.12)(0.15)(12)^2\right] = -21.2 \text{ N}$$
$$E_y = \frac{2.5}{(2)(0.15)}\left[-\frac{1}{2}(0.08)^2(12)(50) + 0 + (0.06)(0.15)(12)^2\right] = -5.20 \text{ N}$$

$$\mathbf{E} = -(21.2 \text{ N})\mathbf{i} - (5.20 \text{ N})\mathbf{j} \blacktriangleleft$$




## PROBLEM 18.104

A 2.5-kg homogeneous disk of radius 80 mm rotates at the constant rate $\omega_1 = 50$ rad/s with respect to arm *ABC*, which is welded to a shaft *DCE*. Knowing that at the instant shown shaft *DCE* has an angular velocity $\omega_2 = (12 \text{ rad/s})\mathbf{k}$ and an angular acceleration $\alpha_2 = (8 \text{ rad/s}^2)\mathbf{k}$, determine (*a*) the couple which must be applied to shaft *DCE* to produce that acceleration, (*b*) the corresponding dynamic reactions at *D* and *E*.

## SOLUTION

Angular velocity of shaft *DCE* and arm *CBA*: $\mathbf{\Omega} = \omega_2 \mathbf{k}$

Angular velocity of disk *A*: $\boldsymbol{\omega} = \omega_1 \mathbf{j} + \omega_2 \mathbf{k}$

Its angular momentum about *A*: $\mathbf{H}_A = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} = \bar{I}_y \omega_1 \mathbf{j} + \bar{I}_z \omega_2 \mathbf{k}$

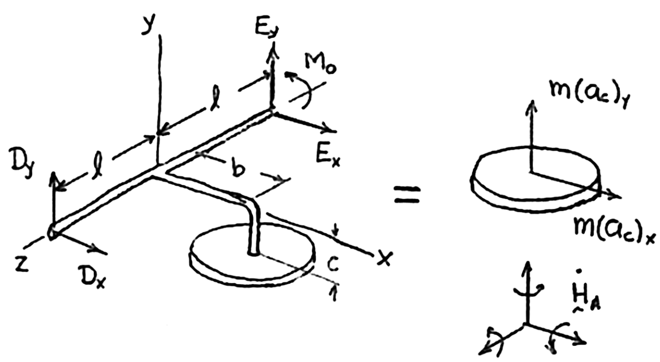
Let the reference frame *Cxyz* be rotating with angular velocity $\mathbf{\Omega}$.

$$\begin{aligned}
\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Axyz} + \mathbf{\Omega} \times \mathbf{H}_A \\
&= \bar{I}_y \dot{\omega}_1 \mathbf{j} + \bar{I}_z \dot{\omega}_2 \mathbf{k} + \omega_2 \mathbf{k} \times (\bar{I}_y \omega_1 \mathbf{j} + \bar{I}_z \omega_2 \mathbf{k}) \\
&= -\bar{I}_y \omega_2 \omega_1 \mathbf{i} + \bar{I}_y \dot{\omega}_1 \mathbf{j} + \bar{I}_z \dot{\omega}_2 \mathbf{k} \\
&= -\frac{1}{2} m r^2 \omega_2 \omega_1 \mathbf{i} + \frac{1}{2} m r^2 \dot{\omega}_1 \mathbf{j} + \frac{1}{4} m r^2 \dot{\omega}_2 \mathbf{k}
\end{aligned}$$

Velocity and acceleration of the mass center *A* of the disk:

$$\mathbf{v}_A = \boldsymbol{\omega}_2 \mathbf{k} \times \mathbf{r}_{A/C} = \omega_2 \mathbf{k} \times (b\mathbf{i} - c\mathbf{j}) = c\omega_2 \mathbf{i} + b\omega_2 \mathbf{j}$$

$$\mathbf{a}_A = \dot{\omega}_2 \mathbf{k} \times \mathbf{r}_{A/C} + \omega_2 \mathbf{k} \times \mathbf{v}_A = (c\dot{\omega}_2 - b\omega_2^2)\mathbf{i} + (b\dot{\omega}_2 + c\omega_2^2)\mathbf{j}$$




$$\Sigma \mathbf{F} = \Sigma \mathbf{F}_{\text{eff}}$$

$$D_x \mathbf{i} + D_y \mathbf{j} + E_x \mathbf{i} + E_y \mathbf{j} = m\mathbf{a}_A$$

Resolve into components.

$$D_x + E_x = m(c\dot{\omega}_2 - b\omega_2^2)$$
$$D_y + E_y = m(b\dot{\omega}_2 + c\omega_2^2)$$
$$\Sigma \mathbf{M}_E = \dot{\mathbf{H}}_E = \dot{\mathbf{H}}_A + \mathbf{r}_{A/E} \times m\mathbf{a}_A = \dot{\mathbf{H}}_A + (b\mathbf{i} - c\mathbf{j} + l\mathbf{k}) \times m\mathbf{a}_A$$
$$2l\mathbf{k} \times (D_x\mathbf{i} + D_y\mathbf{j}) + M_0\mathbf{k} = \dot{\mathbf{H}}_A + m(bl\dot{\omega}_2 - cl\omega_2^2)\mathbf{i} + m(cl\dot{\omega}_2 + bl\omega_2^2)\mathbf{j} + m(b^2 + c^2)\dot{\omega}_2\mathbf{k}$$
$$-2lD_y\mathbf{i} + 2lD_x\mathbf{j} + M_0\mathbf{k} = m\left(-\frac{1}{2}r^2\omega_2\omega_1 + bl\dot{\omega}_2 - cl\omega_2^2\right)\mathbf{i}$$
$$+ m\left(\frac{1}{2}r^2\dot{\omega}_1 + cl\dot{\omega}_2 - bl\omega_2^2\right)\mathbf{j} + m\left(\frac{1}{4}r^2 + b^2 + c^2\right)\dot{\omega}_2\mathbf{k}$$

$$\mathbf{k}: \quad M_0 = m\left(\frac{1}{4}r^2 + b^2 + c^2\right)\dot{\omega}_2$$
$$\mathbf{j}: \quad D_x = \frac{m}{2l}\left(\frac{1}{2}r^2\dot{\omega}_1 + cl\dot{\omega}_2 - bl\omega_2^2\right) \quad E_x = \frac{m}{2l}\left(-\frac{1}{2}r^2\dot{\omega}_1 + cl\dot{\omega}_2 - bl\omega_2^2\right)$$
$$\mathbf{i}: \quad D_y = \frac{m}{2l}\left(\frac{1}{2}r^2\omega_2\omega_1 + bl\dot{\omega}_2 + cl\omega_2^2\right) \quad E_y = \frac{m}{2l}\left(-\frac{1}{2}r^2\omega_2\omega_1 + bl\dot{\omega}_2 + cl\omega_2^2\right)$$

<u>Data</u>: $m = 2.5$ kg, $r = 80$ mm $= 0.08$ m

$b = 120$ mm $= 0.12$ m, $c = 60$ mm $= 0.06$ m, $l = 150$ mm $= 0.15$ m

$\omega_1 = 50$ rad/s, $\dot{\omega}_1 = 0$, $\omega_2 = 12$ rad/s, $\dot{\omega}_2 = \alpha_2 = 8$ rad/s$^2$

(*a*)
$$M_0 = (2.5)\left[\frac{1}{4}(0.08)^2 + (0.12)^2 + (0.06)^2\right](8)$$

$\mathbf{M}_0 = (0.392\ \text{N}\cdot\text{m})\mathbf{k}$ ◀

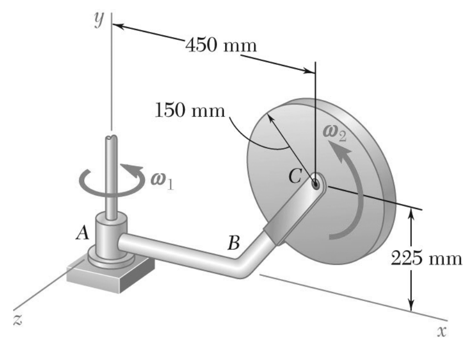
(*b*)
$$D_x = \frac{2.5}{(2)(0.15)}[0 + (0.06)(0.15)(8) - (0.12)(0.15)(12)^2] = -21.0\ \text{N}$$
$$D_y = \frac{2.5}{(2)(0.15)}\left[\frac{1}{2}(0.08)^2(12)(50) + (0.12)(0.15)(8) + (0.06)(0.15)(12)^2\right] = 28.0\ \text{N}$$

$\mathbf{D} = -(21.0\ \text{N})\mathbf{i} + (28.0\ \text{N})\mathbf{j}$ ◀

$$E_x = \frac{2.5}{(2)(0.15)}[-0 + (0.06)(0.15)(8) - (0.12)(0.15)(12)^2] = -21.0\ \text{N}$$
$$E_y = \frac{2.5}{(2)(0.15)}\left[-\frac{1}{2}(0.08)^2(12)(50) + (0.12)(0.15)(8) + (0.06)(0.15)(12)^2\right]$$
$$= -4.00\ \text{N}$$

$\mathbf{E} = -(21.0\ \text{N})\mathbf{i} - (4.00\ \text{N})\mathbf{j}$ ◀

## PROBLEM 18.105




For the disk of Problem 18.99, determine (*a*) the couple $M_1\mathbf{j}$ which should be applied to arm *ABC* to give it an angular acceleration $\alpha_1 = -(7.5 \text{ rad/s}^2)\mathbf{j}$ when $\omega_1 = 5$ rad/s, knowing that the disk rotates at the constant rate $\omega_2 = 15$ rad/s, (*b*) the force-couple system representing the dynamic reaction at *A* at that instant. Assume that *ABC* has a negligible mass.

**PROBLEM 18.99** A thin disk of mass $m = 4$ kg rotates with an angular velocity $\omega_2$ with respect to arm *ABC*, which itself rotates with an angular velocity $\omega_1$ about the *y* axis. Knowing that $\omega_1 = 5$ rad/s and $\omega_2 = 15$ rad/s and that both are constant, determine the force-couple system representing the dynamic reaction at the support at *A*.

## SOLUTION

*Angular velocity of the disk.* $\quad \boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{k} = (5 \text{ rad/s})\mathbf{j} + (15 \text{ rad/s})\mathbf{k}$

*Moments of inertia about principal axes passing through the mass center.*

$$\bar{I}_x = \bar{I}_y = \frac{1}{4}mr^2$$
$$= \frac{1}{4}(4)(0.150 \text{ m})^2 = 0.0225 \text{ kg}\cdot\text{m}^2$$
$$\bar{I}_z = \frac{1}{2}mr^2 = 0.045 \text{ kg}\cdot\text{m}^2$$

*Angular momentum about mass center C.*

$$\mathbf{H}_C = \bar{I}_{x'}\omega_{x'}\mathbf{i} + \bar{I}_{y'}\omega_{y'}\mathbf{j} + I_{z'}\omega_{z'}\mathbf{k}$$
$$= 0 + (0.0225)5\mathbf{j} + (0.045)15\mathbf{k}$$
$$= (0.1125 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (0.6750 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$$

*Rate of change of* $\mathbf{H}_C$. Let the frame *Axyz* be turning with angular velocity $\mathbf{\Omega} = \omega_1\mathbf{j}$.

$$\dot{\mathbf{H}}_C = (\dot{\mathbf{H}}_C)_{Axyz} + \mathbf{\Omega}\times\mathbf{H}_C = \bar{I}_x\dot{\omega}_x\mathbf{i} + \bar{I}_y\dot{\omega}_y\mathbf{j} + \bar{I}_z\dot{\omega}_z\mathbf{k} + \mathbf{\Omega}\times\mathbf{H}_C$$
$$= 0 + (0.0225)(-7.5)\mathbf{j} + 0 + 5\mathbf{j}\times(0.1125\mathbf{j} + 0.675\mathbf{k})$$
$$= -(0.16875 \text{ N}\cdot\text{m})\mathbf{j} + (3.375 \text{ N}\cdot\text{m})\mathbf{i}$$

*Position vector of Point C.* $\quad \mathbf{r}_{C/A} = (0.450 \text{ m})\mathbf{i} + (0.225 \text{ m})\mathbf{j}$

*Velocity of Point C, the mass center of the disk.*

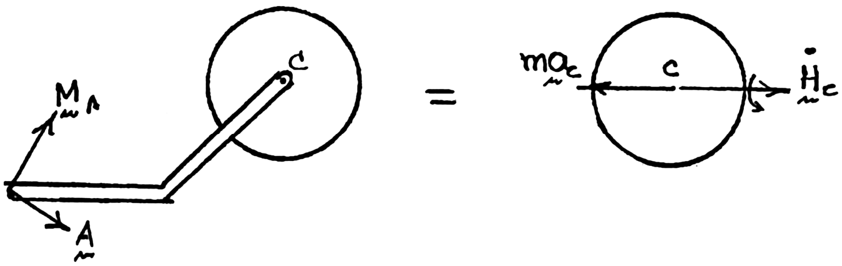
$$\mathbf{v}_C = \omega_1\times\mathbf{r}_{C/A} = 5\mathbf{j}\times(0.45\mathbf{i} + 0.225\mathbf{j}) = -(2.25 \text{ m/s})\mathbf{k}$$

*Acceleration of Point C.*

$$\mathbf{a}_C = \alpha_1\mathbf{j}\times\mathbf{r}_{C/A} + \omega_1\mathbf{j}\times\mathbf{v}_C = (-7.5\mathbf{j})\times(0.45\mathbf{i}+0.225\mathbf{j}) + 5\mathbf{j}\times(-2.25\mathbf{k})$$
$$= (3.3750\text{ m/s}^2)\mathbf{k} - (11.25\text{ m/s}^2)\mathbf{i}$$

$$m\mathbf{a}_C = (4)(-11.25\mathbf{i} + 3.3750\mathbf{k})$$
$$= -(45\text{ N})\mathbf{i} + (13.5\text{ N})\mathbf{k}$$

Free body and kinetic diagrams




Linear components: $\mathbf{A} = m\mathbf{a}_C$ $\quad \mathbf{A} = -(45\text{ N})\mathbf{i} + (13.5\text{ N})\mathbf{k}$

Moments about **A**.

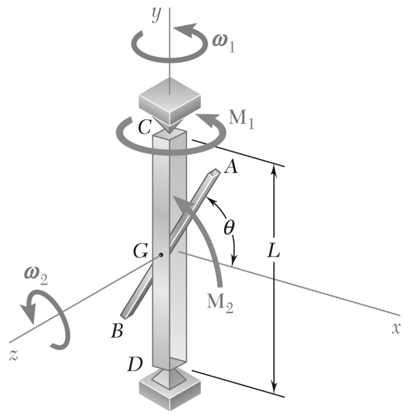
$$\mathbf{M}_A = \mathbf{r}_{C/A}\times m\mathbf{a}_C + \dot{\mathbf{H}}_C$$
$$\mathbf{M}_A = (0.450\mathbf{i} + 0.225\mathbf{j})\times(-45\mathbf{i} + 13.5\mathbf{k}) - 0.16875\mathbf{j} + 3.375\mathbf{i}$$
$$= -6.0750\mathbf{j} + 10.125\mathbf{k} + 3.0375\mathbf{i} - 0.16875\mathbf{j} + 3.375\mathbf{i}$$
$$= 6.4125\mathbf{i} - 6.2438\mathbf{j} + 10.125\mathbf{k}$$

(*a*) Required couple. $\quad \mathbf{M}_1\mathbf{j} = -(6.24\text{ N}\cdot\text{m})\mathbf{j}$ ◀

(*b*) Dynamic reaction. $\quad \mathbf{A} = -(45.0\text{ N})\mathbf{i} + (13.50\text{ N})\mathbf{k}$ ◀

$\mathbf{M}_A = (6.41\text{ N}\cdot\text{m})\mathbf{i} + (10.13\text{ N}\cdot\text{m})\mathbf{k}$ ◀

## PROBLEM 18.106*



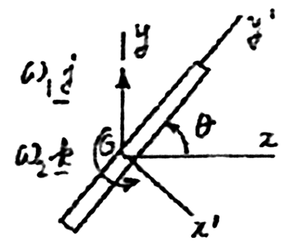

A slender homogeneous rod $AB$ of mass $m$ and length $L$ is made to rotate at the constant rate $\omega_2$ about the horizontal $z$ axis, while frame $CD$ is made to rotate at the constant rate $\omega_1$ about the $y$ axis. Express as a function of the angle $\theta$ (*a*) the couple $\mathbf{M}_1$ required to maintain the rotation of the frame, (*b*) the couple $\mathbf{M}_2$ required to maintain the rotation of the rod, (*c*) the dynamic reactions at the supports $C$ and $D$.

## SOLUTION

Angular momentum $\mathbf{H}_G$:

We resolve the angular velocity $\boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{k}$ into components along the principal axes $Gx'y'z$:

$$\omega_{x'} = -\omega_1 \cos\theta$$
$$\omega_{y'} = \omega_1 \sin\theta$$
$$\omega_z = \omega_2$$




Moments of inertia:

$$\bar{I}_{x'} = \bar{I}_z = \frac{1}{12}mL^2 \quad \bar{I}_{y'} = 0$$

We have

$$H_{x'} = \bar{I}_{x'}\,\omega_{x'} = -\frac{1}{12}mL^2\,\omega_1\cos\theta$$
$$H_{y'} = \bar{I}_{y'}\,\omega_{y'} = 0$$
$$H_z = \bar{I}_z\,\omega_z = \frac{1}{12}mL^2\,\omega_2$$

Computing the components of $\mathbf{H}_G$ along the $x$, $y$, $z$ axes:

$$H_x = H_{x'}\sin\theta = -\frac{1}{12}mL^2\,\omega_1\cos\theta\sin\theta = -\frac{1}{24}mL^2\,\omega_1\sin 2\theta$$
$$H_y = -H_{x'}\cos\theta = +\frac{1}{12}mL^2\,\omega_1\cos^2\theta$$
$$H_z = \frac{1}{12}mL^2\,\omega_2$$

The angular momentum is therefore

$$\mathbf{H}_G = -\frac{1}{24}mL^2\omega_1\sin 2\theta\mathbf{i} + \frac{1}{12}mL^2\omega_1\cos^2\theta\mathbf{j} + \frac{1}{12}mL^2\omega_2\mathbf{k}$$

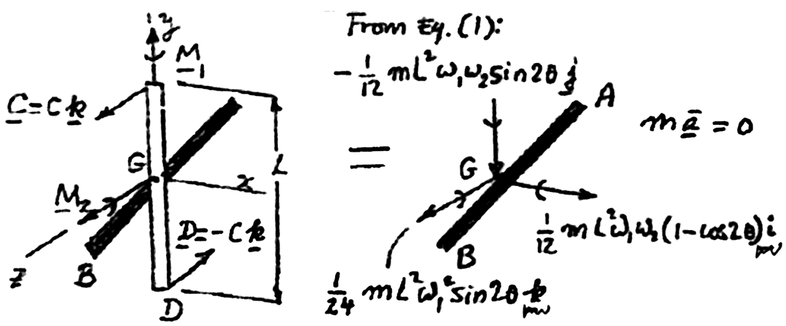
where the reference frame $Gxyz$ rotates with the angular velocity

$$\boldsymbol{\Omega} = \omega_1\mathbf{j}$$

Rate of change of $\mathbf{H}_G$. We note that $\omega_1$ and $\omega_2$ are constant, while $\theta$ varies with $t$, with $\dot{\theta} = \omega_2$.
Eq. (18.22) yields

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G = -\frac{1}{24} mL^2 \omega_1 (2\cos 2\theta \dot{\theta})\mathbf{i} + \frac{1}{12} mL^2 \omega_1 (-2\cos\theta \sin\theta \dot{\theta})\mathbf{j}$$
$$+ \omega_1 \mathbf{j} \times \left( -\frac{1}{24} mL^2 \omega_1 \sin 2\theta \mathbf{i} + \frac{1}{12} mL^2 \omega_2 \mathbf{k} \right)$$
$$= -\frac{1}{12} mL^2 \omega_1 \omega_2 \cos 2\theta \mathbf{i} - \frac{1}{12} mL^2 \omega_1 \omega_2 \sin 2\theta \mathbf{j}$$
$$+ \frac{1}{24} mL^2 \omega_1^2 \sin 2\theta \mathbf{k} + \frac{1}{12} mL^2 \omega_1 \omega_2 \mathbf{i}$$
$$\dot{\mathbf{H}}_G = \frac{1}{12} mL^2 \omega_1 \omega_2 (1 - \cos 2\theta)\mathbf{i} - \frac{1}{12} mL^2 \omega_1 \omega_2 \sin 2\theta \mathbf{j}$$
$$+ \frac{1}{24} mL^2 \omega_1^2 \sin 2\theta \mathbf{k} \qquad (1)$$

Equivalence of external and effective forces.




Equating the moments of the variable couples:

$$L\mathbf{j} \times C\mathbf{k} + M_1 \mathbf{j} + M_2 \mathbf{k} = \dot{\mathbf{H}}_G$$
$$LC\mathbf{i} + M_1 \mathbf{j} + M_2 \mathbf{k} = \frac{1}{12} mL^2 \omega_1 \omega_2 (1 - \cos 2\theta)\mathbf{i} - \frac{1}{12} mL^2 \omega_1 \omega_2 \sin 2\theta \mathbf{j}$$
$$+ \frac{1}{24} mL^2 \omega_1^2 \sin 2\theta \mathbf{k}$$

Equating the coefficients of the unit vectors:

$$\mathbf{j}: \quad M_1 = -\frac{1}{12} mL^2 \omega_1 \omega_2 \sin 2\theta$$

(*a*) Couple $M_1$: $\mathbf{M}_1 = -\frac{1}{12} mL^2 \omega_1 \omega_2 \sin 2\theta \mathbf{j}$ ◀

$$\mathbf{k}: \quad M_2 = \frac{1}{24} mL^2 \omega_1^2 \sin 2\theta$$

## PROBLEM 18.106* (Continued)

(*b*) Couple $M_2$: $\mathbf{M}_2 = \dfrac{1}{24} mL^2 \omega_1^2 \sin 2\theta \mathbf{k}$ ◀

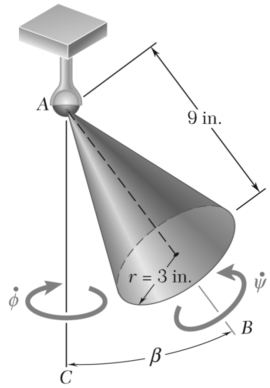
$$\mathbf{i}: \quad LC = \frac{1}{12} mL^2 \omega_1 \omega_2 (1 - \cos 2\theta)$$

$$C = \frac{1}{12} mL\omega_1 \omega_2 (1 - \cos 2\theta) = \frac{1}{6} mL\omega_1 \omega_2 \sin^2 \theta$$

Dynamic reactions.

$\mathbf{C} = \dfrac{1}{6} mL\omega_1 \omega_2 \sin^2 \theta \mathbf{k}$; $\mathbf{D} = -\dfrac{1}{6} mL\omega_1 \omega_2 \sin^2 \theta \mathbf{k}$ ◀

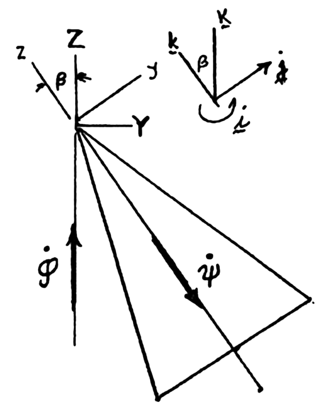
## PROBLEM 18.107




A solid cone of height 9 in. with a circular base of radius 3 in. is supported by a ball-and-socket joint at $A$. Knowing that the cone is observed to precess about the vertical axis $AC$ at the constant rate of 40 rpm in the sense indicated and that its axis of symmetry $AB$ forms an angle $\beta = 40°$ with $AC$, determine the rate at which the cone spins about the axis $AB$.

## SOLUTION

Use principal axes $xyz$ with origin at $A$ as shown.




For the solid cone, $r = 3 \text{ in.} = 0.25 \text{ ft}$

$$h = 9 \text{ in.} = 0.75 \text{ ft} \quad c = \frac{3h}{4} = 0.5625 \text{ ft}$$

$$I = \frac{3}{10}mr^2 \quad I' = \frac{3}{5}m\left(h^2 + \frac{1}{4}r^2\right)$$

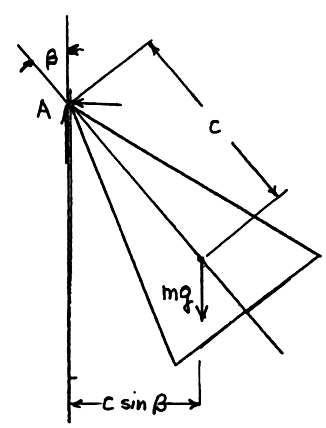
$$I' - I = \frac{3}{5}m\left(h^2 - \frac{1}{4}r^2\right)$$

<u>Angular velocity</u>.

spin: $\dot{\psi}$ about negative $z$ axis

precession: $\dot{\phi}$ about positive $Z$ axis

$$\boldsymbol{\omega} = \dot{\phi}\mathbf{K} - \dot{\psi}\mathbf{k}$$
$$= \dot{\phi}(\cos\beta\mathbf{k} + \sin\beta\mathbf{j}) - \dot{\psi}\mathbf{k}$$
$$\omega_x = 0, \quad \omega_y = \dot{\phi}\sin\beta, \quad \omega_z = \dot{\phi}\cos\beta - \dot{\psi}$$




<u>Angular momentum about fixed Point $A$</u>.

$$\mathbf{H}_A = I'\omega_x\mathbf{i} + I'\omega_y\mathbf{j} + I\omega_z\mathbf{k}$$
$$= I'\dot{\phi}\sin\beta\mathbf{j} + I(\dot{\phi}\cos\beta - \dot{\psi})\mathbf{k}$$

Let frame $Axyz$ be rotating with angular velocity $\mathbf{\Omega}$.

$$\mathbf{\Omega} = \dot{\phi}\mathbf{K} = \dot{\phi}\cos\beta\mathbf{j} + \dot{\phi}\sin\beta\,\mathbf{k}$$

<u>Rate of change of $\mathbf{H}_A$</u>.

$$\dot{\mathbf{H}}_A = \mathbf{\Omega} \times \mathbf{H}_A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \dot{\phi}\sin\beta & \dot{\phi}\cos\beta \\ 0 & I'\dot{\phi}\sin\beta & I(\dot{\phi}\cos\beta - \dot{\psi}) \end{vmatrix}$$
$$= -[(I' - I)\dot{\phi}^2\cos\beta\sin\beta + I\dot{\phi}\dot{\psi}\sin\beta]\mathbf{i}$$

<u>Moment about *A*</u>.

$$\mathbf{M}_A = -mgc\sin\beta\mathbf{i}$$

$\mathbf{M}_A = \dot{\mathbf{H}}_A$ leads to

$$gc = \frac{I' - I}{m}\dot{\phi}^2\cos\beta + \frac{I}{m}\dot{\phi}\dot{\psi}$$

$$= \frac{3}{5}\left(h^2 - \frac{1}{4}r^2\right)\dot{\phi}^2\cos\beta + \frac{3}{10}r^2\dot{\phi}\dot{\psi}$$

$$20gc = (12h^2 - 3r^2)\dot{\phi}^2\cos\beta + 6r^2\dot{\phi}\dot{\psi}$$

$$\dot{\psi} = \frac{20gc - (12h^2 - 3r^2)\dot{\phi}^2\cos\beta}{6r^2\dot{\phi}}$$

<u>Data</u>:

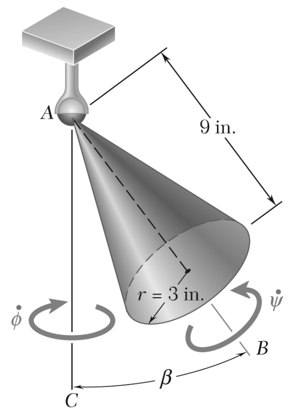
$$\beta = 40°$$

$$\dot{\phi} = 40 \text{ rpm} = 4.1888 \text{ rad/s}$$

$$\dot{\psi} = \frac{(20)(32.2)(0.5625) - [(12)(0.75)^2 - (3)(0.25)^2]4.1888^2\cos 40°}{(6)(0.25)^2(4.1888)}$$

$$= 174.46 \text{ rad/s} \qquad \dot{\psi} = 1666 \text{ rpm} \blacktriangleleft$$

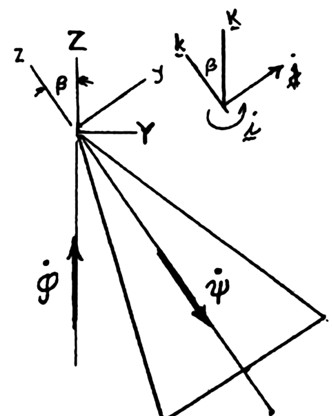
## PROBLEM 18.108

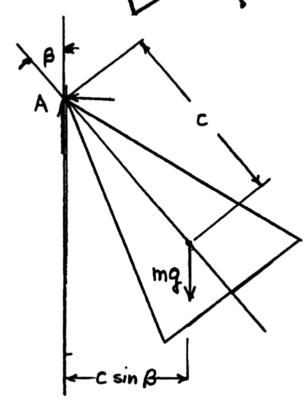



A solid cone of height 9 in. with a circular base of radius 3 in. is supported by a ball-and-socket joint at $A$. Knowing that the cone is spinning about its axis of symmetry $AB$ at the rate of 3000 rpm and that $AB$ forms an angle $\beta = 60°$ with the vertical axis $AC$, determine the two possible rates of steady precession of the cone about the axis $AC$.

## SOLUTION

Use principal axes $xyz$ with origin at $A$ as shown.






For the solid cone,

$$r = 3\text{ in.} = 0.25\text{ ft}$$

$$h = 9\text{ in.} = 0.75\text{ ft} \quad c = \frac{3h}{4} = 0.5625\text{ ft}$$

$$I = \frac{3}{10}mr^2 \quad I' = \frac{3}{5}m\left(h^2 + \frac{1}{4}r^2\right)$$

$$I' - I = \frac{3}{5}m\left(h^2 - \frac{1}{4}r^2\right)$$

<u>Angular velocity</u>.

spin: $\dot\psi$ about negative $z$ axis

precession: $\dot\phi$ about positive $Z$ axis

$$\boldsymbol{\omega} = \dot\phi\mathbf{K} - \dot\psi\mathbf{k}$$

$$= \dot\phi(\cos\beta\,\mathbf{k} + \sin\beta\,\mathbf{j}) - \dot\psi\mathbf{k}$$

$$\omega_x = 0, \quad \omega_y = \dot\phi\sin\beta, \quad \omega_z = \dot\phi\cos\beta - \dot\psi$$

<u>Angular momentum about fixed Point $A$</u>.

$$\mathbf{H}_A = I'\omega_x\mathbf{i} + I'\omega_y\mathbf{j} + I\omega_z\mathbf{k}$$

$$= I'\dot\phi\sin\beta\,\mathbf{j} + I(\dot\phi\cos\beta - \dot\psi)\mathbf{k}$$

Let frame $Axyz$ be rotating with angular velocity $\boldsymbol{\Omega}$.

$$\boldsymbol{\Omega} = \dot\phi\mathbf{K} = \dot\phi\cos\beta\,\mathbf{j} + \dot\phi\sin\beta\,\mathbf{k}$$

<u>Rate of change of $\mathbf{H}_A$</u>.

$$\dot{\mathbf{H}}_A = \boldsymbol{\Omega}\times\mathbf{H}_A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \dot\phi\sin\beta & \dot\phi\cos\beta \\ 0 & I'\dot\phi\sin\beta & I(\dot\phi\cos\beta - \dot\psi) \end{vmatrix}$$

$$= -[(I' - I)\dot\phi^2\cos\beta\sin\beta + I\dot\phi\dot\psi\sin\beta]\mathbf{i}$$

## PROBLEM 18.108 (Continued)

Moment about $A$. $\mathbf{M}_A = -mgc\sin\beta\mathbf{i}$

$\mathbf{M}_A = \dot{\mathbf{H}}_A$ leads to

$$gc = \frac{I' - I}{m}\dot{\phi}^2\cos\beta + \frac{I}{m}\dot{\phi}\dot{\psi}$$

$$= \frac{3}{5}\left(h^2 - \frac{1}{4}r^2\right)\dot{\phi}^2\cos\beta + \frac{3}{10}r^2\dot{\phi}\dot{\psi}$$

$$20gc = (12h^2 - 3r^2)\dot{\phi}^2\cos\beta + 6r^2\dot{\phi}\dot{\psi}$$

$$(12h^2 - 3r^2)\dot{\phi}^2\cos\beta + 6r^2\dot{\psi}\dot{\phi} - 20gc = 0$$

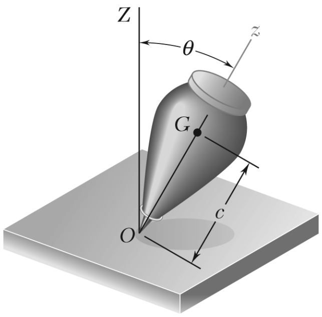
Data: $\dot{\psi} = 3000\text{ rpm} = 314.16\text{ rad/s}$

$\beta = 60°$

$$[(12)(0.75)^2 - (3)(0.25)^2](\cos 60°)\dot{\phi}^2 + (6)(0.25)^2(314.16)\dot{\phi} - (20)(32.2)(0.5625) = 0$$

$$3.28125\dot{\phi}^2 + 117.81\dot{\phi} - 362.25 = 0$$

Solving the quadratic equation, $\dot{\phi} = 2.8488\text{ rad/s}, \quad -38.753\text{ rad/s}$ $\qquad \dot{\phi} = 27.2\text{ rpm}, \quad -370\text{ rpm}$ ◀

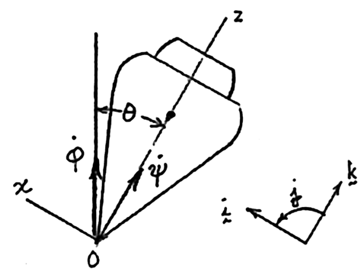



## PROBLEM 18.109

The 85-g top shown is supported at the fixed Point $O$. The radii of gyration of the top with respect to its axis of symmetry and with respect to a transverse axis through $O$ are 21 mm and 45 mm, respectively. Knowing that $c = 37.5$ mm and that the rate of spin of the top about its axis of symmetry is 1800 rpm, determine the two possible rates of steady precession corresponding to $\theta = 30°$.

