

COMBINING INDIVIDUAL SECURITIES INTO PORTFOLIOS

Suppose we have two individual securities with underlying probability distributions like those of Figures 4.1 and 4.2. We're going to construct a portfolio by investing some of our money in one security and the rest in the other. The question we're going to ask in this chapter is, What will the probability distribution of returns to the portfolio in Figure 4.3 look like? Actually, we are going to restrict the question to just two properties of the distribution, the expected rate of return and the variance or standard deviation. If you are willing to accept standard deviation as a reasonable measure of the risk of a portfolio, we're going to find out how to predict the risk and expected rate of return of a *portfolio* based on the characteristics of the securities we put into it.

THE RISK AND EXPECTED RETURN OF A PORTFOLIO

THE PORTFOLIO'S RATE OF RETURN

Let's start with something basic. Suppose we consider a single period of time; let's again say a month. If the individual securities in the portfolio produce various rates of return, what will be the return to the portfolio as a whole? Let's consider a portfolio of two securities and first consider the *dollar* return to the portfolio. We'll assume we have \$1,000 to invest, and we put \$400 of it in security *A* and \$600 in security *B*. In the next month, *A* produces a rate of return of 10 percent (a dollar return of \$40), and *B* produces a rate of return of 6 percent (\$36). What is the *dollar* rate of return to the portfolio? The dollar return to the portfolio is obviously the sum of the dollar returns to the two securities:

$$\$76 = (\$400 \times .10) + (\$600 \times .06)$$

$$\$76 = \$40 + \$36$$

The percentage *rate* of return to the portfolio is given by the dollar return divided by the amount we have invested, which in this case is \$1,000. Dividing both sides of the above equation by \$1,000, we get

$$7.6\% = \frac{\$76}{\$1,000} = \left(\frac{\$400}{\$1,000} \times .10 \right) + \left(\frac{\$600}{\$1,000} \times .06 \right)$$

$$r_P = x_A \times r_A + x_B \times r_B$$

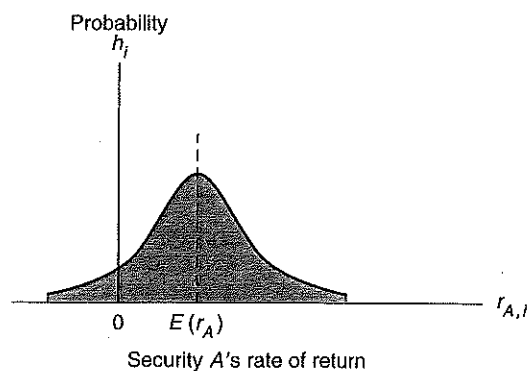


FIGURE 4.1 PROBABILITY DISTRIBUTION FOR SECURITY A.

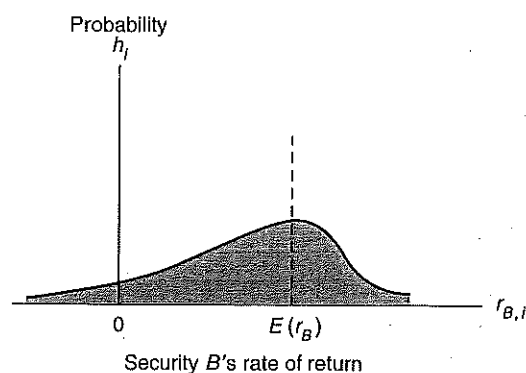


FIGURE 4.2 PROBABILITY DISTRIBUTION FOR SECURITY B.

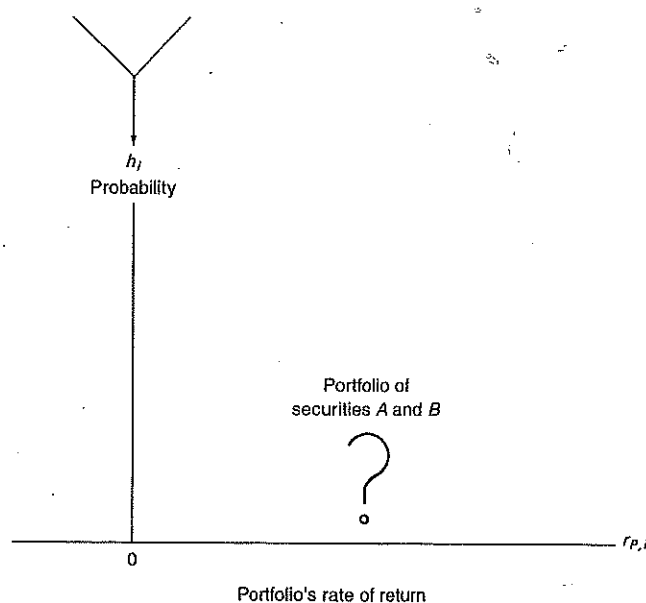


FIGURE 4.3 PROBABILITY DISTRIBUTION FOR A PORTFOLIO OF A AND B.

The term x is the fraction of money you are investing in each security. Thus, the rate of return to our portfolio, in any given period of time, is a weighted average of the rates of return that are being produced by the securities in the portfolio, where we are weighting by the fraction of our own money that we are investing in each security. These fractions are also called **portfolio weights**. When summed, they add up to 100 percent, and they are computed as

$$x_A = \frac{\text{Dollar amount of security A bought (sold short)}}{\text{Your total equity investment in the portfolio}}$$

A portfolio weight can be either positive or negative. A positive weight means you are buying the security; we also refer to this as taking a *long* position in the security. The opposite of taking a long position is taking a short position, or *selling short*. In this case, the portfolio weight is negative because the numerator is negative.

Selling a security short isn't quite the same as selling some security that you happen to own. For example, when you sell security short, you borrow *shares* of security from someone (usually through your broker). You're obligated to return to this person, after a certain period of time, the same number of shares you borrowed. Suppose you borrow 100 shares of Blue Steel security from me. Then you turn around and sell it for \$10 per share, collecting \$1,000. After a period of time, the security falls by \$5 per share. You then go back into the market and buy the 100 shares back for a total of \$500. You now return the 100 shares that you borrowed from me, and the short sale is completed. You have made a profit on this short sale, the difference between the \$1,000 you got for the borrowed security when you sold it and the \$500 it took to buy the security back later in the market. However, if the security paid any dividends between the time you sold it and the time you bought it back, you would have to pay cash, in the amount of these dividends, to me, the person from whom you borrowed the stock.

You might ask, What are you allowed to do with the \$1,000 proceeds between the time you short-sell and the time you buy the security back? If you or I were to sell short, we would have to set the proceeds aside, and, in addition, we would have to set aside some of our own money (perhaps 30 percent, or \$300, in this case) as a margin on the transaction. In any case, in the discussions that follow, we will assume that when you sell short, you are free to use the proceeds to invest in other securities and that no margin is required on the transaction.

Suppose you have \$1,000 of your own money and you sell short \$600 of security *B* and use this money in addition to your own money to buy \$1,600 of security *A*. What are your portfolio weights? You're buying \$1,600 of security *A*, which is 160 percent of your \$1,000 equity investment. Thus, x_A is 1.6. At the same time you're *selling* \$600 of security *B*, which is -60 percent of your equity investment. The sign is negative because you're doing the opposite of buying the security—you're selling it. Note that the two portfolio weights still add up to 100 percent.

