Optimum Receiver and Digital Binary Transmission

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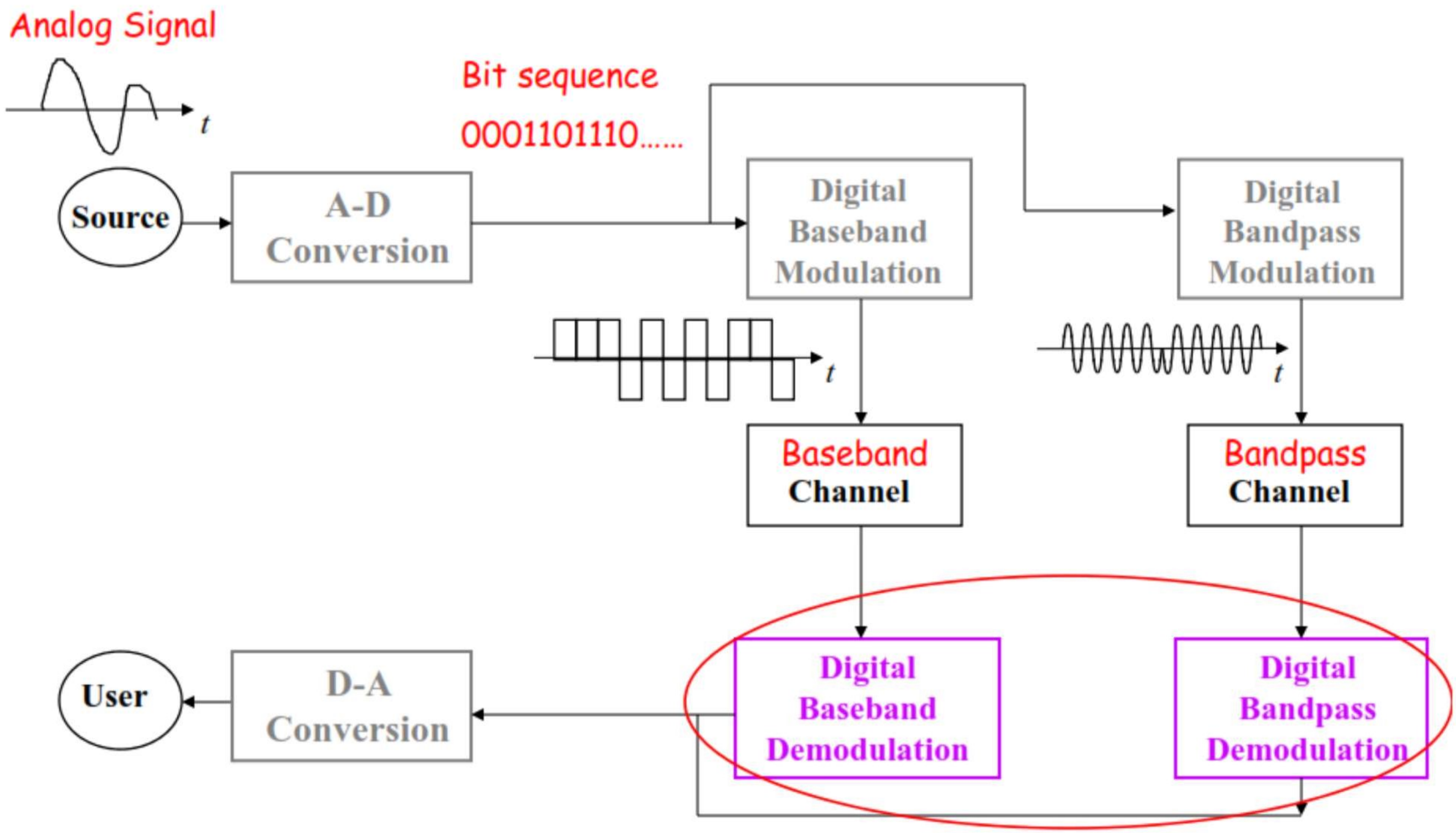
In binary data transmission over a communication channel, logic 1 is represented by a signal $s_1(t)$ and logic 0 by a signal $s_2(t)$. The time allocated for each signal is the symbol duration τ, where τ is related to the data rate by $r_b = 1/\tau$. We have two types of data transmission:

Baseband Data Transmission: Binary data transmission by means of two baseband waveforms (typically, two voltage levels) is referred to as baseband signaling. The spectrum of the transmitted signal occupies the low part of the frequency band (around the zero frequency). No high frequency carrier is used in this mode of transmission.