## SOLUTION

Use principal axes $x$, $y$, $z$ with origin at $O$.



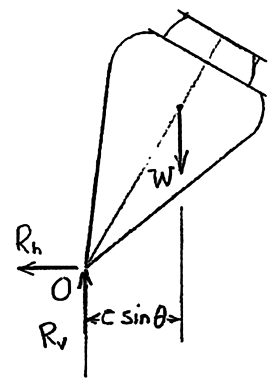

Angular velocity: $\boldsymbol{\omega} = \dot{\varphi}\sin\theta\mathbf{i} + (\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$

$$\omega_x = \dot{\varphi}\sin\theta, \quad \omega_y = 0, \quad \omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$$

Angular momentum about $O$:

$$\mathbf{H}_O = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$
$$= I'\dot{\varphi}\sin\theta\mathbf{i} + I\omega_z\mathbf{k}$$

Let the reference frame $Oxyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k}$.




$$\dot{\mathbf{H}}_O = \boldsymbol{\Omega} \times \mathbf{H}_O$$
$$= (\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k}) \times (I'\dot{\varphi}\sin\theta + I\omega_z\mathbf{k})$$
$$= (I'\dot{\varphi}\sin\theta\cos\theta - I\omega_z\sin\theta)\dot{\varphi}\mathbf{j}$$

$$\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O$$

$$-Wc\sin\theta\mathbf{j} = (I'\dot{\varphi}\sin\theta\cos\theta - I\omega_z\sin\theta)\mathbf{j}$$

$$Wc = (I\omega_z - I'\dot{\varphi}\cos\theta)\dot{\varphi}$$

$$Wc = [I\dot{\psi} - (I' - I)\dot{\varphi}\cos\theta]\dot{\varphi} \qquad (1)$$

<u>Data</u>:

$$m = 85 \text{ g} = 0.085 \text{ kg}$$
$$W = mg = (0.085)(9.81) = 0.83385 \text{ N}$$

$$I = mk_z^2 = (0.085)(0.021)^2 = 37.485 \times 10^{-6} \text{ kg} \cdot \text{m}^2$$
$$I' = mk_x^2 = (0.085)(0.045)^2 = 172.125 \times 10^{-6} \text{ kg} \cdot \text{m}^2$$

$$c = 37.5 \text{ mm} = 0.0375 \text{ m}, \quad \dot{\psi} = 1800 \text{ rpm} = 188.496 \text{ rad/s}$$

$$\theta = 30°$$

## PROBLEM 18.109 (Continued)

Substituting into Eq. (1),

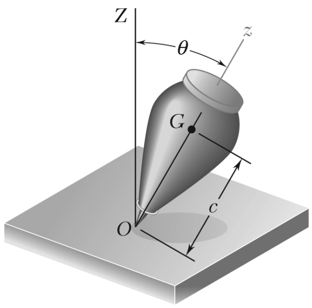
$$(0.83385)(0.0375) = [(37.485\times10^{-6})(188.496) - (134.64\times10^{-6})\dot{\varphi}\cos 30°]\dot{\varphi}$$

$$116.602\times10^{-6}\,\dot{\varphi}^2 - 7.0658\times10^{-3}\,\dot{\varphi} + 31.269\times10^{-3} = 0$$

$$\dot{\varphi} = 30.299 \pm 25.492 \qquad \dot{\varphi} = 4.807 \text{ rad/s}, \quad 55.791 \text{ rad/s}$$

$$\dot{\varphi} = 45.9 \text{ rpm}, \quad 533 \text{ rpm} \blacktriangleleft$$

## PROBLEM 18.110

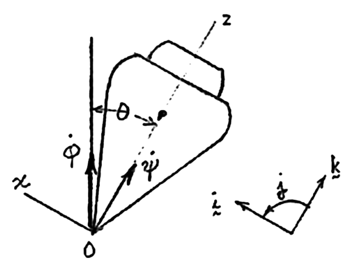



The top shown is supported at the fixed Point $O$ and its moments of inertia about its axis of symmetry and about a transverse axis through $O$ are denoted, respectively, by $I$ and $I'$. (*a*) Show that the condition for steady precession of the top is

$$(I\omega_z - I'\dot{\phi}\cos\theta)\dot{\phi} = Wc$$

where $\dot{\phi}$ is the rate of precession and $\omega_z$ is the rectangular component of the angular velocity along the axis of symmetry of the top. (*b*) Show that if the rate of spin $\dot{\psi}$ of the top is very large compared with its rate of precession $\dot{\phi}$, the condition for steady precession is $I\dot{\psi}\dot{\phi} \approx Wc$. (*c*) Determine the percentage error introduced when this last relation is used to approximate the slower of the two rates of precession obtained for the top of Problem 18.109.

## SOLUTION




Use principal axes $x$, $y$, $z$ with origin at $O$.

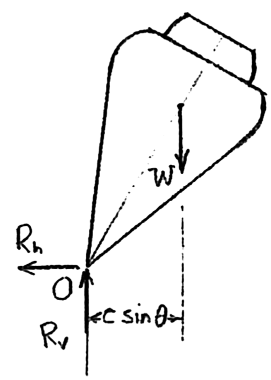
Angular velocity: $\boldsymbol{\omega} = \dot{\varphi}\sin\theta\mathbf{i} + (\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$

$$\omega_x = \dot{\varphi}\sin\theta, \quad \omega_y = 0, \quad \omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$$

Angular momentum about $O$:

$$\mathbf{H}_O = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k} = I'\dot{\varphi}\sin\theta\mathbf{i} + I\omega_z\mathbf{k}$$

Let the reference frame $Oxyz$ be rotating with angular velocity $\mathbf{\Omega} = \dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k}$.




$$\begin{aligned}\dot{\mathbf{H}}_O &= \mathbf{\Omega}\times\mathbf{H}_O \\ &= (\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k})\times(I'\dot{\varphi}\sin\theta + I\omega_z\mathbf{k}) \\ &= (I'\dot{\varphi}\sin\theta\cos\theta - I\omega_z\sin\theta)\dot{\varphi}\mathbf{j}\end{aligned}$$

$$\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O$$

$$-Wc\sin\theta\mathbf{j} = (I'\dot{\varphi}\sin\theta\cos\theta - I\omega_z\sin\theta)\mathbf{j}$$

(*a*) $\qquad Wc = (I\omega_z - I'\dot{\varphi}\cos\theta)\dot{\varphi}$ ◀

$$Wc = [I\dot{\psi} - (I' - I)\dot{\varphi}\cos\theta]\dot{\varphi} \qquad (1)$$

(*b*) For $|\dot{\varphi}| << \dot{\psi}$, $\qquad Wc \approx I\dot{\psi}\dot{\varphi} \qquad (2)$ ◀

(*c*) Data:

$$m = 85 \text{ g} = 0.085 \text{ kg}$$
$$W = mg = (0.085)(9.81) = 0.83385 \text{ N}$$
$$I = mk_z^2 = (0.085)(0.021)^2 = 37.485 \times 10^{-6} \text{ kg} \cdot \text{m}^2$$
$$I = mk_x^2 = (0.085)(0.045)^2 = 172.125 \times 10^{-6} \text{ kg} \cdot \text{m}^2$$
$$c = 37.5 \text{ mm} = 0.0375 \text{ m}, \quad \dot{\psi} = 1800 \text{ rpm} = 188.496 \text{ rad/s}$$
$$\theta = 30°$$

Substituting into Eq. (1),

$$(0.83385)(0.0375) = [(37.485 \times 10^{-6})(188.496) - (134.64 \times 10^{-6})\dot{\varphi}\cos 30°]\dot{\varphi}$$

$$116.602 \times 10^{-6}\dot{\varphi}^2 - 7.0658 \times 10^{-3}\dot{\varphi} + 31.269 \times 10^{-3} = 0$$

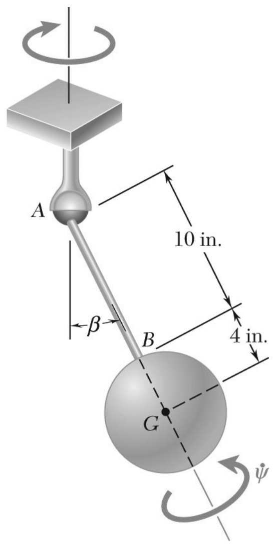
$$\dot{\varphi} = 30.299 \pm 25.492 \qquad \dot{\varphi} = 4.807 \text{ rad/s, } 55.791 \text{ rad/s}$$

From Eq. (2),
$$\dot{\varphi} = \frac{Wc}{I\dot{\psi}} = \frac{(0.83385)(0.0375)}{(37.485 \times 10^{-6})(188.496)} = 4.425 \text{ rad/s}$$

$$\% \text{ error} = \frac{4.425 - 4.807}{4.807} \times 100\%$$

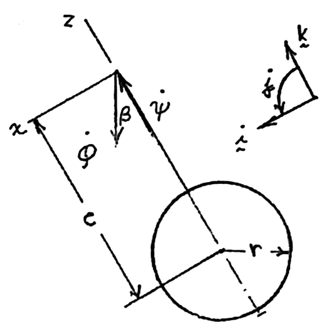
$\%$ error $= -7.95\%$ ◀

## PROBLEM 18.111




A solid aluminum sphere of radius 4 in. is welded to the end of a10-in.-long rod $AB$ of negligible mass which is supported by a ball-and-socket joint at $A$. Knowing that the sphere is observed to precess about a vertical axis at the constant rate of 60 rpm in the sense indicated and that rod $AB$ forms an angle $\beta = 20°$ with the vertical, determine the rate of spin of the sphere about line $AB$.

## SOLUTION

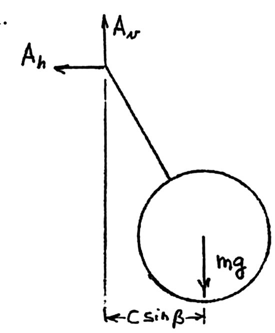



Use principal axes $x$, $y$, $z$ with origin at $A$.

Angular velocity:
$$\boldsymbol{\omega} = \dot{\varphi}\sin\beta\,\mathbf{i} + (\dot{\psi} - \dot{\varphi}\cos\beta)\mathbf{k}$$
$$\omega_x = \dot{\varphi}\sin\beta, \quad \omega_y = 0, \quad \omega_z = \dot{\psi} - \dot{\varphi}\cos\beta$$

Angular momentum about $A$:
$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$
$$= I'\dot{\varphi}\sin\beta\,\mathbf{i} + I(\dot{\psi} - \dot{\varphi}\cos\beta)\mathbf{k}$$

Let the reference frame $Axyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \dot{\varphi}\sin\beta\,\mathbf{i} - \dot{\varphi}\cos\beta\,\mathbf{k}$.




$$\dot{\mathbf{H}}_A = \boldsymbol{\Omega} \times \mathbf{H}_A$$
$$= -\dot{\varphi}[I\dot{\psi}\sin\beta + (I' - I)\dot{\varphi}\sin\beta\cos\beta]\mathbf{j}$$
$$\Sigma\mathbf{M}_A = \dot{\mathbf{H}}_A$$
$$-mgc\sin\beta\,\mathbf{j} = -\dot{\varphi}[I\dot{\psi}\sin\beta + (I' - I)\dot{\varphi}\sin\beta\cos\beta]\mathbf{j}$$
$$mgc = I\dot{\varphi}\dot{\psi} - (I' - I)\dot{\varphi}^2\cos\beta$$
$$\frac{I\dot{\varphi}\dot{\psi}}{mgc} - \frac{(I' - I)\dot{\varphi}^2\cos\beta}{mgc} + 1 = 0 \qquad (1)$$

Data:
$$r = 4\text{ in.} = 0.333\text{ ft}$$
$$c = 10\text{ in.} + 4\text{ in.}$$
$$= 14\text{ in.} = 1.1667\text{ ft},$$
$$g = 32.2\text{ ft/s}^2$$

## PROBLEM 18.111 (Continued)

$$\frac{I}{m} = \frac{2}{5}r^2 = \frac{2}{5}(0.3333)^2 = 0.04444\ \text{ft}^2$$

$$\frac{I'}{m} = \frac{2}{5}r^2 + c^2 = \frac{2}{5}(0.3333)^2 + (1.16667)^2 = 1.4056\ \text{ft}^2$$

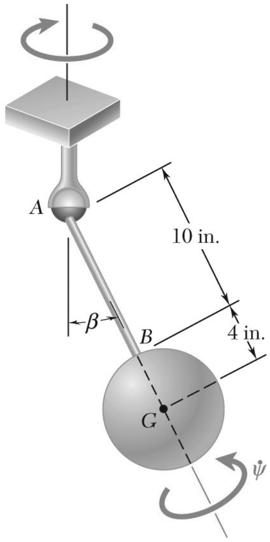
$$\beta = 20°, \quad \dot{\varphi} = 60\ \text{rpm} = 6.2832\ \text{rad/s}$$

Substituting into Eq. (1),

$$\frac{(0.04444)(6.2832)\dot{\psi}}{(32.2)(1.1667)} - \frac{(1.4056 - 0.04444)(6.2832)^2 \cos 20°}{(32.2)(1.1667)} + 1 = 0$$

$$7.4335 \times 10^{-3}\dot{\psi} - 1.34412 + 1 = 0$$

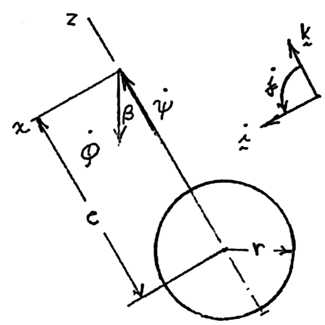
$$\dot{\psi} = 46.293\ \text{rad/s} = 442\ \text{rpm} \qquad \dot{\psi} = 442\ \text{rpm} \blacktriangleleft$$




## PROBLEM 18.112

A solid aluminum sphere of radius 4 in. is welded to the end of a 10-in.-long rod *AB* of negligible mass which is supported by a ball-and-socket joint at *A*. Knowing that the sphere spins as shown about line *AB* at the rate of 600 rpm, determine the angle $\beta$ for which the sphere will precess about a vertical axis at the constant rate of 60 rpm in the sense indicated.

## SOLUTION




Use principal axes *x*, *y*, *z* with origin at *A*.

Angular velocity:

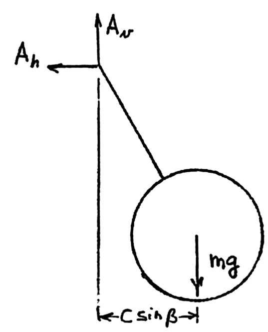
$$\boldsymbol{\omega} = \dot{\varphi}\sin\beta\mathbf{i} + (\dot{\psi} - \dot{\varphi}\cos\beta)\mathbf{k}$$

$$\omega_x = \dot{\varphi}\sin\beta, \quad \omega_y = 0, \quad \omega_z = \dot{\psi} - \dot{\varphi}\cos\beta$$

Angular momentum about *A*:

$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$
$$= I'\dot{\varphi}\sin\beta\mathbf{i} + I(\dot{\psi} - \dot{\varphi}\cos\beta)\mathbf{k}$$

Let the reference frame *Axyz* be rotating with angular velocity $\mathbf{\Omega} = \dot{\varphi}\sin\beta\mathbf{i} - \dot{\varphi}\cos\beta\mathbf{k}$.




$$\dot{\mathbf{H}}_A = \mathbf{\Omega}\times\mathbf{H}_A$$
$$= -\dot{\varphi}[I\dot{\psi}\sin\beta + (I' - I)\dot{\varphi}\sin\beta\cos\beta]\mathbf{j}$$

$$\Sigma\mathbf{M}_A = \dot{\mathbf{H}}_A$$

$$-mgc\sin\beta\mathbf{j} = -\dot{\varphi}[I\dot{\psi}\sin\beta + (I' - I)\dot{\varphi}\sin\beta\cos\beta]\mathbf{j}$$

$$mgc = I\dot{\varphi}\dot{\psi} - (I' - I)\dot{\varphi}^2\cos\beta$$

$$\frac{I\dot{\varphi}\dot{\psi}}{mgc} - \frac{(I' - I)\dot{\varphi}^2\cos\beta}{mgc} + 1 = 0 \qquad (1)$$

<u>Data</u>:

$$r = 4\text{ in.} = \frac{1}{3}\text{ ft}$$

$$c = 10 + 4 = 14\text{ in.}$$

$$= \frac{14}{12} = \frac{7}{6}\text{ ft,}$$

$$g = 32.2\text{ ft/s}^2$$

## PROBLEM 18.112 (Continued)

$$\frac{I}{m} = \frac{2}{5}r^2 = \frac{2}{5}\left(\frac{1}{3}\right)^2 = 0.04444 \text{ ft}^2$$

$$\frac{I'}{m} = \frac{2}{5}r^2 + c^2 = \frac{2}{5}\left(\frac{1}{3}\right)^2 + \left(\frac{7}{6}\right)^2 = 1.4056 \text{ ft}^2$$

$$\dot{\psi} = 600 \text{ rpm} = 62.832 \text{ rad/s} \qquad \dot{\varphi} = -60 \text{ rpm} = -6.2832 \text{ rad/s}$$

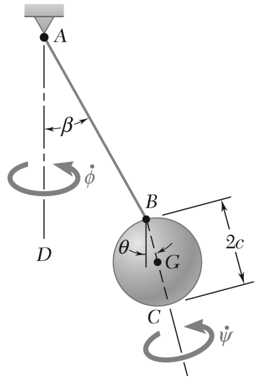
Substituting into Eq. (1),

$$\frac{(0.044444)(-6.2832)(62.832)}{(32.2)(1.16667)} - \frac{(1.40556 - 0.04444)(-6.2832)^2}{(32.2)(1.16667)}\cos\beta + 1 = 0$$

$$-0.46706 - 1.43038\cos\beta + 1 = 0$$

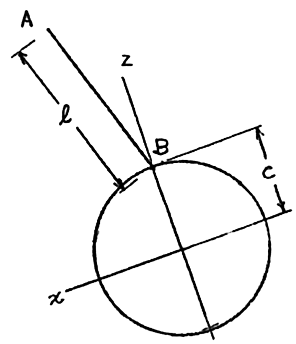
$$\cos\beta = 0.37259 \qquad \beta = 68.1° \blacktriangleleft$$

## PROBLEM 18.113




A solid sphere of radius $c = 3$ in. is attached as shown to cord $AB$. The sphere is observed to precess at the constant rate $\dot{\phi} = 6$ rad/s about the vertical axis $AD$. Knowing that $\beta = 40°$, determine the angle $\theta$ that the diameter $BC$ forms with the vertical when the sphere (*a*) has no spin, (*b*) spins about its diameter $BC$ at the rate $\dot{\psi} = 50$ rad/s, (*c*) spins about $BC$ at the rate $\dot{\psi} = -50$ rad/s.

## SOLUTION




Use principal centroidal axes $x$, $y$, $z$ as shown.

Angular velocity: $\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\mathbf{i} + (\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$

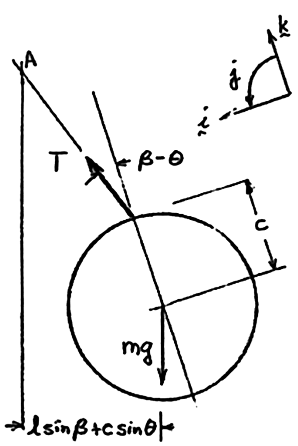
Angular momentum about the mass center $G$:

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$
$$= -I'\dot{\varphi}\sin\theta\mathbf{i} + I(\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$$

Let the reference frame $Gxyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k}$$
$$\dot{\mathbf{H}}_G = \boldsymbol{\Omega}\times\mathbf{H}_G$$
$$= [I\dot{\varphi}\dot{\psi}\sin\theta - (I' - I)\dot{\varphi}^2\sin\theta\cos\theta]\mathbf{j}$$

Acceleration of the mass center:




$$\bar{\mathbf{a}} = (l\sin\beta + c\sin\theta)\dot{\varphi}^2 \longleftarrow$$

$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}:$$

$+\uparrow$: $\quad T\cos\beta - mg = 0, \qquad T = \dfrac{mg}{\cos\beta}$ (1)

$+\longleftarrow$: $\quad T\sin\beta = m\bar{a}$

$$g\tan\beta = (l\sin\beta + c\sin\theta)\dot{\varphi}^2 \tag{2}$$

$$+\circlearrowright\Sigma M_G = Tc\sin(\beta - \theta) = \dot{H}_G$$

$$\frac{mgc\sin(\beta-\theta)}{\cos\beta} = I\dot{\psi}\dot{\varphi}\sin\theta - (I' - I)\dot{\varphi}^2\sin\theta\cos\theta \tag{3}$$

<u>Data</u>: $\quad c = 3\text{ in.} = 0.25\text{ ft}, \quad \beta = 40°, \quad \dot{\varphi} = 6\text{ rad/s}$

$$\frac{I}{m} = \frac{I'}{m} = \frac{2}{5}c^2 = 0.025\text{ ft}^2 \quad I' - I = 0$$

Substituting into Eq. (1),

$$\frac{(32.2)(0.25)\sin(\beta-\theta)}{\cos\beta} = (0.025)(6)\dot{\psi}\sin\theta$$

$$8.05(\sin\beta\cos\theta - \cos\beta\sin\theta) = 0.15\dot{\psi}\sin\theta\cos\beta$$

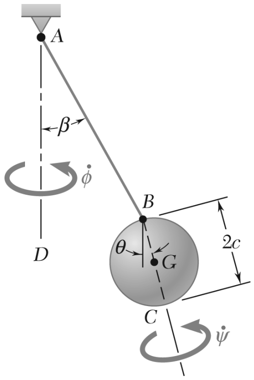
$$\tan\theta = \frac{\tan\beta}{1+0.018634\dot{\psi}}$$

(*a*) $\dot{\psi} = 0,$ $\tan\theta = \tan\beta \quad \theta = \beta$ $\theta = 40.0° \blacktriangleleft$

(*b*) $\dot{\psi} = 50$ rad/s, $\tan\theta = \dfrac{\tan 40°}{1+(0.018634)(50)}$ $\theta = 23.5° \blacktriangleleft$

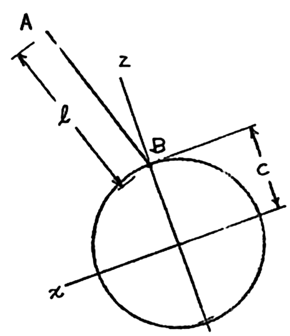
(*c*) $\dot{\psi} = -50$ rad/s, $\tan\theta = \dfrac{\tan 40°}{1+(0.018634)(-50)}$ $\theta = 85.3° \blacktriangleleft$

## PROBLEM 18.114




A solid sphere of radius $c = 3$ in. is attached as shown to a cord $AB$ of length 15 in. The sphere spins about its diameter $BC$ and precesses about the vertical axis $AD$. Knowing that $\theta = 20°$ and $\beta = 35°$, determine ($a$) the rate of spin of the sphere, ($b$) its rate of precession.

## SOLUTION




Use principal centroidal axes $x$, $y$, $z$ as shown.

Angular velocity: $\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\mathbf{i} + (\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$

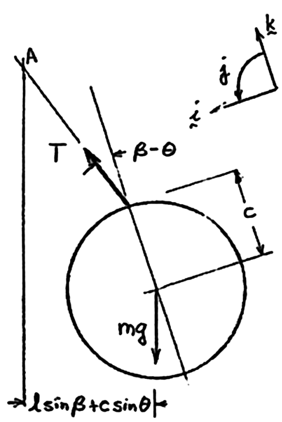
Angular momentum about the mass center $G$:

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$
$$= -I'\dot{\varphi}\sin\theta\mathbf{i} + I(\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$$

Let the reference frame $Gxyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k}$$
$$\dot{\mathbf{H}}_G = \boldsymbol{\Omega}\times\mathbf{H}_G$$
$$= [I\dot{\varphi}\dot{\psi}\sin\theta - (I' - I)\dot{\varphi}^2\sin\theta\cos\theta]\mathbf{j}$$

Acceleration of the mass center:




$$\bar{\mathbf{a}} = (l\sin\beta + c\sin\theta)\dot{\varphi}^2 \longleftarrow$$

$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}:$$

$$+\uparrow: \quad T\cos\beta - mg = 0, \qquad T = \frac{mg}{\cos\beta} \tag{1}$$

$$+\longleftarrow: \quad T\sin\beta = m\bar{a}$$

$$g\tan\beta = (l\sin\beta + c\sin\theta)\dot{\varphi}^2 \tag{2}$$

$$+\circlearrowright\ \Sigma M_G = Tc\sin(\beta - \theta) = \dot{H}_G$$

$$\frac{mgc\sin(\beta-\theta)}{\cos\beta} = I\dot{\psi}\dot{\varphi}\sin\theta - (I' - I)\dot{\varphi}^2\sin\theta\cos\theta \tag{3}$$

<u>Data</u>:

$$c = 3\text{ in.} = 0.25\text{ ft}, \quad l = 15\text{ in.} = 1.25\text{ ft}$$
$$g = 32.2\text{ ft/s}^2$$
$$\frac{I}{m} = \frac{I'}{m} = \frac{2}{5}c^2 = 0.025\text{ ft}^2 \quad I' - I = 0,$$
$$\beta = 35°, \quad \theta = 20°$$

## PROBLEM 18.114 (Continued)

From Equation (2),

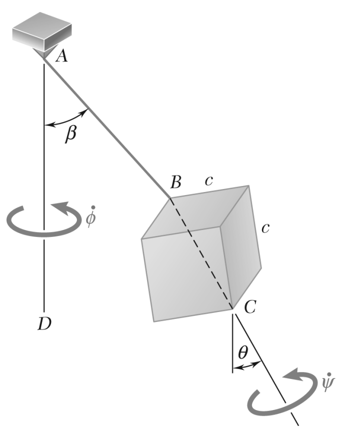
$$(32.2)\tan 35° = (1.25\sin 35° + 0.25\sin 20°)\dot{\varphi}^2$$
$$\dot{\varphi}^2 = 28.096$$
$$\dot{\varphi} = 5.3006 \text{ rad/s}$$

From Equation (3), $$\frac{(32.2)(0.25)\sin 15°}{\cos 35°} = (0.025)\dot{\psi}(5.3006)\sin 20°$$

(*a*) $$\dot{\psi} = 56.119 \text{ rad/s}$$ $\dot{\psi} = 56.1 \text{ rad/s}$ ◀

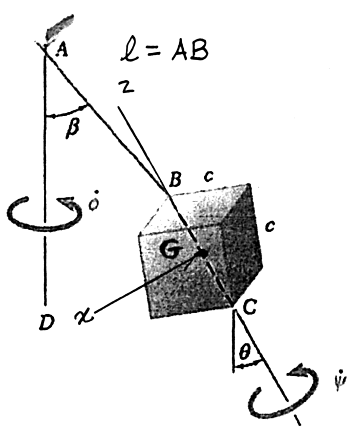
(*b*) $\dot{\varphi} = 5.30 \text{ rad/s}$ ◀

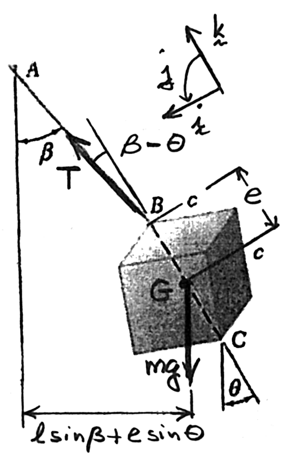
## PROBLEM 18.115




A solid cube of side $c = 80$ mm is attached as shown to cord $AB$. It is observed to spin at the rate $\dot{\psi} = 40$ rad/s about its diagonal $BC$ and to precess at the constant rate $\dot{\phi} = 5$ rad/s about the vertical axis $AD$. Knowing that $\beta = 30°$, determine the angle $\phi$ that the diagonal $BC$ forms with the vertical. (*Hint:* The moment of inertia of a cube about an axis through its center is independent of the orientation of that axis.)

## SOLUTION




Use centroidal axes $x$, $y$, $z$ such that the $z$ axis lies along the body diagonal $BC$ and the $x$ axis lies in the plane of $A$, $B$, $C$, and $D$. Let $e$ be the length $GB$.

$$e = \frac{\sqrt{3}}{2}c$$

Moment of inertia: $$I_x = I_y = I_z = \frac{1}{6}mc^2$$

Angular velocity: $$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\mathbf{i} + (\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$$

Angular momentum about the mass center $G$:

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k} = -I'\dot{\varphi}\sin\theta\mathbf{i} + I(\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$$

Let the reference frame $Gxyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k}$$

$$\dot{\mathbf{H}}_G = \boldsymbol{\Omega} \times \mathbf{H}_G$$

$$= [I\dot{\varphi}\dot{\psi}\sin\theta - (I' - I)\dot{\varphi}^2\sin\theta\cos\theta]\mathbf{j}$$

Acceleration of the mass center:

$$\bar{\mathbf{a}} = (l\sin\beta + e\sin\theta)\dot{\varphi}^2 \longleftarrow$$

$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}:$$

$$+\uparrow: \quad T\cos\beta - mg = 0, \quad T = \frac{mg}{\cos\beta} \qquad (1)$$

$$+\longleftarrow: \quad T\sin\beta = m\bar{a}$$

$$g\tan\beta = (l\sin\beta + e\sin\theta)\dot{\varphi}^2 \qquad (2)$$

$$+\circlearrowleft\ \Sigma M_{G.} = Te\sin(\beta - \theta) = \dot{H}_G$$

$$\frac{mge\sin(\beta - \theta)}{\cos\beta} = I\dot{\psi}\dot{\varphi}\sin\theta - (I' - I)\dot{\varphi}^2\sin\theta\cos\theta \qquad (3)$$

## PROBLEM 18.115 (Continued)

Data:

$$c = 80 \text{ mm} = 0.08 \text{ m} \quad e = \frac{\sqrt{3}}{2}(0.08) = 0.069282 \text{ m}$$

$$\frac{I}{m} = \frac{I'}{m} = \frac{1}{6}c^2 = 1.06667 \times 10^{-3} \text{ m}^2 \quad I' - I = 0$$

$$\beta = 30° \quad \dot{\psi} = 40 \text{ rad/s} \quad \dot{\varphi} = 5 \text{ rad/s}$$

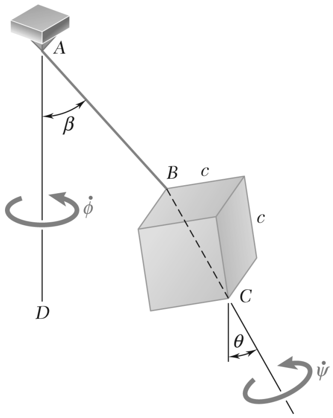
Substituting into Eq. (3),

$$\frac{(9.81)(0.069282)\sin(\beta - \theta)}{\cos\beta} = (1.06667 \times 10^{-3})(40)(5)\sin\theta$$

$$0.67966(\sin\beta\cos\theta - \cos\beta\sin\theta) = 0.21333\sin\theta\cos\beta$$

$$0.89299\sin\theta\cos\beta = 0.67966\sin\beta\cos\theta$$

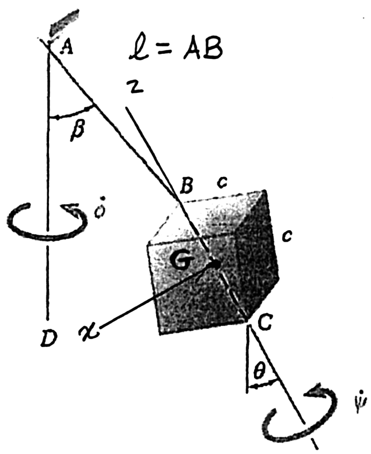
$$\tan\theta = 0.76111\tan\beta = 0.76111\tan 30° = 0.43942 \qquad \theta = 23.7° \blacktriangleleft$$

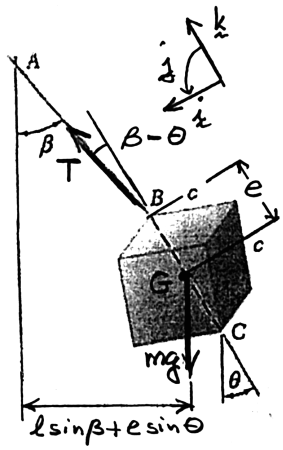



## PROBLEM 18.116

A solid cube of side $c = 120$ mm is attached as shown to a cord $AB$ of length 240 mm. The cube spins about its diagonal $BC$ and precesses about the vertical axis $AD$. Knowing that $\theta = 25°$ and $\beta = 40°$, determine (*a*) the rate of spin of the cube, (*b*) its rate of precession. (See hint of Problem 18.115.)