Now suppose that in the period of time you're holding this position, security *A* produces a 20 percent rate of return and security *B* produces a 10 percent rate of return. What's your rate of return on the "portfolio" of the two securities?

$$r_P = (x_A \cdot r_A) + (x_B \cdot r_B)$$

$$.26 = (1.6 \times .20) + (-.6 \times .10)$$

To verify this, consider that we've got a profit on our long position of \$320 ($.20 \times \$1,600$). We've also got a \$60 loss on the short sale, since we sold security *B* for \$600 and repurchased it for \$660 (a 10 percent increase). Our net profit, therefore, is \$260. Since we initially invested \$1,000 of our own money, this represents a 26 percent rate of return on our investment.

To summarize, we can say that in any given period of time, the rate of return on our portfolio is a weighted average of the rates of return on the securities in the portfolio. In taking the average, the weights are given by the fraction of our own money that we are investing in each security. If we are buying the security in question, the weight assigned to it is positive; if we are short-selling the security, the weight is negative. In any case, the sum of the weights is 100 percent.

THE PORTFOLIO'S EXPECTED RATE OF RETURN

In Appendix 1 at the end of this chapter, we show that the expected rate of return to a portfolio is a simple weighted average of the expected rates of return to the securities that are included in the portfolio. The weights are again equal to the fractions of our own money that we are investing in each security. If there are M securities in the portfolio,

$$E(r_p) = \sum_{j=1}^M x_j E(r_j)$$

Thus, if security A has a 20 percent expected return and B has a 10 percent expected return and we allocate half our money to each security, the expected return to the portfolio is 15 percent:

$$.15 = (.5 \times .2) + (.5 \times .1)$$

On the other hand, suppose we have \$1,000 of our own money, sell short \$1,000 of security B , and use the proceeds, in addition to our own money, to invest \$2,000 in security A . The expected rate of return to our portfolio is now 30 percent:

$$.30 = (2.0 \times .2) + (-1.0 \times .1)$$

Note that if we don't sell short, the expected portfolio return is always somewhere in between the two securities, depending on how much of the two securities we buy. However, if we sell one of the two securities short, our expected return is unlimited on the upside or the downside. We can make the expected return to the portfolio as big as we want it simply by selling short huge amounts of the security with the lower expected rate of return. However, you should also understand that as we increase our expected return, we also usually increase the risk of the portfolio. This brings us to the next section.

THE PORTFOLIO'S VARIANCE

To compute the variance of a portfolio of securities, you need to have the covariance matrix for the securities you are putting in the portfolio. The covariance matrix shows the covariances between each of the securities in the portfolio. For example, the covariance matrix among three securities A , B , and C is as follows:

Security	A	B	C
A	$\text{Cov}(r_A, r_A)$	$\text{Cov}(r_B, r_A)$	$\text{Cov}(r_C, r_A)$
B	$\text{Cov}(r_A, r_B)$	$\text{Cov}(r_B, r_B)$	$\text{Cov}(r_C, r_B)$
C	$\text{Cov}(r_A, r_C)$	$\text{Cov}(r_B, r_C)$	$\text{Cov}(r_C, r_C)$

Each element of the matrix tells you the covariance between the returns to the security given at the top of each column and the returns to the security given at the left of each row. To illustrate, as we go down the diagonal of the matrix from the extreme northwest corner to the extreme southeast corner, we are looking at the covariance between each security and itself. This may seem strange at first, but if you consider the formula for covariance, you will realize that the covariance between a security and itself is simply its own variance:

$$\text{Cov}(r_A, r_A) = \sum_{i=1}^m h_i [r_{A,i} - E(r_A)] [r_{A,i} - E(r_A)] = \sum_{i=1}^m h_i [r_{A,i} - E(r_A)]^2 = \sigma^2(r_A)$$

Thus, all the numbers going down the diagonal of the matrix represent the variances for the individual securities.

It is also true that for each number above the diagonal there is a corresponding and equal number below the diagonal. This is true because

$$\begin{aligned}\text{Cov}(r_A, r_B) &= \sum_{i=1}^m h_i [r_{A,i} - E(r_A)][r_{B,i} - E(r_B)] \\ &= \sum_{i=1}^m h_i [r_{B,i} - E(r_B)][r_{A,i} - E(r_A)] = \text{Cov}(r_B, r_A)\end{aligned}$$

To determine what the variance of a portfolio is going to be, we need to know the portfolio weights for each security, and we need to have estimates (perhaps either sample or subjective estimates) for the numbers in the covariance matrix. We then set up the matrix in the following way:

		x_A A	x_B B	x_C C
Security				
x_A	A	$\sigma^2(r_A)$	$\text{Cov}(r_B, r_A)$	$\text{Cov}(r_C, r_A)$
x_B	B	$\text{Cov}(r_A, r_B)$	$\sigma^2(r_B)$	$\text{Cov}(r_C, r_B)$
x_C	C	$\text{Cov}(r_A, r_C)$	$\text{Cov}(r_B, r_C)$	$\sigma^2(r_C)$

Computing the variance is now a comparatively simple operation. You simply take each of the covariances in the matrix and multiply it by the portfolio weight at the top of the column and then again by the portfolio weight at the left side of the row. For example, in the case of $\text{Cov}(r_A, r_B)$ you would compute the following product:

$$\text{Cov}(r_A, r_B) \cdot x_A \cdot x_B$$

When you have obtained such a product for each element of the matrix, you add up all the products, and the resulting sum is the variance of the portfolio.

By recognizing that each of the elements above the diagonal is paired with an identical element below the diagonal, the formula for the variance of a three-security portfolio is given as follows:

$$\begin{aligned}\sigma^2(r_P) &= x_A^2 \sigma^2(r_A) + x_B^2 \sigma^2(r_B) + x_C^2 \sigma^2(r_C) + 2x_A x_B \text{Cov}(r_A, r_B) \\ &\quad + 2x_A x_C \text{Cov}(r_A, r_C) + 2x_B x_C \text{Cov}(r_B, r_C)\end{aligned}$$

The number of elements in the sum is equal to the square of the number of securities in the portfolio. With three securities there are nine elements, with four there are 16 and so on. Obviously, to compute the standard deviation of the portfolio, you need only take the square root of the preceding equation.

COMBINATION LINES

A *combination line* is drawn on a graph where you are plotting $E(r)$ against $\sigma(r)$. Each point on the line shows you the expected rate of return and standard deviation of a portfolio of two securities with given portfolio weights. Each point on the line represents a different set of portfolio weights in the two securities. Thus, the combination line tells us how the expected return and risk of a two-security portfolio changes as we

change the weights. The combination line is really a plot of the equations for $E(r_P)$ and $\sigma(r_P)$ for a two-security portfolio.

