4.2 Matrix Representation of linear Transformation

* In Section 4.1 : we have seen that for each $A_{m\times n}$ matrix, there is a linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$ s.t $L_A(x) = Ax \quad \forall x \in \mathbb{R}^n$.

* In this section : we will see that for each linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$, there is $m\times n$ matrix A s.t $L(x) = Ax$

Th 4.2.1 If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there is an $m\times n$ matrix A s.t $L(x) = Ax \quad \forall x \in \mathbb{R}^n$. In fact, the j^{th} column vector of A is given by $a_j = L(e_j)$, $j=1,\dots,n$

Proof: • Let $a_j = L(e_j)$, $j = 1, 2, \dots, n$

• Let $A = (a_{ij}) = (a_1 \ a_2 \ \cdots \ a_n)$

• If $x \in \mathbb{R}^n$ is arbitrary, then $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

Hence, $L(x) = x_1 L(e_1) + x_2 L(e_2) + \cdots + x_n L(e_n)$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

$$= (a_1 \ a_2 \ \cdots \ a_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = Ax$$

Exp Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ be a Linear transformation defined by $L(x) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix} \quad \forall x \in \mathbb{R}^3$. Find the (standard) matrix representing L

• $L(e_1) = L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$L(e_2) = L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$L(e_3) = L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ Hence, $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Check $L(x) = Ax = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}$

• Find $L\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ using the matrix A representing L.

$L\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

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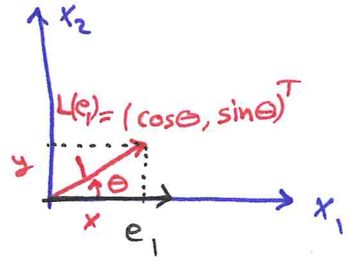
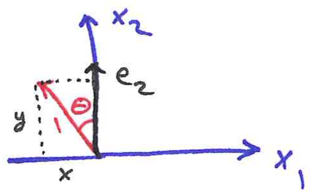
* A is called the standard matrix representation of L if we use the standard basis $(e_1, e_2, \dots, e_n)$ for $\mathbb{R}^n$.

Exp Let L be the linear operator on $\mathbb{R}^2$ that rotates each vector by angle θ counterclockwise direction. Find the matrix A that represent L.



- $L(e_1) = L\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$

$L(e_2) = L\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$



Hence, $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

$\sin\theta = \frac{x}{1}$
$\cos\theta = y$

- Find $L\begin{pmatrix}2\\4\end{pmatrix}$ if $\theta = \frac{\pi}{2}$

- $L\begin{pmatrix}2\\1\end{pmatrix} = L(2e_1 + 4e_2) = 2L(e_1) + 4L(e_2)$

$= 2\begin{pmatrix}\cos\frac{\pi}{2}\\ \sin\frac{\pi}{2}\end{pmatrix} + 4\begin{pmatrix}-\sin\frac{\pi}{2}\\ \cos\frac{\pi}{2}\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}-4\\0\end{pmatrix} = \begin{pmatrix}-4\\2\end{pmatrix}$

- or $L\begin{pmatrix}2\\1\end{pmatrix} = \begin{bmatrix}\cos\frac{\pi}{2} & -\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-4\\2\end{bmatrix}$

Th 4.2.2 (Matrix Representation Th)

Let $E = \{v_1, v_2, \cdots, v_n\}$ be an ordered basis for the vector space V
and $F = \{w_1, w_2, \cdots, w_m\}$ 〃 〃 〃 〃 〃 〃 〃 W.

For each $L: V \to W$ linear transformation, there is $m\times n$ matrix A s.t

$$[L(v)]_F = A\,[v]_E \quad \forall\, v \in V, \text{ where}$$

A is the matrix representing L relative to the ordered bases E and F.

We can find A using either:

[1] $a_j = [L(v_j)]_F$, $j = 1, 2, \cdots, n$ or

[2] the reduced row echelon form of

$(w_1 \;\; w_2 \;\cdots\; w_m \mid L(v_1) \;\; L(v_2) \;\cdots\; L(v_n))$ is $(I \mid A)$

$$\begin{tikzcd}
v \in V \arrow[r, "L"] \arrow[d, leftrightarrow] & w = L(v) \in W \arrow[d, leftrightarrow] \\
x = [v]_E \in \mathbb{R}^n \arrow[r, "A"] & Ax = [w]_F \in \mathbb{R}^m
\end{tikzcd}$$

L maps v into w iff
A maps x into $y = [w]_F$.

Exp Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ be the Linear Transformation defined by

$$L(x) = x_1 b_1 + (x_2+x_3) b_2 \quad \forall x \in \mathbb{R}^3 \text{ where } b_1 = \begin{pmatrix}1\\1\end{pmatrix},\ b_2 = \begin{pmatrix}-1\\1\end{pmatrix}.$$

① Find the matrix A representing L w.r.t the ordered basis $\{e_1, e_2, e_3\}$ and $\{b_1, b_2\}$.