## SOLUTION




Use centroidal axes $x$, $y$, $z$ such that the $z$ axis lies along the body diagonal $BC$ and the $x$ axis lies in the plane of $A$, $B$, $C$, and $D$. Let $e$ be the length $GB$.

$$e = \frac{\sqrt{3}}{2}c$$

Moment of inertia: $$I_x = I_y = I_z = \frac{1}{6}mc^2$$

Angular velocity: $$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k}$$

Angular momentum about the mass center $G$:

$$\mathbf{H}_G = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k} = -I'\dot{\varphi}\sin\theta\,\mathbf{i} + I(\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$$

Let the reference frame $Gxyz$ be rotating with angular velocity

$$\boldsymbol{\Omega} = -\dot{\varphi}\sin\theta\,\mathbf{i} + \dot{\varphi}\cos\theta\,\mathbf{k}$$

$$\dot{\mathbf{H}}_G = \boldsymbol{\Omega} \times \mathbf{H}_G$$

$$= [I\dot{\varphi}\dot{\psi}\sin\theta - (I' - I)\dot{\varphi}^2\sin\theta\cos\theta]\mathbf{j}$$

Acceleration of the mass center:

$$\bar{\mathbf{a}} = (l\sin\beta + e\sin\theta)\dot{\varphi}^2 \longleftarrow$$

$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}:$$

$$+\uparrow:\quad T\cos\beta - mg = 0,\quad T = \frac{mg}{\cos\beta} \qquad (1)$$

$$+\longleftarrow:\quad T\sin\beta = m\bar{a}$$

$$g\tan\beta = (l\sin\beta + e\sin\theta)\dot{\varphi}^2 \qquad (2)$$

$$+\circlearrowleft \Sigma M_G: \quad Te\sin(\beta - \theta) = \dot{H}_G$$

$$\frac{mge\sin(\beta - \theta)}{\cos\beta} = I\dot{\psi}\dot{\varphi}\sin\theta - (I' - I)\dot{\varphi}^2\sin\theta\cos\theta \qquad (3)$$

## PROBLEM 18.116 (Continued)

Data:

$$c = 120 \text{ mm} = 0.12 \text{ m}$$

$$e = \frac{\sqrt{3}}{2}(0.12) = 0.103923 \text{ m}$$

$$l = 240 \text{ mm} = 0.24 \text{ m}$$

$$\frac{I}{m} = \frac{I'}{m} = \frac{1}{6}c^2 = \frac{1}{6}(0.12)^2 = 2.4\times10^3 \,\text{m}^2 \quad I' - I = 0$$

$$\theta = 25° \quad \beta = 40° \quad \theta - \beta = 15°$$

From Eq. (2),

$$(9.81)\tan 40° = (0.24\sin 40° + 0.103923\sin 25°)\dot{\varphi}^2$$

$$\dot{\varphi}^2 = 41.534 \qquad \dot{\varphi} = 6.4447 \text{ rad/s}$$

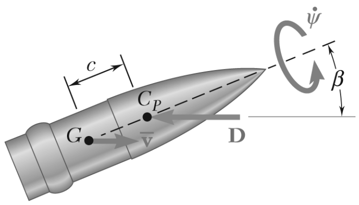
From Eq. (3),

$$\frac{(9.81)(0.103923)\sin 15°}{\cos 40°} = (2.4\times10^{-3})\dot{\psi}(6.4447)\sin 25°$$

(*a*) $\dot{\psi} = 52.694$ rad/s $\qquad \dot{\psi} = 52.7$ rad/s ◀

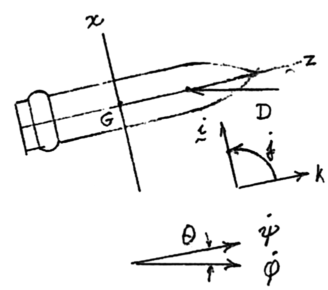
(*b*) $\dot{\varphi} = 6.44$ rad/s ◀

## PROBLEM 18.117




A high-speed photographic record shows that a certain projectile was fired with a horizontal velocity $\bar{\mathbf{v}}$ of 2000 ft/s and with its axis of symmetry forming an angle $\beta = 3°$ with the horizontal. The rate of spin $\dot{\psi}$ of the projectile was 6000 rpm, and the atmospheric drag was equivalent to a force **D** of 25 lb acting at the center of pressure $C_P$ located at a distance $c = 6$ in. from $G$. (*a*) Knowing that the projectile has a weight of 45 lb and a radius of gyration of 2 in. with respect to its axis of symmetry, determine its approximate rate of steady precession. (*b*) If it is further known that the radius of gyration of the projectile with respect to a transverse axis through $G$ is 8 in., determine the exact values of the two possible rates of precession.

## SOLUTION




Choose principal centroidal axes $x$, $y$, $z$ as shown.

The symmetry (spin) axis is the $z$ axis.

Reduce the drag force to a force-couple system at the mass center.

$$\mathbf{F} = (D\sin\beta - W\cos\beta)\mathbf{i} - (D\cos\beta - W\sin\beta)\mathbf{k}$$

$$\mathbf{M}_G = Dc\sin\beta\,\mathbf{j}$$

For the occurrence of steady precession, the precession axis must be parallel to the drag force.

Thus, $\qquad \theta = \beta = 3°.$

(*a*) Using Equation (18.44), $\qquad \Sigma M_G = \dot{\mathbf{H}}_G$

$$Dc\sin\beta = (I\omega_z - I'\dot{\varphi}\cos\theta)\dot{\varphi}\sin\theta$$

Using $\beta = \theta$ and $\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$ gives

$$I\dot{\psi}\dot{\varphi} - (I' - I)\dot{\varphi}^2\cos\theta = Dc \tag{1}$$

Neglecting the quadratic term in $\dot{\varphi}$,

$$I\dot{\psi}\dot{\varphi} \approx Dc \qquad \dot{\varphi} \approx \frac{Dc}{I\dot{\psi}}$$

<u>Data</u>: $\qquad W = 45 \text{ lb}, \quad m = \frac{W}{g} = \frac{45}{32.2} = 1.3975 \text{ slug}$

$$I = mk_z^2 = (1.3975)\left(\frac{2}{12}\right)^2 = 0.0388205 \text{ slug}\cdot\text{ft}^2$$

$$\dot{\psi} = 6000 \text{ rpm} = 628.32 \text{ rad/s}, \quad c = \frac{6}{12} \text{ in.} = 0.5 \text{ ft}$$

$$\dot{\varphi} \approx \frac{(25)(0.5)}{(0.0388205)(628.32)} = 0.51248 \text{ rad/s}$$

$\dot{\varphi} \approx 4.89$ rpm ◀

## PROBLEM 18.117 (Continued)

(*b*) $$I' = mk_x^2 = (1.3975)\left(\frac{8}{12}\right)^2 = 0.62112 \text{ slug}\cdot\text{ft}^2$$

$$I' - I = 0.58230 \text{ slug}\cdot\text{ft}^2$$

Substituting into Equation (1),

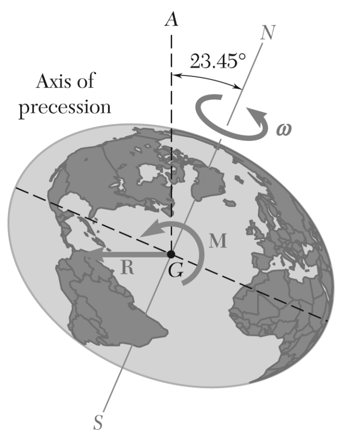
$$(0.0388205)(628.32)\dot{\varphi} - (0.58230)\dot{\varphi}^2 \cos 3° = (25)(0.5)$$

$$0.58150\dot{\varphi}^2 - 24.3912\dot{\varphi} + 12.5 = 0$$

$$\dot{\varphi} = 20.9727 \pm 20.4538$$

$$= 0.51890 \text{ rad/s}, \quad 41.427 \text{ rad/s} \qquad \dot{\varphi} = 4.96 \text{ rpm}, 396 \text{ rpm} ◀$$

## PROBLEM 18.118




If the earth were a sphere, the gravitational attraction of the sun, moon, and planets would at all times be equivalent to a single force **R** acting at the mass center of the earth. However, the earth is actually an oblate spheroid and the gravitational system acting on the earth is equivalent to a force **R** and a couple **M**. Knowing that the effect of the couple **M** is to cause the axis of the earth to precess about the axis *GA* at the rate of one revolution in 25,800 years, determine the average magnitude of the couple **M** applied to the earth. Assume that the average density of the earth is 5.51 g/cm$^3$, that the average radius of the earth is 6370 km, and that $\bar{I} = \frac{2}{5}mR^2$. (*Note:* This forced precession is known as the precession of the equinoxes and is not to be confused with the free precession discussed in Problem 18.123.)

## SOLUTION

$$25{,}800 \text{ years} = (25{,}800 \text{ yr})(365.24 \text{ day/yr})(24 \text{ h/day})(3600 \text{ s/h})$$
$$= 814.16 \times 10^9 \text{ s}$$
$$\dot{\varphi} = \frac{2\pi}{814.16 \times 10^9} = 7.7173 \times 10^{-12} \text{ rad/s}$$
$$\dot{\psi} = \frac{2\pi}{(23.93 \text{ h})(3600 \text{ s/h})} = 72.935 \times 10^{-6} \text{ rad/s}$$

Mass density of Earth: $\rho = 5.51 \text{ g/cm}^3 = 5510 \text{ kg/m}^3$

Radius of Earth: $R = 6370 \text{ km} = 6.370 \times 10^6 \text{ m}$

Mass of Earth: $\frac{4}{3}\pi R^3 \rho = \frac{4}{3}\pi (6.370 \times 10^6)^3 (5510) = 5.9657 \times 10^{24} \text{ kg}$

$$\bar{I} = \frac{2}{5}mR^2 = \frac{2}{5}(5.9657 \times 10^{24})(6.370 \times 10^6)^2$$
$$= 96.827 \times 10^{36} \text{ kg} \cdot \text{m}^2$$

Using Equation (18.44),

$$M = (I\omega_z - I'\dot{\varphi}\cos\theta)\dot{\varphi}\sin\theta$$
$$= [I\dot{\psi} + (I - I')\dot{\varphi}\cos\theta]\dot{\varphi}\sin\theta$$
$$= I\dot{\psi}\dot{\varphi}\sin\theta$$
$$= (96.827 \times 10^{36})(72.935 \times 10^{-6})(7.7173 \times 10^{-12})\sin 23.45°$$
$$= 21.688 \times 10^{21} \text{ N} \cdot \text{m}$$

$$M = 21.7 \times 10^{21} \text{ N} \cdot \text{m} \blacktriangleleft$$

## PROBLEM 18.119

Show that for an axisymmetrical body under no force, the rates of precession and spin can be expressed, respectively, as

$$\dot{\varphi} = \frac{H_G}{I'}$$

and

$$\dot{\psi} = \frac{H_G \cos\theta (I' - I)}{II'}$$

where $H_G$ is the constant value of the angular momentum of the body.

## SOLUTION

By Equations (18.48), $\quad \omega_x = -\frac{H_G \sin\theta}{I'}, \quad \omega_y = 0, \quad \omega_z = \frac{H_G \cos\theta}{I}$

By Equation (18.35) with $\quad \dot{\theta} = 0,$

$$\omega_x = -\dot{\varphi}\sin\theta, \qquad \omega_y = 0, \qquad \omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$$

Eliminating $\omega_x$ and $\omega_z$,

$$\dot{\varphi} = \frac{H_G}{I'}$$

$$\dot{\psi} + \dot{\varphi}\cos\theta = \frac{H_G \cos\theta}{I}$$

$$\dot{\psi} = \frac{H_G \cos\theta}{I} - \dot{\varphi}\cos\theta = \frac{H_G \cos\theta}{I} - \frac{H_G \cos\theta}{I'} = \frac{H_G (I' - I)\cos\theta}{II'}$$

## PROBLEM 18.120

(*a*) Show that for an axisymmetrical body under no force, the rate of precession can be expressed as

$$\dot{\phi} = \frac{I\omega_z}{I'\cos\theta}$$

where $\omega_z$ is the rectangular component of $\boldsymbol{\omega}$ along the axis of symmetry of the body. (*b*) Use this result to check that the condition (18.44) for steady precession is satisfied by an axisymmetrical body under no force.

## SOLUTION

(*a*) Angular velocity of the body: $\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \omega_2\mathbf{k}$

Its angular momentum about *G*: $\mathbf{H}_G = -I'\dot{\varphi}\sin\theta\mathbf{i} + I\omega_2\mathbf{k}$

Let the reference frame $Gxyz$ be rotating with angular velocity $\boldsymbol{\Omega}$, where

$$\boldsymbol{\Omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k}$$

$$\begin{aligned}\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega}\times\mathbf{H}_G \\ &= 0 + (-\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{k})\times(-I'\dot{\varphi}\sin\theta\mathbf{i} + I\omega_z\mathbf{k}) \\ &= I\dot{\varphi}\omega_z\sin\theta - I'\dot{\varphi}^2\sin\theta\cos\theta\end{aligned}$$

For no force, $\dot{\mathbf{H}}_G = 0$

Hence, $(I\omega_z - I'\dot{\varphi}\cos\theta)\dot{\varphi}\sin\theta = 0$ or $I\omega_z - I'\dot{\varphi}\cos\theta = 0$ (1)

Solving for $\dot{\varphi}$, $\dot{\varphi} = \dfrac{I\omega_z}{I'\cos\theta}$

(*b*) Comparing Equation (1) with Equation (18.44) yields $\Sigma\mathbf{M}_0 = 0$, which is the condition for no force.

## PROBLEM 18.121

Show that the angular velocity vector $\boldsymbol{\omega}$ of an axisymmetrical body under no force is observed from the body itself to rotate about the axis of symmetry at the constant rate

$$n = \frac{I' - I}{I'}\omega_z$$

where $\omega_z$ is the rectangular component of $\boldsymbol{\omega}$ along the axis of symmetry of the body.

## SOLUTION

Angular velocity of the body: $\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\mathbf{i} + (\dot{\varphi}\cos\theta + \dot{\psi})\mathbf{k}$

Let the reference frame $Gxyz$ be rotating with angular velocity $\boldsymbol{\Omega}$, where

$$\boldsymbol{\Omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{j}$$

Angular acceleration of the body: $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}_{Gxyz} + \boldsymbol{\Omega}\times\boldsymbol{\omega}$

$$\boldsymbol{\alpha} = 0 + \dot{\varphi}\dot{\psi}\sin\theta\mathbf{j}$$

The rate of change of angular velocity as observed from the body is $-\boldsymbol{\alpha}$.

Assume that $-\boldsymbol{\alpha}$ may be represented as the angular velocity vector rotating with angular velocity $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$.

$$\begin{aligned} -\boldsymbol{\alpha} &= (l\mathbf{i} + m\mathbf{j} + n\mathbf{k})\times(\omega_x\mathbf{i} + \omega_z\mathbf{k}) \\ &= m\omega_z\mathbf{i} + (n\omega_x - l\omega_z)\mathbf{j} - m\omega_x\mathbf{k} \end{aligned}$$

Matching components: $\mathbf{i}$: $0 = m\omega_z \quad m = 0$

$$\mathbf{j}: \quad -\dot{\varphi}\dot{\psi}\sin\theta = n\omega_x - l\omega_z \tag{1}$$

$$\mathbf{k}: \quad 0 = -m\omega_x \quad m = 0$$

From Equation (1), $-\dot{\varphi}\dot{\psi}\sin\theta = -n\dot{\varphi}\sin\theta - l(\dot{\varphi}\cos\theta + \dot{\psi})$

from which $l = 0$ and $n = \dot{\psi}$.

Using Equation (18.44) with $\Sigma\mathbf{M}_0 = 0$ yields $I\omega_z - I'\dot{\varphi}\cos\theta = 0$

or $\dot{\varphi}\cos\theta = \dfrac{I\omega_z}{I'}$

But $\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta = \dot{\psi} + \dfrac{I\omega_z}{I'}$

Solving for $\dot{\psi}$, $\dot{\psi} = \dfrac{I' - I}{I}\omega_z$

Using $\omega_z = \omega_2$ and $n = \dot{\psi}$ yields $n = \dfrac{I' - I}{I'}\omega_z$. ◀

## PROBLEM 18.122

For an axisymmetrical body under no force, prove (*a*) that the rate of retrograde precession can never be less than twice the rate of spin of the body about its axis of symmetry, (*b*) that in Figure 18.24 the axis of symmetry of the body can never lie within the space cone.

## SOLUTION

For no force, $\mathbf{M}_G = 0$

(*a*) Using Equation (18.44), $0 = (\bar{I}\omega_z - \bar{I}'\dot{\varphi}\cos\theta)\dot{\varphi}\sin\theta$

$$\bar{I}\omega_z - \bar{I}'\dot{\varphi}\cos\theta = 0$$

Using $\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$ yields $\bar{I}\dot{\psi} - (\bar{I}' - I)\dot{\varphi}\cos\theta = 0$

$$\dot{\varphi} = \frac{\bar{I}\dot{\psi}}{(\bar{I}' - \bar{I})\cos\theta}$$

$$|\dot{\varphi}| = \frac{\bar{I}\dot{\psi}}{(\bar{I} - \bar{I}')}\sec\theta$$

For a flat disk, $\bar{I}' = \frac{1}{2}\bar{I}$

For any other shape, $\bar{I}' > \frac{1}{2}\bar{I}$

Hence, $\bar{I} - \bar{I}' < \frac{1}{2}I$ and $\frac{\bar{I}}{\bar{I} - \bar{I}'} > 2$

Also, $\sec\theta > 1$: $|\dot{\varphi}| > 2\dot{\psi}$

(*b*) $\tan\gamma = \frac{I}{I'}\tan\theta$ or $\tan\theta = \frac{I'}{I}\tan\gamma > \frac{1}{2}\tan\gamma$

Angle of surface of space cone is $\gamma - \theta$.

Angle of axis is $\theta$.

$$\tan\theta > \frac{1}{2}\tan[\theta + (\gamma - \theta)] = \frac{1}{2}\frac{\tan\theta + \tan(\gamma - \theta)}{1 + \tan\theta\tan(\gamma - \theta)}$$

$$[1 + \tan\theta\tan(\gamma - \theta)]\tan\theta > \frac{1}{2}\tan\theta + \frac{1}{2}\tan(\gamma - \theta)$$

## PROBLEM 18.122 (Continued)

$$\tan\theta + \tan^2\theta\tan(\gamma-\theta) > \frac{1}{2}\tan\theta + \frac{1}{2}\tan(\gamma-\theta)$$

$$\frac{1}{2}\tan\theta > \frac{1}{2}\tan(\gamma-\theta) - \tan^2\theta\tan(\gamma-\theta)$$

$$\frac{1}{2}\tan\theta > \frac{1}{2}\tan(\gamma-\theta)$$

$$\tan\theta > \tan(\gamma-\theta)$$

$$\theta > (\gamma-\theta)$$

The axis lies outside the space cone. ◀

## PROBLEM 18.123

Using the relation given in Problem 18.121, determine the period of precession of the north pole of the earth about the axis of symmetry of the earth. The earth may be approximated by an oblate spheroid of axial moment of inertia $I$ and of transverse moment of inertia $I' = 0.9967I$. (*Note:* Actual observations show a period of precession of the north pole of about 432.5 mean solar days; the difference between the observed and computed periods is due to the fact that the earth is not a perfectly rigid body. The free precession considered here should not be confused with the much slower precession of the equinoxes, which is a forced precession. See Problem 18.118.)

## SOLUTION

Angular velocity of the body: $\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\mathbf{i} + (\dot{\varphi}\cos\theta + \dot{\psi})\mathbf{k}$

Let the reference frame $Gxyz$ be rotating with angular velocity $\mathbf{\Omega}$, where

$$\mathbf{\Omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\varphi}\cos\theta\mathbf{j}$$

Angular acceleration of the body: $\boldsymbol{\alpha} = \dot{\omega}_{Gxyz} + \mathbf{\Omega}\times\boldsymbol{\omega}$

The rate of change of angular velocity as observed from the body is $-\boldsymbol{\alpha}$.

Assume that $-\alpha$ may be represented as the angular velocity vector rotating with angular velocity

$$l\mathbf{i} + m\mathbf{j} + n\mathbf{k}.$$

$$-\boldsymbol{\alpha} = (l\mathbf{i} + m\mathbf{j} + n\mathbf{k})\times(\omega_x\mathbf{i} + \omega_z\mathbf{k})$$
$$= -m\omega_z\mathbf{i} + (n\omega_x - l\omega_z\mathbf{j}) - m\omega_x\mathbf{k}$$

Matching components: $\mathbf{i}$: $0 = m\omega_z \qquad m = 0$

$\mathbf{j}$: $-\dot{\varphi}\dot{\psi}\sin\theta = n\omega_x - l\omega_z$ (1)

$\mathbf{k}$: $0 = -m\omega_x \qquad m = 0$

From Equation (1), $-\dot{\varphi}\dot{\psi}\sin\theta = -n\dot{\varphi}\sin\theta - l(\dot{\varphi}\cos\theta + \psi)$

From which $l = 0$ and $n = \dot{\psi}$.

Using Equation (18.44) with $\Sigma\mathbf{M}_0 = 0$ yields $I\omega_z - I'\dot{\varphi}\cos\theta = 0$

or $$\dot{\varphi}\cos\theta = \frac{I\omega_z}{I'}$$

But $$\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta = \dot{\psi} + \frac{I\omega_z}{I'}$$

Solving for $\dot{\psi}$, $$\dot{\psi} = \frac{I' - I}{I}\omega_z$$

## PROBLEM 18.123 (Continued)

Using $\omega_z = \omega_2$ and $n = \dot{\psi}$ yields $n = \dfrac{I' - I}{I'}\omega_2$

Data for Earth: $I' = 0.9967I$

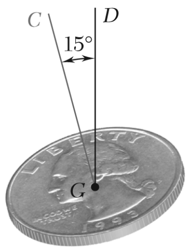
$$I' - I = -0.0033I,$$

$$n = -\frac{0.0033}{0.9967}\omega_2 = -0.003311\omega_2$$

$$\text{period} = \frac{2\pi}{|n|} = \frac{1}{0.003311}\frac{2\pi}{\omega_2} = 302.03\frac{2\pi}{\omega_2}$$

$$= (302.03)(24\text{ h}) = 7248\text{ h}$$

period = 302 days ◀




## PROBLEM 18.124

A coin is tossed into the air. It is observed to spin at the rate of 600 rpm about an axis *GC* perpendicular to the coin and to precess about the vertical direction *GD*. Knowing that *GC* forms an angle of 15° with *GD*, determine (*a*) the angle that the angular velocity $\boldsymbol{\omega}$ of the coin forms with *GD*, (*b*) the rate of precession of the coin about *GD*.

## SOLUTION

Moments of inertia: $I = \frac{1}{2}mr^2$

$I' = \frac{1}{4}mr^2$

Euler angle $\theta$ for steady precession: $\theta = 15°$

For axisymmetric body under no force, Equation (18.49) gives for the body cone angle:

$$\tan\gamma = \frac{I}{I'}\tan\theta = 2\tan 15° \qquad \gamma = 28.187°$$

(*a*) Angular velocity: $\boldsymbol{\omega} = \omega(-i\sin\gamma + \mathbf{k}\cos\gamma)$

Its projection onto the vertical direction is

$$\omega\cos\beta = \boldsymbol{\omega}\cdot\mathbf{k} = \omega(-i\sin\gamma + \mathbf{k}\cos\gamma)\cdot(-\sin\theta\mathbf{i} + \cos\theta\mathbf{k})$$
$$= \omega(\sin\gamma\sin\theta + \cos\gamma\cos\theta) = \omega\cos(\gamma - \theta)$$

$$\cos\beta = \cos(\gamma - \theta) \qquad \beta = |\gamma - \theta| = 28.187° - 15° = 13.187°$$

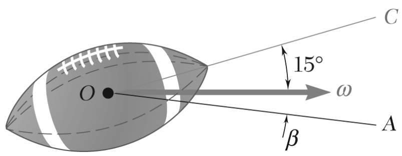
Angle between $\boldsymbol{\omega}$ and vertical direction *GD*: $\beta = 13.19°$ ◀

(*b*)
$$\tan\gamma = -\frac{\omega_x}{\omega_z} = \frac{\dot{\varphi}\sin\theta}{\dot{\varphi}\cos\theta + \dot{\psi}}$$

from which
$$\dot{\varphi} = \frac{\dot{\psi}\sin\gamma}{\sin(\theta - \gamma)} = \frac{\dot{\psi}\sin 28.187°}{-\sin 13.187°} = -2.0705\dot{\psi}$$

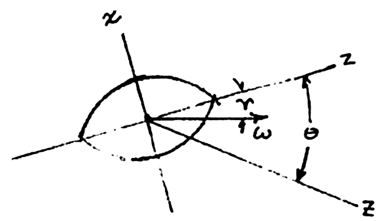
With $\dot{\psi} = 600$ rpm, $\dot{\varphi} = (-2.0705)(600)$ $\qquad |\dot{\varphi}| = 1242$ rpm (retrograde) ◀

## PROBLEM 18.125




The angular velocity vector of a football which has just been kicked is horizontal, and its axis of symmetry $OC$ is oriented as shown. Knowing that the magnitude of the angular velocity is 200 rpm and that the ratio of the axis and transverse moments of inertia is $I/I' = \frac{1}{3}$, determine (*a*) the orientation of the axis of precession $OA$, (*b*) the rates of precession and spin.

## SOLUTION




$$\tan\gamma = -\frac{\omega_x}{\omega_z}$$

$$\gamma = 15°$$

For steady precession with no force,

$$\tan\theta = \frac{I'}{I}\tan\gamma$$

$$= 3\tan 15°$$

$$\theta = 38.794°$$

(*a*) $\beta = \theta - \gamma = 38.794 - 15°$ $\quad\quad \beta = 23.8°$ ◀

(*b*) $\omega_x = -\dot{\varphi}\sin\theta = -\omega\sin\gamma$

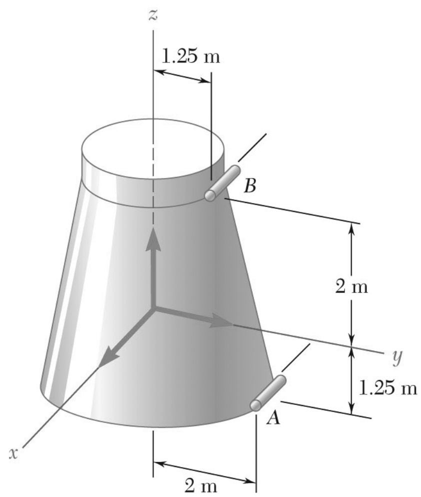
$$\dot{\varphi} = \frac{\omega\sin\gamma}{\sin\theta} = \frac{(200\text{ rpm})\sin 15°}{\sin(38.794°)}$$

$= 82.621$ rpm $\quad\quad$ *precession*: $\dot{\varphi} = 82.6$ rpm ◀

$$\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta = \omega\cos\gamma$$

$$\dot{\psi} = \omega\cos\gamma - \dot{\varphi}\cos\theta$$

$= 200\cos 15° - 82.621\cos 38.794°$ $\quad\quad$ *spin*: $\dot{\psi} = 128.8$ rpm ◀




## PROBLEM 18.126

The space capsule has no angular velocity when the jet at $A$ is activated for 1 s in a direction parallel to the $x$ axis. Knowing that the capsule has a mass of 1000 kg, that its radii of gyration are $\bar{k}_z = \bar{k}_y = 1.00$ m and $\bar{k}_z = 1.25$ m, and that the jet at $A$ produces a thrust of 50 N, determine the axis of precession and the rates of precession and spin after the jet has stopped.

## SOLUTION

Initial angular momentum about the mass center:

$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} = 0$$

Applied impulse at $A$: $\mathbf{A}\,\Delta t = (50\text{ N})(1\text{ s})\mathbf{i} = (50\text{ N}\cdot\text{s})\mathbf{i}$

Its moment about the mass center $G$:

$$\begin{aligned}\mathbf{r}_{A/G} \times \mathbf{A}\,\Delta t &= [(2\text{ m})\mathbf{j} - (1.25\text{ m})\mathbf{k}] \times (50\text{ N}\cdot\text{s})\mathbf{i} \\ &= -(100\text{ N}\cdot\text{m}\cdot\text{s})\mathbf{k} - (62.5\text{ N}\cdot\text{m}\cdot\text{s})\mathbf{j}\end{aligned}$$

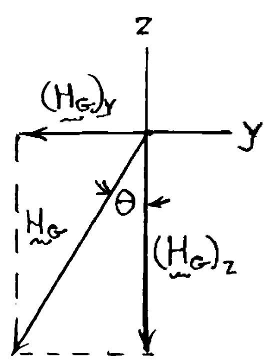
Principle of impulse and momentum. (Moments about $G$)

$$(\mathbf{H}_G)_0 = \mathbf{r}_{A/G} \times \mathbf{A}\,\Delta t = \mathbf{H}_G$$

where $\mathbf{H}_G$ is the final angular momentum about $G$.

$$0 + \mathbf{r}_{G/A} \times \mathbf{A}\,\Delta t = (H_G)_x \mathbf{i} + (H_G)_y \mathbf{j} + (H_G)_z \mathbf{k}$$

Angular momentum vector components:




**i**: $\quad 0 = (H_G)_x$

**j**: $\quad -62.5\text{ N}\cdot\text{m}\cdot\text{s} = (H_G)_y = -62.5\text{ kg}\cdot\text{m}^2/\text{s}$

**k**: $\quad -100\text{ N}\cdot\text{m}\cdot\text{s} = (H_G)_z = -100\text{ kg}\cdot\text{m}^2/\text{s}$

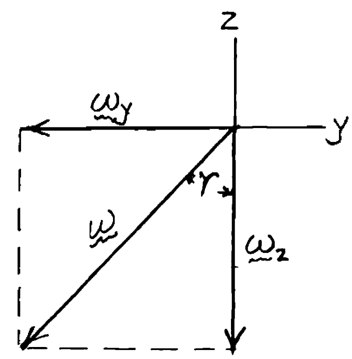
$$\tan\theta = \frac{(H_G)_y}{(H_G)_z} = 0.625$$

$\theta = 32.005°$ ◀

Moments of inertia:

$$\bar{I}_x = m\bar{k}_k^2 = (1000 \text{ kg})(1 \text{ m})^2 = 1000 \text{ kg}\cdot\text{m}^2$$
$$\bar{I}_y = m\bar{k}_y^2 = (1000 \text{ kg})(1 \text{ m})^2 = 1000 \text{ kg}\cdot\text{m}^2$$
$$\bar{I}_z = m\bar{k}_z^2 = (1000 \text{ kg})(1.25 \text{ m})^2 = 1562.5 \text{ kg}\cdot\text{m}^2$$

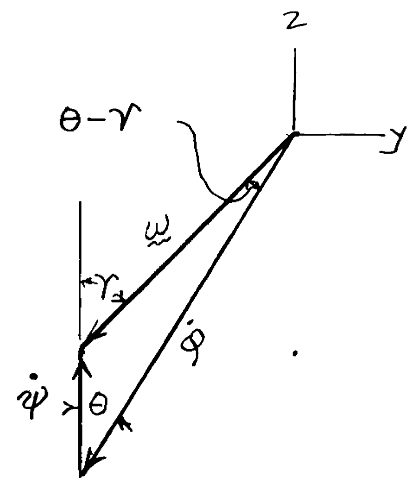
Angular velocity vector components:




$$\omega_x = \frac{(H_G)_x}{\bar{I}_x} = 0$$
$$\omega_y = \frac{(H_G)_y}{\bar{I}_y} = \frac{-62.5}{1000} = -0.0625 \text{ rad/s}$$
$$\omega_z = \frac{(H_G)_z}{\bar{I}_z} = \frac{-100}{1562.5} = -0.0640 \text{ rad/s}$$
$$\tan\gamma = \frac{\omega_y}{\omega_z} = 0.97656 \qquad \gamma = 44.321°$$
$$\omega = \sqrt{\omega_y^2 + \omega_z^2} = 0.089455 \text{ rad/s.}$$

Rates of precession and spin.

The angular velocity is resolved into a component $\dot{\varphi}$ (rate of precession) parallel to $\mathbf{H}_G$ and $\dot{\psi}$ (rate of spin) parallel to the axis of symmetry.




$$\gamma - \theta = 12.316°$$

Law of sines:

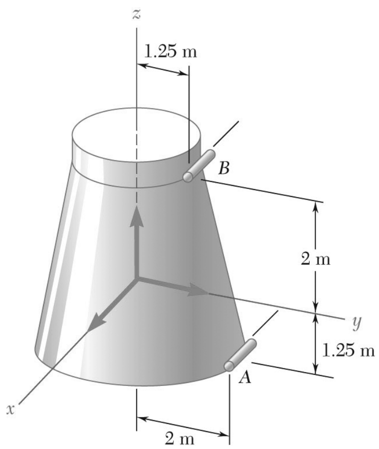
$$\frac{\dot{\varphi}}{\sin\gamma} = \frac{\dot{\psi}}{\sin(\gamma-\theta)} = \frac{\omega}{\sin\theta}$$
$$\dot{\varphi} = \frac{\omega\sin\gamma}{\sin\theta} = \frac{0.089455\sin 44.321°}{\sin 32.005°}$$
$$\dot{\psi} = \frac{\omega\sin(\gamma-\theta)}{\sin\theta} = \frac{0.089455\sin 12.316°}{\sin 32.005°}$$

*Rate of precession*: $\dot{\varphi} = 0.1179 \text{ rad/s}$ $\qquad \dot{\varphi} = 1.126 \text{ rpm}$ ◀

*Rate of spin*: $\dot{\psi} = 0.0360 \text{ rad/s}$ $\qquad \dot{\psi} = 0.344 \text{ rpm}$ ◀

Since $\gamma > \theta$, the precession is <u>retrograde</u>.

## PROBLEM 18.127




The space capsule has an angular velocity $\omega = (0.02 \text{ rad/s})\mathbf{j} + (0.10 \text{ rad/s})\mathbf{k}$ when the jet at $B$ is activated for 1 s in a direction parallel to the $x$ axis. Knowing that the capsule has a mass of 1000 kg, that its radii of gyration are $\bar{k}_z = \bar{k}_y = 1.00 \text{ m}$ and $\bar{k}_z = 1.25 \text{ m}$, and that the jet at $B$ produces a thrust of 50 N, determine the axis of precession and the rates of precession and spin after the jet has stopped.

## SOLUTION

Moments of inertia:

$$\bar{I}_x = m\bar{k}_k^2 = (1000 \text{ kg})(1 \text{ m})^2 = 1000 \text{ kg}\cdot\text{m}^2$$

$$\bar{I}_y = m\bar{k}_y^2 = (1000 \text{ kg})(1 \text{ m})^2 = 1000 \text{ kg}\cdot\text{m}^2$$

$$\bar{I}_z = m\bar{k}_z^2 = (1000 \text{ kg})(1.25 \text{ m})^2 = 1562.5 \text{ kg}\cdot\text{m}^2$$

Initial angular velocity: $(\omega_x)_0 = 0$

$(\omega_y)_0 = 0.02 \text{ rad/s}$

$(\omega_z)_0 = 0.10 \text{ rad/s}$

Initial angular momentum about $G$:

$$\begin{aligned}(\mathbf{H}_G)_0 &= \bar{I}_x(\omega_x)_0\mathbf{i} + \bar{I}_y(\omega_y)_0\mathbf{j} + \bar{I}_z(\omega_z)_0\mathbf{k} \\ &= (1000)(0)\mathbf{i} + (1000)(0.02)\mathbf{j} + (1562.5)(0.10)\mathbf{k} \\ &= (20 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (156.25 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}\end{aligned}$$

Applied impulse at $B$: $\mathbf{B}\,\Delta t = (50 \text{ N})(1 \text{ s})\mathbf{i} = (50 \text{ N}\cdot\text{s})\mathbf{i}$

Its moment about the mass center $G$:

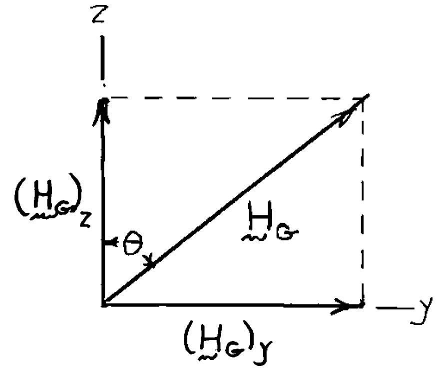
$$\begin{aligned}\mathbf{r}_{B/G} \times \mathbf{B}\,\Delta t &= [(1.25 \text{ m})\mathbf{j} + (2 \text{ m})\mathbf{k}] \times (50 \text{ N}\cdot\text{s})\mathbf{i} \\ &= (100 \text{ N}\cdot\text{m}\cdot\text{s})\mathbf{j} - (62.5 \text{ N}\cdot\text{m}\cdot\text{s})\mathbf{k}\end{aligned}$$

Principle of impulse and momentum. (Moments about $G$):

$$(\mathbf{H}_G)_0 + \mathbf{r}_{B/G} \times \mathbf{B}\,\Delta t = \mathbf{H}_G$$

where $\mathbf{H}_G$ is the final angular momentum about $G$.