Bandpass Data Transmission: The baseband data modulates a high frequency carrier to produce a modulated signal, whose spectrum is centered around the carrier frequency.

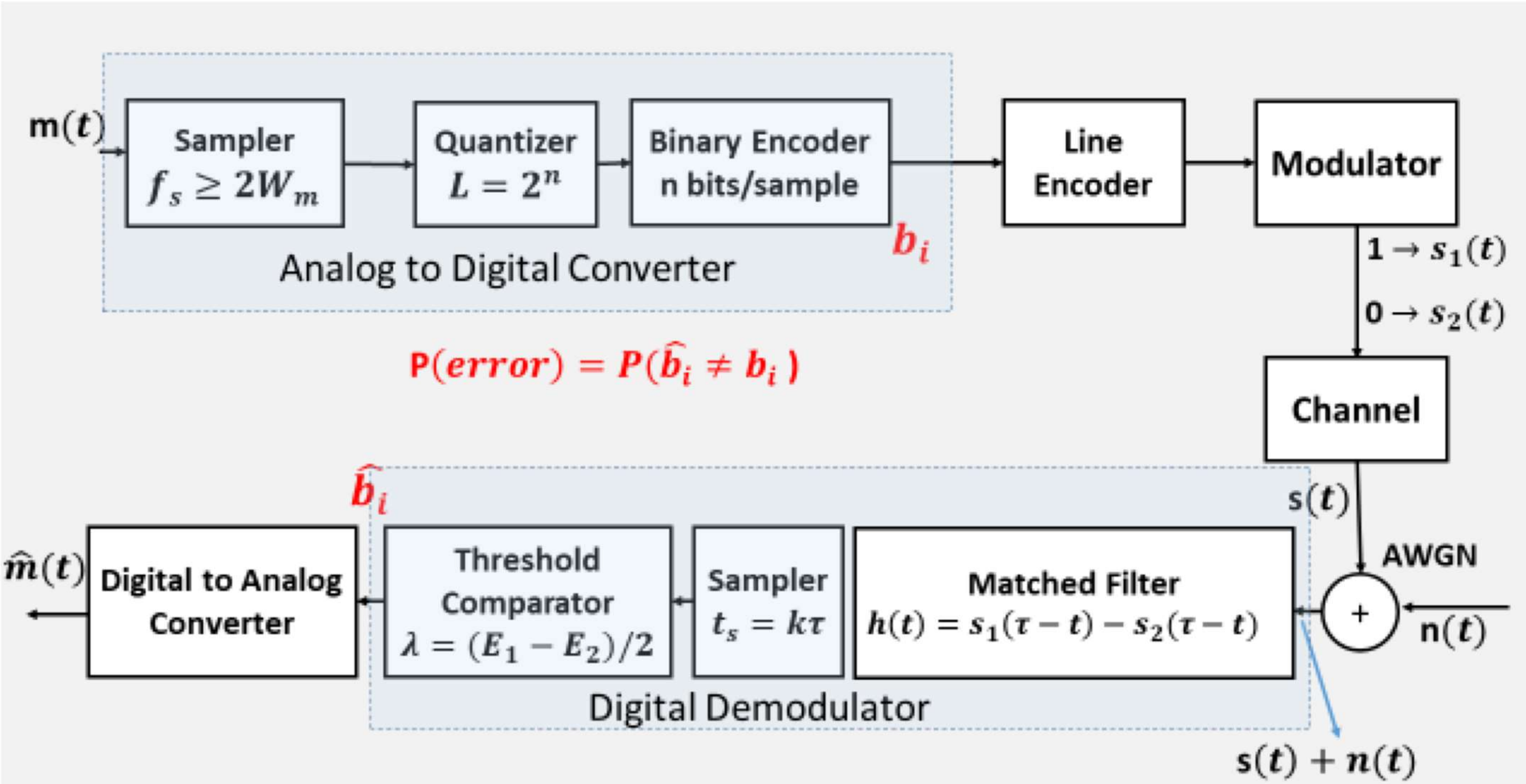
Digital Communication Block Diagram



Assumptions

- The channel noise $n(t)$ is additive white Gaussian (AWGN) with a double-sided PSD of $N_0/2$. Noise is assumed to be added at the front end of the receiver.
- The data component at the front end of the receiver is assumed to be an exact replica of the transmitted signal, in the sense that the transmission bandwidth of the medium is wide enough to reproduce the signal without distortion.
- Bits in different time intervals are assumed independent.
- The signal to be processed by the receiver is the noisy signal $\boldsymbol{y(t) = s_i(t) + n(t)}$