$\boxed{S1}$: $L(e_1) = L\begin{pmatrix}1\\0\\0\end{pmatrix} = 1\, b_1 + 0\, b_2 \quad \Rightarrow a_1 = \begin{pmatrix}1\\0\end{pmatrix}$

$L(e_2) = L\begin{pmatrix}0\\1\\0\end{pmatrix} = 0 b_1 + 1 b_2 \quad \Rightarrow a_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$L(e_3) = L\begin{pmatrix}0\\0\\1\end{pmatrix} = 0\, b_1 + 1\, b_2 \quad \Rightarrow a_3 = \begin{pmatrix}0\\1\end{pmatrix}$

Hence $A = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}$

$\boxed{S2}$ $\left(b_1 \ b_2 \mid L(e_1) \ L(e_2) \ L(e_3)\right) \Rightarrow \left(\begin{array}{cc|ccc} 1 & -1 & 1 & -1 & -1\\ \boxed{1} & 1 & 1 & 1 & 1\end{array}\right)$

$\left(\begin{array}{cc|ccc} 1 & -1 & 1 & -1 & -1\\ 0 & -2 & 0 & -2 & -2\end{array}\right) \Rightarrow \left(\begin{array}{cc|ccc} 1 & \boxed{-1} & 1 & -1 & -1\\ 0 & 1 & 0 & 1 & 1\end{array}\right) \Rightarrow$

$\left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 1\end{array}\right) = (I \mid A)$. Hence, $A = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}$

② If $L: \mathbb{R}^2 \to \mathbb{R}^2$ is a LT defined by $L(\alpha b_1 + \beta b_2) = (\alpha+\beta) b_1 + 2\beta b_2$. Find the matrix A representing L w.r.t $\{b_1, b_2\}$

$\boxed{S1}$ $L(b_1) = L\begin{pmatrix}1\\1\end{pmatrix} = 1\, b_1 + 0\, b_2 \Rightarrow a_1 = \begin{pmatrix}1\\0\end{pmatrix}$

$\alpha = 1$ and $\beta = 0$ to get $L(b_1)$

$\alpha = 0$ and $\beta = 1$ to get $L(b_2)$

$L(b_2) = L\begin{pmatrix}-1\\1\end{pmatrix} = 1 b_1 + 2\, b_2 \Rightarrow a_2 = \begin{pmatrix}1\\2\end{pmatrix}$

Hence, $A = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$

$\boxed{S2}$ $\left(b_1 \ b_2 \mid L(b_1) \ L(b_2)\right) = \left(\begin{array}{cc|cc} 1 & -1 & 1 & -1\\ 1 & 1 & 1 & 3\end{array}\right) \Rightarrow$

$\left(\begin{array}{cc|cc} 1 & -1 & 1 & -1\\ 0 & -2 & 0 & -4\end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & -1 & 1 & -1\\ 0 & 1 & 0 & 2\end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1\\ 0 & 1 & 0 & 2\end{array}\right) = (I \mid A)$

Hence, $A = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$

Exp Let $D: P_3 \to P_2$ be the LT defined by $D(p) = p'$.

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① Find the matrix representation of D w.r.t the ordered basis $[x^2, x, 1]$ and $[x, 1]$ for P_3 and P_2 resp.

$$D(x^2) = 2x = \boxed{2}(x) + \boxed{0}(1) \Rightarrow a_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$D(x) = 1 = \boxed{0}(x) + \boxed{1}(1) \Rightarrow a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$D(1) = 0 = \boxed{0}(x) + \boxed{0}(1) \Rightarrow a_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

② If $p(x) = ax^2 + bx + c \in P_3$, then find $[D(p(x))]_{[x,1]}$

Note that $[p(x)]_{[x^2, x, 1]} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$[D(p(x))]_{[x,1]} = A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a \\ b \end{pmatrix}$$

Thus, $D(ax^2 + bx + c) = \boxed{2a}(x) + \boxed{b}(1)$

Exp Let $L: \mathbb{R}^2 \to \mathbb{R}^3$ be a LT defined by $L(x) = \begin{pmatrix} x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$.

Find the matrix representations of L w.r.t the ordered bases $\{u_1, u_2\} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$ and $\{b_1, b_2, b_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$$\left(b_1 \; b_2 \; b_3 \;\middle|\; L(u_1) \; L(u_2) \; L(u_3) \right) = \left(\begin{array}{ccc|cc} 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 2 \end{array} \right)$$

$$= \left(\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{array} \right)$$

A

This means $L(u_1) = -b_1 + 4b_2 - b_3 \Rightarrow a_1 = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$

$L(u_2) = -3b_1 + 2b_2 + 2b_3 \Rightarrow a_2 = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$