Angular momentum vector components:




$$\mathbf{i}: \quad 0 + 0 = (H_G)_x = 0$$

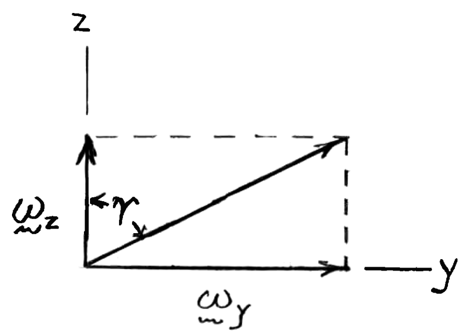
$$\mathbf{j}: \quad 20\ \text{kg}\cdot\text{m}^2/\text{s} + 100\ \text{N}\cdot\text{m}\cdot\text{s} = (H_G)_y = 120\ \text{kg}\cdot\text{m}^2/\text{s}$$

$$\mathbf{k}: \quad 156.25\ \text{kg}\cdot\text{m}^2/\text{s} - 62.5\ \text{N}\cdot\text{m}\cdot\text{s} = (H_G)_z = 93.75\ \text{kg}\cdot\text{m}^2/\text{s}$$

$$\tan\theta = \frac{(H_G)_y}{(H_G)_z} = 1.28$$

$\theta = 52.001°$ ◀

Angular velocity vector components:




$$\omega_x = \frac{(H_G)_x}{\bar{I}_x} = 0$$

$$\omega_y = \frac{(H_G)_y}{\bar{I}_y} = \frac{120}{1000} = 0.12\ \text{rad/s}$$

$$\omega_z = \frac{(H_G)_z}{\bar{I}_z} = \frac{93.75}{1562.5} = 0.060\ \text{rad/s}$$

$$\tan\gamma = \frac{\omega_y}{\omega_z} = 2.0000 \qquad \gamma = 63.435°$$

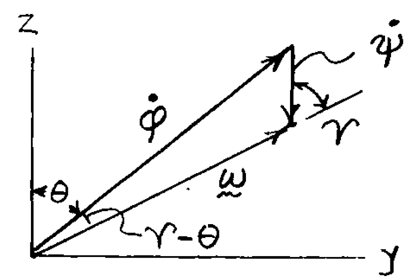
$$\omega = \sqrt{\omega_y^2 + \omega_z^2} = 0.134164\ \text{rad/s}$$

Rates of precession and spin.

The angular velocity is resolved into a component $\dot{\varphi}$ (rate of precession) parallel to $\mathbf{H}_G$ and $\dot{\psi}$ (rate of spin) parallel to the axis of symmetry.

$$\gamma - \theta = 11.434°$$

Law of sines:




$$\frac{\dot{\varphi}}{\sin\gamma} = \frac{\dot{\psi}}{\sin(\gamma-\theta)} = \frac{\omega}{\sin\theta}$$

$$\dot{\varphi} = \frac{\omega\sin\gamma}{\sin\theta} = \frac{0.134164\sin 63.435°}{\sin 52.001°}$$

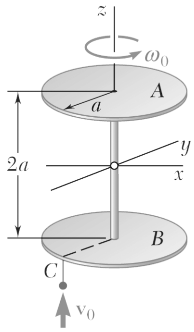
$$\dot{\psi} = \frac{\omega\sin(\gamma-\theta)}{\sin\theta} = \frac{0.134164\sin 11.434°}{\sin 52.001°}$$

*Rate of precession*: $\dot{\varphi} = 0.1523\ \text{rad/s}$ ◀

*Rate of spin*: $\dot{\psi} = 0.0338\ \text{rad/s}$ ◀

Since $\gamma > \theta$, the precession is retrograde.

## PROBLEM 18.128




Solve Sample Problem 18.6, assuming that the meteorite strikes the satellite at $C$ with a velocity $\mathbf{v}_0 = (2000\text{ m/s})\mathbf{i}$.

**PROBLEM 18.6** A space satellite of mass $m$ is known to be dynamically equivalent to two thin disks of equal mass. The disks are of radius $a$ = 800 mm and are rigidly connected by a light rod of length $2a$. Initially the satellite is spinning freely about its axis of symmetry at the rate $\omega_0 = 60$ rpm. A meteorite, of mass $m_0 = m/1000$ and traveling with a velocity $\mathbf{v}_0$ of 2000 m/s relative to the satellite, strikes the satellite and becomes embedded at $C$. Determine ($a$) the angular velocity of the satellite immediately after impact, ($b$) the precession axis of the ensuing motion, ($c$) the rates of precession and spin of the ensuing motion.

## SOLUTION

($a$) *Angular velocity after impact.*

From Sample Problem 18.6:

$$I = I_z = \frac{1}{2}ma^2 \quad I' = I_x = I_y = \frac{5}{4}ma^2$$

Conservation of angular momentum: Angular momentum after impact:

$$\begin{aligned}\mathbf{H}_G &= \mathbf{r}_C \times m_0\mathbf{v}_0 + I\omega_0\mathbf{k} \\ &= (-a\mathbf{j} - a\mathbf{k}) \times m_0 v_0 \mathbf{i} + I\omega_0\mathbf{k} \\ &= -am_0v_0\mathbf{j} + (I\omega_0 + am_0v_0)\mathbf{k}\end{aligned}$$

Data: $a = 800\text{ mm} = 0.8\text{ m}$

$$\omega_0 = 60\text{ rpm} = 2\pi\text{ rad/s}$$

$$v_0 = 2000\text{ m/s}, \quad m_0 = \frac{m}{1000}$$

$$\omega_x = \frac{(H_G)_x}{I_x} = 0$$

$$\omega_y = \frac{(H_G)_y}{I_y} = -\frac{am_0v_0}{\frac{5}{4}ma^2} = -\frac{4}{5}\frac{m_0}{m}\frac{v_0}{a} = -\frac{4}{5}\left(\frac{1}{1000}\right)\frac{2000}{0.8} = -2\text{ rad/s}$$

$$\omega_z = \frac{(H_G)_z}{I_z} = \omega_0 + \frac{am_0v_0}{\frac{1}{2}ma^2} = \omega_0 + 2\frac{m_0}{m}\frac{v_0}{a}$$

$$= 2\pi + 2\left(\frac{1}{1000}\right)\frac{2000}{0.8} = 11.2832\text{ rad/s}$$

$$\boldsymbol{\omega} = -(2.00\text{ rad/s})\mathbf{j} + (11.28\text{ rad/s})\mathbf{k}$$

$$\omega = \sqrt{(2)^2 + (11.2832)^2} = 11.4591\text{ rad/s} = 109.426\text{ rpm} \qquad \omega = 109.4\text{ rpm} ◀$$

$$\tan\gamma = \left|\frac{\omega_y}{\omega_z}\right| = \frac{2}{11.2832} \qquad \gamma = 10.0515° \qquad \gamma_x = 90°, \quad \gamma_y = 100.05°, \quad \gamma_z = 10.05° \blacktriangleleft$$

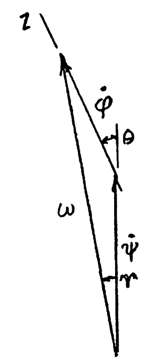
(*b*) *Precession axis.*

$$\tan\theta = \frac{I'}{I}\tan\gamma$$

$$= \left(\frac{5}{4}\right)(2)\frac{2}{11.2832}$$

$$\theta = 23.900° \qquad \theta_x = 90°, \quad \theta_y = 113.9°, \quad \theta_z = 23.9° \blacktriangleleft$$

(*c*) *Rates of precession and spin.*




$$\theta - \gamma = 13.8484°$$

Law of sines.

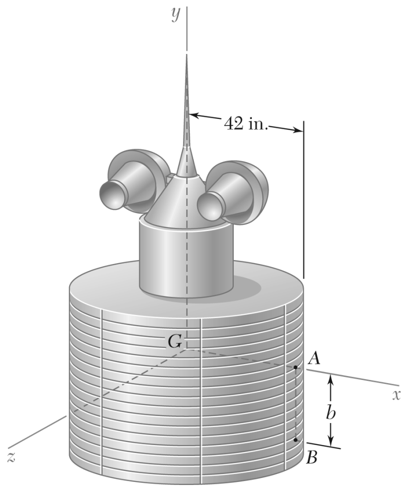
$$\frac{\dot{\varphi}}{\sin\gamma} = \frac{\dot{\psi}}{\sin(\theta-\gamma)} = \frac{\omega}{\sin\theta}$$

$$\dot{\varphi} = \frac{\omega\sin\gamma}{\sin\theta} = \frac{109.4\sin 10.05°}{\sin 23.9°}$$

*precession*: $\dot{\varphi} = 47.1$ rpm $\blacktriangleleft$

$$\dot{\psi} = \frac{\omega\sin(\theta-\gamma)}{\sin\theta} = \frac{109.4\sin 13.85°}{\sin 23.9°}$$

*spin*: $\dot{\psi} = 64.6$ rpm $\blacktriangleleft$




## PROBLEM 18.129

An 800-lb geostationary satellite is spinning with an angular velocity $\omega_0 = (1.5 \text{ rad/s})\mathbf{j}$ when it is hit at $B$ by a 6-oz meteorite traveling with a velocity $\mathbf{v}_0 = -(1600 \text{ ft/s})\mathbf{i} + (1300 \text{ ft/s})\mathbf{j} + (4000 \text{ ft/s})\mathbf{k}$ relative to the satellite. Knowing that $b = 20$ in. and that the radii of gyration of the satellite are $\bar{k}_x = \bar{k}_z = 28.8$ in. and $\bar{k}_y = 32.4$ in., determine the precession axis and the rates of precession and spin of the satellite after the impact.

## SOLUTION

Mass of satellite: $$m = \frac{W}{g} = \frac{800}{32.2} = 24.845 \text{ lb}\cdot\text{s}^2/\text{ft}$$

Principal moments of inertia: $$\bar{I}_x = m k_x^2 = (24.845)\left(\frac{28.8}{12}\right)^2$$
$$= 143.106 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$
$$\bar{I}_y = m k_y^2 = (24.845)\left(\frac{32.4}{12}\right)^2$$
$$= 181.118 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$
$$\bar{I}_z = \bar{I}_x = 143.106 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Mass of meteorite: $$m' = \frac{6}{(16)(32.2)} = 0.011649 \text{ lb}\cdot\text{s}^2/\text{ft}$$

Initial momentum of meteorite:
$$m'\mathbf{v}_0 = (0.011649)(-1600\mathbf{i} + 1300\mathbf{j} + 4000\mathbf{k})$$
$$= -(18.633 \text{ lb}\cdot\text{s})\mathbf{i} + (15.140 \text{ lb}\cdot\text{s})\mathbf{j} + (46.584 \text{ lb}\cdot\text{s})\mathbf{k}$$

Assume that the position of the mass center of the satellite plus the meteorite is essentially that of the satellite alone.

Position of Point $B$ relative to the mass center:
$$\mathbf{r}_{B/G} = \left(\frac{42}{12}\mathbf{i} - \frac{20}{12}\mathbf{j}\right)$$
$$= (3.5 \text{ ft})\mathbf{i} - (1.66667 \text{ ft})\mathbf{j}$$

Angular velocity of satellite before impact:
$$\boldsymbol{\omega}_0 = (1.5 \text{ rad/s})\mathbf{j} \qquad (\omega_0)_x = (\omega_0)_z = 0, \quad (\omega_0)_y = 1.5 \text{ rad/s}$$

Angular momentum of satellite-meteorite system before impact:

$$(\mathbf{H}_G)_0 = \bar{I}_y \omega_0 \mathbf{j} + \mathbf{r}_{B/G} \times (m'\mathbf{v}_0)$$

$$= (181.118)(1.5)\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3.5 & -1.66667 & 0 \\ -18.633 & 15.140 & 46.584 \end{vmatrix}$$

$$= -(77.64 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{i} + (108.637 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{j} + (21.935 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{k}$$

*Principle of impulse and momentum for satellite-meteorite system.* Moments about $G$:

$$(\mathbf{H}_G)_1 = (\mathbf{H}_G)_0 = \mathbf{H}_G$$

*Angular velocity immediately after impact.*

$$\boldsymbol{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$$

Neglect the mass of the meteorite.

$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k}$$

$$\omega_x = \frac{(H_G)_x}{\bar{I}_x} = \frac{-77.64}{143.106} = -0.54253 \text{ rad/s}$$

$$\omega_y = \frac{(H_G)_y}{\bar{I}_y} = \frac{108.637}{181.118} = 0.59981 \text{ rad/s}$$

$$\omega_z = \frac{(H_G)_z}{\bar{I}_z} = \frac{21.934}{143.106} = 0.15327 \text{ rad/s}$$

$$\boldsymbol{\omega} = -(0.54253 \text{ rad/s})\mathbf{i} + (0.59981 \text{ rad/s})\mathbf{j} + (0.15327 \text{ rad/s})\mathbf{k}$$

$$\omega = \sqrt{(0.54253)^2 + (0.59981)^2 + (0.15327)^2} = 0.82317 \text{ rad/s}$$

$$H_G = \sqrt{(77.64)^2 + (108.637)^2 + (21.935)^2} = 135.319 \text{ lb}\cdot\text{s}\cdot\text{ft}$$

*Motion after impact.* Since the moments of inertia $\bar{I}_x$ and $\bar{I}_z$ are equal, the body moves as an axisymmetrical body with the $y$ axis as the symmetry axis.

Moment of inertia about the symmetry axis: $\quad I = \bar{I}_y = 181.118 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$

Moment of inertia about a transverse axis through $G$: $\quad I' = \bar{I}_x = \bar{I}_z = 143.106 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$

The motion is a steady precession $\dot{\phi}$ about the precession axis together with a steady spin $\dot{\psi}$ about the spin or symmetry axis. Since $I > I'$, the precession is retrograde.

*Precession axis*. The precession axis is directed along the angular momentum vector $\mathbf{H}_G$, which remains fixed. Immediately after impact, its direction cosines relative to the body axes $x$, $y$, $z$ are:

$$\cos\theta_x = \frac{(H_G)_x}{H_G} = \frac{-77.64}{135.319} = -0.57376 \qquad \theta_x = 125.0° ◀$$

$$\cos\theta_y = \frac{(H_G)_y}{H_G} = \frac{108.637}{135.319} = 0.80282 \qquad \theta_y = 36.6° ◀$$

$$\cos\theta_z = \frac{(H_G)_z}{H_G} = \frac{21.935}{135.319} = 0.16210 \qquad \theta_z = 80.7° ◀$$

The angle $\theta$ between the spin axis ($y$ axis) and the precession axis remains constant.

$$\theta = \theta_y = 36.600°$$

The angle $\gamma$ between the angular velocity vector and the spin axis ($y$ axis) is

$$\cos\gamma = \frac{\omega_y}{\omega} = \frac{0.59981}{0.82317} \qquad \gamma = 43.226°$$

The angle $\gamma$ could also have been calculated from

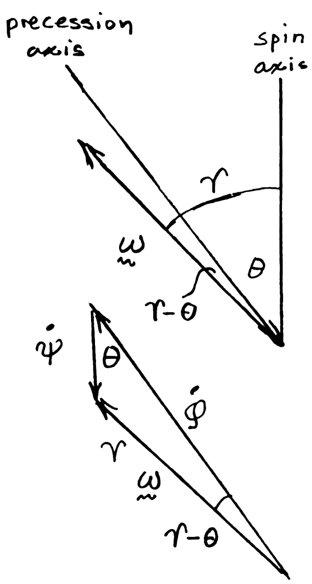
$$\tan\gamma = \frac{I}{I'}\tan\beta = \frac{181.118}{143.106}\tan 36.600°$$

The angle between the precession axis and the angular velocity vector is

$$\gamma - \theta = 6.626°$$

*Rates of precession and spin.*

Set up the triangle of vector addition for the components of angular velocity. Apply the law of sines.



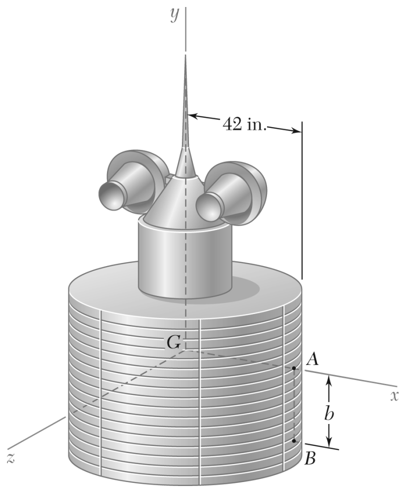

$$\frac{\omega}{\sin\theta} = \frac{\dot\psi}{\sin(\gamma-\theta)} = \frac{\dot\varphi}{\sin\gamma}$$

$$\frac{0.82317}{\sin 36.600°} = \frac{\dot\psi}{\sin 6.626°} = \frac{\dot\varphi}{\sin 43.226°}$$

Rate of precession: $\dot\varphi = 0.946$ rad/s ◀

Rate of spin: $\dot\psi = 0.1593$ rad/s ◀

Since $\gamma > \theta$, precession is retrograde. ◀




## PROBLEM 18.130

Solve Problem 18.129, assuming that the meteorite hits the satellite at $A$ instead of $B$.

**PROBLEM 18.129** An 800-lb geostationary satellite is spinning with an angular velocity $\omega_0 = (1.5 \text{ rad/s})\mathbf{j}$ when it is hit at $B$ by a 6-oz meteorite traveling with a velocity $\mathbf{v}_0 = -(1600 \text{ ft/s})\mathbf{i} + (1300 \text{ ft/s})\mathbf{j} + (4000 \text{ ft/s})\mathbf{k}$ relative to the satellite. Knowing that $b = 20$ in. and that the radii of gyration of the satellite are $\bar{k}_x = \bar{k}_z = 28.8$ in. and $\bar{k}_y = 32.4$ in., determine the precession axis and the rates of precession and spin of the satellite after the impact.

## SOLUTION

Mass of satellite: $$m = \frac{W}{g} = \frac{800}{32.2} = 24.845 \text{ lb}\cdot\text{s}^2/\text{ft}$$

Principal moments of inertia: $$\bar{I}_x = m k_x^2 = (24.845)\left(\frac{28.8}{12}\right)^2 = 143.106 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_y = m k_y^2 = (24.845)\left(\frac{32.4}{12}\right)^2 = 181.118 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

$$\bar{I}_z = \bar{I}_x = 143.106 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Mass of meteorite: $$m' = \frac{6}{(16)(32.2)} = 0.011649 \text{ lb}\cdot\text{s}^2/\text{ft}$$

Initial momentum of meteorite:

$$m'\mathbf{v}_0 = (0.011649)(-1600\mathbf{i} + 1300\mathbf{j} + 4000\mathbf{k})$$
$$= -(18.633 \text{ lb}\cdot\text{s})\mathbf{i} + (15.140 \text{ lb}\cdot\text{s})\mathbf{j} + (46.584 \text{ lb}\cdot\text{s})\mathbf{k}$$

Assume that the position of the mass center of the satellite plus the meteorite is essentially that of the satellite alone.

Position of Point $A$ relative to the mass center:

$$\mathbf{r}_{A/G} = \frac{42}{12}\mathbf{i} = (3.5 \text{ ft})\mathbf{i}$$

Angular velocity of satellite before impact:

$$\boldsymbol{\omega}_0 = (1.5 \text{ rad/s})\mathbf{j}, \quad (\omega_0)_x = (\omega_0)_z = 0, \quad (\omega_0)_y = 1.5 \text{ rad/s}$$

Angular momentum of satellite-meteorite system before impact:

$$(\mathbf{H}_G)_0 = \bar{I}_y\omega_0\mathbf{j} + \mathbf{r}_{A/G} \times (m'\mathbf{v}_0)$$

$$= (181.118)(1.5)\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3.5 & 0 & 0 \\ -18.633 & 15.140 & 46.584 \end{vmatrix}$$

$$= (108.637 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{j} + (52.99 \text{ lb}\cdot\text{s}\cdot\text{ft})\mathbf{k}$$

*Principle of impulse and momentum for satellite-meteorite system.* Moments about $G$:

$$(\mathbf{H}_G)_1 = (\mathbf{H}_G)_0 = \mathbf{H}_G$$

*Angular velocity immediately after impact.*

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

Neglect the mass of the meteorite.

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$

$$\omega_x = \frac{(H_G)_x}{\bar{I}_x} = 0$$

$$\omega_y = \frac{(H_G)_y}{\bar{I}_y} = \frac{108.637}{181.118} = 0.59981 \text{ rad/s}$$

$$\omega_z = \frac{(H_G)_z}{\bar{I}_z} = \frac{52.99}{143.106} = 0.37028 \text{ rad/s}$$

$$\boldsymbol{\omega} = (0.59981 \text{ rad/s})\mathbf{j} + (0.37028 \text{ rad/s})\mathbf{k}$$

$$\omega = \sqrt{(0.59981)^2 + (0.37028)^2} = 0.70490 \text{ rad/s}$$

$$H_G = \sqrt{(108.637)^2 + (52.99)^2} = 120.872 \text{ lb}\cdot\text{s}\cdot\text{ft}$$

*Motion after impact*. Since the moments of inertia $\bar{I}_x$ and $\bar{I}_z$ are equal, the body moves as an axisymmetrical body with the $y$ axis as the symmetry axis.

Moment of inertia about the symmetry axis:

$$I = \bar{I}_y = 181.118 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

Moment of inertia about a transverse axis through $G$:

$$I' = \bar{I}_x = \bar{I}_z = 143.106 \text{ lb}\cdot\text{s}^2\cdot\text{ft}$$

The motion is a steady precession $\dot{\phi}$ about the precession axis together with a steady spin $\dot{\psi}$ about the spin or symmetry axis. Since $I > I'$, the precession is retrograde.

*Precession axis*. The precession axis is directed along the angular momentum vector $\mathbf{H}_G$, which remains fixed. Immediately after impact, its direction cosines relative to the body axes $x$, $y$, $z$ are:

$$\cos\theta_x = \frac{(H_G)_x}{H_G} = 0 \qquad \theta_x = 90.0° \blacktriangleleft$$

$$\cos\theta_y = \frac{(H_G)_y}{H_G} = \frac{108.637}{120.872} = 0.89878 \qquad \theta_y = 26.0° \blacktriangleleft$$

$$\cos\theta_z = \frac{(H_G)_z}{H_G} = \frac{52.99}{120.872} = 0.43840 \qquad \theta_z = 64.0° \blacktriangleleft$$

The angle $\theta$ between the spin axis ($y$ axis) and the precession axis remains constant.

$$\theta = \theta_y = 26.002°$$

The angle $\gamma$ between the angular velocity vector and the spin axis ($y$ axis) is

$$\cos\gamma = \frac{\omega_y}{\omega} = \frac{0.59981}{0.70490} \qquad \gamma = 31.689°$$

The angle $\gamma$ could also have been calculated from

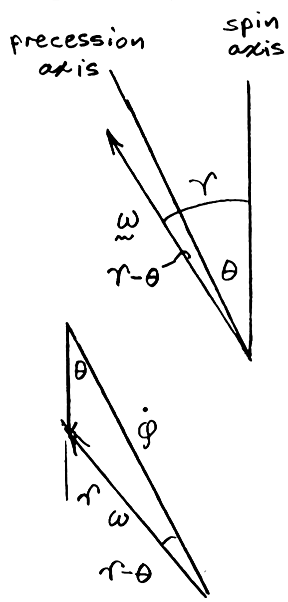
$$\tan\gamma = \frac{I}{I'}\tan\theta = \frac{181.118}{143.106}\tan 26.002°$$

The angle between the precession axis and the angular velocity vector is

$$\gamma - \theta = 5.687°$$

*Rates of precession and spin.*

Set up the triangle of vector addition for the components of angular velocity. Apply the law of sines.



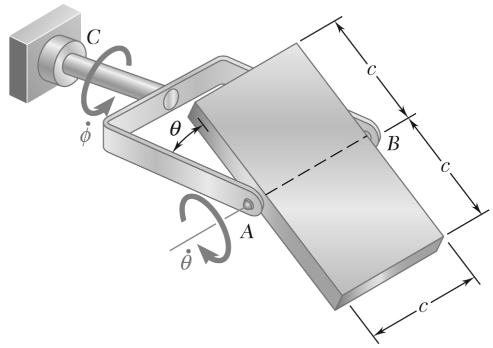

$$\frac{\omega}{\sin\theta} = \frac{\dot{\psi}}{\sin(\gamma-\theta)} = \frac{\dot{\varphi}}{\sin\gamma}$$

$$\frac{0.70490}{\sin 26.002°} = \frac{\dot{\psi}}{\sin 5.687°} = \frac{\dot{\varphi}}{\sin 31.689°}$$

Rate of precession: $\dot{\varphi} = 0.844$ rad/s ◀

Rate of spin: $\dot{\psi} = 0.1593$ rad/s ◀

Since $\gamma > \theta$, the precession is retrograde. ◀




## PROBLEM 18.131

A homogeneous rectangular plate of mass $m$ and sides $c$ and $2c$ is held at $A$ and $B$ by a fork-ended shaft of negligible mass, which is supported by a bearing at $C$. The plate is free to rotate about $AB$, and the shaft is free to rotate about a horizontal axis through $C$. Knowing that, initially, $\theta_0 = 40°$, $\dot{\theta}_0 = 0$, and $\dot{\phi}_0 = 10$ rad/s, determine for the ensuring motion ($a$) the range of values of $\theta$, ($b$) the minimum value of $\dot{\phi}$, ($c$) the maximum value of $\dot{\theta}$.

## SOLUTION

Let the fixed $Z$ axis lie along the axle of the fork-ended shaft. Let the axes $Gxyz$ be attached at the mass center with $x$ perpendicular to the plate, $y$ along the axle $AB$ and $z$ parallel to the long edges of the plate.

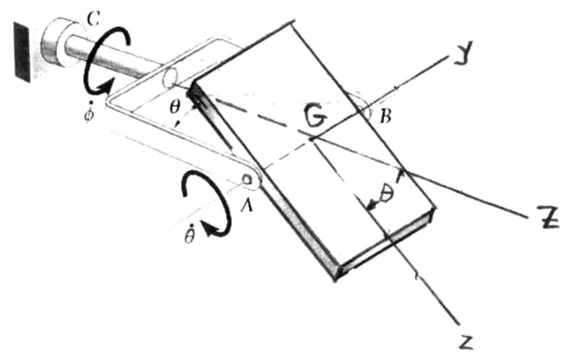
Angular velocity vector: $\dot{\boldsymbol{\omega}} = \dot{\phi}\mathbf{k} + \dot{\theta}\mathbf{j}$

$$= \dot{\phi}\cos\theta\mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\phi}\sin\theta\mathbf{k}$$

<u>Conservation of angular momentum</u>.

Since plate is free to rotate about $Z$ axis,

$$H_Z = \text{constant} \tag{1}$$




But

$$H_Z = H_x\cos\theta + H_z\sin\theta$$

$$H_Z = I_x\omega_x\cos\theta + I_z\omega_z\sin\theta$$

$$= \frac{1}{12}mc^2\dot{\phi}\cos^2\theta + \frac{5}{12}mc^2\dot{\phi}\sin^2\theta$$

$$= \frac{1}{12}mc^2\dot{\phi}(\cos^2\theta + 5\sin^2\theta)$$

$$= \frac{1}{12}mc^2\dot{\phi}(1 + 4\sin^2\theta)$$

Using the initial conditions, Eq. (1) yields

$$\dot{\phi}(1+4\sin^2\theta) = \dot{\phi}_0(1+4\sin^2\theta_0) \tag{2}$$

<u>Conservation of energy</u>.

Since no work is done, we have $T = \text{constant}$ (3)

where

$$T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

$$T = \frac{1}{2}\left(\frac{1}{12}mc^2\dot{\phi}^2\cos^2\theta + \frac{1}{3}mc^2\dot{\theta}^2 + \frac{5}{12}mc^2\dot{\phi}^2\sin^2\theta\right)$$

$$= \frac{1}{24}mc^2[4\dot{\theta}^2 + \dot{\phi}^2(\cos^2\theta + 5\sin^2\theta)]$$

$$= \frac{1}{24}mc^2[4\dot{\theta}^2 + \dot{\phi}^2(1+4\sin^2\theta)]$$

## PROBLEM 18.131 (Continued)

Using the initial conditions, including $\dot{\theta}_0 = 0$, Eq. (3) yields

$$4\dot{\theta}^2 + \dot{\phi}^2(1+4\sin^2\theta) = \dot{\phi}_0^2(1+4\sin^2\theta_0) \tag{4}$$

(*a*) With $\theta_0 = 40°$ and $\dot{\phi}_0 = 10$ rad/s in. Eqs. (2) and (4),

$$\dot{\phi}(1+4\sin^2\theta) = 26.527$$

$$\dot{\phi} = \frac{26.527}{1+4\sin^2\theta} \tag{2$'$}$$

$$4\dot{\theta}^2 + \dot{\phi}^2(1+4\sin^2\theta) = 265.27$$

Eliminate $\dot{\phi}$ and solve for $\dot{\theta}^2$:

$$4\dot{\theta}^2 = 265.27 - \frac{(26.527)^2}{1+4\sin^2\theta} \tag{5}$$

For $\quad \dot{\theta}^2 = 0, \quad 1+4\sin^2\theta = \dfrac{(26.527)^2}{265.27} = 2.6527$

$$\sin^2\theta = 0.4132 \qquad \sin\theta = 0.6428$$

From which $\quad \theta = 40°$ and $140°$ $\qquad 40° < \theta < 140°$ ◀

(*b*) From Eq. (2$'$), $\dot{\phi}$ is minimum for $\theta = 90°$.

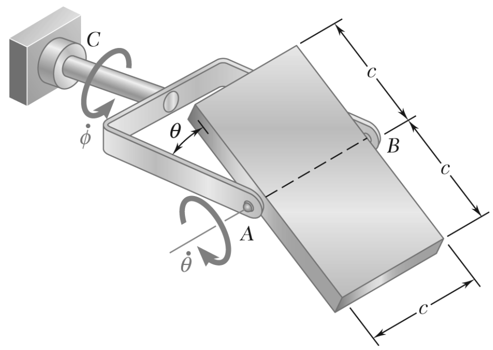
$\dot{\phi}_{\min} = 5.3054$ $\qquad \dot{\phi}_{\min} = 5.31$ rad/s ◀

(*c*) From Eq. (5),

$$\dot{\theta}^2 = 66.318 - \frac{175.92}{1+4\sin^2\theta}$$

$\dot{\theta}^2$ is maximum for $\theta = 90°$.

$\dot{\theta}_{\max}^2 = 31.134 \text{ rad}^2/\text{s}^2$ $\qquad \dot{\theta}_{\max} = 5.58$ rad/s ◀




## PROBLEM 18.132

A homogeneous rectangular plate of mass $m$ and sides $c$ and $2c$ is held at $A$ and $B$ by a fork-ended shaft of negligible mass which is supported by a bearing at $C$. The plate is free to rotate about $AB$, and the shaft is free to rotate about a horizontal axis through $C$. Initially the plate lies in the plane of the fork $(\theta_0 = 0)$ and the shaft has an angular velocity $\dot{\phi}_0 = 10$ rad/s. If the plate is slightly disturbed, determine for the ensuring motion $(a)$ the minimum value of $\dot{\phi}$, $(b)$ the maximum value of $\dot{\theta}$.

## SOLUTION

Let the fixed $Z$ axis lie along the axle of the fork-ended shaft. Let the axes $Gxyz$ be attached at the mass center with $x$ perpendicular to the plate, $y$ along the axle $AB$ and $z$ parallel to the long edges of the plate.

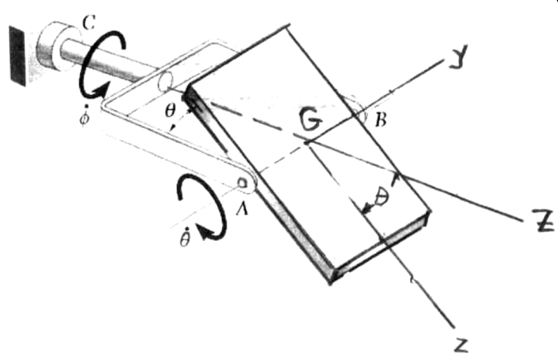
Angular velocity vector: $\quad \dot{\boldsymbol{\omega}} = \dot{\varphi}\mathbf{k} + \dot{\theta}\mathbf{j}$

$$= \dot{\varphi}\cos\theta\,\mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\varphi}\sin\theta\,\mathbf{k}$$

<u>Conservation of angular momentum</u>.

Since plate is free to rotate about $Z$ axis,

$$H_Z = \text{constant} \tag{1}$$




But $\quad H_Z = H_x \cos\theta + H_z \sin\theta$

$$H_Z = I_x\omega_x \cos\theta + I_z\omega_z \sin\theta$$

$$= \frac{1}{12}mc^2\dot{\phi}\cos^2\theta + \frac{5}{12}mc^2\dot{\phi}\sin^2\theta$$

$$= \frac{1}{12}mc^2\dot{\phi}(\cos^2\theta + 5\sin^2\theta)$$

$$= \frac{1}{12}mc^2\dot{\phi}(1 + 4\sin^2\theta)$$

Using the initial conditions, Eq. (1) yields

$$\dot{\phi}(1 + 4\sin^2\theta) = \dot{\phi}_0(1 + 4\sin^2\theta_0) \tag{2}$$

<u>Conservation of energy</u>.