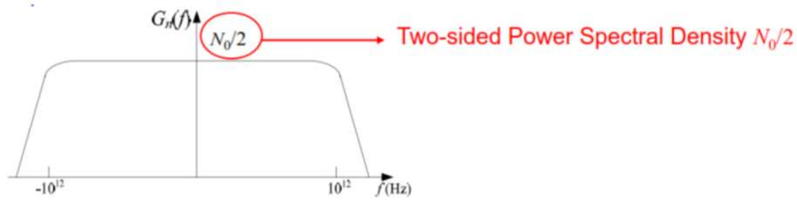
Based on $y(t)$, the task of the receiver is to decide whether a 1 or a 0 was transmitted during each transmission slot τ with minimum probability of error.



Digital Communication System

Thermal Noise

- **Thermal Noise**: caused by the random motion of electrons within electronic devices
- Thermal noise, $n(t)$, is modeled as a wise sense stationary (WSS) Gaussian random process. wide
- The thermal noise has a power spectrum that is constant from dc to approximately 10^{12} Hz; hence, $n(t)$ can be approximately regarded as a white process.
- Thermal noise is superimposed (added) on the transmitted signal. The received signal is $y(t) = s(t) + n(t)$.

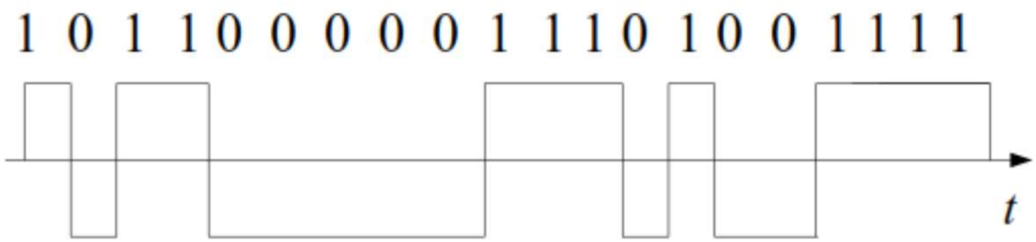


- The mean value of the thermal noise $n(t)$ is zero.
- At any given time t_0 the probability density function of $n(t_0)$ follows the Gaussian distribution; $N(0, \sigma_0^2)$; where $\sigma_0^2 = E(n(t))^2$ is the noise power.

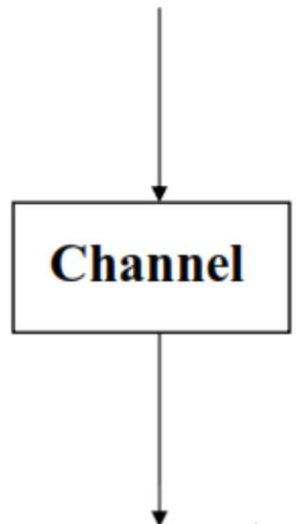
$$f_n(n) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-n^2/2\sigma_0^2}$$

Effect of Noise and Channel on Received Data

Transmitted signal
$s(t)$



- Suppose that channel bandwidth is properly chosen such that most frequency components of the transmitted signal can pass through the channel.



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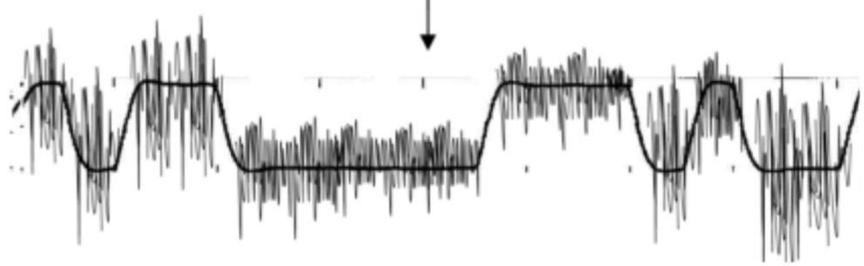
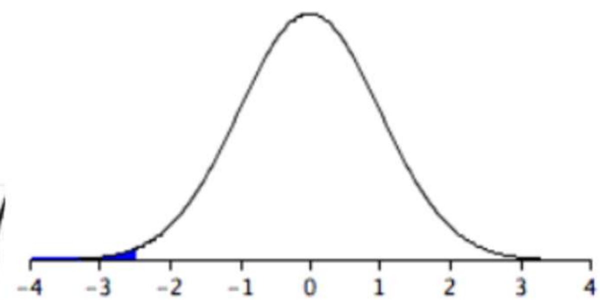
Received signal
$y(t)$





Basic Elements of the Receiver

To decide on whether logic 1 or logic 0 was transmitted during a given time slot τ, the received signal (transmitted signal and noise) passes through three basic units.