By recognizing that in a portfolio of two securities $x_B = (1 - x_A)$, the equations for expected return and standard deviation are given by

$$E(r_P) = x_A E(r_A) + (1 - x_A) E(r_B)$$

and

$$\sigma(r_P) = [x_A^2 \sigma^2(r_A) + (1 - x_A)^2 \sigma^2(r_B) + 2x_A(1 - x_A) \text{Cov}(r_A, r_B)]^{1/2}$$

As shown in Chapter 3

$$\text{Cov}(r_A, r_B) = \rho_{AB} \sigma(r_A) \sigma(r_B)$$

Therefore, we can rewrite the equation for the standard deviation as

$$\sigma(r_P) = [x_A^2 \sigma^2(r_A) + (1 - x_A)^2 \sigma^2(r_B) + 2x_A(1 - x_A) \rho_{AB} \sigma(r_A) \sigma(r_B)]^{1/2} \quad (4.1)$$

To illustrate the concept of a combination line, suppose we have two securities with the following characteristics:

	Security	
	A	B
$E(r)$	.10	.04
$\sigma(r)$	.05	.10

To construct a combination line for the two securities, we need to make an assumption about the degree to which they are correlated. Let's first assume the correlation coefficient between the two securities is zero. Filling in the preceding numbers for the corresponding values in our equations for expected return and standard deviation gives

$$E(r_P) = x_A \times .10 + (1 - x_A) \times .04$$

$$\sigma(r_P) = [x_A^2 \times .05^2 + (1 - x_A)^2 \times .10^2]^{1/2}$$

Note that the covariance term has disappeared from the formula for standard deviation because the correlation is presumed to be zero.

Assume that you have \$1,000 to invest, you short-sell \$500 of security B and you use the proceeds in addition to your equity to invest \$1,500 in security A. Your portfolio weight in A is, thus, 1.5. Substituting this value into the formulas, we get the following values for the expected rate of return and standard deviation:

$$E(r_P) = 1.50 \times .10 - .50 \times .04 = .13$$

$$\sigma(r_P) = [1.50^2 \times .05^2 + (-.50)^2 \times .10^2]^{1/2} = .09$$

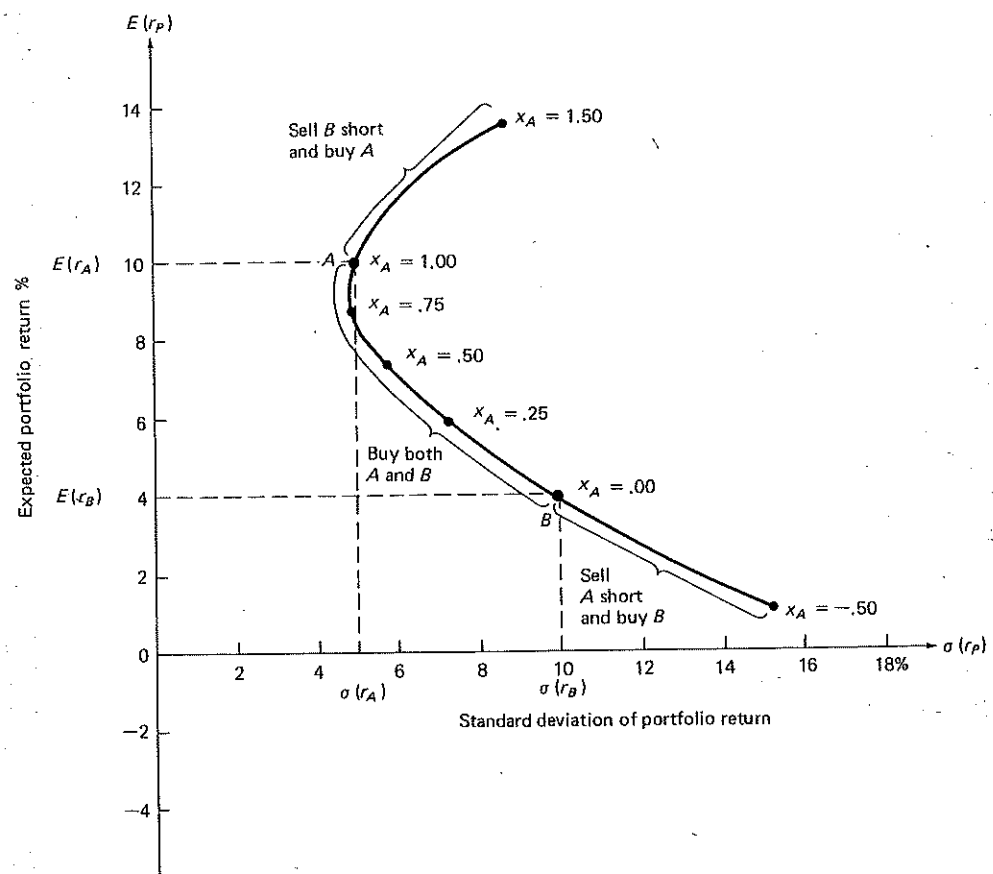
By making the same computations for other values for x_A , we obtain the following schedule:

x_A	$E(r_P)$	$\sigma(r_P)$
1.50	.130	.090
.75	.085	.045
.50	.070	.056
.25	.055	.076
-.50	.010	.152

The points in the preceding schedule are plotted in Figure 4.4. If still more such points were plotted for different values for x_A , they would trace out the bullet-shaped curve drawn in the diagram. This curve is called the *combination line* for the two securities. It shows what happens to the risk and expected return to a portfolio of the two securities as the portfolio weights are shifted from one value to another. The two securities are positioned at points A and B. At these points, x_A is equal to 1.00 and $(1 - x_A)$ is equal to .00, respectively. For points along the curve to the northeast of point A, we are short-selling B and investing in A. For points on the curve between points A and B, we are taking positive positions in both. For points to the southeast of B, we are short-selling A and investing in B. While these points represent unattractive portfolios, in the sense that risk is high and expected return is low, nevertheless they are available, given the assumed position of the two securities. You also should understand that the curve extends out indefinitely toward the northeast and toward the southeast. The more of B we short-sell, the farther we move on the curve to the northeast, and the more of A we short-sell, the farther we move to the southeast.

To see what is happening as we move along the combination line, consider Figure 4.5. In this figure, probability distributions are shown for three portfolios along the combination line, portfolios C, D, and E. Note that as you move from C to D to E,

FIGURE 4.4 COMBINATION LINE BETWEEN SECURITIES A AND B FOR THE CASE OF ZERO CORRELATION.



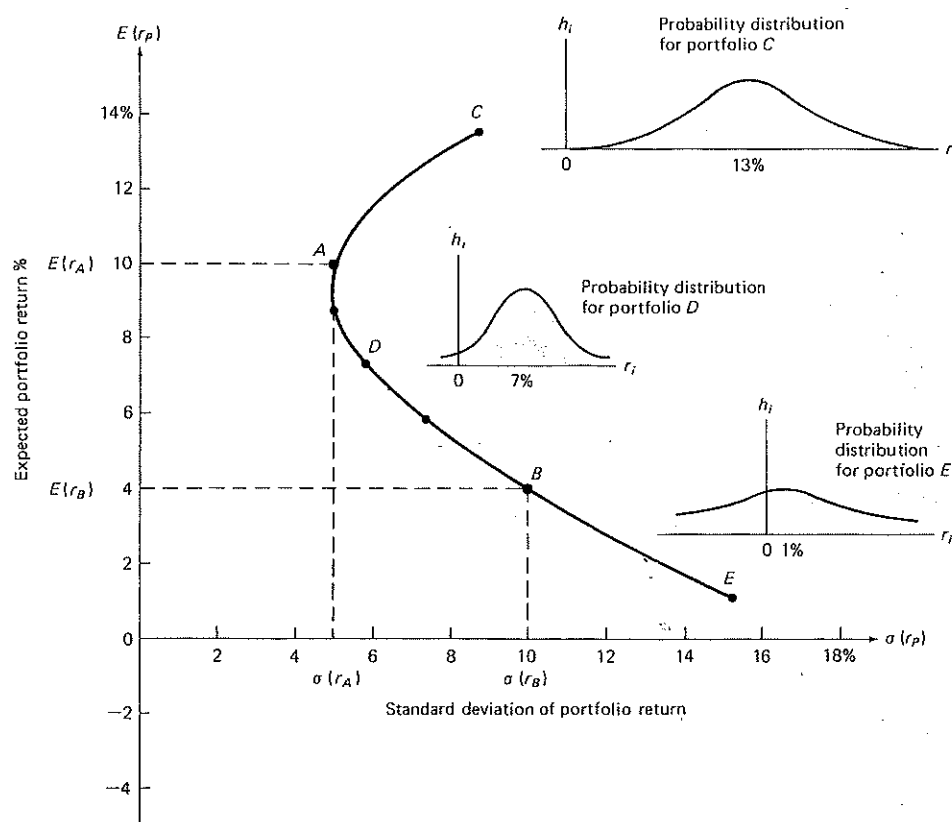


FIGURE 4.5 PROBABILITY DISTRIBUTIONS CORRESPONDING TO POSITIONS ON THE COMBINATION LINE.