Since no work is done, we have $T = \text{constant}$ (3)

where $\quad T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$

$$T = \frac{1}{2}\left(\frac{1}{12}mc^2\dot{\phi}^2\cos^2\theta + \frac{1}{3}mc^2\dot{\theta}^2 + \frac{5}{12}mc^2\dot{\phi}^2\sin^2\theta\right)$$

$$= \frac{1}{24}mc^2[4\dot{\theta}^2 + \dot{\phi}^2(\cos^2\theta + 5\sin^2\theta)]$$

$$= \frac{1}{24}mc^2[4\dot{\theta}^2 + \dot{\phi}^2(1 + 4\sin^2\theta)]$$

## PROBLEM 18.132 (Continued)

Using the initial conditions, including $\dot{\theta}_0 = 0$, Eq. (3) yields

$$4\dot{\theta}^2 + \dot{\phi}^2(1+4\sin^2\theta) = \dot{\phi}_0^2(1+4\sin^2\theta_0) \tag{4}$$

(*a*) With $\qquad \theta_0 = 0, \quad \dot{\phi}_0 = 10 \text{ rad/s}$

Eq. (2) yields $\qquad \dot{\phi} = \dfrac{10}{1+4\sin^2\theta}$

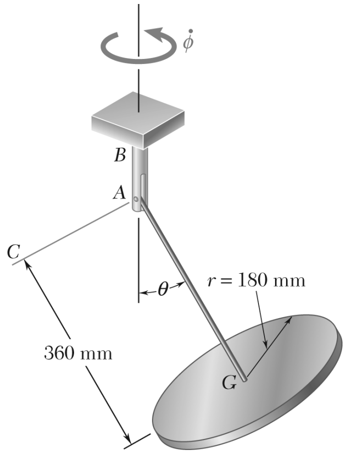
$\dot{\phi}$ is minimum for $\theta = 90°$: $\qquad \dot{\phi}_{\min} = 2.00 \text{ rad/s}$ ◀

(*b*) Eq. (4) yields $\qquad 4\dot{\theta}^2 = 100 - \dot{\phi}^2(1+4\sin^2\theta) = 100\left(1 - \dfrac{1}{1+4\sin^2\theta}\right)$

$\dot{\theta}^2$ is largest for $\theta = 90°$:

$$4\dot{\theta}_{\max}^2 = 100\left(1 - \frac{1}{5}\right)$$

$\dot{\theta}_{\max}^2 = 20 \qquad \dot{\theta}_{\max} = 4.47 \text{ rad/s}$ ◀



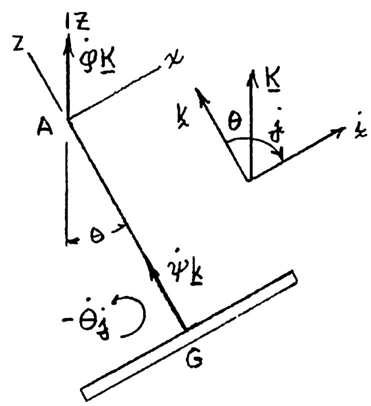

## PROBLEM 18.133

A homogeneous disk of radius 180 mm is welded to a rod *AG* of length 360 mm and of negligible mass which is connected by a clevis to a vertical shaft *AB*. The rod and disk can rotate freely about a horizontal axis *AC*, and shaft *AB* can rotate freely about a vertical axis. Initially, rod *AG* is horizontal $(\theta_0 = 90°)$ and has no angular velocity about *AC*. Knowing that the maximum value $\dot{\phi}_m$ of the angular velocity of shaft *AB* in the ensuing motion is twice its initial value $\dot{\phi}_0$, determine (*a*) the minimum value of $\theta$, (*b*) the initial angular velocity $\dot{\phi}_0$ of shaft *AB*.

## SOLUTION

Let the *Z* axis be vertical.

For principal axes *xyz* with origin at *A*, the principal moments of inertia are




$$I' = I_x = I_y$$

$$= m\left[\frac{1}{4}a^2 + (2a)^2\right] = \frac{17}{4}ma^2$$

$$I = I_z = \frac{1}{2}ma^2$$

Angular velocity components:

$$\omega_x = \dot{\varphi}\sin\theta$$

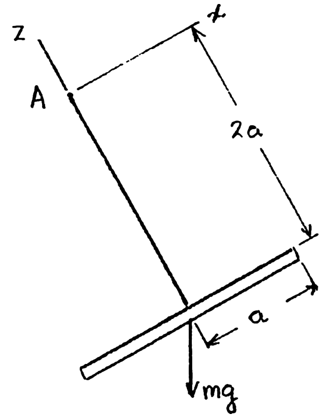
$$\omega_y = -\dot{\theta}$$

$$\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$$

Angular momentum about *A*:

$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= I'\dot{\varphi}\sin\theta\mathbf{i} - I'\dot{\theta}\mathbf{j} + I\omega_z\mathbf{k}$$




Kinetic energy:

$$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

$$T = \frac{1}{2}I'(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I\omega_z^2$$

Potential energy:

$$V = -2mga\cos\theta$$

Conservation of angular momentum about fixed *Z* axis:

$$\mathbf{H}_A \cdot \mathbf{K} = \mathbf{H}_A \cdot (\mathbf{i}\sin\theta + \mathbf{k}\cos\theta)$$

$$= I'\dot{\varphi}\sin^2\theta + I\omega_z\cos\theta$$

$$= \frac{17}{4}ma^2\dot{\varphi}\sin^2\theta + \frac{1}{2}ma^2\omega_z\cos\theta = \alpha \qquad (1)$$

where $\alpha$ is a constant.

## PROBLEM 18.133 (Continued)

Conservation of energy: $T + V = E$, where $E$ is a constant.

$$\frac{17}{8} ma^2 (\dot{\varphi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{4} ma^2 \omega_z^2 - 2mga \cos\theta = E \qquad (2)$$

Constraint of clevis: $\dot{\psi} = 0 \qquad \omega_z = \dot{\varphi}\cos\theta$

(*a*) From Eq. (1), $\frac{17}{4} ma^2 \dot{\varphi}_m \sin^2 \theta_m + \frac{1}{2} ma^2 \dot{\varphi}_m \cos^2 \theta_m = \frac{17}{4} ma^2 \dot{\varphi}_0 \sin^2 90° + \frac{1}{2} ma^2 \dot{\varphi}_0 \cos^2 90°$

$$\frac{17}{4} \sin^2 \theta_m + \frac{1}{2} \cos^2 \theta_m = \frac{17}{4} \frac{\dot{\varphi}_0}{\dot{\varphi}_m} = \left(\frac{17}{4}\right)\left(\frac{1}{2}\right) = \frac{17}{8}$$

$$\frac{17}{4} \sin^2 \theta_m + \frac{1}{2}(1 - \sin^2 \theta_m) = \frac{17}{8}$$

$$\sin^2 \theta_m = \frac{13}{30}$$

$$\sin\theta_m = 0.65828$$

$$\theta_m = 41.169° \qquad \theta_m = 41.2° \blacktriangleleft$$

(*b*) At the minimum value of $\theta$, $\dot{\theta} = 0$

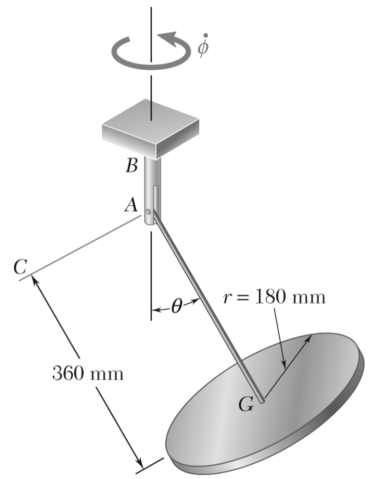
From Eq. (2), $\frac{17}{8} ma^2 (\dot{\varphi}_m^2 \sin^2 \theta_m + 0) + \frac{1}{4} ma^2 \dot{\varphi}_m^2 \cos^2 \theta_m - 2mga\cos\theta_m$

$$= \frac{17}{8} ma^2 (\dot{\varphi}_0^2 \sin^2 \theta_0 + 0) + \frac{1}{4} ma^2 \dot{\varphi}_0^2 \cos^2 90° - 2mga \cos 90°$$

$$\left[\left(\frac{17}{8}\right)(2)^2 \sin^2 \theta_m + \left(\frac{1}{4}\right)(2)^2 \cos^2 \theta_m - \frac{17}{8}\right] ma^2 \dot{\varphi}_0^2 = 2mga\cos\theta_m$$

$$2.1250\, ma^2 \dot{\varphi}_0^2 = 1.5055 mga$$

$$\dot{\varphi}_0^2 = 0.70849 \frac{g}{a} = 0.70849 \frac{9.81}{0.18} = 38.613 \qquad \dot{\varphi}_0 = 6.21 \text{ rad/s} \blacktriangleleft$$



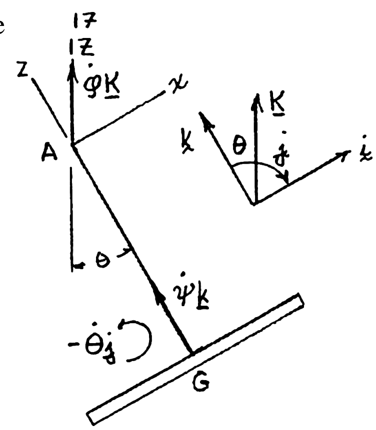

## PROBLEM 18.134

A homogeneous disk of radius 180 mm is welded to a rod *AG* of length 360 mm and of negligible mass which is connected by a clevis to a vertical shaft *AB*. The rod and disk can rotate freely about a horizontal, axis *AC*, and shaft *AB* can rotate freely about a vertical axis. Initially, rod *AG* is horizontal $(\theta_0 = 90°)$ and has no angular velocity about *AC*. Knowing that the smallest value of $\theta$ in the ensuing motion is 30°, determine (*a*) the initial angular velocity of shaft *AB*, (*b*) its maximum angular velocity.

## SOLUTION

Let the *Z* axis be vertical.

For principal axes *x*, *y*, *z* with origin at *A*, the principal moments of inertia are




$$I' = I_x = I_y$$

$$= m\left[\frac{1}{4}a^2 + (2a)^2\right] = \frac{17}{4}ma^2$$

$$I = I_z = \frac{1}{2}ma^2$$

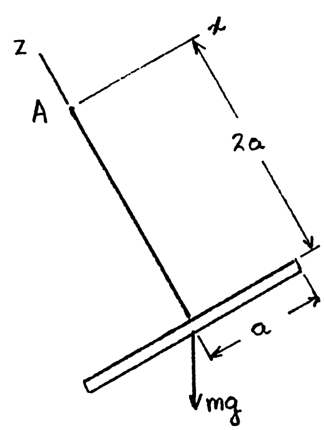
Angular velocity components: $\omega_x = \dot{\varphi}\sin\theta$

$$\omega_y = -\dot{\theta}$$

$$\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$$

Angular momentum about *A*: $\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$

$$= I'\dot{\varphi}\sin\theta\mathbf{i} - I'\dot{\theta}\mathbf{j} + I\omega_z\mathbf{k}$$




Kinetic energy: $T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$

$$T = \frac{1}{2}I'(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I\omega_z^2$$

Potential energy: $V = -2mga\cos\theta$

Conservation of angular momentum about fixed *Z* axis:

$$\mathbf{H}_A \cdot \mathbf{K} = \mathbf{H}_A \cdot (\mathbf{i}\sin\theta + \mathbf{k}\cos\theta)$$

$$= I'\dot{\varphi}\sin^2\theta + I\omega_z\cos\theta$$

$$= \frac{17}{4}ma^2\dot{\varphi}\sin^2\theta + \frac{1}{2}ma^2\omega_z\cos\theta = \alpha \qquad (1)$$

where $\alpha$ is a constant

## PROBLEM 18.134 (Continued)

Conservation of energy: $T + V = E$, where $E$ is a constant.

$$\frac{17}{8}ma^2(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{4}ma^2\omega_z^2 - 2mga\cos\theta = E \qquad (2)$$

Constraint of clevis: $\dot{\psi} = 0 \qquad \omega_z = \dot{\varphi}\cos\theta$

($a$) At $\theta = \theta_m = 30°$ and at $\theta = \theta_0 = 90°$, $\dot{\theta} = 0.$

From Eq. (1), $\frac{17}{4}ma^2\dot{\varphi}_m \sin^2\theta_m + \frac{1}{2}ma^2\dot{\varphi}_m \cos^2\theta_m = \frac{17}{4}ma^2\dot{\varphi}_0 \sin^2\theta_0 + 0$

$$\left[\left(\frac{17}{4}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2\right]ma^2\dot{\varphi}_m = \frac{17}{4}ma^2\dot{\varphi}_0$$

$$\dot{\varphi}_m = \frac{68}{23}\dot{\varphi}_0$$

From Eq. (2), $\frac{17}{8}ma^2\left(\frac{68}{23}\dot{\varphi}_0\right)^2 \sin^2 30° + \frac{1}{4}ma^2\left(\frac{68}{23}\dot{\varphi}_0\right)^2 \cos^2 30° - 2mga\cos 30°$

$$= \frac{17}{32}ma^2\dot{\varphi}_0^2 \sin^2 90° + \frac{1}{4}ma^2\dot{\varphi}_0^2 \cos^2 90° - 2mga\cos 90°$$

$$\left[\left(\frac{17}{32} + \frac{3}{16}\right)\left(\frac{68}{23}\right)^2 - \frac{17}{8}\right]ma^2\dot{\varphi}_0^2 = 2mga\cos 30°$$

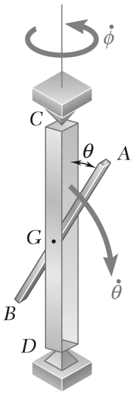
$$\dot{\varphi}_0^2 = 0.41660\frac{g}{a}$$

$$= 0.41660\ \frac{9.81}{0.18}$$

$$= 22.705$$

$$\dot{\varphi}_0 = 4.7649 \text{ rad/s} \qquad \dot{\varphi}_0 = 4.76 \text{ rad/s} \blacktriangleleft$$

($b$) $$\dot{\varphi}_m = \frac{68}{23}(4.7649) \qquad \dot{\varphi}_m = 14.09 \text{ rad/s} \blacktriangleleft$$



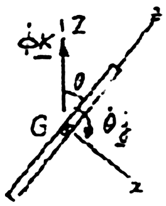

## PROBLEM 18.135

The slender homogeneous rod $AB$ of mass $m$ and length $L$ is free to rotate about a horizontal axle through its mass center $G$. The axle is supported by a frame of negligible mass which is free to rotate about the vertical $CD$. Knowing that, initially, $\theta = \theta_0$, $\dot{\theta} = 0$, and $\dot{\phi} = \dot{\phi}_0$, show that the rod will oscillate about the horizontal axle and determine (*a*) the range of values of angle $\theta$ during this motion, (*b*) the maximum value of $\dot{\theta}$, (*c*) the minimum value of $\dot{\phi}$.

## SOLUTION

Angular velocity.

Using the coordinate axes $x, y, z$ shown (with $y$ running *into* the paper), we have




$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\phi}\cos\theta\mathbf{k} \tag{1}$$

Moments of inertia.

For slender rod of length $L$ are mass $m$:

$$\bar{I}_x = \bar{I}_y = \frac{1}{12}mL^2$$

$$\bar{I}_z = 0 \tag{2}$$

Conservation of energy.

Since the $x$, $y$, $z$ axes are principal axis, we use

$$T = \frac{1}{2}\left(\bar{I}_x\omega_x^2 + \bar{I}_y\omega_y^2 + \bar{I}_z\omega_z^2\right)$$

$$= \frac{1}{2}\left(\frac{1}{12}mL^2\dot{\phi}^2\sin^2\theta + \frac{1}{12}mL^2\dot{\theta}^2 + 0\right)$$

$$T = \frac{1}{24}mL^2(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) \tag{3}$$

Using a datum through $G$, we have $V = 0$.

$$T + V = \text{constant}: \quad \frac{1}{24}mL^2(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) = \text{constant}$$

Recalling the initial conditions $\theta = \theta_0$, $\dot{\theta} = 0$, $\dot{\varphi} = \dot{\phi}_0$, we determine the constant and write

$$\dot{\phi}^2\sin^2\theta + \dot{\theta}^2 = \dot{\phi}_0^2\sin^2\theta_0 \tag{4}$$

Conservation of angular momentum.

Since the only forces exerted on the rod are its weight and the reaction at $G$, we have $\Sigma \mathbf{M}_G = 0$.

Using a fixed reference frame $GXYZ$, with $Z$ directed vertically upward, we have from Eq. (18.2)

$$\dot{\mathbf{H}}_G = \Sigma \mathbf{M}_G = 0$$

or, integrating with respect to the frame $GXYZ$,

$$\mathbf{H}_G = \text{constant}$$

Considering the vertical component of $\mathbf{H}_G$,

$$H_Z = \text{constant}$$

But

$$H_Z = H_z \cos\theta - H_x \sin\theta = \bar{I}_z \omega_z \cos\theta - \bar{I}_x \omega_x \sin\theta$$
$$= 0 - \frac{1}{12} mL^2 (-\dot{\phi} \sin\theta) \sin\theta$$

Thus,

$$H_Z = \frac{1}{12} m L^2 \dot{\phi} \sin^2\theta = \text{constant}$$

Recalling the initial conditions, we obtain

$$\dot{\phi} \sin^2\theta = \dot{\phi}_0 \sin^2\theta_0 \tag{5}$$

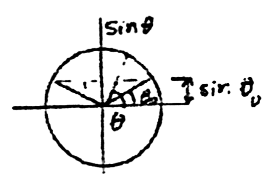
Solving Eq. (5) for $\dot{\phi}$ and substituting into Eq. (4):

$$\left(\frac{\dot{\phi}_0 \sin^2\theta_0}{\sin^2\theta}\right)^2 \sin^2\theta + \dot{\theta}^2 = \dot{\phi}_0 \sin^2\theta_0$$

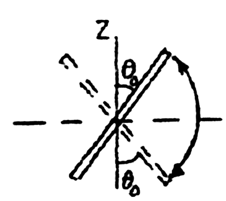
$$\dot{\theta}^2 = \dot{\phi}_0^2 \sin^2\theta_0 \left(1 - \frac{\sin^2\theta_0}{\sin^2\theta}\right) \tag{6}$$

(*a*) Range of values of $\theta$.

Since $\dot{\theta} \geq 0$, we must have




$$1 - \frac{\sin^2\theta_0}{\sin^2\theta} \geq 0$$
$$\sin^2\theta \geq \sin^2\theta_0$$
$$|\sin\theta| \geq |\sin\theta_0|$$




From trigonometric circle, we conclude that the range is

$$\theta_0 \leq \theta \leq 180° - \theta_0 \blacktriangleleft$$

Rod oscillates about axle and about horizontal line (dashed line).

## PROBLEM 18.135 (Continued)

(*b*) Maximum value of $\dot{\theta}$.

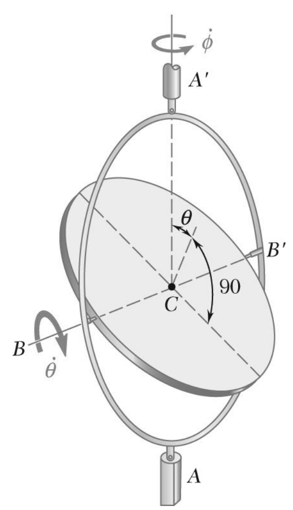
Referring to Eq. (6), we note that this occurs when $\sin\theta = 1$, that is, when $\theta = 90°$. We have

$$\dot{\theta}^2_{\max} = \dot{\phi}^2_0 \sin^2\theta_0 (1 - \sin^2\theta_0)$$
$$= \dot{\phi}^2_0 \sin^2\theta_0 \cos^2\theta_0 \qquad \dot{\theta}_{\max} = \dot{\phi}_0 \sin\theta_0 \cos\theta_0 \blacktriangleleft$$

(*c*) Minimum value of $\dot{\phi}_0$.

Referring to Eq. (5), we note that $\dot{\phi}$ is minimum when $\sin\theta = 1$, that is, when $\theta = 90°$.
We have

$$\dot{\phi}_{\min} = \dot{\phi}_0 \sin^2\theta_0 \blacktriangleleft$$




## PROBLEM 18.136

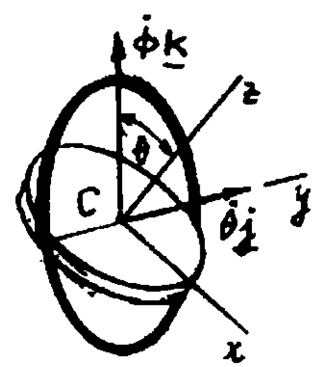
The gimbal $ABA'B'$, is of negligible mass and may rotate freely about the vertical $AA'$. The uniform disk of radius $a$ and mass $m$ may rotate freely about its diameter $BB'$, which is also the horizontal diameter of the gimbal. (*a*) Applying the principle of conservation of energy, and observing that, since $\Sigma M_{AA'} = 0,$ the component of the angular momentum of the disk along the fixed axis $AA'$ must be constant, write two first-order differential equations defining the motion of the disk. (*b*) Given the initial conditions $\theta_0 \neq 0, \dot{\phi}_0 = 0,$ and $\dot{\theta}_0 = 0,$ express the rate of nutation $\dot{\theta}$ as a function of $\theta$. (*c*) Show that the angle $\theta$ will never be larger than $\theta_0$ during the ensuing motion.

## SOLUTION

We use a reference frame *Cxyz* attached to the disk as shown. The angular velocity of the disk is

$$\omega = \dot{\phi}\mathbf{k} + \dot{\theta}\mathbf{j}$$

$$\omega = -\dot{\phi}\sin\theta\,\mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\phi}\cos\theta\,\mathbf{k}$$




(*a*) Conservation of energy: $T + V = \text{constant}$

Since $V = \text{constant}$, we have

$$T = \text{constant}.$$

For principal centroidal axes and $\overline{v} = 0$, the kinetic energy is given by

$$T = \frac{1}{2}(I_z\omega_z^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

But $I_x = I_y = \frac{1}{4}ma^2, \quad I_z = \frac{1}{2}ma^2$

Using the components of $\boldsymbol{\omega}$ computed above:

$$T = \frac{1}{8}ma^2\dot{\phi}^2\sin^2\theta + \frac{1}{8}ma^2\dot{\theta}^2 + \frac{1}{4}ma^2\omega s^2\theta$$

$$T = \frac{1}{8}ma^2[(1+\cos^2\theta)\dot{\phi}^2 + \dot{\theta}^2]$$

We have therefore $(1+\cos^2\theta)\dot{\phi}^2 + \dot{\theta}^2 = \text{constant}$ (1) ◀

## PROBLEM 18.136 (Continued)

We now determine the angular momentum $\mathbf{H}_C$:

$$\mathbf{H}_C = I_z\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= -\frac{1}{4}ma^2\dot{\phi}\sin\theta\mathbf{i} + \frac{1}{4}ma^2\dot{\theta}\mathbf{j} + \frac{1}{2}ma^2\dot{\phi}\cos\theta\mathbf{k}$$

Since $H_z =$ constant, we write

$$H_z = \mathbf{H}_C \cdot \mathbf{K} = \text{constant}$$

$$\frac{1}{4}ma^2(-\dot{\phi}\sin\theta\mathbf{i} + \dot{\theta}\mathbf{j} + 2\dot{\phi}\cos\theta\mathbf{k}) \cdot \mathbf{K} = \text{constant}$$

Since $\mathbf{i} \cdot \mathbf{K} = -\sin\theta$, $\mathbf{j} \cdot \mathbf{K} = 0$, $\mathbf{k} \cdot \mathbf{K} = \cos\theta$, we have

$$\dot{\phi}\sin^2\theta + 2\dot{\phi}\cos^2\theta = \text{constant}$$

$$\dot{\phi}(1+\cos^2\theta) = \text{constant} \qquad (2) \blacktriangleleft$$

(*b*) We determine the constants in (1) and (2) from the initial conditions $\theta_0, \dot{\phi}_0, \dot{\theta}_0 = 0$, and write

$$(1+\cos^2\theta)\dot{\phi}^2 + \dot{\theta}^2 = (1+\cos^2\theta_0)\dot{\phi}_0^2 \qquad (1')$$

$$\dot{\phi}(1+\cos^2\theta) = \dot{\phi}_0(1+\cos^2\theta_0) \qquad (2')$$

Solving (2′) for $\dot{\phi}$:

$$\dot{\phi} = \dot{\phi}_0\frac{1+\cos^2\theta_0}{1+\cos^2\theta}$$

Substituting into (1′):

$$\frac{(1+\cos^2\theta_0)^2}{1+\cos^2\theta}\dot{\phi}_0^2 + \dot{\theta}^2 = (1+\cos^2\theta_0)\dot{\phi}_0^2$$

$$\dot{\theta}^2 = \dot{\phi}_0^2(1+\cos^2\theta_0)\left(1 - \frac{1+\cos^2\theta_0}{1+\cos^2\theta_0}\right)$$

$$\dot{\theta} = \dot{\phi}_0\sqrt{\frac{(1+\cos^2\theta_0)(\cos^2\theta - \cos^2\theta_0)}{1+\cos^2\theta}} \qquad \blacktriangleleft$$

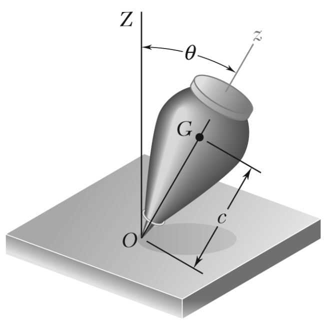
(*c*) For $\dot{\theta}$ to be real, we need $\cos^2\theta_0 - \cos^2\theta_0 \geq 0$

Thus $\qquad |\cos\theta| \geq |\cos\theta_0|$

Assuming that the axes have been chosen so that $\theta_0 \leq 90°$, we must have

$$\cos\theta \geq \cos\theta_0 \qquad \theta \leq \theta_0 \blacktriangleleft$$

## PROBLEM 18.137*




The top shown is supported at the fixed Point $O$. Denoting by $\phi$, $\theta$, and $\psi$ the Eulerian angles defining the position of the top with respect to a fixed frame of reference, consider the general motion of the top in which all Eulerian angles vary.

($a$) Observing that $\Sigma M_Z = 0$ and $\Sigma M_z = 0$, and denoting by $I$ and $I'$, respectively, the moments of inertia of the top about its axis of symmetry and about a transverse axis through $O$, derive the two first-order differential equations of motion

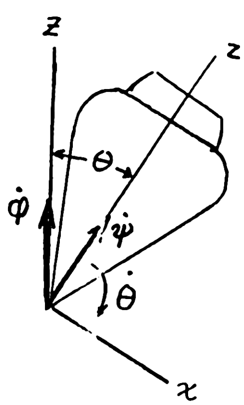
$$I'\dot{\phi}\sin^2\theta + I(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \alpha$$

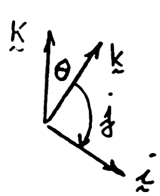
$$I(\dot{\psi} + \dot{\phi}\cos\theta) = \beta$$

where $\alpha$ and $\beta$ are constants depending upon the initial conditions. These equations express that the angular momentum of the top is conserved about both the $Z$ and $z$ axes, i.e., that the rectangular component of $\mathbf{H}_O$ along each of these axes is constant.

($b$) Use Eqs. (1) and (2) to show that the rectangular component $\omega_z$ of the angular velocity of the top is constant and that the rate of precession $\dot{\phi}$ depends upon the value of the angle of nutation $\theta$.

## SOLUTION




Use a rotating frame of reference with the $y$ axis pointing into the paper.

Angular velocity of the frame:

$$\mathbf{\Omega} = -\dot{\phi}\sin\theta\mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\phi}\cos\theta\mathbf{k}$$

Angular velocity of the top:

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\mathbf{i} + \dot{\theta}\mathbf{j} + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k}$$

Its angular momentum about $O$:

$$\mathbf{H}_O = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$
$$= -I'\dot{\phi}\sin\theta\mathbf{i} + I'\dot{\theta}\mathbf{j} + I(\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k}$$

where $\quad I_x = I_y = I' \quad$ and $\quad I_z = I.$

The moment $\mathbf{M}_O$ about $O$ is due to the weight $mg$.

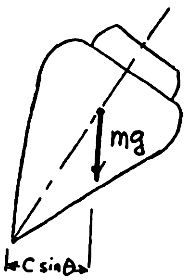
$$\mathbf{M}_0 = mgc\sin\theta\mathbf{j}$$

($a$) Since the fixed $Z$ axis refers to a Newtonian frame of reference and $(M_O)_Z = 0$, it follows that $(H_O)_Z$ is constant. Thus,

$$(H_O)_Z = \mathbf{H}_O \cdot \mathbf{K} = \mathbf{H}_O \cdot (-\mathbf{i}\sin\theta + \mathbf{k}\cos\theta)$$
$$= I'\dot{\phi}\sin^2\theta + I(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \alpha \qquad (1) ◀$$

where $\alpha$ is a constant.

## PROBLEM 18.137* (Continued)




$$\mathbf{M}_O = \dot{\mathbf{H}}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \mathbf{\Omega} \times \mathbf{H}_O$$

$$\begin{aligned} mgc\sin\theta\,\mathbf{j} = {} & -I'\frac{d}{dt}(\dot{\phi}\sin\theta)\,\mathbf{i} + I'\ddot{\theta}\,\mathbf{j} + I\frac{d}{dt}(\dot{\psi} + \dot{\phi}\cos\theta)\,\mathbf{k} \\ & + [I(\dot{\psi} + \dot{\phi}\cos\theta)\,\dot{\theta} - I'\dot{\phi}\dot{\theta}\cos\theta]\mathbf{i} \\ & + [I(\dot{\psi} + \dot{\phi}\cos\theta)\,\dot{\phi}\sin\theta - I'\dot{\phi}^2\sin\theta\cos\theta]\mathbf{j} \end{aligned}$$

$z$-components: $\quad 0 = \dfrac{d}{dt}(\dot{\psi} + \dot{\phi}\cos\theta)$

Integrating, $\quad I(\dot{\psi} + \dot{\phi}\cos\theta) = \beta \quad (2)$ ◀

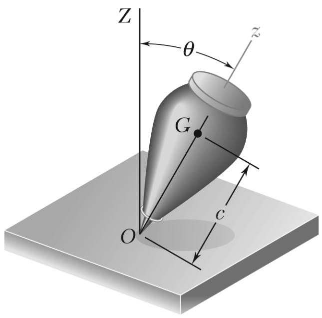
where $\beta$ is a constant.

(*b*) Since $\quad \omega_z = \dot{\psi} + \dot{\phi}\cos\theta \qquad \omega_z = \dfrac{\beta}{I} = \text{constant} \quad (3)$ ◀

From Eqs. (1) and (2), $\quad I'\dot{\phi}\sin^2\theta + \beta\cos\theta = \alpha \qquad \dot{\phi} = \dfrac{\alpha - \beta\cos\theta}{I'\sin^2\theta} \quad (4)$ ◀

which is a function of $\theta$.

## PROBLEM 18.138*




(*a*) Applying the principle of conservation of energy, derive a third differential equation for the general motion of the top of Problem 18.137.

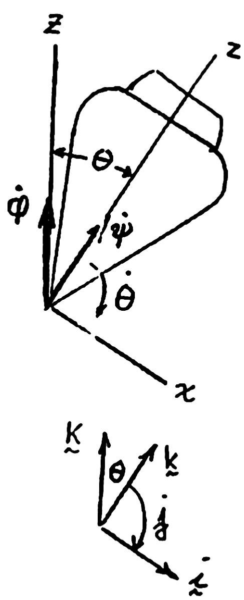
(*b*) Eliminating the derivatives $\dot{\phi}$ and $\dot{\psi}$ from the equation obtained and from the two equations of Problem 18.139, show that the rate of nutation $\dot{\theta}$ is defined by the differential equation $\dot{\theta}^2 = f(\theta)$, where

$$f(\theta) = \frac{1}{I'}\left(2E - \frac{\beta^2}{I} - 2mgc\cos\theta\right) - \left(\frac{\alpha - \beta\cos\theta}{I'\sin\theta}\right)^2$$

(*c*) Further show, by introducing the auxiliary variable $x = \cos\theta$, that the maximum and minimum values of $\theta$ can be obtained by solving for $x$ the cubic equation

$$\left(2E - \frac{\beta^2}{I} - 2mgcx\right)(1 - x^2) - \frac{1}{I'}(\alpha - \beta x)^2 = 0$$

## SOLUTION




(*a*) Angular velocity of the top:

$$\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\theta}\mathbf{j} + (\dot{\psi} + \dot{\varphi}\cos\theta)\mathbf{k}$$

Kinetic energy:

$$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

$$= \frac{1}{2}I'(\dot{\varphi}\sin\theta)^2 + \frac{1}{2}I'\dot{\theta}^2 + \frac{1}{2}I_z\omega_z^2$$

Potential energy: $V = mgc\cos\theta$

Principle of conservation of energy: $T + V = E$

$$\frac{1}{2}I'(\dot{\varphi}\sin\theta)^2 + \frac{1}{2}I'\dot{\theta}^2 + \frac{1}{2}I\omega_z^2 + mgc\cos\theta = E \quad ◀$$

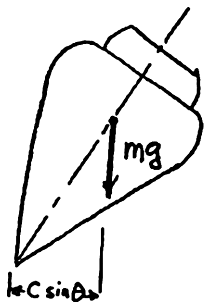
(*b*) Solving for $\dot{\theta}^2$,

$$\dot{\theta}^2 = \frac{1}{I'}(2E - I_z\omega_z^2 - 2mgc\cos\theta) - (\dot{\varphi}\sin\theta)^2 \qquad \text{(A)}$$

Equation (2) of Problem 18.137, with $\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta$ gives

$$I_z\omega_z^2 = \frac{\beta^2}{I} \qquad \text{(B)}$$

## PROBLEM 18.138* (Continued)




Equation (1) of Problem 18.137 gives

$$I'\dot{\varphi}\sin^2\theta + \beta\cos\theta = \alpha \tag{C}$$

$$\dot{\varphi} = \frac{\alpha - \beta\cos\theta}{I'\sin^2\theta} \tag{D}$$

Substituting Equations (D) and (B) into Equation (A),

$$\dot{\theta}^2 = f(\theta)$$

where $$f(\theta) = \frac{1}{I'}\left(2E - \frac{\beta^2}{I} - 2mgc\cos\theta\right) - \left(\frac{\alpha - \beta\cos\theta}{I'\sin\theta}\right)^2 \qquad (1) ◀$$

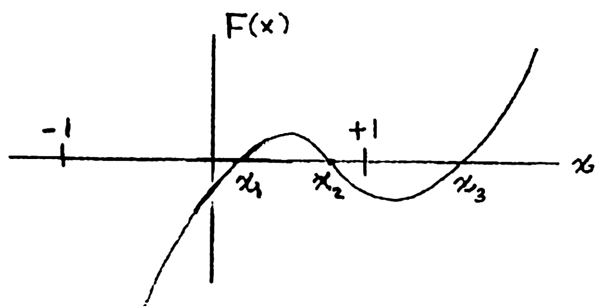
(*c*) Maximum and minimum values of $\theta$ occur when $f(\theta) = 0$. Setting $\cos\theta = x$ and $\sin^2\theta = 1 - x^2$ in Equation (1), and letting $f(\theta) = 0$ gives

$$\frac{1}{I'}\left(2E - \frac{\beta^2}{I} + 2mgcx\right) - \frac{(\alpha - \beta x)^2}{(I')^2(1 - x^2)} = 0$$

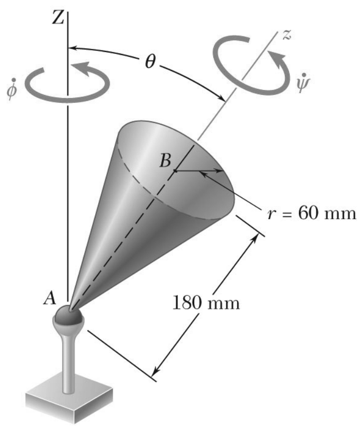
Multiplying by $I'(1 - x^2)$ gives the cubic equation $F(x) = 0$:

$$\left(2E - \frac{\beta^2}{I} - 2mgcx\right)(1 - x^2) - \frac{1}{I'}(\alpha - \beta x)^2 = 0 \qquad (2) ◀$$

Solving this equation will yield three values of $x$. The two values lying between $-1$ and $+1$ correspond to the maximum and minimum values of $\theta$.