Filter: The optimum filter, which we will also call the matched filter.

Sampler: Samples the received signal (data component plus noise) at some time $t = t_0 = \tau = symbol\ duration.$

Threshold comparator: If the sampled value is larger than a given threshold, λ, digit 1 is declared true, otherwise digit 0 is declared true.

There are three design elements at the receiver

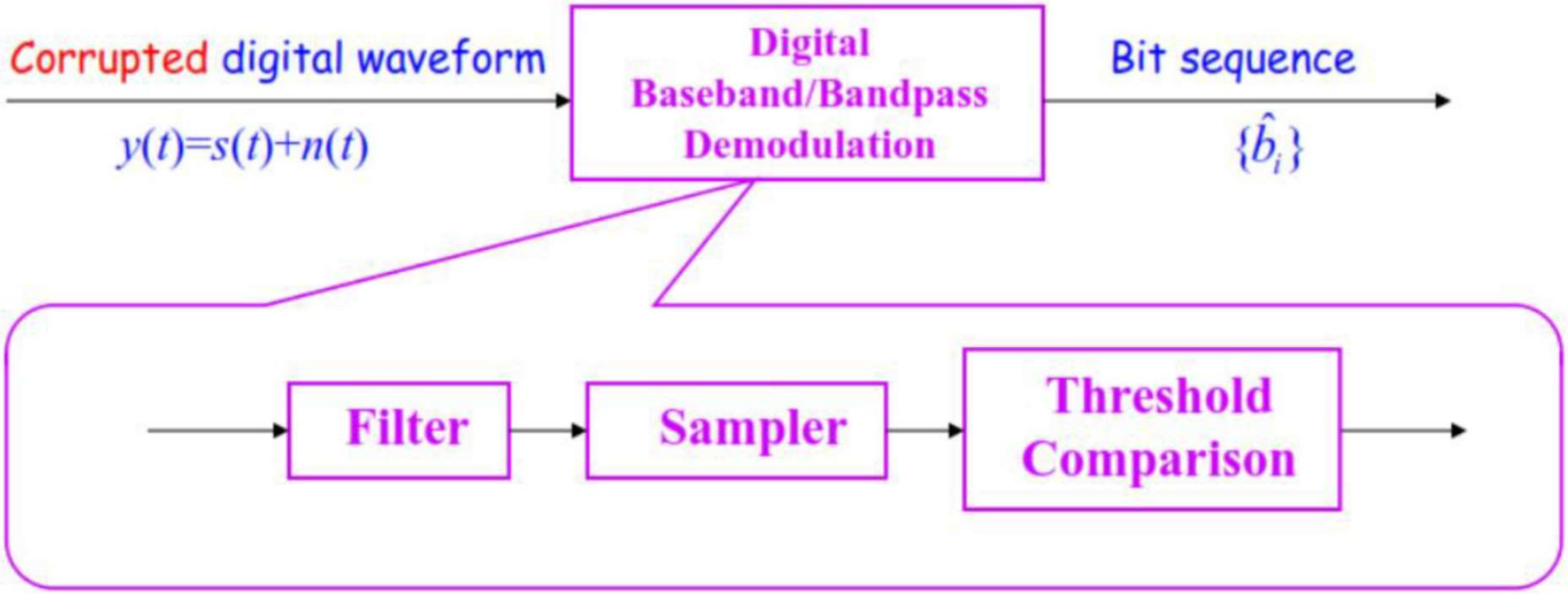
a. The impulse response $h(t)$ of the filter
b. The sampling time t_0
c. The threshold λ

Basic Elements of the Receiver

These parameters should be chosen so as to minimize the average probability of error (or bit error rate BER), defined as