the expected rate of return becomes smaller and smaller. The standard deviation of the distribution grows smaller as you move from portfolio C to D, but then it increases in magnitude as you go from portfolio D to E.

THE CASES OF PERFECT POSITIVE AND NEGATIVE CORRELATION

The combination line of Figure 4.4 is drawn for an assumed 0 value for the correlation coefficient between the two securities. If we assumed a different value for the correlation coefficient, the schedule of $\sigma(r_p)$ values would change, and we would obtain a different combination line. In fact, there is a family of combination lines, one line for each assumed value for the correlation coefficient.

Suppose, for example, we assume the two securities are perfectly positively correlated. The assumed relationship between the two is depicted in Figure 4.6. If the securities are perfectly positively correlated, pairs of returns produced by them must all fall on a straight line with a positive slope, as they do with the points labeled 1, 2, and 3. If we are dealing with a case of perfect correlation, the slope of the line must reflect the relative standard deviations for the securities. In our example, security B has twice the standard deviation of A. A line with a slope equal to 2.00 is consistent with these relative standard deviations. In this case, a change in the return on security A is always

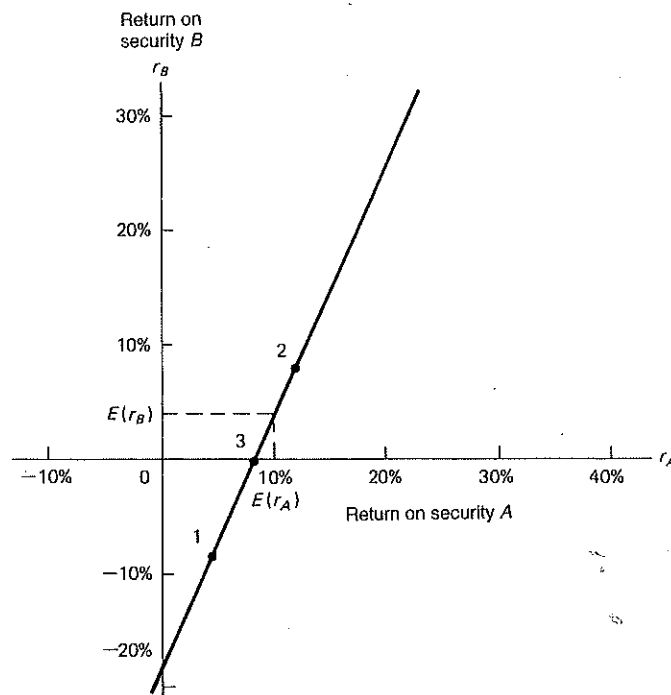


FIGURE 4.6 RELATIONSHIP BETWEEN SECURITIES A AND B WITH PERFECT POSITIVE CORRELATION.

accompanied by a change in the return on *B* that is twice as great. The line must intercept the vertical axis at a value that is consistent with the relative expected rates of return on the two securities. Denoting the slope of the line as a_1 and its intercept as a_0 , the relationship between the expected returns is given by

$$E(r_B) = a_0 + a_1 E(r_A)$$

Since a_1 is equal to 2.00 in this case and since the expected returns to *A* and *B* are 10 percent and 4 percent, respectively, we have

$$.04 = a_0 + 2.00 \times .10$$

or

$$a_0 = -.16$$

Now consider the equation for the standard deviation of a two-security portfolio as given by Equation (4.1). When the correlation coefficient is assumed to be 1.00, the terms in parentheses become a perfect square in the sense that $[x_A \sigma(r_A) + (1 - x_A) \sigma(r_B)]$ multiplied by itself produces the expression in brackets. Thus, for the case of perfect positive correlation, we can say the standard deviation of a portfolio is a simple weighted average of the standard deviations of the securities we are putting in the portfolio:

$$\sigma(r_P) = x_A \sigma(r_A) + (1 - x_A) \sigma(r_B)$$

Technically speaking, since the standard deviation is always a positive number (it's the square root of the variance), the portfolio's standard deviation is equal to the absolute value of the right-hand side of the equation.

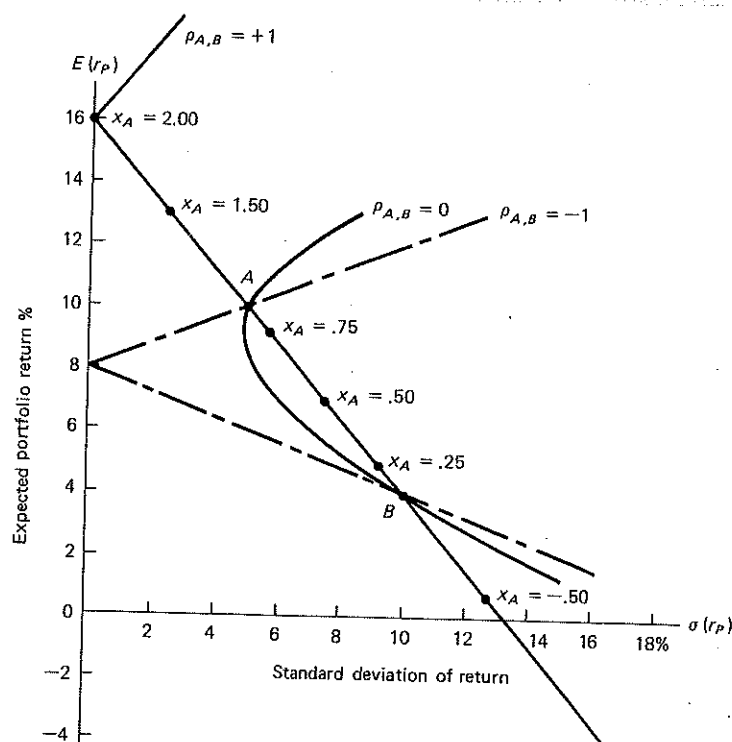
Given the preceding expression, we can again develop a schedule of values for $E(r_P)$ and $\sigma(r_P)$, given different values for x_A :

x_A	$E(r_P)$	$\sigma(r_P)$
3.00	.220	.0500
2.00	.160	.0000
1.50	.130	.0250
.75	.085	.0625
.50	.070	.0750
.25	.055	.0875
-.50	.010	.1250

These points are plotted on the solid linear combination line of Figure 4.7. Note that you can achieve a riskless portfolio by selling B short in an amount equal to 100 percent of your equity investment and using the proceeds to add to your equity investment in A . To see this, consider the three pairs of returns labeled 1, 2, and 3 in Figure 4.6. Assume these pairs are observed in three successive months:

	Month		
	1	2	3
Return to security A	.04	.12	.08
Return to security B	-.08	.08	.00

FIGURE 4.7 COMBINATIONS FOR THE CASES OF PERFECT POSITIVE, PERFECT NEGATIVE, AND ZERO CORRELATION.