## PROBLEM 18.139




A solid cone of height 180 mm with a circular base of radius 60 mm is supported by a ball and socket at $A$. The cone is released from the position $\theta_0 = 30°$ with a rate of spin $\dot{\psi}_0 = 300$ rad/s, a rate of precession $\dot{\phi}_0 = 20$ rad/s, and a zero rate of nutation. Determine (*a*) the maximum value of $\theta$ in the ensuing motion, (*b*) the corresponding values of the rates of spin and precession. [*Hint:* Use Eq. (2) of Prob. 18.138; you can either solve this equation numerically or reduce it to a quadratic equation, since one of its roots is known.]

## SOLUTION

Data: $r = 60 \text{ mm} = 0.06 \text{ m}, \quad h = 180 \text{ mm} = 0.18 \text{ m}, \quad c = \frac{3}{4}h = 0.135 \text{ m},$

$$\theta_0 = 30°, \quad \dot{\psi}_0 = 300 \text{ rad/s}, \quad \dot{\phi}_0 = 20 \text{ rad/s}, \quad \dot{\theta}_0 = 0$$

Calculate the following:

$$\frac{I}{m} = \frac{3}{10}r^2 = \frac{3}{10}(0.06)^2 = 1.08 \times 10^{-3} \text{ m}^2$$

$$\frac{I'}{m} = \frac{3}{5}\left(\frac{1}{4}r^2 + h^2\right) = \frac{3}{5}\left[\frac{1}{4}(0.06)^2 + (0.18)^2\right] = 19.98 \times 10^{-3} \text{ m}^2$$

Initially, $\omega_x = -\dot{\phi}_0 \sin\theta_0 = -20\sin 30° = -10 \text{ rad/s}, \quad \omega_y = \dot{\theta}_0 = 0$

$$\omega_z = \dot{\psi}_0 + \dot{\phi}_0 \cos\theta_0 = 300 + 20\cos 30° = 317.32 \text{ rad/s}$$

$$\frac{T}{m} = \frac{1}{2}\frac{I'}{m}(\omega_x^2 + \omega_y^2) + \frac{1}{2}\frac{I}{m}\omega_z^2$$

$$= \frac{1}{2}(19.98 \times 10^{-3})(10^2 + 0) + \frac{1}{2}(1.08 \times 10^{-3})(317.32)^2 = 55.3728 \text{ m}^2/\text{s}^2$$

$$\frac{V}{m} = gc\cos\theta_0 = (9.81)(0.135)\cos 30° = 1.1469 \text{ m}^2/\text{s}^2$$

$$\frac{2E}{m} = 2(55.3728 + 1.1469) = 113.0394 \text{ m}^2/\text{s}^2$$

$$\frac{\beta}{m} = \frac{I}{m}\omega_z = (1.08 \times 10^{-3})(317.32) = 0.342706 \text{ m}^2/\text{s}$$

$$\frac{\alpha}{m} = \frac{I'}{m}\dot{\phi}_0 \sin^2\theta_0 + \beta\cos\theta_0$$

$$= (19.98 \times 10^{-3})(20)\sin^2 30° + 0.342706\cos 30°$$

$$= 0.396692 \text{ m}^2/\text{s}$$

After dividing by $m$, Equation (2) of Problem 18.138 becomes

$$F(x) = \left(\frac{2E}{m} - \frac{\frac{\beta^2}{m^2}}{\frac{I}{m}} - 2gcx\right)(1 - x^2) - \frac{m}{I'}\left(\frac{\alpha}{m} - \frac{\beta x}{m}\right)^2 = 0$$

$$\left[113.0394 - \frac{(0.342706)^2}{1.08 \times 10^{-3}} - (2)(9.81)(0.135)x\right](1 - x^2) - \frac{(0.396692 - 0.342706x)^2}{19.98 \times 10^{-3}} = 0$$

$$(4.29198 - 2.6487x)(1 - x^2) - 5.87825(1.157529 - x)^2 = 0$$

($a$) Roots are: $x = \cos\theta = 0.68170, \quad 0.86603, \quad 2.2919$

$$\theta_{\max} = \cos^{-1}(0.68170) = 47.023°$$

$\theta_{\max} = 47.0°$ ◀

($b$) By Equation (4) of Problem 18.137,

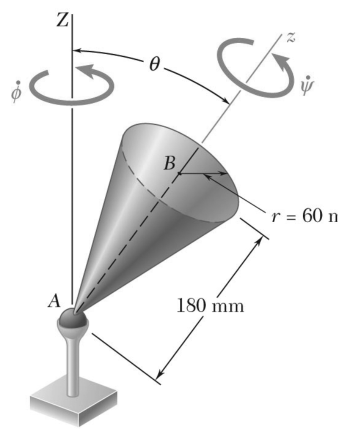
$$\dot{\phi} = \frac{\alpha - \beta\cos\theta}{I'\sin^2\theta} = \frac{0.396692 - (0.342706)(0.68170)}{(19.98 \times 10^{-3})\sin^2 47.023°} = 15.2474$$

$$\omega_z = 317.32 \text{ rad/s}$$

$$\dot{\psi} = \omega_z - \dot{\phi}\cos\theta = 317.32 - (15.2474)(0.68170)$$

*spin*: $\dot{\psi} = 307$ rad/s ◀

*precession*: $\dot{\phi} = 15.25$ rad/s ◀




## PROBLEM 18.140

A solid cone of height 180 mm with a circular base of radius 60 mm is supported by a ball and socket at $A$. The cone is released from the position $\theta_0 = 30°$ with a rate of spin $\dot{\psi}_0 = 300$ rad/s, a rate of precession $\dot{\phi}_0 = -4$ rad/s, and a zero rate of nutation. Determine (*a*) the maximum value of $\theta$ in the ensuing motion, (*b*) the corresponding values of the rates of spin and precession, (*c*) the value of $\theta$ for which the sense of the precession is reversed. (See hint of Problem 18.139.)

## SOLUTION

Data: $r = 60 \text{ mm} = 0.06 \text{ m}, \quad h = 180 \text{ mm} = 0.18 \text{ m}, \quad c = \dfrac{3}{4}h = 0.135 \text{ m},$

$$\theta_0 = 30°, \quad \dot{\psi}_0 = 300 \text{ rad/s}, \quad \dot{\phi}_0 = -4 \text{ rad/s}, \quad \dot{\theta}_0 = 0$$

Calculate the following:

$$\frac{I}{m} = \frac{3}{10}r^2 = \frac{3}{10}(0.06)^2 = 1.08 \times 10^{-3} \text{ m}^2$$

$$\frac{I'}{m} = \frac{3}{5}\left(\frac{1}{4}r^2 + h^2\right) = \frac{3}{5}\left[\frac{1}{4}(0.06)^2 + (0.18)^2\right] = 19.98 \times 10^{-3} \text{ m}^2$$

Initially, $\omega_x = -\dot{\phi}_0 \sin\theta_0 = -(-4)\sin 30° = 2 \text{ rad/s}, \qquad \omega_y = \dot{\theta}_0 = 0$

$$\omega_z = 300 + (-4)\cos 30° = 296.536 \text{ rad/s}$$

$$\frac{T}{m} = \frac{1}{2}\frac{I'}{m}(\omega_x^2 + \omega_y^2) + \frac{1}{2}\frac{I}{m}\omega_z^2$$

$$= \frac{1}{2}(19.98 \times 10^{-3})((2)^2 + 0) + \frac{1}{2}(1.08 \times 10^{-3})(296.536)^2 = 47.5241 \text{ m}^2/\text{s}^2$$

$$\frac{V}{m} = gc\cos\theta = (9.81)(0.135)\cos 30° = 1.1469 \text{ m}^2/\text{s}^2$$

$$\frac{2E}{m} = 2(47.5241 + 1.1469) = 97.3419 \text{ m}^2/\text{s}^2$$

$$\frac{\beta}{m} = \frac{I}{m}\omega_z = (1.08 \times 10^{-3})(296.536) = 0.320259 \text{ m}^2/\text{s}$$

$$\frac{\alpha}{m} = \frac{I'}{m}\dot{\phi}_0 \sin^2\theta_0 + \beta\cos\theta_0$$

$$= (19.98 \times 10^{-3})(-4)\sin^2 30° + 0.320259\cos 30°$$

$$= 0.257372 \text{ m}^2/\text{s}$$

After dividing by $m$, Equation (2) of Problem 18.138 becomes

$$F(x) = \left( \frac{2E}{m} - \frac{\frac{\beta^2}{m^2}}{\frac{I}{m}} - 2gcx \right)(1 - x^2) - \frac{m}{I'}\left( \frac{\alpha}{m} - \frac{\beta x}{m} \right)^2 = 0$$

$$\left[ 97.3419 - \frac{(0.320259)^2}{1.08 \times 10^{-3}} - (2)(9.81)(0.135)x \right](1 - x^2) - \frac{(0.257372 - 0.320259x)^2}{19.98 \times 10^{-3}} = 0$$

$$(2.37324 - 2.6487x)(1 - x^2) - 5.13342(0.80364 - x)^2 = 0$$

(*a*) Roots are: $x = \cos\theta = 0.23732, \quad 0.86603, \quad 1.73$

$$\theta_{\max} = \cos^{-1}(0.23732) = 76.272°$$

$\theta_{\max} = 76.3°$ ◀

(*b*) By Equation (4) of Problem 18.137,

$$\dot{\phi} = \frac{\alpha - \beta\cos\theta}{I'\sin^2\theta} = \frac{0.257372 - (0.320259)(0.23732)}{(19.98 \times 10^{-3})\sin^2 76.272°} = 9.6192 \text{ rad/s}$$

$$\omega_z = 296.536 \text{ rad/s}$$

$$\dot{\psi} = \omega_z - \dot{\phi}\cos\theta = 296.536 - (9.6192)(0.23732)$$

*spin*: $\dot{\psi} = 294$ rad/s ◀

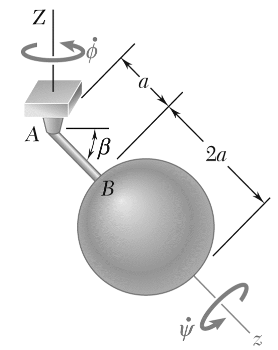
*precession*: $\dot{\phi} = 9.62$ rad/s ◀

(*c*)

$$\dot{\phi} = \frac{\alpha - \beta\cos\theta}{I'\sin^2\theta} = 0$$

$$\cos\theta = \frac{\alpha}{\beta} = \frac{0.257372}{0.320259}$$

$\theta = 36.5°$ ◀




## PROBLEM 18.141*

A homogeneous sphere of mass $m$ and radius $a$ is welded to a rod $AB$ of negligible mass, which is held by a ball-and-socket support at $A$. The sphere is released in the position $\beta = 0$ with a rate of precession $\dot{\phi}_0 = \sqrt{17g/11a}$ with no spin or nutation. Determine the largest value of $\beta$ in the ensuing motion.

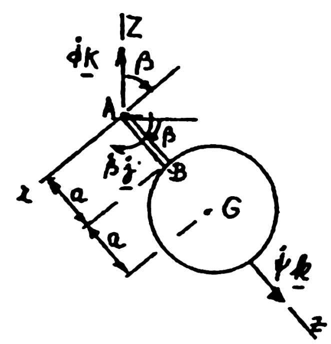
## SOLUTION

Conservation of angular momentum about the $Z$ and $z$ axes.

Since the only external forces are the weight of the sphere and the reaction at $A$, a reasoning similar to that used in Problem 18.109 shows that the angular momentum is conserved about the $Z$ and $z$ axes.

Choosing the principal axes $Axyz$ shown (with $y$ horizontal and pointing into the paper), we have

$$\boldsymbol{\omega} = -\dot{\phi}\cos\beta\mathbf{i} + \dot{\beta}\mathbf{j} + (\dot{\psi} - \dot{\phi}\sin\beta)\mathbf{k}$$




The moments of inertia are

$$I_z = \frac{2}{5}ma^2$$

$$I_x = I_y = \frac{2}{5}ma^2 + m(2a)^2 = \frac{22}{5}ma^2$$

Angular momentum about $A$:

$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$\mathbf{H}_A = -\frac{22}{5}ma^2\dot{\phi}\cos\beta\mathbf{i} + \frac{22}{5}ma^2\dot{\beta}\mathbf{j} + \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)\mathbf{k}$$

We write $\mathbf{H}_z = \text{constant}$, or $\mathbf{H}_A \cdot \mathbf{K} = \text{constant}$

Since $\mathbf{i}\cdot\mathbf{K} = -\cos\beta$, $\quad \mathbf{j}\cdot\mathbf{K} = 0$, $\quad \mathbf{k}\cdot\mathbf{K} = -\sin\beta$

$$\mathbf{H}_0 \cdot \mathbf{K} = -\frac{22}{5}ma^2\dot{\phi}\cos\beta(-\cos\beta) + \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)(-\sin\beta) = \text{constant}$$

With the initial conditions $\dot{\phi} = \dot{\phi}_0$, $\dot{\psi} = \dot{\beta} = 0$, $\beta = 0$, we find that the constant is $\frac{22}{5}ma^2\dot{\phi}_0$. Thus,

$$11\dot{\phi}\cos^2\beta - (\dot{\psi} - \dot{\phi}\sin\beta)\sin\beta = 11\dot{\phi}_0 \qquad (1)$$

We now write $H_z = \text{constant}$:

$$H_z = \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta) = \text{constant}$$

and, from the initial conditions, we find that the constant is zero. Thus,

$$\dot{\psi} - \dot{\phi}\sin\beta = 0 \tag{2}$$

<u>Conservation of energy</u>.

We have

$$T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$
$$T = \frac{1}{2}\left[\frac{22}{5}ma^2\dot{\phi}^2\cos^2\beta + \frac{22}{5}ma^2\dot{\beta}^2 + \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)^2\right]$$

and selecting the datum at $\beta = 0$:

$$V = -2mga\sin\beta$$

$$T + V = \text{constant:}\quad \frac{1}{2}\left[\frac{22}{5}ma^2\dot{\phi}^2\cos^2\beta + \frac{22}{5}ma^2\dot{\beta}^2 + \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)^2\right] - 2mga\sin\beta = \text{constant}$$

From the initial conditions $\dot{\phi} = \dot{\phi}_0,\ \dot{\psi} = \dot{\beta} = 0,\ \beta = 0,$ we find that the constant is $\frac{11}{5}ma^2\dot{\phi}_0^2$. Thus,

$$11\dot{\phi}^2\cos^2\beta + 11\dot{\beta}^2 + (\dot{\psi} - \dot{\phi}\sin\beta)^2 - 10\frac{g}{a}\sin\beta = 11\dot{\phi}_0^2 \tag{3}$$

Substituting for $\dot{\psi} - \dot{\phi}\sin\beta$ from Eq. (2) into Eqs. (1) and (3):

Eq. (1): $$11\dot{\phi}\cos^2\beta = 11\dot{\phi}_0 \tag{1$'$}$$

Eq. (3): $$11\dot{\phi}^2\cos^2\beta + 11\dot{\beta}^2 - 10\frac{g}{a}\sin\beta = 11\dot{\phi}_0^2 \tag{3$'$}$$

Solving $(1')$ for $\dot{\phi}$,

$$\dot{\phi} = \dot{\phi}_0\sec^2\beta \tag{4}$$

Substituting for $\dot{\phi}$ from Eq. (4) into Eq. $(3')$:

$$11(\dot{\phi}_0\sec^2\theta)^2\cos^2\beta + 11\dot{\beta}^2 - 10\frac{g}{a}\sin\beta = 11\dot{\phi}_0^2 \tag{5}$$

For the maximum value of $\beta$, we have $\dot{\beta} = 0$ and Eq. (5) yields

$$\dot{\phi}_0^2\left(\frac{1}{\cos^2\theta} - 1\right) = \frac{10}{11}\frac{g}{a}\sin\beta$$

$$\dot{\phi}_0^2 = \frac{10}{11}\frac{g}{a}\frac{\cos^2\beta}{\sin\beta} \tag{6}$$

## PROBLEM 18.141* (Continued)

Given data: $\dot{\phi}_0^2 = \frac{17}{11}\frac{g}{a} \cdot$ Substituting into Eq. (6),

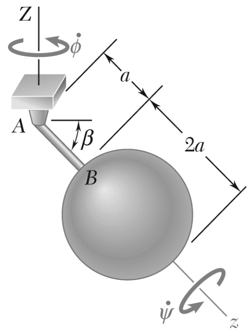
$$\frac{17}{11}\frac{g}{a} = \frac{10}{11}\frac{g}{a}\frac{\cos^2\beta}{\sin\beta} \qquad \cos^2\beta = 1.7\sin\beta$$

Letting $\cos^2\beta = 1 - \sin^2\beta$, we have

$$\sin^2\beta + 1.7\sin\beta - 1 = 0$$

Solving the quadratic,

$$\sin\beta = -2.162 \text{ (impossible)} \quad \text{and} \quad \sin\beta = 0.46244 \qquad \beta_{\max} = 27.5° \blacktriangleleft$$




## PROBLEM 18.142*

A homogeneous sphere of mass $m$ and radius $a$ is welded to a rod $AB$ of negligible mass, which is held by a ball-and-socket support at $A$. The sphere is released in the position $\beta = 0$ with a rate of precession $\dot\phi = \dot\phi_0$ with no spin or nutation. Knowing that the largest value of $\beta$ in the ensuing motion is 30°, determine (*a*) the rate of precession $\dot\phi_0$ of the sphere in its initial position, (*b*) the rates of precession and spin when $\beta = 30°$.

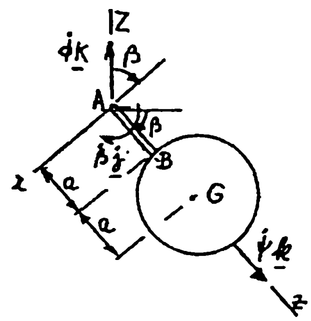
## SOLUTION

Conservation of angular momentum about the $Z$ and $z$ axes.

Since the only external forces are the weight of the sphere and the reaction at $A$, a reasoning similar to that used in Problem 18.109 shows that the angular momentum is conserved about the $Z$ and $z$ axes.

Choosing the principal axes $Axyz$ shown (with $y$ horizontal and pointing into the paper), we have

$$\boldsymbol{\omega} = -\dot\phi\cos\beta\mathbf{i} + \dot\beta\mathbf{j} + (\dot\psi - \dot\phi\sin\beta)\mathbf{k}$$




The moments of inertia are

$$I_z = \frac{2}{5}ma^2$$

$$I_x = I_y = \frac{2}{5}ma^2 + m(2a)^2 = \frac{22}{5}ma^2$$

Angular momentum about $A$:

$$\mathbf{H}_A = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$\mathbf{H}_A = -\frac{22}{5}ma^2\dot\phi\cos\beta\mathbf{i} + \frac{22}{5}ma^2\dot\beta\mathbf{j} + \frac{2}{5}ma^2(\dot\psi - \dot\phi\sin\beta)\mathbf{k}$$

We write $\mathbf{H}_Z = \text{constant}$, or $\mathbf{H}_A \cdot \mathbf{K} = \text{constant}$

Since $\mathbf{i}\cdot\mathbf{K} = -\cos\beta$, $\quad \mathbf{j}\cdot\mathbf{K} = 0$, $\quad \mathbf{k}\cdot\mathbf{K} = -\sin\beta$

$$\mathbf{H}_0 \cdot \mathbf{K} = -\frac{22}{5}ma^2\dot\phi\cos\beta(-\cos\beta)$$

$$+\frac{2}{5}ma^2(\dot\psi - \dot\phi\sin\beta)(-\sin\beta) = \text{constant}$$

With the initial conditions $\dot\phi = \dot\phi_0$, $\dot\psi = \dot\beta = 0$, $\beta = 0$, we find that the constant is $\frac{22}{5}ma^2\dot\phi_0$. Thus,

$$11\dot\phi\cos^2\beta - (\dot\psi - \dot\phi\sin\beta)\sin\beta = 11\dot\phi_0 \tag{1}$$

We now write $H_z = \text{constant}$:

$$H_z = \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta) = \text{constant}$$

and, from the initial conditions, we find that the constant is zero. Thus,

$$\dot{\psi} - \dot{\phi}\sin\beta = 0 \tag{2}$$

<u>Conservation of energy</u>.

We have

$$T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

$$T = \frac{1}{2}\left[\frac{22}{5}ma^2\dot{\phi}^2\cos^2\beta + \frac{22}{5}ma^2\dot{\beta}^2 + \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)^2\right]$$

and, selecting the datum at $\beta = 0$:

$$V = -2mga\sin\beta$$

$$T + V = \text{constant:} \quad \frac{1}{2}\left[\frac{22}{5}ma^2\dot{\phi}^2\cos^2\beta + \frac{22}{5}ma^2\dot{\beta}^2 + \frac{2}{5}ma^2(\dot{\psi} - \dot{\phi}\sin\beta)^2\right] - 2mga\sin\beta = \text{constant}$$

From the initial conditions $\dot{\phi} = \dot{\phi}_0$, $\dot{\psi} = \dot{\beta} = 0$, $\beta = 0$, we find that the constant is $\frac{11}{5}ma^2\dot{\phi}_0^2$. Thus,

$$11\dot{\phi}^2\cos^2\beta + 11\dot{\beta}^2 + (\dot{\psi} - \dot{\phi}\sin\beta)^2 - 10\frac{g}{a}\sin\beta = 11\dot{\phi}_0^2 \tag{3}$$

Substituting for $\dot{\psi} - \dot{\phi}\sin\beta$ from Eq. (2) into Eqs. (1) and (3),

Eq. (1): $$11\dot{\phi}\cos^2\beta = 11\dot{\phi}_0 \tag{1′}$$

Eq. (3): $$11\dot{\phi}^2\cos^2\beta + 11\dot{\beta}^2 - 10\frac{g}{a}\sin\beta = 11\dot{\phi}_0^2 \tag{3′}$$

Solving (1′) for $\dot{\phi}$,

$$\dot{\phi} = \dot{\phi}_0\sec^2\beta \tag{4}$$

Substituting for $\dot{\phi}$ from Eq. (4) into Eq. (3′),

$$11(\dot{\phi}_0\sec^2\theta)^2\cos^2\beta + 11\dot{\beta}^2 - 10\frac{g}{a}\sin\beta = 11\dot{\phi}_0^2 \tag{5}$$

For the maximum value of $\beta$, we have $\dot{\beta} = 0$ and Eq. (5) yields

$$\dot{\phi}_0^2\left(\frac{1}{\cos^2\theta} - 1\right) = \frac{10}{11}\frac{g}{a}\sin\beta$$

$$\dot{\phi}_0^2 = \frac{10}{11}\frac{g}{a}\frac{\cos^2\beta}{\sin\beta} \tag{6}$$

## PROBLEM 18.142* (Continued)

($a$) Making $\beta = 30°$ in Eq. (6), we have

$$\dot{\phi}_0^2 = \frac{10}{11}\frac{g}{a}\frac{0.75}{0.5} = \frac{15}{11}\frac{g}{a} \qquad \dot{\phi}_0 = \sqrt{\frac{15}{11}\frac{g}{a}} \blacktriangleleft$$

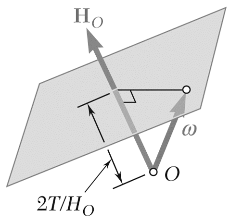
($b$) Substituting for $\dot{\phi}_0$ in Eq. (4), and making $\beta = 30°$:

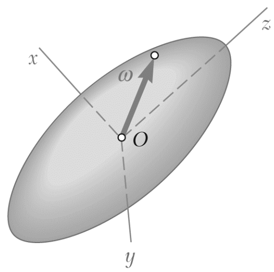
$$\dot{\phi} = \dot{\phi}_0 \sec^2 30° = \sqrt{\frac{15}{11}\frac{g}{a}}\left(\frac{4}{3}\right) = \sqrt{\frac{15(16)}{11(9)}\frac{g}{a}} \qquad \dot{\phi} = 2\sqrt{\frac{20}{33}\frac{g}{a}} \blacktriangleleft$$

Substituting for in Eq. (2), and making $\beta = 30°$:

$$\dot{\psi} = \dot{\phi}\sin 30° = \frac{1}{2}\dot{\phi} \qquad \dot{\psi} = \sqrt{\frac{20}{33}\frac{g}{a}} \blacktriangleleft$$

## PROBLEM 18.143*




Consider a rigid body of arbitrary shape which is attached at its mass center $O$ and subjected to no force other than its weight and the reaction of the support at $O$.

($a$) Prove that the angular momentum $\mathbf{H}_O$ of the body about the fixed Point $O$ is constant in magnitude and direction, that the kinetic energy $T$ of the body is constant, and that the projection along $\mathbf{H}_O$ of the angular velocity $\boldsymbol{\omega}$ of the body is constant.

($b$) Show that the tip of the vector $\boldsymbol{\omega}$ describes a curve on a fixed plane in space (called the *invariable plane*), which is perpendicular to $\mathbf{H}_O$ and at a distance $2T/H_O$ from $O$.

($c$) Show that with respect to a frame of reference attached to the body and coinciding with its principal axes of inertia, the tip of the vector $\boldsymbol{\omega}$ appears to describe a curve on an ellipsoid of equation

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T = \text{constant}$$

The ellipsoid (called the *Poinsot ellipsoid*) is rigidly attached to the body and is of the same shape as the ellipsoid of inertia, but of a different size.

## SOLUTION

($a$) From Equation (18.27), $\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O$

Since $\Sigma\mathbf{M}_O = 0, \quad \dot{\mathbf{H}}_O = 0.$ $\qquad \mathbf{H}_O = \text{constant} \quad (1)$ ◀

Conservation of energy: $T + V = \text{constant}$

Since $V = 0,$ $\qquad T = \text{constant} \quad (2)$ ◀

For a rigid body rotating about Point $O$,

$$T = \frac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right)$$

$$\mathbf{H}_O = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

$$\mathbf{H}_O \cdot \omega = I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T$$

Let $\beta$ be the angle between the vectors $\mathbf{H}_O$ and $\boldsymbol{\omega}$.

$$\mathbf{H}_O \cdot \boldsymbol{\omega} = H_O\omega\cos\beta$$

The projection of $\boldsymbol{\omega}$ along $\mathbf{H}_O$ is $\omega\cos\beta$

$$\omega\cos\beta = \frac{2T}{H_O} = \text{constant}$$

$\omega\cos\beta = \text{constant} \quad (3)$ ◀

## PROBLEM 18.143* (Continued)

($b$) $\omega \cos \beta$ is the perpendicular distance from the invariable plane. This distance is equal to $\frac{2T}{H_O}$.

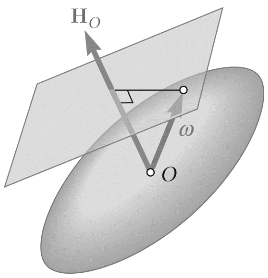
($c$) For a frame of reference attached to the body, the moments of inertia with respect of orthogonal axes of the frame do not change.

$$I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 = 2T \tag{4}$$

Let
$$a_1 = \sqrt{\frac{2T}{I_x}}, \quad b_1 = \sqrt{\frac{2T}{I_y}}, \quad c_1 = \sqrt{\frac{2T}{I_z}} \tag{5}$$

Then
$$\frac{\omega_x^2}{a_1^2} + \frac{\omega_y^2}{b_1^2} + \frac{\omega_z^2}{c_1^2} = 1 \tag{6}$$

which is the equation of an ellipsoid.




## PROBLEM 18.144*

Referring to Problem 18.143, (*a*) prove that the Poinsot ellipsoid is tangent to the invariable plane, (*b*) show that the motion of the rigid body must be such that the Poinsot ellipsoid appears to roll on the invariable plane. [*Hint:* In part *a,* show that the normal to the Poinsot ellipsoid at the tip of $\boldsymbol{\omega}$ is parallel to $\mathbf{H}_O$. It is recalled that the direction of the normal to a surface of equation $F(x, y, z) = \text{constant}$ at a Point *P* is the same as that of **grad** *F* at Point *P*.]

## SOLUTION

(*a*) From Problem 18.143, the equation of the Poinsot ellipsoid is

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T = \text{constant}$$

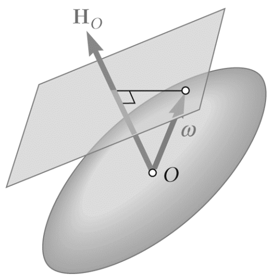
Let $$F(\omega_x, \omega_y, \omega_z) = I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2$$

$$\textbf{grad}\ F = \frac{\partial F}{\partial \omega_x}\mathbf{i} + \frac{\partial F}{\partial \omega_y}\mathbf{j} + \frac{\partial F}{\partial \omega_z}\mathbf{k}$$

$$= 2I_x\omega_x\mathbf{i} + 2I_y\omega_y\mathbf{j} + 2I_z\omega_z\mathbf{k} = 2\mathbf{H}_O$$

The direction of the normal at a point on the surface of the ellipsoid is parallel to **grad** *F*, which in turn is parallel to $\mathbf{H}_O$. Since $\mathbf{H}_O$ is normal also to the invariable plane, it follows that the Poinsot ellipsoid is tangent to the invariable plane at the point common to the plane and the ellipsoid.

(*b*) The Poinsot elipsoid moves with the body. Thus, its angular velocity is $\boldsymbol{\omega}$, the angular velocity of the body. Since Point *O* is regarded as fixed, the angular velocity vector lies along the axis of rotation, or the locus of points of zero velocity. Thus, the velocity of the point of contact of the Poinsot ellipsoid with the invariable plane is zero. The Poinsot ellipsoid rolls without slipping on the invariable plane.




## PROBLEM 18.145*

Using the results obtained in Problems 18.143 and 18.144, show that for an axisymmetrical body attached at its mass center $O$ and under no force other than its weight and the reaction at $O$, the Poinsot ellipsoid is an ellipsoid of revolution and the space and body cones are both circular and are tangent to each other. Further show that (*a*) the two cones are tangent externally, and the precession is direct, when $I < I'$, where $I$ and $I'$ denote, respectively, the axial and transverse moment of inertia of the body, (*b*) the space cone is inside the body cone, and the precession is retrograde, when $I > I'$.

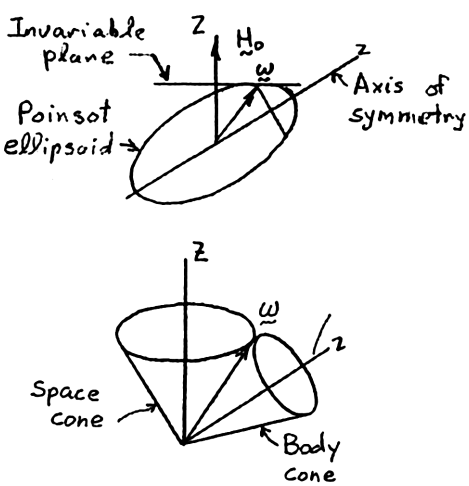
## SOLUTION

Let $I_x = I_y = I'$ and $I_z = I$ so that the $z$ axis is the symmetry axis. Then, the equation of the Poinsot ellipsoid (Equation (4) of Problem 18.143) becomes

$$I'\left(\omega_x^2 + \omega_y^2\right) + I\omega_z^2 = 2T = \text{constant}$$

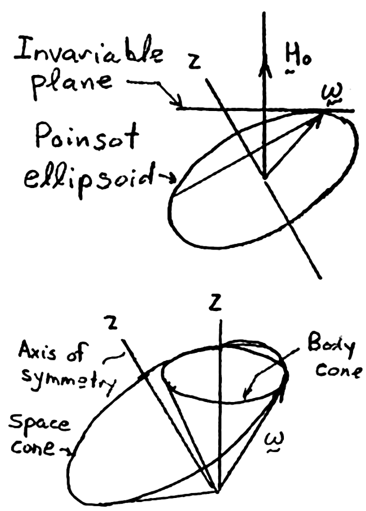
which is the equation of an *ellipsoid of revolution*. It follows that the tip of $\boldsymbol{\omega}$ describes circles on both the Poinsot ellipsoid and on the invariable plane, and that the vector $\boldsymbol{\omega}$ itself describes *circular body and space cones*. The Poinsot ellipsoid, the invariable plane, and the body and space cones are shown below for cases *a* and *b*.

(*a*) $I < I'$



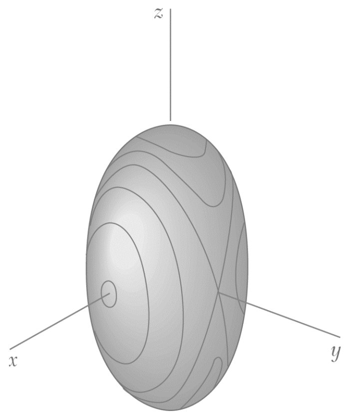

*Direct Precession*

(*b*) $I > I'$




*Retrograde Precession*

## PROBLEM 18.146*




Refer to Problems 18.143 and 18.144.