$$P_b = P(b_i \neq \hat{b_i})$$

$$P_b = \Pr\{\hat{b}_i = 1,\, b_i = 0\} + \Pr\{\hat{b}_i = 0,\, b_i = 1\}$$



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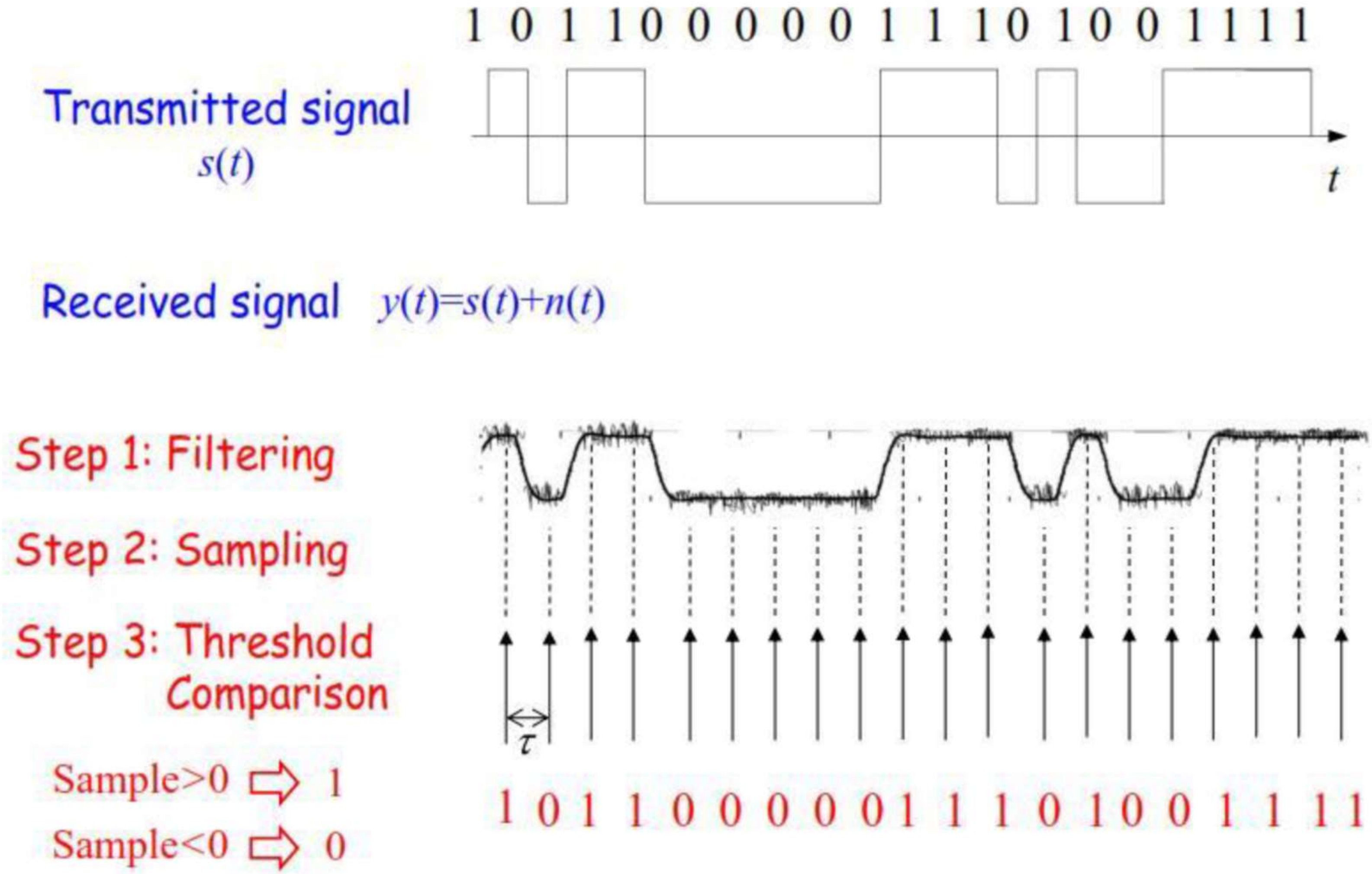


Theorem on the Optimum Binary Receiver

Consider a binary communication system, corrupted by AWGN with power spectral density $N_0/2$, where the equally probable binary digits 1 and 0 are represented by the signals $s_1(t)$ and $s_2(t)$, respectively. The transmission time for each signal is τ sec. The optimum receiver elements, i.e., the elements that minimize the receiver probability of error are given by

Impulse response of the matched filter: $\boldsymbol{h(t) = s_1(\tau - t) - s_2(\tau - t)}$, $0 \le t \le \tau$

