1.10: Chebyshev's Inequality

Thm6: let $u(X)$ be a nonnegative function of the Random variable X If $E(u(X))$ exists, then for every positive constant c.

$$\Pr(u(X) \geq c) \leq \frac{E(u(X))}{c}.$$

Proof:

let X be continuous Random variable

let $c > 0$

let $A = \{x : u(x) \geq c\}$ Then $A^* = \{x : u(x) < c\}$.

$$\rightsquigarrow E(u(X)) = \int_{-\infty}^{\infty} u(x) f(x)\, dx = \int_{A} u(x) f(x)\, dx + \int_{A^*} u(x) f(x)\, dx.$$

$$\rightsquigarrow E(u(X)) - \int_{A} u(x) f(x)\, dx = \int_{A^*} u(x) f(x)\, dx \quad \text{-----} \quad ①$$

But $u(x) \geq 0$, $f(x) \geq 0$ for all $x \in A^*$

$$\Rightarrow \int_{A^*} u(x) f(x)\, dx.$$

on ① $E(u(X)) - \int_{A} u(x) f(x)\, dx \geq 0$

so $E(u(X)) \geq \int_{A} u(x) f(x)\, dx \quad \text{-----} \quad ②$

But $u(x) \geq c \quad \forall x \in A$.

$\Rightarrow u(x) f(x) \geq c f(x) \quad \forall x \in A \quad (f(x) \geq 0 \quad \forall x \in A)$

$$\Rightarrow \int_{A} u(x) f(x)\, dx \geq \int_{A} c f(x)\, dx = c \int_{A} f(x)\, dx = c \Pr(X \in A).$$

$\rightarrow$

$$\Rightarrow \int_A u(x)\, f(x)\, dx \geq c\, Pr(u(x) \geq c) \;.\;\dots\dots \text{③}$$

$$\text{②} + \text{③} \rightarrow \quad E(u(x)) \geq c\, Pr(u(x) \geq c)$$

$$\Rightarrow \frac{E(u(x))}{c} \geq Pr(u(x) \geq c) \;. \quad c>0$$

$$\Rightarrow Pr(u(x) \geq c) \leq \frac{E(u(x))}{c} \;, \quad c>0 \;. \quad \blacksquare$$

Thm 7: Chebyshev's Inequality.

let the Random variable X have a dist. of probability, about which we assume only that there is a variance σ^2. This of course implies that there is a mean μ. then for every $K>0$:

$$Pr(|X-\mu| \geq K\sigma) \leq \frac{1}{K^2} \;.$$

or equivalently, $Pr(|X-\mu| < K\sigma) \geq 1 - \frac{1}{K^2}$.

proof : $u(x) = (x-\mu)^2$, $c = K^2\sigma^2$, $K>0$, $\sigma>0$.

From Thm 6 : $Pr(u(x) \geq c) \leq \frac{E(u(x))}{c}$

$$Pr((x-\mu)^2 \geq K^2\sigma^2) \leq \frac{E((x-\mu)^2)}{K^2\sigma^2}$$

$$Pr(|x-\mu| \geq K\sigma) \leq \frac{\sigma^2}{K^2\sigma^2} = \frac{1}{K^2} \;.$$

$$1 - Pr(|x-\mu| < K\sigma) \leq \frac{1}{K^2} \Rightarrow Pr(|x-\mu| < K\sigma) \geq 1 - \frac{1}{K^2}$$

example 1: let X is a Random variable with p.d.f.

$$f(x) = \begin{cases} \frac{1}{2\sqrt{3}} , & -\sqrt{3} < x < \sqrt{3} \\ 0 , & \text{elsewhere} \end{cases}$$

continuous.

1. Find μ.

$$\mu = E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\sqrt{3}}^{\sqrt{3}} x \frac{1}{2\sqrt{3}}\, dx = 0$$

2. Find σ :

$$\cdot\ E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \frac{1}{2\sqrt{3}}\, dx = \frac{1}{2\sqrt{3}} \left(\frac{x^3}{3} \Big|_{-\sqrt{3}}^{\sqrt{3}} \right) = \frac{1}{2\sqrt{3}} . 2\sqrt{3} = 1$$

$$\cdot\ \sigma^2 = E(x^2) - (E(x))^2 = 1 - 0 = 1$$

$$\cdot\ \sigma = \sqrt{1} = 1 .$$

3. $$Pr\left(|X| \geq \tfrac{3}{2}\right) = 1 - Pr\left(|X| < \tfrac{3}{2}\right) = 1 - \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2\sqrt{3}}\, dx$$

$$= 1 - \frac{1}{2\sqrt{3}} \left(\frac{3}{2} - -\frac{3}{2} \right)$$

$$= 1 - \left(\frac{1}{2\sqrt{3}} . 3 \right)$$

$$= \frac{3}{2\sqrt{3}} = 0.134 .$$

4. Find an upperbound for $Pr(|X| \geq \frac{2}{3})$ using chebyshev's inequality :

$$Pr(|X-\mu| \geq K\delta) \leq \frac{1}{K^2}$$

$$Pr(|X-0| \geq \frac{3}{2} \cdot 1) \leq \frac{1}{(\frac{3}{2})^2}$$

$$Pr(|X| \geq \frac{3}{2}) \leq \frac{4}{9} = 0.4444 .$$

5. $P(|x| \geq 2) = 0$

because $\sqrt{3} = 1.7$, $2 > \sqrt{3}$ لو طلعت برا ال density ، يعني تعتبر elsewhere يعني 0 "

6. Find an upperbound for $Pr(|X| \geq 2)$ using chebyshev's inequality :

$$Pr(|X-0| \geq 2 \cdot 1) \leq \frac{1}{2^2}$$ هاي ineq. مش exactly 100% .

$$Pr(|X| \geq 2) \leq \frac{1}{4} = 0.26$$

check exp 2.

ch 1 done :)