Now consider the corresponding returns to a portfolio with weights that are adjusted to $x_A = 2.00$ and $x_B = -1.00$ at the beginning of each month:

$$\begin{aligned} r_{P,t} &= x_A \cdot r_{A,t} + x_B \cdot r_{B,t} \\ .16 &= 2.00 \times .04 + (-1.00) \times (-.08) \\ .16 &= 2.00 \times .12 + (-1.00) \times .08 \\ .16 &= 2.00 \times .08 + (-1.00) \times .00 \end{aligned}$$

The portfolio's return is perfectly stable at 16 percent. That is so no matter what pairs of returns we take from the line of Figure 4.6. With these weights we have created a portfolio that produces a constant rate of return period after period. We can always do this by selling one of the two short to some degree, provided the standard deviations are not equal and the securities are perfectly positively correlated.

Now suppose, instead, the two securities are perfectly *negatively* correlated. In this case, all pairs of returns must come from a straight line that has a negative slope. Again, the slope of the line must be equal to -2.00 because B has twice the standard deviation of A . The intercept of the line again must be consistent with the expected returns on the securities. Thus,

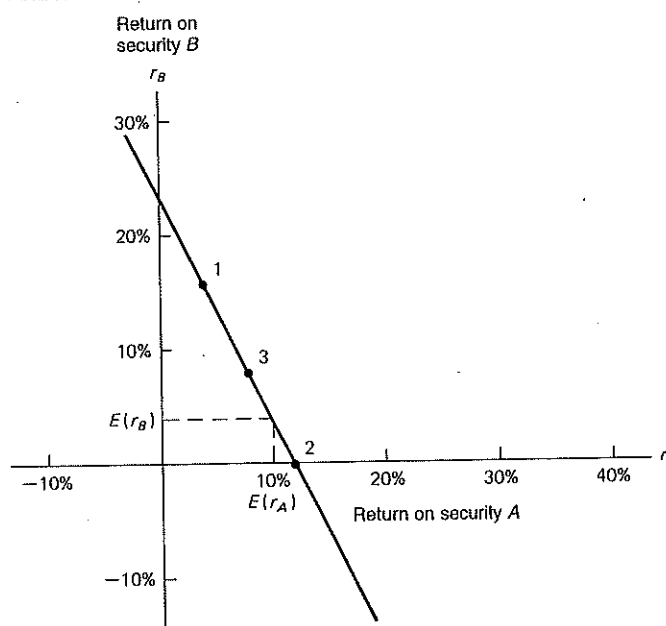
$$.04 = a_0 - 2.00 \times .10$$

and

$$a_0 = .24$$

The relationship between A and B with perfect negative correlation is drawn in Figure 4.8.

FIGURE 4.8 RELATIONSHIP BETWEEN SECURITIES A AND B WITH PERFECT NEGATIVE CORRELATION.



Consider again Equation (4.1) for the standard deviation of a two-security portfolio. If we assume a value of -1.00 for the correlation coefficient, the terms in brackets again become a perfect square in the sense that the term $[x_A\sigma(r_A) - (1 - x_A)\sigma(r_B)]$, when multiplied by itself, produces the bracketed expression. Thus, in this case, the formula for the standard deviation of a two-security portfolio reduces to

$$\sigma(r_P) = x_A\sigma(r_A) - (1 - x_A)\sigma(r_B)$$

This expression is almost the same as the expression for the case of perfect positive correlation, except that a negative sign now separates the two terms on the right-hand side. The standard deviation is again equal to the absolute value of the right-hand side of the expression.

We can again derive a schedule, graphed as the broken line in Figure 4.7, showing $E(r_P)$ and $\sigma(r_P)$ values corresponding to various values for x_A :

x_A	$E(r_P)$	$\sigma(r_P)$
3.000	.220	.3500
2.000	.160	.2000
1.500	.130	.1250
.667	.080	.0000
.250	.055	.0875
-.500	.010	.1750

With perfect negative correlation, we can create a riskless portfolio by taking positive positions in both. Whenever the return on security B increases, the return on A decreases, so if we invest positive amounts in both, variability in their returns will tend to cancel. Note, however, that when the return on A changes by a given amount, the return on B changes by twice the amount. Consequently, to make the canceling of returns complete, we must invest twice as much in A as we do in B . This is the case when the portfolio weights are $x_A = .667$ and $x_B = .333$.

To see this, consider the three pairs of returns plotted in Figure 4.8. Again, assume they are realized in three successive months:

	<i>Month</i>		
	1	2	3
Return on security A	.04	.12	.08
Return on security B	.16	.00	.08

If, at the beginning of each month, we adjust to portfolio weights equal to $x_A = .667$ and $x_B = .333$, the portfolio will produce an 8 percent return in each of the three periods:

$$r_{P,t} = x_A \cdot r_{A,t} + x_B \cdot r_{B,t}$$

$$.08 = .667 \times .04 + .333 \times .16$$

$$.08 = .667 \times .12 + .333 \times .00$$

$$.08 = .667 \times .08 + .333 \times .08$$

No matter which pair of returns are drawn from the line of Figure 4.8, with these weights the portfolio will always produce an 8 percent rate of return.

Contrast the risk associated with short selling under conditions of perfect positive and negative correlation. If we are short in either security, the risk of our portfolio is

higher if the correlation is negative. Remember, if you are short, it is better for the return to be low than high. With perfect negative correlation, when the return on one security is high relative to its mean, the return on the other is invariably low relative to its mean. If you are short in the high and long in the low, you're suffering in both parts of your portfolio. On the other hand, if you're long in the high and short in the low, you're prospering in both parts. It's a feast or famine situation! It's also highly risky.

We have the opposite situation with perfect positive correlation. Here, when the return on one security is high, the return on the other will also be high. You will prosper from your long position but suffer from your short. Moreover, when the returns on both securities are low, you will suffer from your long but prosper from your short. The returns from your long and short positions will cancel each other, and the return to your overall position will be stabilized.

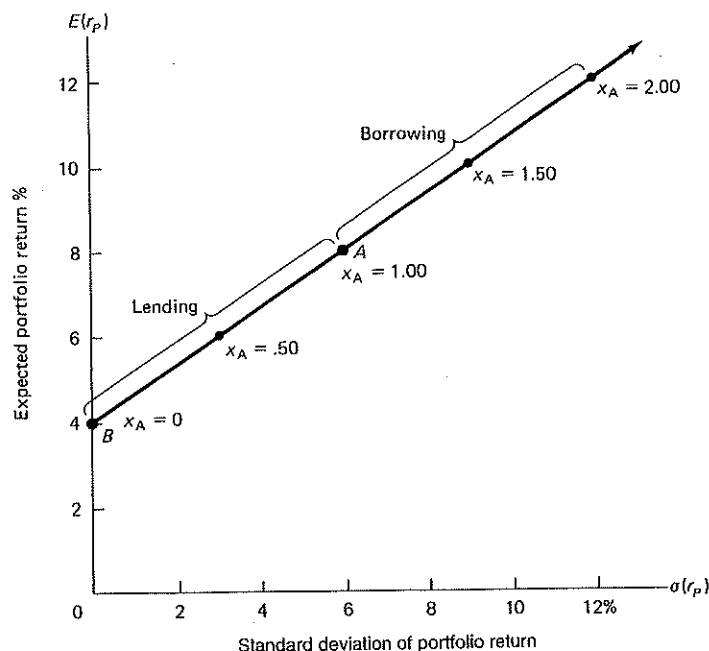
BORROWING AND LENDING AT A RISK-FREE RATE

Consider Figure 4.9. Two investments are plotted. A risky security is plotted at point A, with an expected return of 8 percent and a standard deviation of 6 percent, and a risk-free security (perhaps a treasury bill) is plotted at point B. Assume that the return on the bill is guaranteed by the government, so the probability of getting a 4 percent rate of return on the bill is 100 percent. The standard deviation of the returns on the bill is, therefore, equal to 0 percent.