Optimum sampling time: $\boldsymbol{t_s = \tau}$

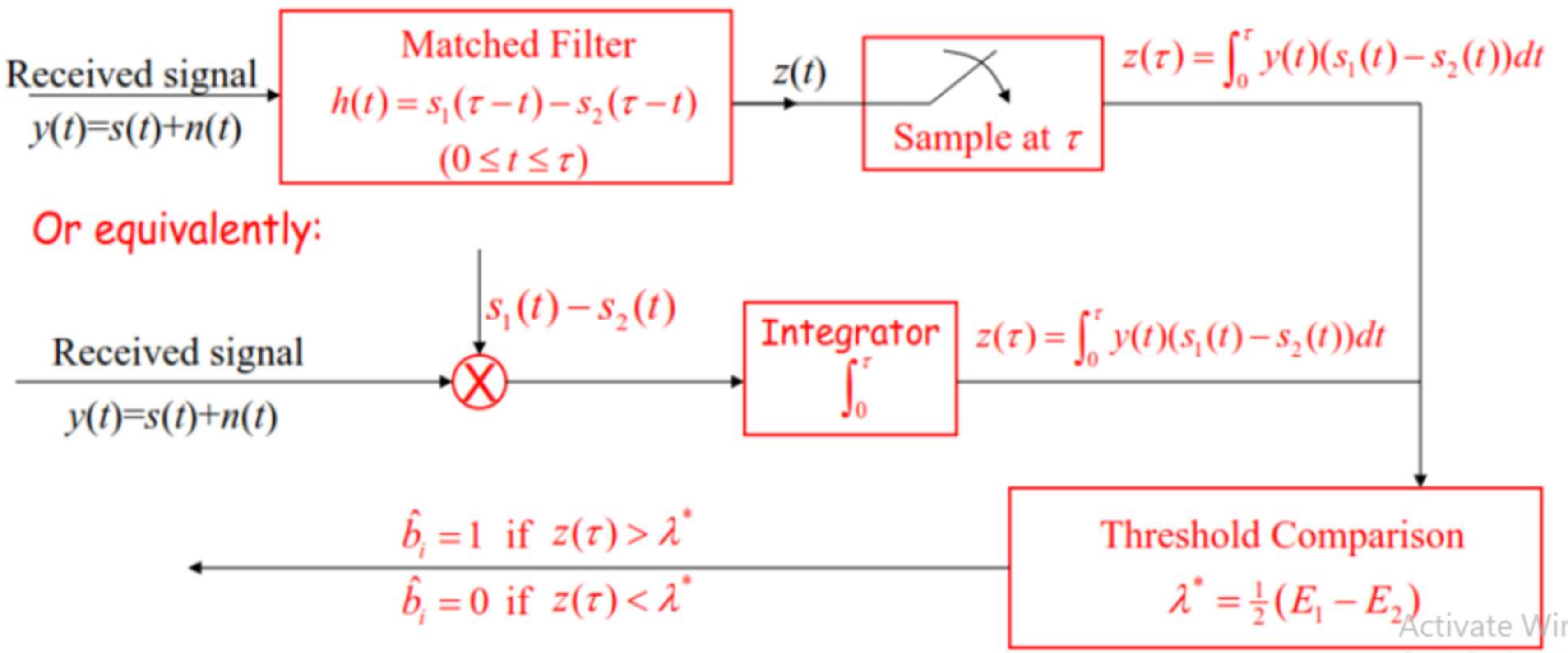
Optimum threshold of comparator: $\boldsymbol{\lambda^* = \frac{1}{2}(E_1 - E_2)}$, $E_k = \int_0^\tau (s_k(t))^2 dt\,, k = 1,2$

When these elements are used, the system minimum probability of error is

$$\boldsymbol{P_b^* = Q\left(\sqrt{\frac{\int_0^\tau (s_1(t) - s_2(t))^2 dt}{2N_0}}\right)}$$

Theorem on the Optimum Binary Receiver

The structure of the optimum receiver is depicted in the figure below. Note that the receiver can be implemented in terms of the matched filter and, equivalently, in terms of a correlator (a multiplier followed by an integrator).