(*a*) Show that the curve (called *polhode*) described by the tip of the vector $\boldsymbol{\omega}$ with respect to a frame of reference coinciding with the principal axes of inertia of the rigid body is defined by the equations

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T = \text{constant} \quad (1)$$

$$I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2 = H_O^2 = \text{constant} \quad (2)$$

and that this curve can, therefore, be obtained by intersecting the Poinsot ellipsoid with the ellipsoid defined by Eq. (2).

(*b*) Further show, assuming $I_x > I_y > I_z$, that the polhodes obtained for various values of $H_O$ have the shapes indicated in the figure.

(*c*) Using the result obtained in part *b*, show that a rigid body under no force can rotate about a fixed centroidal axis if, and only if, that axis coincides with one of the principal axes of inertia of the body, and that the motion will be stable if the axis of rotation coincides with the major or minor axis of the Poinsot ellipsoid ($z$ or $x$ axis in the figure) and unstable if it coincides with the intermediate axis ($y$ axis).

## SOLUTION

(*a*) Equation (1) expresses conservation of energy as shown in the solution to Problem 18.143. It is the equation of the Poinsot ellipsoid.

Let $$a_1 = \sqrt{\frac{2T}{I_x}}, \quad b_1 = \sqrt{\frac{2T}{I_y}}, \quad c_1 = \sqrt{\frac{2T}{I_z}}$$

Then $$\frac{\omega_x^2}{a_1^2} + \frac{\omega_y^2}{b_1^2} + \frac{\omega_z^2}{c_1^2} = 1 \quad (3)$$

which is the equation of an ellipsoid.

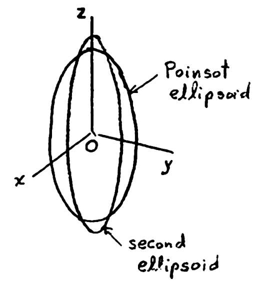
Equation (2) expresses the constancy of $H_O^2 = (H_O)_x^2 + (H_O)_y^2 + (H_O)_z^2$, the square of the magnitude of the angular momentum vector.

Let $$a_2 = \frac{H_O}{I_x}, \quad b_2 = \frac{H_O}{I_y}, \quad c_2 = \frac{H_O}{I_z}$$

Then $$\frac{\omega_x^2}{a_2^2} + \frac{\omega_y^2}{b_2^2} + \frac{\omega_z^2}{c_2^2} = 1 \quad (4)$$

which is the equation of a second ellipsoid.

Since the coordinates $\omega_x$, $\omega_y$, $\omega_z$ of the tip of the vector $\boldsymbol{\omega}$ must satisfy both Equations (1) and (2), the curve described by the tip of $\boldsymbol{\omega}$ is the intersection of the two ellipsoids.

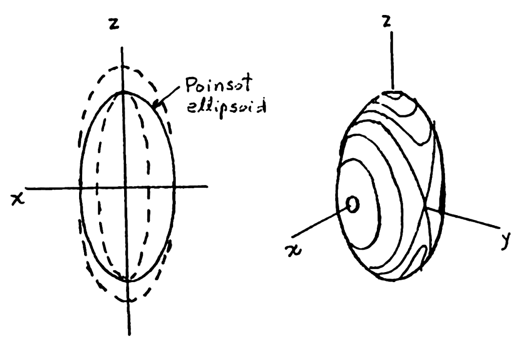



($b$) Assume $I_x > I_y > I_z$.

Then $$a_1 < b_1 < c_1 \quad \text{and} \quad a_2 < b_2 < c_2.$$

Thus, for both ellipsoids, the minor axis is directed along the $x$ axis, the intermediate axis along the $y$ axis, and the major axis along the $z$ axis. However, because the ratio of the major to minor semiaxis is $\sqrt{\frac{I_x}{I_z}}$ for the Poinsot ellipsoid and is $\frac{I_x}{I_z}$ for the second ellipsoid, the deviation from a spherical shape is more pronounced in the second ellipsoid.

The largest ellipsoid of the second type to be in contact with the Poinsot ellipsoid will lie outside that ellipsoid and touch it at its points of intersection with the $x$ axis, and the smallest will lie inside the Poinsot ellipsoid and touch it at its points of intersection with the $z$ axis (see left-hand sketch). All ellipsoids of the second type comprised between these two will intersect the Poinsot ellipsoid along the curves called polhodes, as shown in the right-hand figure.

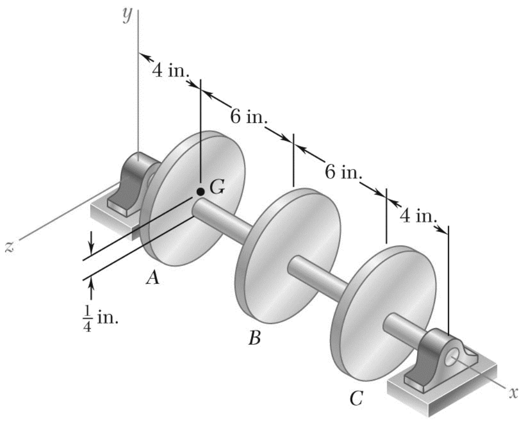



Note that the ellipsoid of the second type, which has the same intermediate axis as the Poinsot ellipsoid, intersects that ellipsoid along two ellipses whose planes contain the $y$ axis. These curves are not polhodes, since the tip of $\boldsymbol{\omega}$ will not describe them, but they separate the polhodes into four groups. Two groups loop around the minor axis ($x$ axis) and the other two around the major axis ($z$ axis).

($c$) If the body is set to spin about one of the principal axes, the Poinsot ellipsoid will remain in contact with the invariable plane at the same point (on the $x$, $y$, or $z$ axis); the rotation is steady. In any other case, the point of contact will be located on one of the polhodes, and the tip of $\boldsymbol{\omega}$ will start describing that polhode, while the Poinsot ellipsoid rolls on the invariable plane.

A rotation about the *minor* or the *major* axis ($x$ or $z$ axis) is *stable*. If that motion is disturbed, the tip of $\boldsymbol{\omega}$ will move to a very small polhode surrounding that axis and stay close to its original position.

On the other hand, a rotation about the *intermediate* axis ($y$ axis) is *unstable*. If that motion is disturbed, the tip of $\boldsymbol{\omega}$ will move to one of the polhodes located near that axis and start describing it, departing completely from its original position and causing the body to tumble.

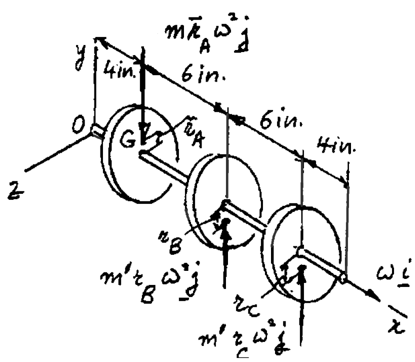



## PROBLEM 18.147

Three 25-lb rotor disks are attached to a shaft which rotates at 720 rpm. Disk $A$ is attached eccentrically so that its mass center is $\frac{1}{4}$in. from the axis of rotation, while disks $B$ and $C$ are attached so that their mass centers coincide with the axis of rotation. Where should 2-lb weights be bolted to disks $B$ and $C$ to balance the system dynamically?

## SOLUTION

The system is dynamically balanced if the effective forces are equivalent to zero. Let Points $A$, $B$, and $C$ be the points where the disks are attached to the shaft and let Point $G$ be the mass center of disk $A$. Let $m$ be the mass of each rotor and $m'$ the magnitude of each added mass.



Since $\omega = \text{constant},$

$$\dot{H}_G = 0$$

for each disk. We treat the added masses as particles so that their moments of inertia about their mass centers are negligible. The effective force of disk $A$ is.

$$(F_A)_{\text{eff}} = -m\bar{r}_A\omega^2\mathbf{j}$$

The effective forces of the added masses are

$$m'r_B\omega^2\mathbf{j} \quad \text{and} \quad m'r_C\omega^2\mathbf{j},$$

respectively for the masses added to disks $B$ and $C$.

$$\Sigma F_{\text{eff}} = 0: \quad -m\bar{r}_A\omega^2\mathbf{j} + m'r_B\omega^2\mathbf{j} + m'r_C\omega^2\mathbf{j} = 0$$

$$r_B + r_C = \frac{m}{m'}\bar{r}_A \tag{1}$$

$$\Sigma(M_0)_{\text{eff}} = 0 \quad 4\mathbf{i}\times(-m\bar{r}_A\omega^2\mathbf{j}) + 10\mathbf{i}\times(m'r_B\omega^2\mathbf{j}) + 16\mathbf{i}\times(m'r_C\omega^2\mathbf{j})$$

$$10r_B + 16r_2 = 4\frac{m}{m'}\bar{r}_A \tag{2}$$

## PROBLEM 18.147 (Continued)

Data: $\bar{r}_A = \frac{1}{4}\text{ in.} = 0.25\text{ in.} \qquad \frac{m}{m'} = \frac{25\text{ lb}}{2\text{ lb}} = 12.5$

$$r_A + r_B = 3.125\text{ in.} \tag{1}'$$

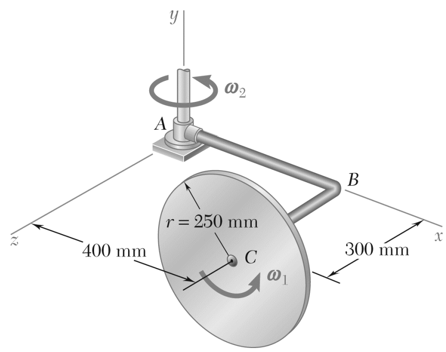
$$10r_A + 16r_B = 12.5\text{ in.} \tag{2}'$$

Solving the simultaneous equations.

$$r_B = 6.25\text{ in.} \qquad r_C = -3.125$$

Placement of added masses.

On $B$: $6\frac{1}{4}$ in. below shaft. On $C$: $3\frac{1}{8}$ in. above shaft ◀

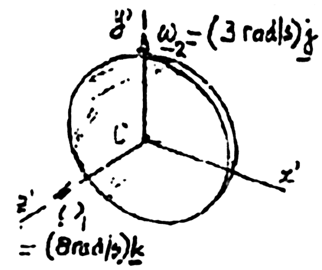



## PROBLEM 18.148

A homogeneous disk of mass $m = 5$ kg rotates at the constant rate $\omega_1 = 8$ rad/s with respect to the bent axle *ABC*, which itself rotates at the constant rate $\omega_2 = 3$ rad/s about the *y* axis. Determine the angular momentum $\mathbf{H}_C$ of the disk about its center *C*.

## SOLUTION

Using frame $Cx'y'z'$:




$$\bar{I}_{x'} = \bar{I}_{y'} = \frac{1}{4}mr^2$$

$$\bar{I}_{z'} = \frac{1}{2}mr^2$$

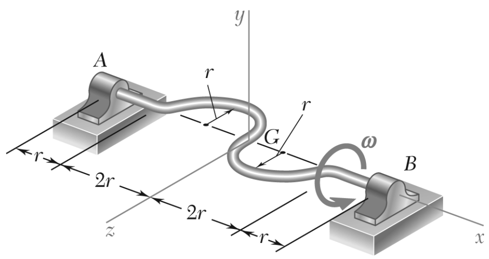
$$\mathbf{H}_C = \bar{I}_{y'}\omega_2\mathbf{j} + I_{z'}\omega_1\mathbf{k}$$

$$= \frac{1}{4}mr^2(\omega_2\mathbf{j} + 2\omega_1\mathbf{k})$$

$$\mathbf{H}_C = \frac{1}{4}(5 \text{ kg})(0.25 \text{ m})^2[(3 \text{ rad/s})\mathbf{j} + 2(8 \text{ rad/s})\mathbf{k}]$$

$$\mathbf{H}_C = (0.234 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (1.250 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k} \blacktriangleleft$$

## PROBLEM 18.149



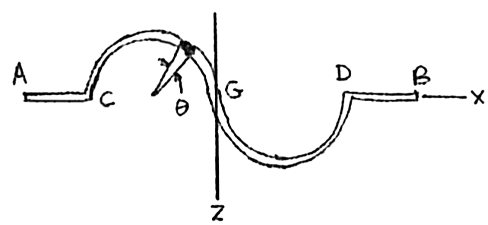

A rod of uniform cross section is used to form the shaft shown. Denoting by $m$ the total mass of the shaft and knowing that the shaft rotates with a constant angular velocity $\boldsymbol{\omega}$, determine (*a*) the angular momentum $\mathbf{H}_G$ of the shaft about its mass center $G$, (*b*) the angle formed by $\mathbf{H}_G$ and the axis $AB$, (*c*) the angular momentum of the shaft about Point $A$.

## SOLUTION

Length of rod: $L = 2r + 2\pi r = 8.2832r$

Angular velocity: $\boldsymbol{\omega} = \omega\mathbf{i}$

Angular momentum: $\mathbf{H}_G = \bar{I}_x\omega\mathbf{i} - \bar{I}_{xz}\omega\mathbf{k}$




Calculation of $\bar{I}_x$ and $\bar{I}_{xz}$:

Let $\rho =$ mass per unit length $= \dfrac{m}{L}$.

For portions $AC$ and $DB$,

$$I_x = 0, \quad I_{xz} = 0$$

For portion $CG$, use polar coordinate $\theta$.

$$x = -r(1-\cos\theta), \quad z = -r\sin\theta \quad dm = \rho r d\theta$$

$$I_x = \int z^2 dm = \int_0^\pi r^2 \sin^2\theta \rho r d\theta = \frac{\pi}{2}\rho r^3$$

$$I_{xz} = \int xz dm = \int_0^\pi r^2(1-\cos\theta)\sin\theta\rho r d\theta = 2\rho r^3$$

Likewise, for portion $GD$, $I_x = \dfrac{\pi}{2}\rho r^3, \quad I_{xz} = 2\rho r^3$

Total: $\bar{I}_x = \pi\rho r^3 = \dfrac{\pi m r^3}{L}, \quad \bar{I}_{xz} = 4\rho r^3 = \dfrac{4mr^3}{L}$

(*a*) <u>Angular momentum $\mathbf{H}_G$.</u> $\mathbf{H}_G = \dfrac{\pi m r^3}{L}\omega\mathbf{i} - \dfrac{4mr^3}{L}\omega\mathbf{k} = mr^2\omega(0.37927\mathbf{i} - 0.48291\mathbf{k})$

$$\mathbf{H}_G = mr^2\omega(0.379\mathbf{i} - 0.483\mathbf{k}) \blacktriangleleft$$

$$H_G = mr^2\omega(0.37927^2 + 0.48291^2)^{1/2} = 0.61404\, mr^2\omega$$

(*b*) Angle formed by $\mathbf{H}_G$ and the axis $AB$.

$$\mathbf{H}_G \cdot \mathbf{i} = 0.37927mr^2\omega$$

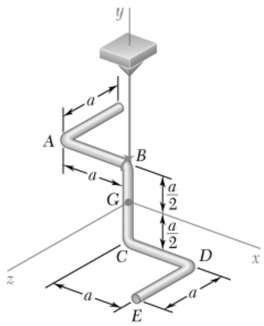
$$\cos\theta = \frac{\mathbf{H}_G \cdot \mathbf{i}}{H_G} = \frac{0.37927mr^2\omega}{0.61404mr^2\omega} = 0.61766 \qquad \theta = 51.9° \blacktriangleleft$$

(*c*) Angular momentum about Point $A$:

$$\mathbf{H}_A = \mathbf{H}_G + \mathbf{r}_{G/A} \times (m\overline{v})$$

$$\mathbf{H}_A = \mathbf{H}_G + 0 \qquad \mathbf{H}_A = mr^2\omega(0.379\mathbf{i} - 0.483\mathbf{k}) \blacktriangleleft$$

## PROBLEM 18.150




A uniform rod of mass $m$ and length $5a$ is bent into the shape shown and is suspended from a wire attached at Point $B$. Knowing that the rod is hit at Point $A$ in the negative $y$ direction and denoting the corresponding impulse by $-(F\Delta t)\mathbf{j}$, determine immediately after the impact ($a$) the velocity of the mass center $G$, ($b$) the angular velocity of the rod.

## SOLUTION

Moments and products of inertia:

| Part | $m$ | $I_x$ | $I_y$ | $I_z$ | $I_{xy}$ |
|---|---|---|---|---|---|
| $OA$ | $\frac{1}{5}m$ | $\left(\frac{1}{5}m\right)\times\left(\frac{1}{12}a^2+\frac{1}{4}a^2+\frac{1}{4}a^2\right)$ | $\left(\frac{1}{5}m\right)\times\left(\frac{1}{12}a^2+a^2+\frac{1}{4}a^2\right)$ | $\left(\frac{1}{5}m\right)\left(a^2+\frac{1}{4}a^2\right)$ | $\left(\frac{1}{5}m\right)\left(-\frac{1}{2}a^2\right)$ |
| $AB$ | $\frac{1}{5}m$ | $\left(\frac{1}{5}m\right)\left(\frac{1}{4}a^2\right)$ | $\left(\frac{1}{5}m\right)\left(\frac{1}{3}a^2\right)$ | $\left(\frac{1}{5}m\right)\times\left(\frac{1}{12}a^2+\frac{1}{4}a^2+\frac{1}{4}a^2\right)$ | $\left(\frac{1}{5}m\right)\left(-\frac{1}{4}a^2\right)$ |
| $BC$ | $\frac{1}{5}m$ | $\left(\frac{1}{5}m\right)\left(\frac{1}{12}a^2\right)$ | 0 | $\left(\frac{1}{5}m\right)\left(\frac{1}{12}a^2\right)$ | 0 |
| $CD$ | $\frac{1}{5}m$ | $\left(\frac{1}{5}m\right)\left(\frac{1}{4}a^2\right)$ | $\left(\frac{1}{5}m\right)\left(\frac{1}{3}a^2\right)$ | $\left(\frac{1}{5}m\right)\times\left(\frac{1}{12}a^2+\frac{1}{4}a^2+\frac{1}{4}a^2\right)$ | $\left(\frac{1}{5}m\right)\left(-\frac{1}{4}a^2\right)$ |
| $DE$ | $\frac{1}{5}m$ | $\left(\frac{1}{5}m\right)\times\left(\frac{1}{12}a^2+\frac{1}{4}a^2+\frac{1}{4}a^2\right)$ | $\left(\frac{1}{5}m\right)\times\left(\frac{1}{12}a^2+a^2+\frac{1}{4}a^2\right)$ | $\left(\frac{1}{5}m\right)\left(a^2+\frac{1}{4}a^2\right)$ | $\left(\frac{1}{5}m\right)\left(-\frac{1}{2}a^2\right)$ |
| $\Sigma$ | $m$ | $0.35ma^2$ | $0.66667ma^2$ | $0.75ma^2$ | $-0.3ma^2$ |

$$I_{xz} = \left(\frac{1}{5}m\right)\left(\frac{1}{2}a^2\right) + \left(\frac{1}{5}m\right)\left(\frac{1}{2}a^2\right) = 0.2ma^2$$

$$I_{yz} = \left(\frac{1}{5}m\right)\left(-\frac{1}{4}a^2\right) + \left(\frac{1}{5}m\right)\left(-\frac{1}{4}a^2\right) = -0.1ma^2$$

Angular momentum about the mass center.

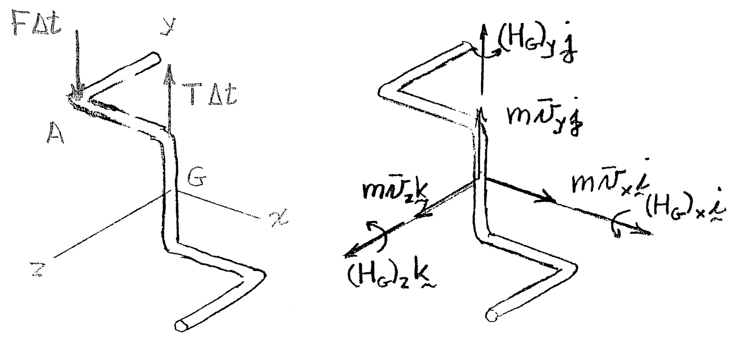
$$(H_G)_x = I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z = 0.35ma^2\omega_x + 0.3ma^2\omega_y - 0.2ma^2\omega_z$$

$$(H_G)_y = -I_{xy}\omega_x + I_y\omega_y - I_{yz}\omega_z = 0.3ma^2\omega_x + 0.66667ma^2\omega_y + 0.1ma^2\omega_z$$

$$(H_G)_z = -I_{xz}\omega_x - I_{yz}\omega_z + I_z\omega_z = -0.2ma^2\omega_x + 0.1ma^2\omega_y + 0.75ma^2\omega_z$$

Constraint of the supporting cable: $\bar{v}_y = 0$

Impulse-momentum principle: Before impact, $\bar{\mathbf{v}} = 0$, $\mathbf{H}_G = 0$.




(*a*) Linear momentum: $\mathbf{F}(\Delta t) + T\Delta t\mathbf{j} = m\bar{\mathbf{v}}$ Resolve into components.

$$0 = m\bar{v}_x, \qquad -F\Delta t + T\Delta t\mathbf{j} = 0, \qquad 0 = m\bar{v}_z$$

$$\bar{v}_x = 0, \qquad T\Delta t = F\Delta t, \qquad \bar{v}_z = 0 \qquad\qquad \bar{\mathbf{v}} = 0 \blacktriangleleft$$

(*b*) Angular momentum, moments about $G$: $\mathbf{r}_{A/G} \times \mathbf{F}\Delta t = \mathbf{H}_G$

$$\left(\frac{a}{2}\mathbf{j} - a\mathbf{i}\right) \times [-(F\Delta t)\mathbf{j}] = (aF\Delta t)\mathbf{k} = (H_G)_x\mathbf{i} + (H_G)_y\mathbf{j} + (H_G)_z\mathbf{k}$$

Using expressions for $(H_G)_x$, $(H_G)_y$, and $(H_G)_z$ and resolving into components,

$$\mathbf{i}: \quad 0 = 0.35ma^2\omega_x + 0.3ma^2\omega_y - 0.2ma^2\omega_z$$

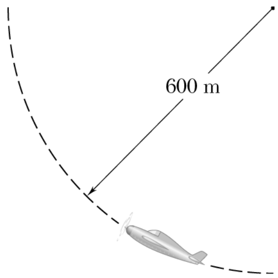
$$\mathbf{j}: \quad 0 = 0.3ma^2\omega_x + 0.66667ma^2\omega_y + 0.1ma^2\omega_z$$

$$\mathbf{k}: \quad aF\Delta t = -0.2ma^2\omega_x + 0.1ma^2\omega_y + 0.75ma^2\omega_z$$

Solving, $\omega_x = 2.5\dfrac{(F\Delta t)}{ma}$, $\omega_y = -1.454\dfrac{(F\Delta t)}{ma}$, $\omega_z = 2.19\dfrac{(F\Delta t)}{ma}$

$$\boldsymbol{\omega} = \left(\frac{F\Delta t}{ma}\right)(2.50\mathbf{i} - 1.454\mathbf{j} + 2.19\mathbf{k}) \blacktriangleleft$$

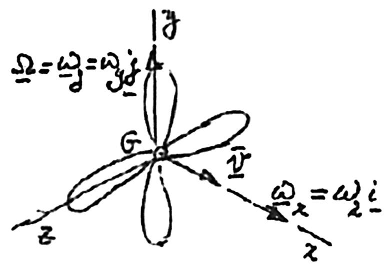
## PROBLEM 18.151




A four-bladed airplane propeller has a mass of 160 kg and a radius of gyration of 800 mm. Knowing that the propeller rotates at 1600 rpm as the airplane is traveling in a circular path of 600-m radius at 540 km/h, determine the magnitude of the couple exerted by the propeller on its shaft due to the rotation of the airplane.

## SOLUTION

We assume senses shown for $\boldsymbol{\omega}_x$, $\boldsymbol{\omega}_y$, and $\mathbf{v}$.



$$\bar{v} = 540 \text{ km/h} = 150 \text{ m/s}$$

$$\omega_x = 1600 \text{ rpm}\left(\frac{2\pi \text{ rad}}{60 \text{ s}}\right)$$

$$= 167.55 \text{ rad/s}$$

$$\omega_y = \frac{\bar{v}}{\rho} = \frac{150 \text{ m/s}}{600 \text{ m}} = 0.25 \text{ rad/s}$$

$$\bar{I}_x = m\bar{k}^2 = (160 \text{ kg})(0.8 \text{ m})^2 = 102.4 \text{ kg} \cdot \text{m}^2$$

Angular momentum about $G$:

$$\mathbf{H}_G = \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j}$$

Eq. (18.22),

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \mathbf{\Omega} \times \mathbf{H}_G = 0 + \omega_y \mathbf{j} \times (\bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j})$$

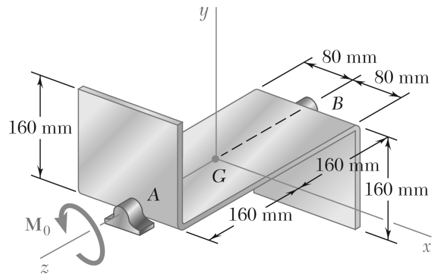
$$\dot{\mathbf{H}}_G = -\bar{I}_x \omega_x \omega_y \mathbf{k} = -(102.4 \text{ kg} \cdot \text{m}^2)(167.55 \text{ rad/s})(0.25 \text{ rad/s})\mathbf{k}$$

$$\dot{\mathbf{H}}_G = -(4289 \text{ N} \cdot \text{m})\mathbf{k} = -(4.29 \text{ kN} \cdot \text{m})\mathbf{k}$$

The couple exerted *on* the propeller, therefore, must be $\mathbf{M} = \dot{\mathbf{H}}_G = -(4.29 \text{ kN} \cdot \text{m})\mathbf{k}$, and the couple exerted *by* the propeller on its shaft is $-\mathbf{M} = (4.29 \text{ kN} \cdot \text{m})\mathbf{k}$.

Magnitude of couple. $4.29 \text{ kN} \cdot \text{m}$ ◀

## PROBLEM 18.152




A 2.4-kg piece of sheet steel with dimensions $160 \times 640$ mm was bent to form the component shown. The component is at rest $(\omega = 0)$ when a couple $\mathbf{M}_0 = (0.8\ \text{N}\cdot\text{m})\mathbf{k}$ is applied to it. Determine (*a*) the angular acceleration of the component, (*b*) the dynamic reactions at *A* and *B* immediately after the couple is applied.

## SOLUTION

$$m = 2.4\ \text{kg}, \quad b = 160\ \text{mm} = 0.16\ \text{m}$$

Area of sheet metal:
$$A = b^2 + (2b)b + b^2 = 4b^2 = 0.1024\ \text{m}^2$$

Let
$$\rho = \frac{m}{A} = \frac{2.4}{0.1024}$$
$$= 23.4375\ \text{kg/m}^2$$
$$= \text{mass per unit area}$$

Angular velocity and angular acceleration:
$$\boldsymbol{\omega} = \omega\mathbf{k}$$
$$\alpha = \alpha\mathbf{k}$$

Angular momentum about *G*:
$$\mathbf{H}_G = -I_{xz}\omega\mathbf{i} - I_{yz}\omega\mathbf{j} + I_z\omega\mathbf{k}$$
$$= -I_{yz}\omega\mathbf{j} + I_z\omega\mathbf{k}$$

Let the frame of reference *Gxyz* be rotating with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega} = \omega\mathbf{k}$

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G$$
$$= -I_{yz}\alpha\mathbf{j} + I_z\alpha\mathbf{k} + I_{yz}\omega^2\mathbf{i}$$

Required moment and product of inertia: $I_{\text{mass}} = \rho I_{\text{area}}$

| Part | $A$ | $I_z$ | $I_{yz}$ |
|---|---|---|---|
| plate *A* | $b^2$ | $\frac{1}{6}b^2 + b^2\left(\frac{1}{2}b\right)^2$ | $b^2(b)\left(\frac{1}{2}b\right)$ |
| plate *AB* | $2b^2$ | $\frac{1}{12}(2b)b^3$ | 0 |
| plate *B* | $b^2$ | $\frac{1}{6}b^2 + b^2\left(\frac{1}{2}b\right)^2$ | $b^2(-b)\left(-\frac{1}{2}b\right)$ |
| $\Sigma$ | $4b^2$ | $b^4$ | $b^4$ |

$$I_z = \rho b^4$$
$$= (23.4375)(0.16)^4$$
$$= 0.01536\ \text{kg}\cdot\text{m}^2$$
$$I_{yz} = \rho b^4$$
$$= 0.01536\ \text{kg}\cdot\text{m}^2$$

Since the mass center lies on the rotation axis, $\bar{\mathbf{a}} = 0$

$$\Sigma\mathbf{F} = \mathbf{A} + \mathbf{B} = m\bar{\mathbf{a}}$$

$$\mathbf{B} = -\mathbf{A}$$

$$\begin{aligned}\Sigma\mathbf{M}_G &= M_0\mathbf{k} + b\mathbf{k}\times\mathbf{A} + (-b\mathbf{k})\times\mathbf{B} \\ &= M_0\mathbf{k} + 2b\mathbf{k}\times(A_x\mathbf{i} + A_y\mathbf{j}) \\ &= M_0\mathbf{k} + 2bA_x\mathbf{j} - 2bA_y\mathbf{i}\end{aligned}$$

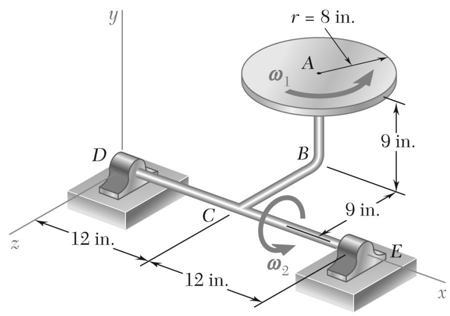
$\Sigma\mathbf{M}_G = \dot{\mathbf{H}}_G$ Resolve into components.

(*a*) $\mathbf{k}$: $M_0 = I_z\alpha \quad \alpha = \dfrac{M_0}{I_z} = \dfrac{0.8}{0.01536} = 52.083 \text{ rad/s}^2$ $\qquad \alpha = 52.1 \text{ rad/s}^2$ ◀

(*b*) $\mathbf{j}$: $2bA_x = -I_{yz}\alpha \quad A_x = -\dfrac{I_{yz}\alpha}{2b} = -\dfrac{(0.01536)(52.083)}{(2)(0.16)} = -2.50 \text{ N}$

$\mathbf{i}$: $-2bA_y = I_{yz}\omega^2 = 0 \qquad A_y = 0$ $\qquad \mathbf{A} = -(2.50 \text{ N})\mathbf{i}$ ◀

$\mathbf{B} = (2.50 \text{ N})\mathbf{i}$ ◀




## PROBLEM 18.153

A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm $ABC$, which is welded to a shaft $DCE$ rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the dynamic reactions at $D$ and $E$.