Now consider again Equation (4.1) for the standard deviation of a portfolio of two securities. If we set $\sigma(r_B) = 0$, the second and third terms in brackets drop out, leaving

$$\sigma(r_P) = [x_A^2 \sigma^2(r_A)]^{1/2}$$

FIGURE 4.9 EFFECT OF BORROWING AND LENDING ON RISK AND EXPECTED RETURN.



This reduces to

$$\sigma(r_P) = x_A \sigma(r_A)$$

Given the preceding expression, we can again produce a schedule showing the risk and expected return to our portfolio, given various values for x_A :

x_A	$E(r_P)$	$\sigma(r_P)$
.00	.040	.000
.50	.060	.030
1.00	.080	.060
1.50	.100	.090
2.00	.120	.120

These points are plotted in Figure 4.9 on the combination line between the two investments. When one of the two investments is risk free, as it is here, the combination line is always a straight line. Correlation is obviously meaningless when one of the investments has no variability in its return.

If you take a position between points A and B , you are investing positive amounts in both the risky security and the risk-free bill. In this case, you are said to be *lending* to the person from whom you bought the bill. When you take positions on the combination line to the northeast of point A , you are said to be *borrowing*, because you are selling the bill to raise money to add to your investment in A . The more you borrow, the farther out on the combination line you go, increasing your risk as well as your expected rate of return.

Remember that if there is a risk-free borrowing and lending opportunity, such as the one in Figure 4.9, you can attain any position on a straight line extending out from the risk-free rate through any investment opportunity in $E(r)$, $\sigma(r)$ space.

To see some examples of security volatility and correlation, you can go to the Web site www.TheNewFinance.com. Once there, go to the area labeled Modern Investment Theory. Install PManager into your computer. Then go to the area labeled Sessions. Copy the session called Combination Lines into the directory in your computer called Optimize. You will find it on your C: drive.

Now go to the Programs section of Windows and run PManager. Go to Open under File, and open the file Combination Lines. Open the window Expected Returns under Optimize, and you will see the expected returns for two assets, A and B as 10 percent and 5 percent, respectively. Open the window Correlation Matrix under Optimize, and you will see the stocks' assumed standard deviations (red numbers) and the assumed correlation between their returns (black). Finally, open the window Exp. Ret/Asset Frontier under Optimize. Then highlight Tile under Window. In the window labeled Efficient Frontier, you will see the combination line between the two assets (without short selling) under the assumed values for the expected return, correlation, and standard deviations. Now assign a correlation coefficient of 0.00 to the two assets and note the change in the combination line. That part of the combination falling below the minimum standard deviation position on the line is not shown.

Now enter a zero standard deviation for the asset with lower expected return. Note that the combination line is now linear, as per the discussion in the previous section.

By changing the assumptions again and again, you will get a better understanding of combination lines.

❖ SUMMARY

In this chapter, we examined how individual securities combine into a portfolio. We concentrated on determining the expected rate of return and variance (or standard deviation) of a portfolio based on the characteristics of the securities we put into the portfolio.

The expected portfolio return is simply a weighted average of the expected rates of return of the securities in it. The weights are the fractions of our own investment we commit to each security in our portfolio. If you buy (go long) a security, the weight for the security is positive; if you sell the security short, the weight is negative.

The portfolio variance is determined on the basis of a covariance matrix for the individual securities in the portfolio. For each covariance element in the matrix, we multiply the covariance by the two portfolio weights for the associated securities. Then we sum up all the products to obtain the variance of the portfolio.

A combination line shows you what happens to the expected return and standard deviation of a two-security portfolio as we change the portfolio weights in the two securities. Combination lines are linear for the cases of perfect positive or perfect negative correlation between the returns on the two securities. For correlation coefficients between these limits, combination lines are curved. When one of the two investments is risk free, the combination line is a straight line extending out from the risk-free rate and passing through the position of the other risky investment.

To this point we have only learned how to determine the positions of portfolio opportunities available to us, given a selection of individual investments. We haven't learned how to determine the *best* opportunities. That is left to the next chapter.

APPENDIX 1

Formulas for the Expected Rate of Return and Variance of a Portfolio

EXPECTED RATE OF RETURN TO A PORTFOLIO

We know that the rate of return to a portfolio in any i th state of nature is given by

$$r_{P,i} = x_A r_{A,i} + x_B r_{B,i} \quad (1.A.1)$$

The statistical definition for the expected rate of return to a portfolio is given by

$$E(r_P) = \sum_{i=1}^n h_i r_{P,i} \quad (1.A.2)$$

Substituting Equation (1.A.1) into (1.A.2), we obtain

$$E(r_P) = \sum_{i=1}^n h_i (x_A r_{A,i} + x_B r_{B,i}) \quad (1.A.3)$$

Since the sum of X plus Y is equal to the sum of X plus the sum of Y , we can bring the summation sign into the parentheses and factor x out of the sums

$$\sum_{i=1}^n x_A h_i r_{A,i} \text{ and } \sum_{i=1}^n x_B h_i r_{B,i}$$

$$E(r_P) = x_A \sum_{i=1}^n h_i r_{A,i} + x_B \sum_{i=1}^n h_i r_{B,i}$$

Recognizing the formulas for expected rates of return on securities A and B , we obtain

$$E(r_P) = x_A E(r_A) + x_B E(r_B)$$

Generalizing to a portfolio of M stocks, we obtain

$$E(r_P) = \sum_{j=1}^M x_j E(r_j)$$

VARIANCE OF RETURN TO A PORTFOLIO

The statistical definition of portfolio variance is given by

$$\sigma^2(r_P) = \sum_{i=1}^n h_i [r_{P,i} - E(r_P)]^2 \quad (1.A.4)$$

Substituting Equations (1.A.1) and (1.A.2) into (1.A.4), we obtain

$$\sigma^2(r_P) = \sum_{i=1}^n h_i \{ (x_A r_{A,i} + x_B r_{B,i}) - [x_A E(r_A) + x_B E(r_B)] \}^2$$

Bringing together terms involving the individual securities and factoring out the portfolio weights, we obtain

$$\sigma^2(r_P) = \sum_{i=1}^n h_i \{ x_A [r_{A,i} - E(r_A)] + x_B [r_{B,i} - E(r_B)] \}^2$$

The squared term tells us to multiply the two underscored terms in brackets. Doing so, we obtain two squared terms and two cross-product terms:

$$\begin{aligned} \sigma^2(r_P) = & x_A^2 \sum_{i=1}^n h_i [r_{A,i} - E(r_A)]^2 + x_B^2 \sum_{i=1}^n h_i [r_{B,i} - E(r_B)]^2 \\ & + 2x_A x_B \sum_{i=1}^n h_i [r_{A,i} - E(r_A)][r_{B,i} - E(r_B)] \end{aligned}$$

The first two terms are the variances of securities A and B . The second two are the covariances between the two. Thus,

$$\sigma^2(r_P) = x_A^2 \sigma^2(r_A) + x_B^2 \sigma^2(r_B) + 2x_A x_B \text{Cov}(r_A, r_B)$$

❖ QUESTION SET 1

1. Explain the concept of short selling.
2. Suppose you purchase \$1,000 of security A , purchase \$500 of security B , and borrow \$500. If these transactions constitute your entire portfolio, what are the portfolio weights for each component of the portfolio?