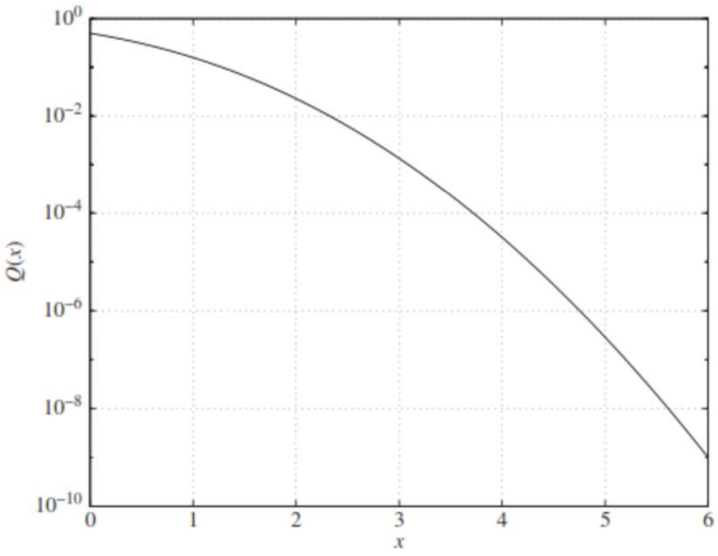


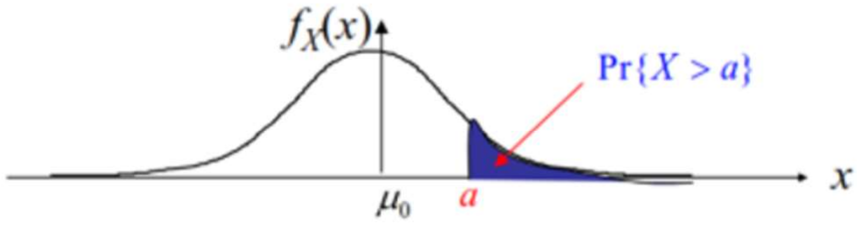
The Q-Function

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx$$

- $Q(\alpha)$ is a decreasing function of α.
- For $X \sim \mathcal{N}(\mu_0, \sigma_0^2)$,

$$\Pr\{X > a\} = \int_a^{\infty} f_X(x)dx = \int_a^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right\} dx = Q\left(\frac{a-\mu_0}{\sigma_0}\right)$$





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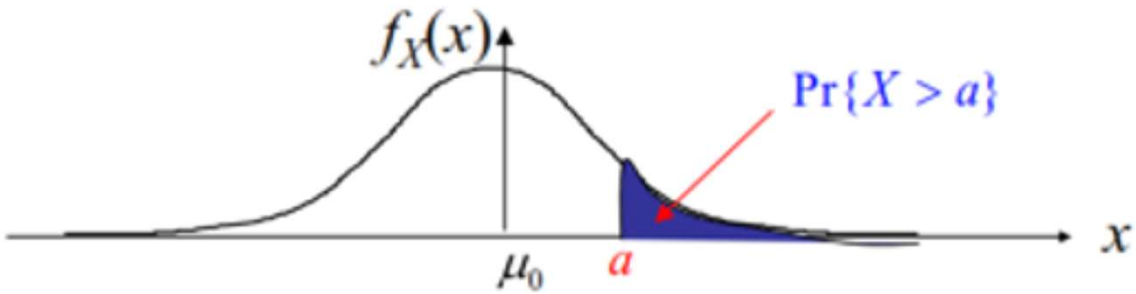
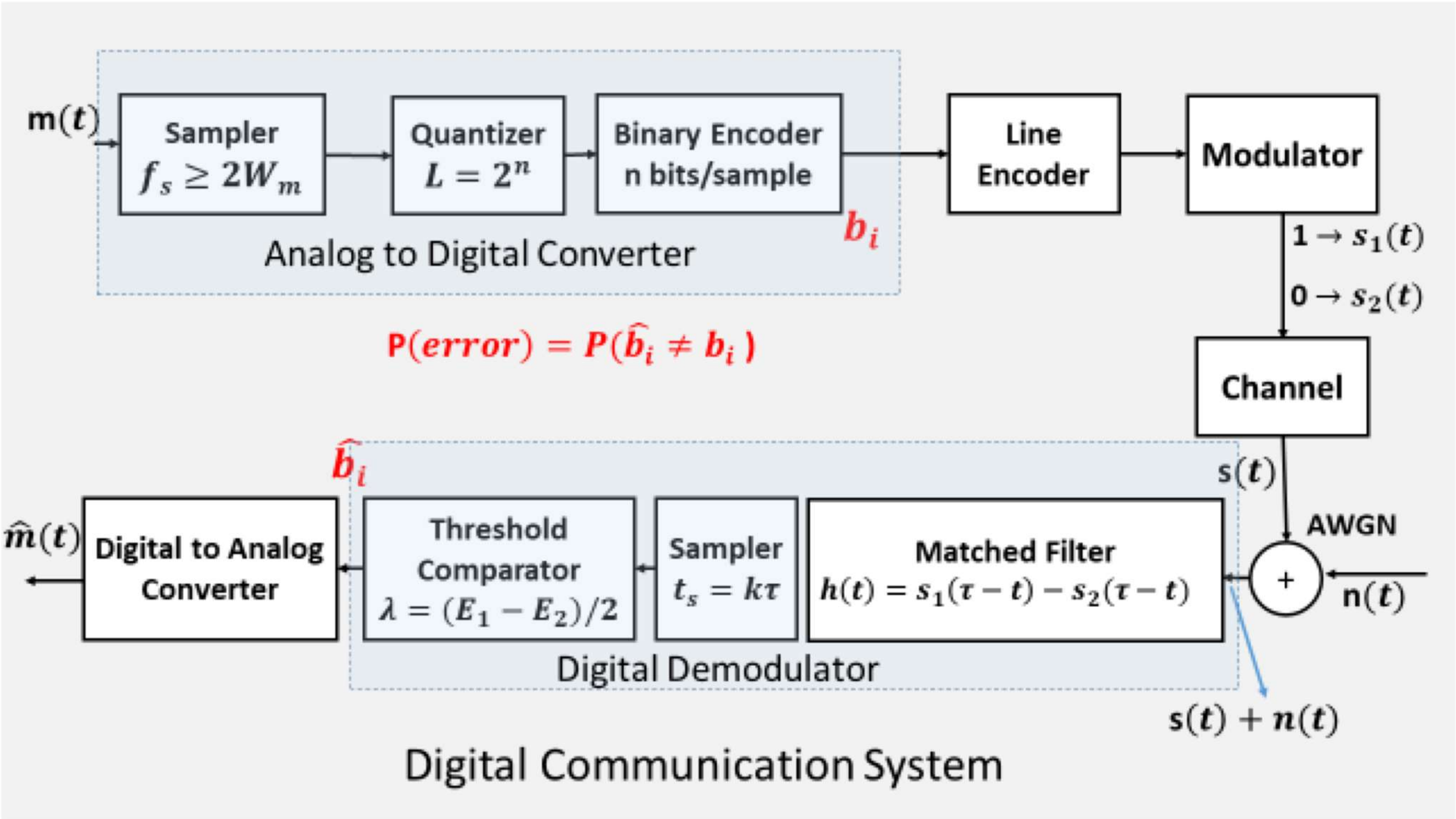