## SOLUTION

Angular velocity of shaft $DE$ and arm $CBA$: $\quad \mathbf{\Omega} = \omega_2 \mathbf{i}$

Angular velocity of disk $A$: $\quad \boldsymbol{\omega} = \omega_2 \mathbf{i} + \omega_1 \mathbf{j}$

Angular velocity about its mass center $A$:

$$\begin{aligned}\mathbf{H}_A &= \bar{I}_x \omega_x \mathbf{i} + \bar{I}_y \omega_y \mathbf{j} + \bar{I}_z \omega_z \mathbf{k} \\ &= \bar{I}_x \omega_2 \mathbf{i} + \bar{I}_y \omega_1 \mathbf{j} \\ &= \frac{1}{4} m r^2 \omega_2 \mathbf{i} + \frac{1}{2} m r^2 \omega_1 \mathbf{j}\end{aligned}$$

Let the reference frame $Oxyz$ be rotating with angular velocity $\mathbf{\Omega} = \omega_2 \mathbf{i}$.

$$\begin{aligned}\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Oxyz} + \mathbf{\Omega} \times \mathbf{H}_A \\ &= \frac{1}{4} m r^2 \dot{\omega}_2 \mathbf{i} + \frac{1}{2} m r^2 \dot{\omega}_1 \mathbf{j} + \omega_2 \mathbf{i} \times \mathbf{H}_A \\ &= \frac{1}{4} m r^2 \dot{\omega}_2 \mathbf{i} + \frac{1}{2} m r^2 \dot{\omega}_1 \mathbf{j} + \frac{1}{2} m r^2 \omega_1 \omega_2 \mathbf{k}\end{aligned}$$

Velocity of Point $A$:

$$\begin{aligned}\mathbf{v}_A &= \omega_2 \mathbf{i} \times \mathbf{r}_{A/O} \\ &= \omega_2 \mathbf{i} \times (-b\mathbf{k} + c\mathbf{j}) \\ &= b\omega_2 \mathbf{j} + c\omega_2 \mathbf{k}\end{aligned}$$

Acceleration of Point $A$:

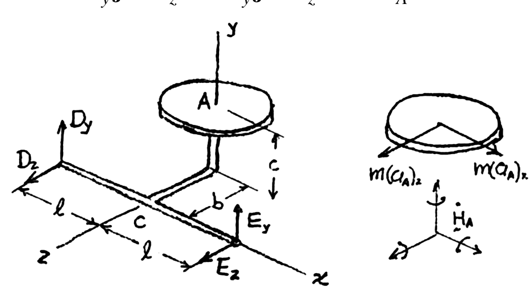
$$\mathbf{a}_A = \dot{\omega}_2 \mathbf{j} \times \mathbf{r}_{A/O} + \omega_2 \mathbf{j} \times \mathbf{v}_A$$

$$\mathbf{a}_A = (b\dot{\omega}_2 - c\omega_2^2)\mathbf{j} + (c\dot{\omega}_2 + b\omega_2)\mathbf{k}$$

Consider the system of particles consisting of the shaft, the arm, and the disk. Neglect the mass of the arm.

$$\Sigma \mathbf{F} = m\mathbf{a}_A$$

$$D_y \mathbf{j} + D_z \mathbf{k} + E_y \mathbf{j} + E_z \mathbf{k} = m\mathbf{a}_A$$



Resolve into components.

$$D_y + E_y = m(b\dot{\omega}_2 - c\omega_2^2)$$

$$D_z + E_z = m(c\dot{\omega}_2 + b\omega_2^2)$$

$$\Sigma \mathbf{M}_D = \dot{\mathbf{H}}_D = \dot{\mathbf{H}}_A + \mathbf{r}_{A/D} \times m\mathbf{a}_A$$

$$(M_0)\mathbf{i} + 2l\mathbf{i} \times (E_y\mathbf{j} + E_z\mathbf{k}) = \dot{\mathbf{H}}_A + (l\mathbf{i} + c\mathbf{j} - b\mathbf{k}) \times m\mathbf{a}_A$$

$$(M_0)\mathbf{i} - 2lE_z\mathbf{j} + 2lE_y\mathbf{k} = m\left(\frac{1}{4}r^2 + b^2 + c^2\right)\dot{\omega}_2\mathbf{i} + m\left(\frac{1}{2}r^2\dot{\omega}_1 - lc\dot{\omega}_2 - lb\omega_2^2\right)\mathbf{j}$$

$$+ m\left(\frac{1}{2}r^2\omega_1\omega_2 + lb\dot{\omega}_2 - lc\omega_2^2\right)\mathbf{k}$$

**i**: $$M_0 = m\left(\frac{1}{4}r^2 + b^2 + c^2\right)\dot{\omega}_2$$

**k**: $$E_y = \frac{m}{2l}\left(\frac{1}{2}r^2\omega_1\omega_2 + lb\dot{\omega}_2 - lc\omega_2^2\right)$$

$$D_y = \frac{m}{2l}\left(-\frac{1}{2}r^2\omega_1\omega_2 + lb\dot{\omega}_2 - lc\omega_2^2\right)$$

**j**: $$E_z = \frac{m}{2l}\left(lc\dot{\omega}_2 + lb\omega_2^2 - \frac{1}{2}r^2\dot{\omega}_1\right)$$

$$D_z = \frac{m}{2l}\left(lc\dot{\omega}_2 + lb\omega_2^2 + \frac{1}{2}r^2\dot{\omega}_1\right)$$

<u>Data</u>: $W = 6$ lb. $\quad m = \dfrac{6}{32.2} = 0.186335 \quad r = 8$ in. $= 0.66667$ ft

$b = c = 9$ in. $= 0.75$ ft $\quad l = 12$ in. $= 1.0$ ft

$\omega_1 = 16$ rad/s, $\quad \dot{\omega}_1 = 0, \quad \omega_2 = 8$ rad/s, $\quad \dot{\omega}_2 = 0$

$$D_y = \frac{0.186335}{(2)(1.0)}\left[-\left(\frac{1}{2}\right)(0.66667)^2(16)(8) + 0 - (1.0)(0.75)(8)^2\right] = -7.12 \text{ lb}$$

$$D_z = \frac{0.186335}{(2)(1.0)}[0 + (1.0)(0.75)(8)^2 + 0] = 4.47 \text{ lb}$$

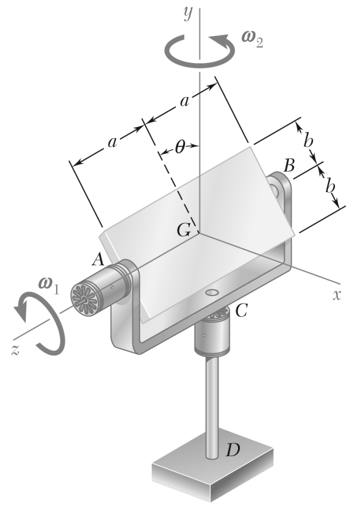
$\mathbf{D} = -(7.12 \text{ lb})\mathbf{j} + (4.47 \text{ lb})\mathbf{k}$ ◀

$$E_y = \frac{0.186335}{(2)(1.0)}\left[\left(\frac{1}{2}\right)(0.66667)^2(16)(8) + 0 - (1.0)(0.75)(8)^2\right] = -1.822 \text{ lb}$$

$$E_z = \frac{0.186335}{(2)(1.0)}[0 + (1.0)(0.75)(8)^2 + 0] = 4.47 \text{ lb}$$

$\mathbf{E} = -(1.822 \text{ lb})\mathbf{j} + (4.47 \text{ lb})\mathbf{k}$ ◀

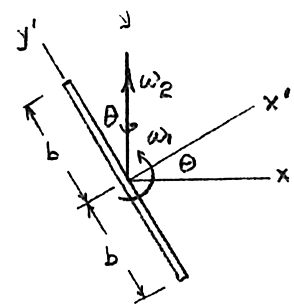
## PROBLEM 18.154




A 48-kg advertising panel of length $2a = 2.4$ m and width $2b = 1.6$ m is kept rotating at a constant rate $\omega_1$ about its horizontal axis by a small electric motor attached at $A$ to frame $ACB$. This frame itself is kept rotating at a constant rate $\omega_2$ about a vertical axis by a second motor attached at $C$ to the column $CD$. Knowing that the panel and the frame complete a full revolution in 6 s and 12 s, respectively, express, as a function of the angle $\theta$, the dynamic reaction exerted on column $CD$ by its support at $D$.

## SOLUTION

Use principal axes $x'$, $y'$, $z'$ as shown.




Moments of inertia: $\bar{I}_{x'} = \frac{1}{3}m(a^2 + b^2)$

$\bar{I}_{y'} = \frac{1}{3}ma^2, \quad \bar{I}_{z'} = \frac{1}{3}mb^2$

Kinematics: $\dot{\theta} = \omega_1$

$$\omega_{x'} = \omega_2 \sin\theta, \quad \omega_{y'} = \omega_2 \cos\theta, \quad \omega_{z'} = \omega_1$$

$$\dot{\omega}_{x'} = \omega_1\omega_2 \cos\theta, \quad \dot{\omega}_{y'} = -\omega_1\omega_2 \sin\theta, \quad \dot{\omega}_{z'} = 0$$

Since the acceleration of the mass center is zero, the resultant force acting on the column $CD$ is zero.

$\mathbf{R} = 0$ ◀

Euler's equations of motion for the plate:

$$\Sigma M_{x'} = \bar{I}_{x'}\dot{\omega}_{x'} - (\bar{I}_{y'} - \bar{I}_{z'})\omega_{y'}\omega_{z'}$$
$$= \bar{I}_{x'}\omega_1\omega_2 \cos\theta - (\bar{I}_{y'} - \bar{I}_{z'})\omega_1\omega_2 \cos\theta$$
$$= (\bar{I}_{x'} + \bar{I}_{z'} - \bar{I}_{y'})\omega_1\omega_2 \cos\theta = \frac{2}{3}mb^2\omega_1\omega_2 \cos\theta$$

$$\Sigma M_{y'} = \bar{I}_{y'}\dot{\omega}_{y'} - (\bar{I}_{z'} - \bar{I}_{x'})\omega_{x'}\omega_{z'}$$
$$= -\bar{I}_y\omega_1\omega_2 \sin\theta - (\bar{I}_z - \bar{I}_x)\omega_1\omega_2 \sin\theta$$
$$= (\bar{I}_{x'} - \bar{I}_{y'} - \bar{I}_{z'})\omega_1\omega_2 \sin\theta = 0$$

$$\Sigma M_{z'} = I_{z'}\dot{\omega}_{z'} - (\bar{I}_{x'} - \bar{I}_y)\omega_{x'}\omega_{y'}$$
$$= 0 - (\bar{I}_{x'} - \bar{I}_y)\omega_2^2 \sin\theta\cos\theta$$
$$= -\frac{1}{3}mb^2\omega_2^2 \sin\theta\cos\theta$$

$$\Sigma \mathbf{M} = \frac{2}{3}mb^2\omega_1\omega_2 \cos\theta \mathbf{i}' - \frac{1}{3}mb^2\omega_2^2 \sin\theta\cos\theta \mathbf{k}$$

$$= \frac{2}{3}mb^2\omega_1\omega_2 \cos\theta(\cos\theta \mathbf{i} + \sin\theta \mathbf{j}) - \frac{1}{3}mb^2\omega_2^2 \sin\theta\cos\theta \mathbf{k}$$

Resolve into components: $\Sigma M_x = \frac{2}{3}mb^2\omega_1\omega_2 \cos^2\theta$

$$\Sigma M_y = \frac{2}{3}mb^2\omega_1\omega_2 \sin\theta\cos\theta$$

$$\Sigma M_z = -\frac{1}{3}mb^2\omega_2^2 \sin\theta\cos\theta$$

Data: $m = 48 \text{ kg}, \qquad b = 0.8 \text{ m}$

$$\omega_1 = \frac{2\pi}{6} = 1.0472 \text{ rad/s}, \quad \omega_2 = \frac{2\pi}{12} = 0.5236 \text{ rad/s}$$

$$\Sigma M_x = \frac{2}{3}(48)(0.8)^2(1.0472)(0.5236)\cos^2\theta$$

$$= 11.23\cos^2\theta \text{ N}\cdot\text{m}$$

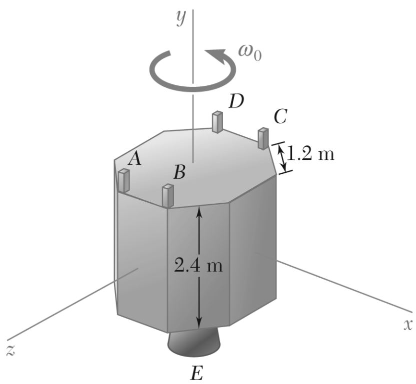
$$\Sigma M_y = \frac{2}{3}(48)(0.8)^2(1.0472)(0.5236)\sin\theta\cos\theta$$

$$= 11.23\sin\theta\cos\theta \text{ N}\cdot\text{m}$$

$$\Sigma M_z = -\frac{1}{3}(48)(0.8)^2(0.5236)^2 \sin\theta\cos\theta$$

$$= -2.81\sin\theta\cos\theta \text{ N}\cdot\text{m}$$

$$\mathbf{M}_D = (11.23 \text{ N}\cdot\text{m})\cos^2\theta \mathbf{i} + (11.23 \text{ N}\cdot\text{m})\sin\theta\cos\theta \mathbf{j} - (2.81 \text{ N}\cdot\text{m})\sin\theta\cos\theta \mathbf{k} \blacktriangleleft$$




## PROBLEM 18.155

A 2500-kg satellite is 2.4 m high and has octagonal bases of sides 1.2 m. The coordinate axes shown are the principal centroidal axes of inertia of the satellite, and its radii of gyration are $k_x = k_z = 0.90$ m and $k_y = 0.98$ m. The satellite is equipped with a main 500-N thruster $E$ and four 20-N thrusters $A$, $B$, $C$, and $D$, which can expel fuel in the positive $y$ direction. The satellite is spinning at the rate of 36 rev/h about its axis of symmetry $Gy$, which maintains a fixed direction in space, when thrusters $A$ and $B$ are activated for 2 s. Determine ($a$) the precession axis of the satellite, ($b$) its rate of precession, ($c$) its rate of spin.

## SOLUTION

$$\omega_0 = (36 \text{ rev/h})\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right)$$

$$= 0.062832 \text{ rad/s}$$

$$m = 2500 \text{ kg}$$

$$I_x = mk_x^2 = (2500)(0.90)^2 = 2025 \text{ kg}\cdot\text{m}^2$$

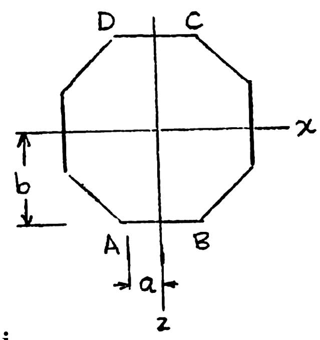
$$I_z = I_x = 2025 \text{ kg}\cdot\text{m}^2$$

$$I_y = mk_y^2 = (2500)(0.98)^2 = 2401 \text{ kg}\cdot\text{m}^2$$

$$(\mathbf{H}_G)_0 = I_y\omega_0\mathbf{j} = (2401)(0.062832)\mathbf{j} = (150.86 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j}$$

$$a = 0.6 \text{ m}, \quad b = 0.6 + 1.2 \cos 45° = 1.4485$$

When thrusters $A$ and $B$ are activated,




$$\mathbf{M}_G = -b(F_A + F_B)\mathbf{i}$$

$$= -(1.4485)(40)\mathbf{i}$$

$$= -(57.941 \text{ N}\cdot\text{m})\mathbf{i}$$

Angular momentum after 2 s:

$$\mathbf{H}_G = (\mathbf{H}_G)_0 + \mathbf{M}_G(\Delta t)$$

$$= 150.86\mathbf{j} + (-57.941)(2)\mathbf{i}$$

$$= -(115.88 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} + (150.86 \text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j}$$

$$\omega_x = \frac{H_x}{I_x} = -\frac{115.88}{2025} = -0.057225 \text{ rad/s} = -32.788 \text{ rev/h}$$

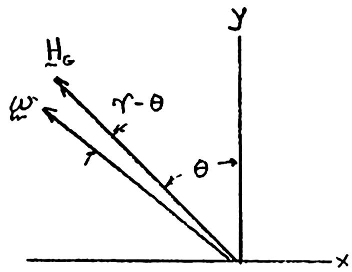
$$\omega_y = \frac{H_y}{I_y} = 36 \text{ rev/h}$$

$$\omega_z = \frac{H_z}{I_z} = 0$$

## PROBLEM 18.155 (Continued)

(*a*) Precession axis:

$$\tan\theta = -\frac{H_x}{H_y} = \frac{115.88}{150.86} \quad \theta = 37.529^\circ \qquad \theta_x = 52.5^\circ,\ \ \theta_y = 37.5^\circ,\ \ \theta_z = 90^\circ \blacktriangleleft$$




$$\tan\gamma = -\frac{\omega_x}{\omega_y}$$

$$= \frac{32.788}{36}$$

$$\gamma = 42.327^\circ$$

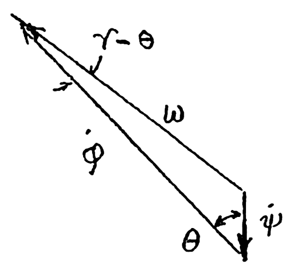
$$\gamma - \theta = 4.798^\circ$$

$$\omega = \sqrt{\omega_x^2 + \omega_y^2}$$

$$= 48.693 \text{ rev/h}$$

<u>Law of sines</u>.

$$\frac{\dot\varphi}{\sin\gamma} = \frac{\dot\psi}{\sin(\gamma-\theta)} = \frac{\omega}{\sin\theta}$$

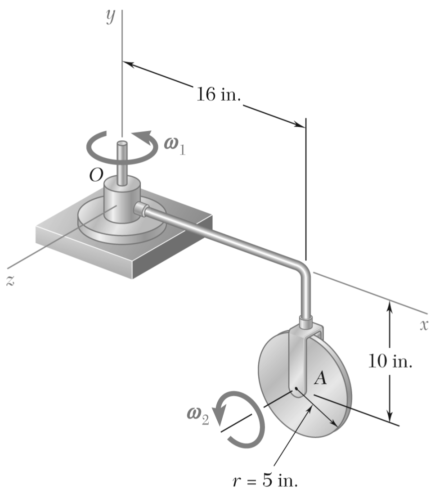



(*b*)
$$\dot\varphi = \frac{48.693\sin 42.327^\circ}{\sin 37.529^\circ}$$

$$\dot\varphi = 53.8 \text{ rev/h} \blacktriangleleft$$

(*c*)
$$\dot\psi = \frac{48.693\sin 4.798^\circ}{\sin 37.529^\circ} \qquad \dot\psi = 6.68 \text{ rev/h} \blacktriangleleft$$

## PROBLEM 18.156




A thin disk of weight $W = 8$ lb rotates with an angular velocity $\omega_2$ with respect to arm $OA$, which itself rotates with an angular velocity $\omega_1$ about the $y$ axis. Determine (*a*) the couple $M_1\mathbf{j}$ which should be applied to arm $OA$ to give it an angular acceleration $\boldsymbol{\alpha}_1 = (6 \text{ rad/s}^2)\mathbf{j}$ with $\omega_1 = 4$ rad/s, knowing that the disk rotates at the constant rate $\omega_2 = 12$ rad/s, (*b*) the force-couple system representing the dynamic reaction at $O$ at that instant. Assume that arm $OA$ has a negligible mass.

## SOLUTION

Angular velocity of arm $OA$: $\quad \boldsymbol{\Omega} = \omega_1\mathbf{j}$

Angular velocity of disk $A$: $\quad \boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{k}$

Angular momentum about its mass center $A$:

$$\begin{aligned}\mathbf{H}_A &= \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k} \\ &= \bar{I}_y\omega_1\mathbf{j} + \bar{I}_z\omega_2\mathbf{k} \\ &= \frac{1}{4}mr^2\omega_1\mathbf{j} + \frac{1}{2}mr^2\omega_2\mathbf{k}\end{aligned}$$

Let the reference $Oxyz$ be rotating with angular velocity $\boldsymbol{\Omega} = \omega_2\mathbf{j}$.

$$\begin{aligned}\dot{\mathbf{H}}_A &= (\dot{\mathbf{H}}_A)_{Oxyz} + \boldsymbol{\Omega}\times\mathbf{H}_A \\ &= \frac{1}{4}mr^2\dot{\omega}_1\mathbf{j} + \frac{1}{2}mr^2\dot{\omega}_2\mathbf{k} + \omega_1\mathbf{j}\times\mathbf{H}_A \\ &= \frac{1}{2}mr^2\omega_1\omega_2\mathbf{i} + \frac{1}{4}mr^2\dot{\omega}_1\mathbf{j} + \frac{1}{2}mr^2\dot{\omega}_2\mathbf{k}\end{aligned}$$

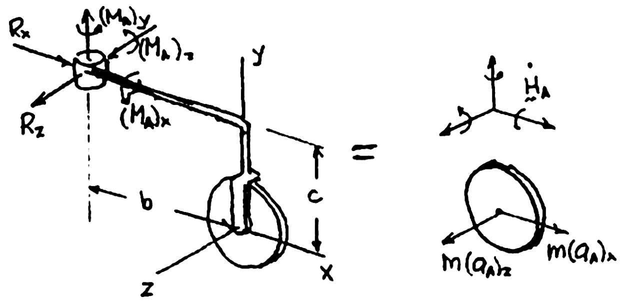
Velocity of Point $A$:

$$\begin{aligned}\mathbf{v}_A &= \omega_1\mathbf{j}\times\mathbf{r}_{A/O} \\ &= \omega_1\mathbf{j}\times(b\mathbf{i} - c\mathbf{j}) \\ &= -b\omega_1\mathbf{k}\end{aligned}$$

Acceleration of Point $A$:

$$\begin{aligned}\mathbf{a}_A &= \dot{\omega}_1\mathbf{j}\times\mathbf{r}_{A/O} + \omega_1\mathbf{j}\times\mathbf{v}_A \\ &= -b\omega_1^2\mathbf{i} - b\dot{\omega}_1\mathbf{k}\end{aligned}$$

Consider the system of particles consisting of the arm $OA$ and disk $A$. Neglect the mass of the arm.




$$\Sigma \mathbf{F} = m\mathbf{a}_A:$$

$$R_x\mathbf{i} + R_z\mathbf{k} = -mb\omega_1^2\mathbf{i} - mb\dot{\omega}_1\mathbf{k}$$

$$R_x = -mb\omega_1^2$$

$$R_z = -mb\dot{\omega}_1$$

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O = \dot{\mathbf{H}}_A + \mathbf{r}_{A/O} \times m\mathbf{a}_A$$

$$(M_O)_x\mathbf{i} + (M_O)_y\mathbf{j} + (M_O)_z\mathbf{k} = m\left(\frac{1}{2}r^2\omega_1\omega_2 + bc\dot{\omega}_1\right)\mathbf{i}$$

$$+ m\left(\frac{1}{4}r^2 + b^2\right)\dot{\omega}_1\mathbf{j} + m\left(\frac{1}{2}r^2\dot{\omega}_2 - bc\omega_1^2\right)\mathbf{k}$$

Data:

$$W = 8 \text{ lb}$$

$$m = \frac{8}{32.2} = 0.24845 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$r = 5 \text{ in.} = 0.41667 \text{ ft}$$

$$b = 16 \text{ in.} = 1.33333 \text{ ft}$$

$$c = 10 \text{ in.} = 0.83333 \text{ ft}$$

$$\omega_1 = 4 \text{ rad/s}$$

$$\dot{\omega}_1 = 6 \text{ rad/s}^2$$

$$\omega_2 = 12 \text{ rad/s}$$

$$\dot{\omega}_2 = 0$$

($a$) Required couple: The required couple is the $y$-component of the couple at Point $O$.

$$(\mathbf{M}_O)_y = (0.24845)\left[\frac{1}{4}(0.41667)^2 + (1.33333)^2\right](6)\mathbf{j} \qquad (\mathbf{M}_O)_y = (2.71 \text{ lb}\cdot\text{ft})\mathbf{j} \blacktriangleleft$$

## PROBLEM 18.156 (Continued)

(*b*) Dynamic reaction at Point *O*.

$$R_x = -(0.24845)(1.33333)(4)^2 = -5.30 \text{ lb}, \qquad R_y = 0$$

$$R_z = -(0.24845)(1.33333)(6) = -1.988 \text{ lb}$$

$$\mathbf{R} = -(5.30 \text{ lb})\mathbf{i} - (1.988 \text{ lb})\mathbf{k} \blacktriangleleft$$

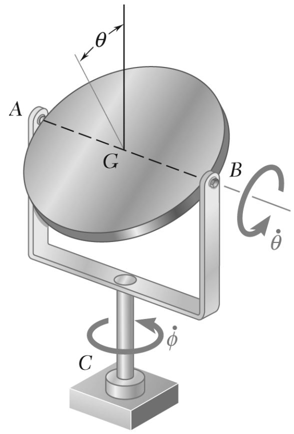
$$(M_O)_x = (0.24845)\left[\left(\frac{1}{2}\right)(0.41667)^2(4)(12) + (1.33333)(0.83333)(6)\right]$$

$$= 2.69 \text{ lb} \cdot \text{ft}$$

$$(M_O)_z = (0.24845)[0 - (1.33333)(0.83333)(4)^2]$$

$$= -4.42 \text{ lb} \cdot \text{ft}$$

$$\mathbf{M}_O = (2.69 \text{ lb} \cdot \text{ft})\mathbf{i} - (4.42 \text{ lb} \cdot \text{ft})\mathbf{k} \blacktriangleleft$$

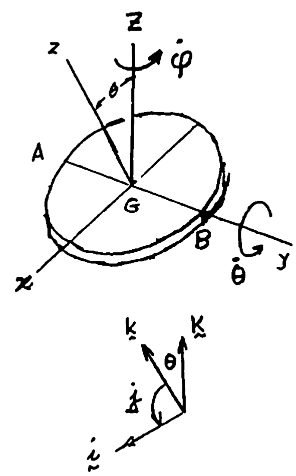



## PROBLEM 18.157

A homogeneous disk of mass $m$ connected at $A$ and $B$ to a fork-ended shaft of negligible mass which is supported by a bearing at $C$. The disk is free to rotate about its horizontal diameter $AB$ and the shaft is free to rotate about a vertical axis through $C$. Initially, the disk lies in a vertical plane $(\theta_0 = 90°)$ and the shaft has an angular velocity $\dot{\phi}_0 = 8$ rad/s. If the disk is slightly disturbed, determine for the ensuring motion (*a*) the minimum value of $\dot{\phi}$, (*b*) the maximum value of $\dot{\theta}$.

## SOLUTION

Place the origin at the center of mass and let $Oxyz$ be a principal axis frame of reference with the $y$ axis directed along the moving axle $AB$. Let the $Z$ axis lie along the fixed axle. Useful unit vectors are **i**, **j** and **k** along the $x$, $y$, $z$ axes and **K** along the $Z$ axis.




$$\mathbf{K} = -\mathbf{i}\sin\theta + \mathbf{k}\cos\theta$$

Angular velocity:

$$\boldsymbol{\omega} = \dot{\varphi}\mathbf{k} + \dot{\theta}\mathbf{j}$$

$$\boldsymbol{\omega} = -\dot{\varphi}\sin\theta\mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\varphi}\cos\theta\mathbf{k}$$

Moments of inertia:

$$I_x = e_1 I_y, \qquad I_z = e_2 I_y$$

<u>Angular momentum about $O$.</u>

$$\mathbf{H}_O = I_x\omega_x\mathbf{i} + I_y\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

$$= I_y(-e_1\dot{\varphi}\sin\theta\,\mathbf{i} + \dot{\theta}\,\mathbf{j} + e_2\dot{\varphi}\cos\theta\,\mathbf{k})$$

The moment about the fixed $Z$ axis is zero, hence, $\mathbf{H}_O \cdot \mathbf{K} = \text{constant}$.

$$\mathbf{H}_O \cdot \mathbf{K} = I_y(e_1\sin^2\theta + e_2\cos^2\theta)\dot{\varphi} = I_y C_1$$

$$\dot{\varphi} = \frac{C_1}{e_1\sin^2\theta + e_2\cos^2\theta}$$

$$C_1 = (e_1\sin^2\theta_0 + e_2\cos^2\theta_0)\dot{\varphi}_0$$

Twice the kinetic energy:

$$2T = I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = \text{constant}$$

$$2T = I_y[(e_1\sin^2\theta + e_2\cos^2\theta)\dot{\varphi}^2 + \dot{\theta}^2]$$

$$= I_y(C_1\dot{\varphi} + \dot{\theta}^2)$$

$$= I_y C_2$$

$$\dot{\theta}^2 = C_2 - C_1\dot{\varphi}$$

$$C_2 = \dot{\theta}_0^2 + C_1\dot{\varphi}_0$$

## PROBLEM 18.157 (Continued)

Data:

$$e_1 = 1$$
$$e_2 = 2$$
$$e_1 \sin^2\theta + e_2 \cos^2\theta = 1 + \cos^2\theta$$
$$\theta_0 = 90°$$
$$\dot{\theta}_0 = 0$$
$$\dot{\varphi}_0 = 16 \text{ rad/s}$$

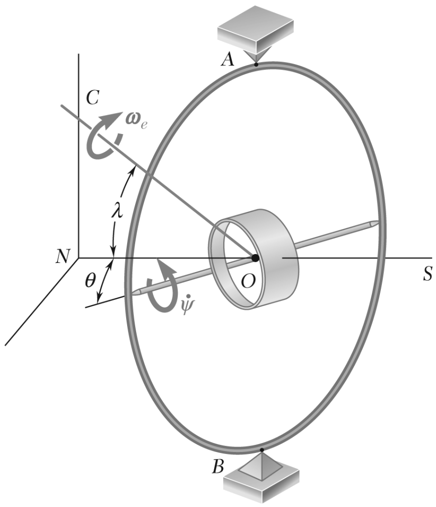
$$C_1 = (1 + \cos^2 90°)(8)$$
$$= 8 \text{ rad/s}$$
$$\dot{\varphi} = \frac{8}{1 + \cos\theta}$$

(*a*) $$\dot{\varphi}_{\min} = \frac{8}{1 + \cos\theta} = 4$$ $\dot{\varphi}_{\min} = 4.00$ rad/s ◀

$$C_2 = 0 + (8)(8) = 64 \text{ (rad/s)}^2$$
$$\dot{\theta}^2 = C_2 - C_1\dot{\varphi}$$

(*b*) $$(\dot{\theta}^2)_{\max} = C_2 - C_1\dot{\varphi}_{\min}$$
$$= 64 - (8)(4)$$
$$= 32 \text{ (rad/s)}^2$$ $\dot{\theta}_{\max} = 5.66$ rad/s ◀

## PROBLEM 18.158




The essential features of the gyrocompass are shown. The rotor spins at the rate $\dot{\psi}$ about an axis mounted in a single gimbal, which may rotate freely about the vertical axis $AB$. The angle formed by the axis of the rotor and the plane of the meridian is denoted by $\theta$, and the latitude of the position on the earth is denoted by $\lambda$. We note that line $OC$ is parallel to the axis of the earth, and we denote by $\boldsymbol{\omega}_e$ the angular velocity of the earth about its axis.

(*a*) Show that the equations of motion of the gyrocompass are

$$I'\ddot{\theta} + I\omega_z\omega_e \cos\lambda \sin\theta - I'\omega_e^2 \cos^2\lambda \sin\theta \cos\theta = 0$$

$$I\dot{\omega}_z = 0$$

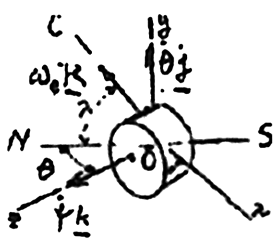
where $\omega_z$ is the rectangular component of the total angular velocity $\boldsymbol{\omega}$ along the axis of the rotor, and $I$ and $I'$ are the moments of inertia of the rotor with respect to its axis of symmetry and a transverse axis through $O$, respectively.

(*b*) Neglecting the term containing $\omega_e^2$, show that for small values of $\theta$, we have

$$\ddot{\theta} + \frac{I\omega_z\omega_e \cos\lambda}{I'}\theta = 0$$

and that the axis of the gyrocompass oscillates about the north-south direction.

## SOLUTION




(*a*) Angular momentum about $O$.

We select a frame of reference $Oxyz$ attached to the gimbal. The angular velocity of $Oxyz$ with respect to a Newtonian frame is $\boldsymbol{\Omega} = \omega_e\mathbf{K} + \dot{\theta}\mathbf{j}$

where $\quad \mathbf{K} = -\cos\lambda \sin\theta\mathbf{i} + \sin\lambda\mathbf{j} + \cos\lambda\cos\theta\mathbf{k}$

Thus, $\quad \boldsymbol{\Omega} = -\omega_e \cos\lambda \sin\theta\mathbf{i} + (\dot{\theta} + \omega_e \sin\lambda)\mathbf{j} + \omega_e \cos\lambda\cos\theta\mathbf{k} \qquad (1)$

The angular velocity $\boldsymbol{\omega}$ of the rotor is obtained by adding its spin $\dot{\psi}\mathbf{k}$ to $\boldsymbol{\Omega}$. Setting

$$\dot{\psi} + \omega_e \cos\lambda\cos\theta = \omega_z.$$

We have $\quad \boldsymbol{\omega} = -\omega_e \cos\lambda \sin\theta\mathbf{i} + (\dot{\theta} + \omega_e \sin\lambda)\mathbf{j} + \omega_z\mathbf{k} \qquad (2)$

The angular momentum $\mathbf{H}_O$ of the rotor is

$$\mathbf{H}_O = I_x \omega_z \mathbf{i} + I_y \omega_y \mathbf{j} + I_z \omega_z \mathbf{k}$$

where $I_x = I_y = I'$ and $I_z = I$. Recalling Eq. (2), we write

$$\mathbf{H}_O = -I'\omega_e \cos\lambda \sin\theta \mathbf{i} + I'(\dot{\theta} + \omega_e \sin\lambda)\mathbf{j} + I\omega_z \mathbf{k} \quad (3)$$

<u>Equations of motion</u>.

Eq. (18.28): $\Sigma \mathbf{M}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \mathbf{\Omega} \times \mathbf{H}_O$ or, from Eqs. (1) and (3):

$$\Sigma \mathbf{M}_O = -I'\omega_e \cos\lambda \cos\theta \dot{\theta} \mathbf{i} + I'\ddot{\theta}\mathbf{j} + I\dot{\omega}_z \mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\omega_e \cos\lambda \sin\theta & \dot{\theta} + \omega_e \sin\lambda & \omega_e \cos\lambda \cos\theta \\ -I'\omega_e \cos\lambda \sin\theta & I'(\dot{\theta} + \omega_e \sin\lambda) & I\omega_z \end{vmatrix} \quad (4)$$

We observe that the rotor is free to spin about the $z$ axis and free to rotate about the $y$ axis. Therefore, the $y$ and $z$ components of $\Sigma \mathbf{M}_O$ must be zero. It follows that the coefficients of $\mathbf{j}$ and $\mathbf{k}$ at the right-hand member of Eq. (4) must also be zero.

Setting the coefficient of $\mathbf{j}$ in the right-hand member of Eq. (4) equal to zero,

$$I'\ddot{\theta} + (-I'\omega_e \cos\lambda \sin\theta)(\omega_e \cos\lambda \cos\theta) - (-\omega_e \cos\lambda \sin\theta) I\omega_z = 0$$

$$I'\ddot{\theta} + I\omega_z \omega_e \cos\lambda \sin\theta - I'\omega_e^2 \cos^2\lambda \sin\theta \cos\theta = 0 \quad (5) ◀$$

Q. E. D

Setting the coefficient of $\mathbf{k}$ equal to zero,

$$I\dot{\omega}_z + (-\omega_e \cos\lambda \sin\theta) I'(\theta + \omega_e \sin\lambda) - (-I'\omega_e \cos\lambda \sin\theta)(\dot{\theta} + \omega_e \sin\lambda) = 0$$

Observing that the last two terms cancel out, we have

$$I\dot{\omega}_z = 0 \qquad \text{Q. E. D.} \quad (6) ◀$$

(*b*) It follows from Eq. (6) that $\omega_z = \text{constant}$ (7)

Rewrite Eq. (5) as follows: $I'\ddot{\theta} + (I\omega_z - I'\omega_e \cos\lambda \cos\theta)\omega_e \cos\lambda \sin\theta = 0$

It is evident that $\omega_z >>> \omega_e$. We can therefore neglect the second term in the parenthesis and write

$$I'\ddot{\theta} + I\omega_z \omega_e \cos\lambda \sin\theta = 0$$

or

$$\ddot{\theta} + \frac{I\omega_z \omega_e \cos\lambda}{I'} \sin\theta = 0 \quad (8)$$

where the coefficient of $\sin\theta$ is a constant. The rotor, therefore, oscillates about the line *NS* as a simple pendulum. For small oscillations, $\sin\theta \approx \theta,$ and Eq. (8) yields

$$\ddot{\theta} + \frac{I\omega_z\omega_e\cos\lambda}{I'}\theta = 0 \qquad \text{Q. E. D.} \qquad (9) \blacktriangleleft$$

Eq. (9) is the equation of simple harmonic motion with period

$$\tau = 2\pi\sqrt{\frac{I'}{I\omega_z\omega_e\cos\lambda}} \qquad (10)$$

Since its rotor oscillates about the line *NS*, the gyrocompass can be used to determine the direction of that line. We should note, however, that for values of $\lambda$ close to 90° or –90°, the period of oscillation becomes very large and the line about which the rotor oscillates cannot be determined. The gyrocompass, therefore, cannot be used in the polar regions.