3. Compute the variance and expected return of the portfolio in Question 2, given the following additional information:

	<i>A</i>	<i>B</i>
Variance	.25	.49
$E(r)$	.10	.16

The correlation of *A* with *B* is .7. Borrowing takes place at a risk-free interest rate of .05.

4. Write out the formula that would be required to compute the variance on a five-stock portfolio.

Assume the following information for Questions 5 through 7:

Stock	$E(r)$	Std. Dev.	Correlation Coefficients
1	.05	.20	1 with 2 = -.2
2	.10	.10	1 with 3 = .3
3	.20	.15	1 with 4 = .5
4	.15	.30	2 with 3 = .2
			2 with 4 = -.5
			3 with 4 = 0

A portfolio is formed as follows: Sell short \$2,000 of security 1 and buy \$3,000 of security 2, \$2,000 of security 3, and \$3,000 of security 4. The cash provided by the owner of the portfolio is \$2,000, and any additional funds required to finance the portfolio are borrowed at a risk-free interest rate of 5 percent. There are no restrictions on the use of short sale proceeds.

5. Compute the portfolio weights for each component of the portfolio.
 6. Compute the expected return of the portfolio.
 7. Compute the standard deviation of the portfolio.
 8. Consider two securities, *A* and *B*, which have the following characteristics:

	<i>A</i>	<i>B</i>
$E(r)$	.12	.06
Std. dev.	.12	.06

Correlation coefficient of *A* with *B* = -1.0. Compute the expected returns and standard deviations of each of the following portfolios of *A* and *B*. Also plot securities and the portfolios of *A* and *B* on a graph with expected return and standard deviation on the axes.

Portfolio 1: $x_A = 2, x_B = -1$

Portfolio 2: $x_A = .5, x_B = .5$

Portfolio 3: $x_A = \frac{1}{3}, x_B = \frac{2}{3}$

Portfolio 4: $x_A = -.5, x_B = 1.5$

9. Consider two securities with the following characteristics:

	Security <i>X</i>	Security <i>Y</i>
Expected return	.10	.14
Standard deviation	.25	.30

Suppose you build a portfolio with equal dollar amounts in the two securities. Compute the expected return and variance of the portfolio under each of the following assumptions about the correlation between returns on X and Y :

- Correlation = 1
- Correlation = 0
- Correlation = -1

10. Assume that two securities have a correlation coefficient of -1.0.
 - a. What would be the lowest possible standard deviation that could be achieved by constructing a portfolio of these two securities?
 - b. Use your answer to part (a) and Equation (4.1) to derive an expression for the lowest standard deviation portfolio weights for the securities. (The weights for the securities will be a function of the standard deviations of the two securities.)
11. Two securities, L and M , are perfectly negatively correlated. L 's standard deviation is .6 and M 's standard deviation is .8. Find the portfolio of L and M that will result in the lowest possible standard deviation.
12. What does a combination line for two securities tell you?
13. Suppose you construct a combination line for two assets, with one of the assets having a zero standard deviation. What feature of this particular combination line is not true for any arbitrary combination line?
14. If two securities were perfectly positively correlated, would it be possible to construct a portfolio of the two securities with zero standard deviation? Explain.

❖ QUESTION SET 2

1. Define a portfolio weight.
2. How can short selling increase a portfolio's expected return?
3. If there is a positive covariance between securities A and B , and security A produces a return above its expected rate of return, what does that do to your expectations for security B ?
4. What does a combination line show?
5. Every time a certain airline security sees a 1 percent jump in rate of return, a certain food company sees a .5 percent increase. What does this tell you about the correlation between the securities? What do you expect the combination line to look like?
6. Given that you can invest in only two securities, describe how to put together a portfolio of a risk-free and a risky security.

❖ ANSWERS TO QUESTION SET 2

1. A portfolio weight is the fraction of your total equity investment in a particular security.
2. Short selling increases portfolio expected return if the proceeds from a short sale are invested in an investment with a higher expected rate of return than the security that was sold short.
3. You would expect security B also to produce a return above its expected rate of return, although not necessarily in the same degree as security A .
4. A combination line shows the relationship between the expected return and the standard deviation for a portfolio of two assets. It indicates, for differing portfo-

lio weights for each of the two assets, how the expected rate of return and standard deviation change.

5. The securities are perfectly positively correlated. The combination line is a straight line.
6. You invest in one security with a guaranteed rate of return, such as a government-backed bond. The other investment is made in a risky security. If you invest positive amounts in both securities, you are said to be lending; if you invest more than 100 percent of your equity in the risky security, you are said to be borrowing. The more you borrow from the risk-free rate security, the greater the variability associated with the return to your equity investment.

❖ PROBLEM SET

1. You have \$10,000 invested, 30 percent of which is invested in Company X, which has an expected rate of return of 15 percent, and 70 percent of which is invested in Company Y, with an expected return of 9 percent.
 - a. What is the dollar return to your portfolio?
 - b. What is the expected percentage rate of return?
2. Suppose you invest in four securities equally. Company A has an expected return of 20 percent, Company B has an expected return of 10 percent, Company C has an expected return of 12 percent, and Company D has an expected return of 9 percent. You have a total of \$40,000 invested. What is the expected rate of return on your portfolio?
3. As an investor, you saw an opportunity to invest in a new security with excellent growth potential. Wanting to invest more than you had, which was only \$1,000, you sold another security short with an expected rate of return of 5 percent. The total amount you sold short was \$4,000, and your total amount invested in the growth security, which had an expected rate of return of 24 percent, was thus \$5,000. Assuming no margin requirements, what is your expected rate of return on this portfolio?
4. Assume the investor in Problem 2 wants to determine how risky his portfolio is and wants you to compute the portfolio variance. If the respected correlations and variances of the securities are as follows, what is the variance of the portfolio? Remember that $\text{Cov}(r_A, r_B) = \rho_{A,B}\sigma(r_A)\sigma(r_B)$.

Correlations:	A	B	C	D
B	.50			
C	.60	.30		
D	-.30	-.20	-.10	
Variances:	.04	.16	.02	.10

5. Draw a combination line for a portfolio of two securities that have the following characteristics, assuming no correlation between the two:

Stock	A	B
$E(r)$	.12	.02
$\sigma(r)$	.08	.10

6. In Problem 5, suppose you have \$10,000 to invest and would like to sell \$5,000 in security B short to invest in A. Calculate the expected rate of return and standard deviation.