Matched Filter and Performance of the Optimum Receiver



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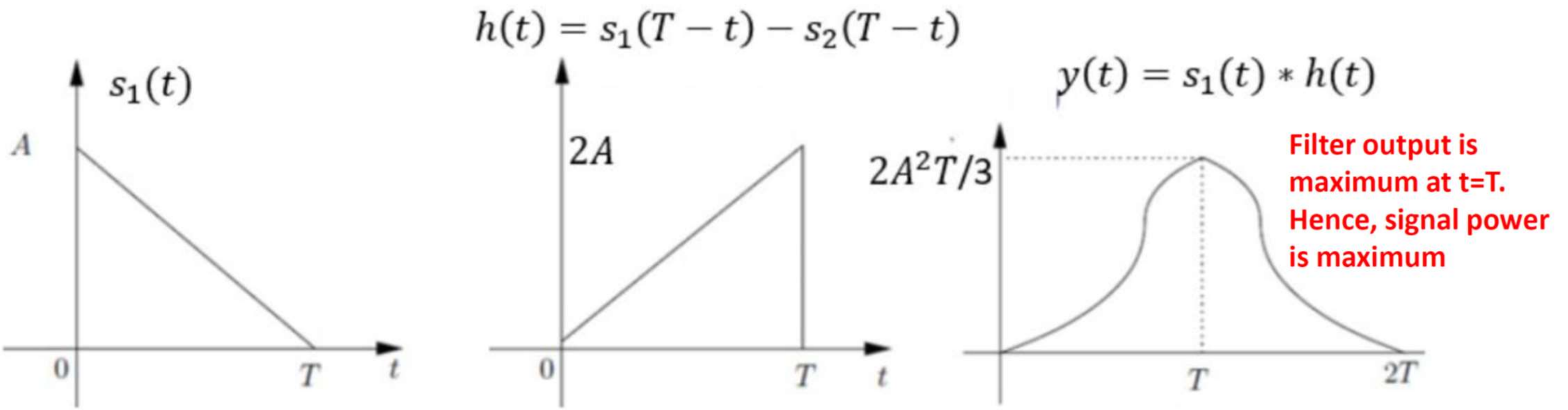
Output of a Matched Filter

Example 2: The next figure shows a signaling scheme where $s_2(t) = -s_1(t)$. The impulse response of the matched filter is $h(t) = s_1(T-t) - s_2(T-t)$. The figure shows the filter output when $s_1(t)$ is applied to the filter. Note that the output attains its maximum value at time t=T, which is the sampling time chosen to maximize the output SNR.



Matched Filter Derived from Signals