❖ ANSWERS TO PROBLEM SET

1. a. The dollar rate of return is the percentage of the portfolio invested in a security multiplied by its expected rate of return. Thus, of the \$10,000 invested Company X—30 percent of total with 15 percent rate of return:

$$.30 \times \$10,000 \times .15 = \$450$$

Company Y—70 percent with a 9 percent rate of return:

$$.70 \times \$10,000 \times .09 = \$630$$

The total dollar return is $\$450 + \$630 = \$1,080$.

- b. The expected percentage rate of return is the dollar return divided by the amount invested:

$$r = \frac{\text{Dollar return}}{\text{Total amount invested}}$$

$$r = \frac{\$1,080}{\$10,000} = 10.8\%$$

2. The expected rate of return is the weighted average of the expected rates in the portfolio:

$$E(r_P) = \sum_{j=1}^M x_j E(r_j)$$

The portfolio weights are first determined by the formula

$$x_A = \frac{\text{Dollar amount of A bought}}{\text{Total equity investment}}$$

Since you have invested equally in four securities and your total investment is \$40,000, the portfolio weights are equal ($x_A = x_B = x_C = x_D$) and are determined:

$$x_A = \frac{\$10,000}{\$40,000} = .25$$

Plugging in the expected returns on the individual securities and the expected rate of return on the portfolio is

$$\begin{aligned} r_P &= (x_A \times r_A) + (x_B \times r_B) + (x_C \times r_C) + (x_D \times r_D) \\ &= (.25 \times .20) + (.25 \times .10) + (.25 \times .12) + (.25 \times .09) \\ &= .1275 = 12.75\% \end{aligned}$$

3. Computing the portfolio weights for each security with the formula

$$x_A = \frac{\text{Dollar amount in A bought (sold short)}}{\text{Total equity investment}}$$

we find

$$x_A = \frac{-4,000}{1,000} = -4.0$$

$$x_B = \frac{5,000}{1,000} = 5.0$$

$$\begin{aligned} r_P &= (x_A \times r_A) + (x_B \times r_B) \\ &= (-4.0 \times .05) + (5.0 \times .24) \\ &= 1.0 = 100\% \end{aligned}$$

You can, thus, expect to double your money in one year if you have guessed correctly in choosing to invest in the growth security.

4. To compute the variance, you need to make a covariance matrix. Using the square roots of the variances and the correlations given, the covariances are calculated:

$$\text{Cov}(r_J, r_K) = \rho_{J,K} \times \sigma(r_J) \times \sigma(r_K)$$

$$\text{Cov}(r_A, r_B) = .500 \times .200 \times .400 = .040$$

$$\text{Cov}(r_A, r_C) = .600 \times .200 \times .141 = .017$$

$$\text{Cov}(r_A, r_D) = -.300 \times .200 \times .316 = -.019$$

$$\text{Cov}(r_B, r_C) = .300 \times .400 \times .141 = .017$$

$$\text{Cov}(r_B, r_D) = -.200 \times .400 \times .316 = -.025$$

$$\text{Cov}(r_C, r_D) = -.100 \times .141 \times .316 = -.004$$

Plugging in the given variances and the portfolio weights, the covariance matrix is as follows:

		.25 A	.25 B	.25 C	.25 D
Stock					
.25	A	.04	.040	.017	-.019
.25	B	.040	.16	.017	-.025
.25	C	.017	.017	.02	-.004
.25	D	-.019	-.025	-.004	.10

Multiplying each covariance by the weight at the top of the column and at the left of the row and summing, we get

$$.25 \times .25 \times .04 = .0025$$

$$.25 \times .25 \times .040 = .0025$$

$$.25 \times .25 \times .017 = .0011$$

$$.25 \times .25 \times -.019 = -.0012$$

$$.25 \times .25 \times .040 = .0025$$

$$.25 \times .25 \times .160 = .0100$$

$$.25 \times .25 \times .017 = .0011$$

$$.25 \times .25 \times -.025 = -.0016$$

$$.25 \times .25 \times .017 = .0011$$

$$.25 \times .25 \times .017 = .0011$$

$$.25 \times .25 \times .020 = .0013$$

$$.25 \times .25 \times -.004 = -.0003$$

$$.25 \times .25 \times -.019 = -.0012$$

$$.25 \times .25 \times -.025 = -.0016$$

$$.25 \times .25 \times -.004 = -.0003$$

$$.25 \times .25 \times .100 = .0063$$

$$\text{Total portfolio variance} = .0233$$

After computing the variance for a four-security portfolio, aren't you glad there are computers for most other portfolios?

5. Assuming no correlation between the securities, we use the following equations:

$$E(r_p) = x_A E(r_A) + (1 - x_A) E(r_B)$$

$$\sigma(r_p) = [x_A^2 \sigma^2(r_A) + (1 - x_A)^2 \sigma^2(r_B)]^{1/2}$$

Using the given characteristics:

Stock	A	B
$E(r)$	.12	.02
$\sigma(r)$	.08	.10

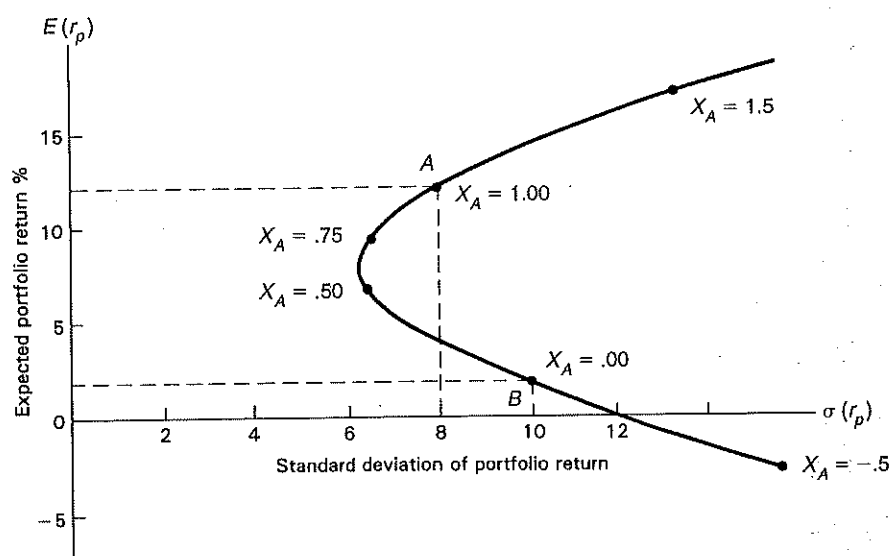
$$E(r) = x_A \times .12 + (1 - x_A) \times .02$$

$$\sigma(r_p) = [x_A^2 \times .08^2 + (1 - x_A)^2 \times .1^2]^{1/2}$$

For a series of values for A , we can make a schedule:

x_A	$E(r_p)$	$\sigma(r_p)$
1.5	.170	.130
1.0	.120	.080
.75	.095	.065
.50	.070	.064
.00	.020	.100
-.50	-.030	.155

The graph of the combination line is shown in the following figure:



6. Expected return:

$$E(r_p) = x_A E(r_A) + x_B E(r_B)$$

$$\frac{15,000}{10,000} .12 - \frac{5,000}{10,000} .02 =$$

$$.18 - .01 = .17$$

Standard deviation:

$$(x_A^2 \sigma^2(r_A) + x_B^2 \sigma^2(r_B))^{1/2} = \sigma(r_P)$$

$$[(1.5)^2 \times (.08)^2 + (-.5)^2 \times (.10)^2]^{1/2} = .130$$

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